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Adaptive control of nonminimum-phase systems using shifted Laurent series

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ABSTRACT
We present a direct discrete-time output-feedback adaptive control algorithm for single-input, single-output systems that are possibly unstable and nonminimum phase. The plant modeling information is given by impulse response components, and the plant is modelled within the algorithm by a truncated shifted Laurent series. A shifted Laurent series is a Laurent series at a point different from the origin in the complex plane and about infinity. The shifted Laurent series is analysed, including its convergence and its relationship to other Laurent series. In particular, we provide a technique for constructing a truncated shifted Laurent series using impulse response components. Numerical examples show that retrospective cost adaptive control can achieve asymptotic command following for a class of exponentially unstable, nonminimum-phase systems.

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Adaptive control; nonminimum-phase zero; unstable system; Laurent series; Markov parameters; impulse response

1. Introduction

Adaptive control is motivated by control applications that are not amenable to reliable modelling prior to operation. Such systems may involve physical processes that are either difficult to model or may be subject to unpredictable changes. Although robust control techniques can be used to account for prior uncertainty, adaptive controllers have the ability to modify the control law in response to the actual plant dynamics, commands, and disturbances. Consequently, an adaptive controller can be viewed as a robust control law that can circumvent the loss of performance due to prior uncertainty.

The benefits of adaptive control must contend with a host of challenges that are well documented in the literature. For example, adaptive control laws are largely confined to systems that are either full-state feedback, positive real, or minimum phase (Anderson et al., 1986; Astrom & Wittenmark, 1994; Ioannou & Sun, 1996; Narendra & Annaswamy, 1989; Tao, 2003). In addition, adaptive control laws may lack robustness to noise and unmodelled dynamics, see Rohrs, Valavani, Athans, and Stein (1985), while others are prone to bursting, see Datta and Ioannou (1994), Hsu and Costa (1987). Nevertheless, the promise of adaptive control in applications with highly uncertain dynamics is evident in the recent literature, see Hovakimyan and Cao (2010).

In the present paper, we revisit retrospective cost adaptive control (RCAC), which was proposed in Venugopal and Bernstein (2000), and subsequently developed in Hoagg, Santillo, and Bernstein (2008), Santillo and Bernstein (2010), Hoagg and Bernstein (2012). As shown in Hoagg and Bernstein (2012), RCAC is applicable to single-input and single-output (SISO) systems with output feedback and nonminimum-phase (NMP) zeros. The multiple-input and multiple-output (MIMO) case is more complicated, see Sumer and Bernstein (2015).

In Hoagg and Bernstein (2012), the required modelling information is given by the first nonzero Markov parameter and knowledge of the NMP zeros of the plant. This information is used implicitly and approximately in Santillo and Bernstein (2010), where the plant modelling information consists of a finite number of Markov parameters (that is, impulse response components) that serve as coefficients of a finite impulse response (FIR) model of the plant.

In particular, the Markov parameters are the coefficients of the truncated Laurent series of the plant at the origin and about infinity (that is, in a punctured plane that excludes the disk centred at the origin whose radius is the spectral radius of the plant), and these coefficients thus implicitly model each zero whose magnitude is greater than the plant spectral radius. Furthermore, if the plant is asymptotically stable, then this Laurent series has two additional features. First, this Laurent series converges uniformly to the plant transfer function in a region that contains the unit circle, which implies that this Laurent series captures the frequency response of the plant. Second, the sequence of coefficients of this Laurent series converges to zero, which implies that the Laurent series

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can be approximated using truncation. Thus, knowledge of a sufficient number of Markov parameters can provide information about the NMP zeros and frequency response of the plant; the former are generally easy to estimate through identification, while the latter may be difficult to estimate.

One drawback of the use of Markov parameters is the fact that, for unstable plants, they fail to capture NMP zeros whose magnitude is less than the plant spectral radius, fail to capture the frequency response of the plant, and are unbounded. This observation suggests that it may be advantageous to work with the coefficients of a Laurent series in a punctured plane that excludes a shifted disk that is not centred at the origin in the complex plane. In some cases, such a disk may exist that excludes all NMP zeros but may include all plant poles, thus rendering the plant 'asymptotically stable' relative to the shifted disk. The challenge, however, is the fact that, although impulse response coefficients may be known or estimated (even for an unstable plant, perhaps by impulsing the system or by means of a suitable system identification method such as OKID in Juang & Phan, 1994), it is not clear how one might go about obtaining estimates of the coefficients of a *shifted* Laurent series, that is, the Laurent series corresponding to a punctured plane that excludes a disk that is not centred at the origin.

The main contribution of the present paper is a technique for obtaining estimates of the coefficients of a shifted Laurent series based on estimates of the Markov parameters. In particular, this paper provides a technique for constructing a model of the plant for use within RCAC that is given by a truncated shifted Laurent series rather than the 'usual' Laurent series whose coefficients are the plant impulse response parameters. It turns out that the model constructed in this manner is an infinite impulse response (IIR) filter rather than an FIR filter as in Santillo and Bernstein (2010), where Markov parameters are used directly. This technique also provides an alternative to the IIR filter construction used in Hoagg and Bernstein (2012).

The numerical examples in this paper show that the IIR filter constructed for RCAC by this technique can provide improved performance relative to the use of an FIR filter. Intuitively speaking, the reason for this improvement appears to be due to the fact that the radius of convergence relative to the excluded disk for a shifted Laurent series may be smaller than the radius of convergence relative to the origin-centred excluded disk. Consequently, the plant may appear to be 'more stable' relative to the former disk.

Much of the technical material in the present paper consists of a detailed development of properties of shifted Laurent series, including convergence properties and the relationship between different Laurent series. Although Laurent series is a classical topic in complex analysis, see Gamelin (2001), the relationship between different Laurent series in different punctured planes does not appear to be considered in the literature, and thus the contribution of the present paper includes a detailed treatment of this material.

The contents of the paper are as follows. Section 2 describes the adaptive control problem, while Section 3 presents the RCAC algorithm. Section 4 reviews Laurent series and considers the boundedness and convergence of its coefficients. Section 5 presents the relationship between the truncated Laurent series and the plant in terms of their NMP zeros, frequency response, and impulse response, as well as the advantages of the shifted Laurent series over the 'usual' Laurent series whose coefficients are the plant impulse response parameters. Section 5 provides a technique for obtaining the truncated shifted Laurent series using Markov parameters, while Section 7 presents the assumption needed to use the truncated shifted Laurent series as the plant model within RCAC. Section 8 provides numerical examples of command following, including performance comparison with Santillo and Bernstein (2010) for asymptotically stable NMP systems, and with Hoagg and Bernstein (2012) for exponentially unstable NMP systems. Technical proofs are relegated to appendices.

2. Problem formulation

Consider the SISO discrete-time system

$$x(k+1) = Ax(k) + Bu(k) + D_1 w(k), \tag{1}$$

$$y(k) = Cx(k) + D_2 w(k), \tag{2}$$

$$z(k) = E_1 x(k) + E_0 w(k), \tag{3}$$

where $x(k) \in \mathbb{R}^{l_x}$ is the state, $y(k) \in \mathbb{R}$ is the output, $u(k) \in \mathbb{R}$ is the input, $w(k) \in \mathbb{R}^{l_w}$ is the exogenous signal, and $z(k) \in \mathbb{R}$ is the performance variable. The components of w can represent either command signals to be followed, disturbances to be rejected, or both, depending on the choice of D_1, D_2, and E_0. In addition, we define the transfer function from u to z as G_{zu}.

The goal is to develop an adaptive output feedback controller that minimises the performance variable z in the presence of the exogenous signal w with limited modelling information about (1)–(3) for possibly unstable and NMP systems. We assume that measurements of the output y and the performance z are available for feedback. However, we assume that a direct measurement of the exogenous signal w is not available. The required modelling information is described in Section 7.

3. Algorithm statement

3.1 Control law

We use a strictly proper time-series controller of order n_c of the form

$$u(k)=\sum_{i=1}^{n_c} M_i(k)u(k-i)+\sum_{i=1}^{n_c} N_i(k)z(k-i), \quad (4)$$

where, for all $i = 1, \ldots, n_c$, $M_i(k) \in \mathbb{R}$ and $N_i(k) \in \mathbb{R}$. The controller (4) can be represented as

$$u(k)=\phi^{\mathrm{T}}(k)\theta(k), \quad (5)$$

where

$$\theta(k) \triangleq [N_1(k) \;\cdots\; N_{n_c}(k)\; M_1(k) \;\cdots\; M_{n_c}(k)]^{\mathrm{T}} \in \mathbb{R}^{2n_c} \quad (6)$$

and

$$\phi(k) \triangleq [z(k-1) \;\cdots\; z(k-n_c)\; u(k-1) \;\cdots\; u(k-n_c)]^{\mathrm{T}} \in \mathbb{R}^{2n_c}. \quad (7)$$

3.2 Retrospective performance

For $\hat{\theta} \in \mathbb{R}^{2n_c}$ and $n_f \geq 1$, we define the *retrospective performance*

$$\hat{z}(\hat{\theta},k) \triangleq \sum_{i=0}^{n_f} a_i z(k-i)+\sum_{i=1}^{n_f} b_i[\phi^{\mathrm{T}}(k-i)\hat{\theta}-u(k-i)], \quad (8)$$

where $a_0 \triangleq 1$ and, for all $i = 1, \ldots, n_f$, $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$. By defining the FIR filters $\alpha(\mathbf{q}) \triangleq \sum_{i=0}^{n_f} a_i\mathbf{q}^{-i}$ and $\beta(\mathbf{q}) \triangleq \sum_{i=1}^{n_f} b_i\mathbf{q}^{-i}$, where $\mathbf{q}$ is the forward shift operator, (8) can be written as

$$\hat{z}(\hat{\theta},k)=\alpha(\mathbf{q})z(k)+\beta(\mathbf{q})[\hat{u}(k)-u(k)], \quad (9)$$

where

$$\hat{u}(k) \triangleq \phi^{\mathrm{T}}(k)\hat{\theta} \quad (10)$$

is the retrospective control. The retrospective performance $\hat{z}(\hat{\theta},k)$ can be interpreted as the performance assuming that $\hat{\theta}$ was used in the past, while $\alpha(\mathbf{q})$ and $\beta(\mathbf{q})$ can be interpreted as a model of G_{zu}. For use below, we define the filtered regressor

$$\Psi(k) \triangleq \beta(\mathbf{q})\phi(k)=\sum_{i=1}^{n_f} b_i\phi(k-i) \in \mathbb{R}^{2n_c}. \quad (11)$$

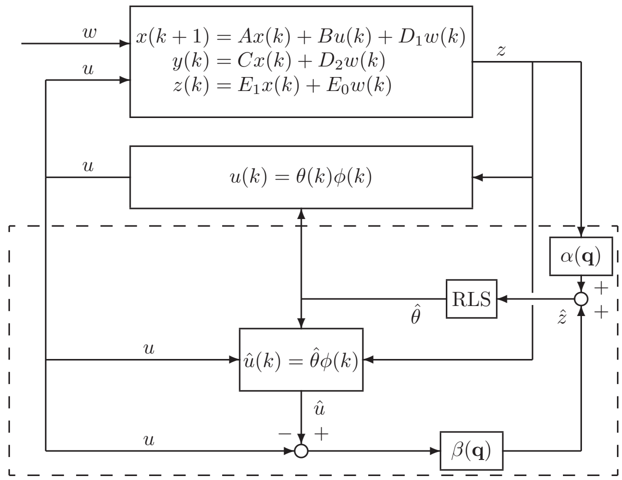

Figure 1. Closed-loop system with RCAC algorithm given by (5), (9), (10), (13), and (14). The adaptation block is marked by the box in dashed-line.

3.3 Cumulative retrospective cost optimisation

The cumulative retrospective cost function is defined by

$$J(\hat{\theta},k) \triangleq \sum_{j=0}^{k} \lambda^{k-i}R_z\hat{z}^2(\hat{\theta},k)+\lambda^k(\hat{\theta}-\theta(0))^{\mathrm{T}}R_\theta(\hat{\theta}-\theta(0)), \quad (12)$$

where $R_z > 0$, $R_\theta \in \mathbb{R}^{2n_c\times 2n_c}$ is positive definite, and $\lambda \in (0, 1]$ is the forgetting factor. The next result follows from standard recursive-least-squares theory (Goodwin & Sin, 1984).

Theorem 3.1: *Let $P(0) = R_\theta^{-1}$ and $\theta(0) \in \mathbb{R}^{2n_c}$. Then, for all $k \geq 0$, the unique global minimiser of (12) is given by $\hat{\theta} = \theta(k)$, where*

$$\theta(k+1)=\theta(k)-\frac{P(k)\Psi(k)\hat{z}(\theta(k),k)}{\lambda R_z^{-1}+\Psi^{\mathrm{T}}(k)P(k)\Psi(k)}, \quad (13)$$

$$P(k+1)=\frac{1}{\lambda}\left[P(k)-\frac{P(k)\Psi(k)\Psi^{\mathrm{T}}(k)P(k)}{\lambda R_z^{-1}+\Psi^{\mathrm{T}}(k)P(k)\Psi(k)}\right]. \quad (14)$$

The closed-loop system with RCAC algorithm given by (5), (9), (10), (13), and (14) is shown in Figure 1. The objective of this paper is to construct $\alpha(\mathbf{q})$ and $\beta(\mathbf{q})$.

4. Review of Laurent series

For $\nu \in \mathbb{R}$, $r_1 \in [0, \infty)$, and $r_2 \in (0, \infty]$, $\mathbb{P}(\nu, r_1) \triangleq \{\mathbf{z} \in \mathbb{C} : |\mathbf{z} - \nu| > r_1\}$ is the open punctured plane centred at $\mathbf{z} = \nu$ with inner radius r_1, $\mathbb{D}(\nu, r_2) \triangleq \{\mathbf{z} \in \mathbb{C} : |\mathbf{z} - \nu| < r_2\}$ is the open disk centred at $\mathbf{z} = \nu$ with outer radius r_2, and $\mathbb{A}(\nu, r_1, r_2) \triangleq \{\mathbf{z} \in \mathbb{C} : r_1 < |\mathbf{z} - \nu| < r_2\}$ is the open annulus centred at $\mathbf{z} = \nu$ with inner radius r_1 and outer radius r_2, where $r_1 < r_2$. In addition, for $\nu \in \mathbb{R}$ and $r \in [0, \infty)$, $\mathbb{S}(\nu, r) \triangleq \{\mathbf{z} \in \mathbb{C} : |\mathbf{z} - \nu| = r\}$ is the circle centred at $\mathbf{z} = \nu$ with radius r. For $\mathcal{S} \subset \mathbb{C}$, $\mathrm{cl}\,\mathcal{S}$ is the closure of $\mathcal{S}$.

Let G be a proper rational function, $\nu \in \mathbb{R}$, and $\rho \in [0, \infty)$. Recall (Gamelin, 2001, p. 168) that, if G is analytic in $\mathbb{P}(\nu, \rho)$, then G has a unique, absolutely convergent Laurent series in $\mathbb{P}(\nu, \rho)$ of the form

$$G(\mathbf{z}) = \sum_{i=0}^{\infty} \frac{L_i}{(\mathbf{z} - \nu)^i}, \tag{15}$$

where, for all $i \geq 0$, $L_i \in \mathbb{R}$. In the case $\nu = 0$, (15) can be written as

$$G(\mathbf{z}) = \sum_{i=0}^{\infty} \frac{H_i}{\mathbf{z}^i}, \tag{16}$$

where $H_0, H_1, H_2, \ldots$ is the impulse response of the linear discrete-time system whose transfer function is G.

Definition 4.1: Let G be a proper rational function, and let $\nu \in \mathbb{R}$. Then, the *spectral radius of G relative to* ν is defined as

$$\rho(G, \nu) \triangleq \max\{|\mathbf{z} - \nu| : \mathbf{z} \text{ is a pole of } G\},$$

and the *inner spectral radius of G relative to* ν is defined as

$$\rho_{\mathrm{i}}(G, \nu) \triangleq \min\{|\mathbf{z} - \nu| : \mathbf{z} \text{ is a pole of } G\}.$$

Note that $\mathbb{P}(\nu, \rho(G, \nu))$ is the largest open punctured plane centred at ν within which G is analytic, while $\mathbb{D}(\nu, \rho_{\mathrm{i}}(G, \nu))$ is the largest open disk centred at ν within which G is analytic. Note that $\rho_{\mathrm{i}}(G, \nu) \leq \rho(G, \nu)$ and $\rho(G, 0)$ is the spectral radius of G.

Proposition 4.1: *Let G be a proper rational function, and let $\nu \in \mathbb{R}$. Consider the Laurent series of G in $\mathbb{P}(\nu, \rho(G, \nu))$ given by (15). Then the following statements hold:*

(1) $\rho(G, \nu) = \limsup_{i\to\infty} |L_i|^{1/i}$.
(2) $\rho(G, \nu) \in [0, 1)$ *if and only if* $\lim_{i\to\infty} L_i = 0$.
(3) $\rho(G, \nu) = 1$ *and G has no repeated poles in* $\mathbb{S}(\nu, 1)$ *if and only if* $0 < \limsup_{i\to\infty} |L_i| < \infty$.
(4) *Either* $\rho(G, \nu) \in (1, \infty)$ *or both* $\rho(G, \nu) = 1$ *and G has at least one repeated pole in* $\mathbb{S}(\nu, 1)$ *if and only if* $\limsup_{i\to\infty} |L_i| = \infty$.

Proof: See Appendix 2. ■

Example 4.1: Consider the Lyapunov-stable transfer function $G(\mathbf{z}) = 1/(\mathbf{z}^2 + 1)$. Then $\rho(G, 0) = 1$, $H_0 = 0$, and, for all $i \geq 1$,

$$H_i = \begin{cases} 0, & i \text{ odd}, \\ (-1)^{(i-2)/2}, & i \text{ even}. \end{cases}$$

Thus $\lim_{i\to\infty}|H_i|$ does not exist. Hence 'lim sup' in 3) of Proposition 4.1 cannot be replaced by 'lim'. Furthermore, $\lim_{i\to\infty}|H_i|^{1/i}$ does not exist. Hence 'lim sup' in 1) of Proposition 4.1 cannot be replaced by 'lim'.

Example 4.2: Consider the exponentially unstable transfer function $G(\mathbf{z}) = 1/(\mathbf{z}^2 + 4)$. Then $\rho(G, 0) = 2 > 1$, $H_0 = 0$, and, for all $i \geq 1$,

$$H_i = \begin{cases} 0, & i \text{ odd}, \\ (-1)^{(i-2)/2} 2^{i-2}, & i \text{ even}. \end{cases}$$

Thus $\lim_{i\to\infty}|H_i|$ does not exist. Hence 'lim sup' in 4) of Proposition 4.1 cannot be replaced by 'lim'.

Example 4.3: Consider the unstable transfer function $G(\mathbf{z}) = 1/(\mathbf{z} - 3/2)$. For $\nu = 0$, $\rho(G, 0) = 3/2 > 1$, $H_0 = 0$, and, for all $i \geq 1$, $H_i = (3/2)^{i-1}$. Thus, $\limsup_{i\to\infty} |H_i| = \infty$, in accordance with 4) of Proposition 4.1. For $\nu = 1/2$, $\rho(G, 1/2) = 1$, $L_0 = 0$, and, for all $i \geq 1$, $L_i = 1$. Thus, $0 < \limsup_{i\to\infty} |L_i| < \infty$, in accordance with 3) of Proposition 4.1. For $\nu = 1$, $\rho(G, 1) = 1/2 < 1$, $L_0 = 0$, and, for all $i \geq 1$, $L_i = (1/2)^{i-1}$. Thus, $\lim_{i\to\infty} L_i = 0$, in accordance with 2) of Proposition 4.1. Note that $\rho(G, \nu) = \limsup_{i\to\infty} |L_i|^{1/i}$ holds in all three cases, in accordance with 1) of Proposition 4.1.

5. Analysis of the truncated Laurent series

Let G be a proper rational function, let $\nu \in \mathbb{R}$, and consider the Laurent series (15) of G in $\mathbb{P}(\nu, \rho(G, \nu))$. For all $\mathbf{z} \in \{\mathbf{z} \in \mathbb{C} : \mathbf{z} \neq \nu\}$ and $n \geq 1$, we define the truncated Laurent series

$$G_{\nu,n}(\mathbf{z}) \triangleq \mathcal{L}_{\nu,n}\{G(\mathbf{z})\} \triangleq \sum_{i=0}^{n} \frac{L_i}{(\mathbf{z} - \nu)^i}. \tag{17}$$

Note that the proper rational function $G_{\nu,n}$ is the transfer function of an IIR filter in the case $\nu \neq 0$ and an FIR filter in the case $\nu = 0$. For all $z \in \mathbb{P}(\nu, \rho(G, \nu))$,

$$\lim_{n\to\infty} G_{\nu,n}(\mathbf{z}) = G(\mathbf{z}), \tag{18}$$

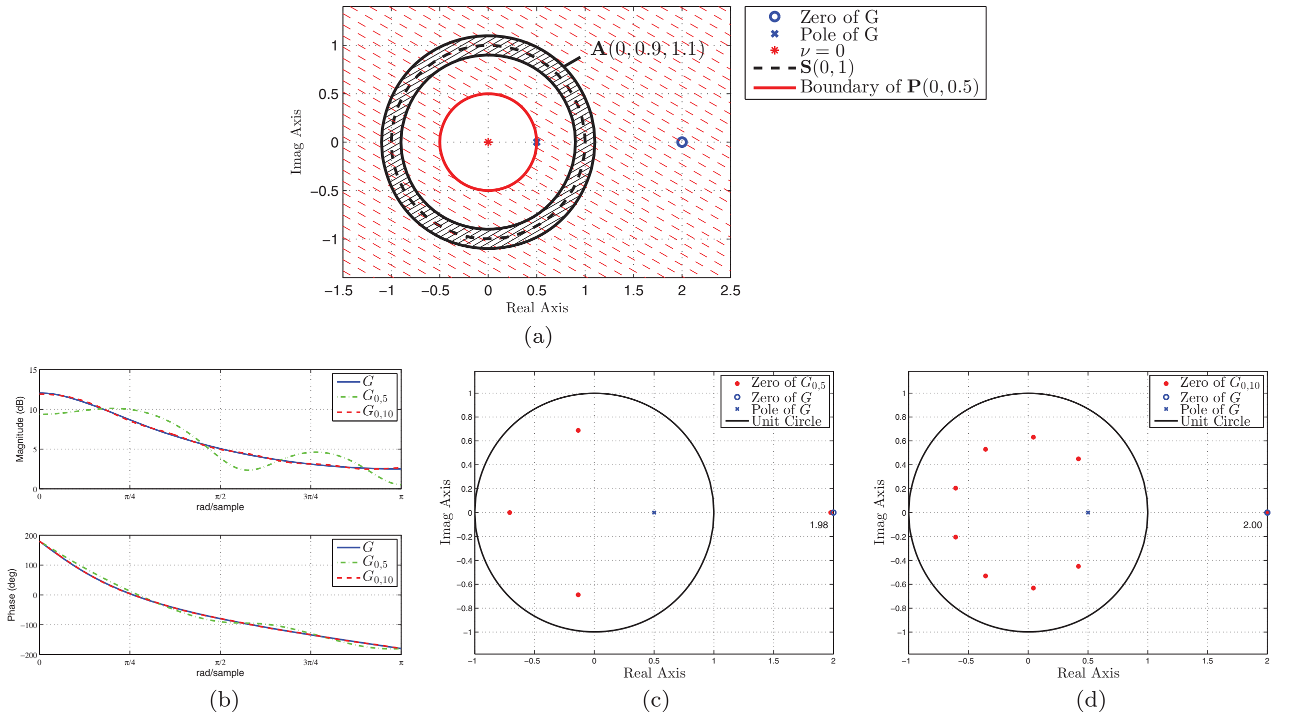

Figure 2. Example 5.1 with G given by (19) and $\nu = 0$. Since $\mathbb{S}(0, 1) \subset \mathrm{cl}\,\mathbb{A}(0, 0.9, 1.1) \subset \mathbb{P}(0, 0.5)$, as shown in (a), $\theta \mapsto G_{0,n}(e^{j\theta})$ converges uniformly to $\theta \mapsto G(e^{j\theta})$ on $(-\pi, \pi]$ as $n \to \infty$, as shown in (b). Furthermore, for large n, $G_{0,n}$ has a zero that is close to the NMP zero 2 of G. In particular, $G_{0,5}$ has a zero at 1.98, as shown in (c), and $G_{0,10}$ has a zero close to 2.00, as shown in (d).

and $G_{\nu,n}(\mathbf{z})$ converges absolutely as $n \to \infty$. Furthermore, for $\rho_1, \rho_2 \in \mathbb{R}$ such that $\rho(G, \nu) < \rho_1 < \rho_2$, $G_{\nu,n}$ converges uniformly to G on $\mathrm{cl}\,\mathbb{A}(\nu, \rho_1, \rho_2)$ as $n \to \infty$. In addition, for all $\mathbf{z} \notin \mathrm{cl}\,\mathbb{P}(\nu, \rho(G, \nu))$, $\lim_{n\to\infty} G_{\nu,n}(\mathbf{z})$ does not exist, where a limit exists if and only if it is a real or complex number.

The next example illustrates the fact that, if G is asymptotically stable, then, for large n, $G_{0,n}$ approximates G uniformly on the unit circle, that is, the frequency response of $G_{0,n}$ uniformly approximates the frequency response of G. Furthermore, in $\mathbb{P}(0, \rho(G, 0))$, the zeros of $G_{0,n}$ approximate the zeros of G.

Example 5.1: Consider the asymptotically stable, NMP transfer function

$$G(\mathbf{z}) = \frac{\mathbf{z} - 2}{(\mathbf{z} - 0.5)^2}. \tag{19}$$

Since $\mathbb{S}(0, 1) \subset \mathrm{cl}\,\mathbb{A}(0, 0.9, 1.1) \subset \mathbb{P}(0, 0.5)$, as shown in Figure 2(a), the Laurent series of G in $\mathbb{P}(0, 0.5)$ converges uniformly to G on the unit circle. Therefore, $\theta \mapsto G_{0,n}(e^{j\theta})$ converges uniformly to $\theta \mapsto G(e^{j\theta})$ on $(-\pi, \pi]$ as $n \to \infty$, as shown in Figure 2(b). Furthermore, for large n, $G_{0,n}$ has a zero that is close to the NMP zero of G. Figure 2(c) and 2(d) shows the poles and zeros of G and $G_{0,n}$ for $n = 5$ and $n = 10$, respectively.

The next example shows that, if G is exponentially unstable, then, for all $\mathbf{z} \in \mathbb{S}(0, 1)$, $G_{0,n}(\mathbf{z})$ does not approximate $G(\mathbf{z})$. However, for a properly chosen $\nu \neq 0$, $G_{\nu,n}$ can approximate G uniformly on a portion of the unit circle. Thus, on a subset of $\mathbb{S}(0, 1)$, $G_{\nu,n}$ can provide a better approximation of G than $G_{0,n}$.

Example 5.2: Consider the exponentially unstable, NMP transfer function

$$G(\mathbf{z}) = \frac{\mathbf{z} - 2}{(\mathbf{z} - 1.2)(\mathbf{z} - 0.5)}. \tag{20}$$

Since G has a pole outside the closed unit disk, it follows that $\mathbb{P}(0, \rho(G, 0)) \cap \mathbb{S}(0, 1) = \varnothing$, where $\rho(G, 0) = 1.2$, as shown in Figure 3(a). Hence, for all $\mathbf{z} \in \mathbb{S}(0, 1)$, the Laurent series of G in $\mathbb{P}(0, \rho(G, 0))$ does not converge at $\mathbf{z}$. Therefore, for all $\theta \in (-\pi, \pi]$, $G_{0,n}(e^{j\theta})$ does not converge to $G(e^{j\theta})$ as $n \to \infty$, as shown in Figure 3(b). However, as in Example 5.1, for large n, $G_{0,n}$ has a zero that is close to the NMP zero of G. Note that (19) is asymptotically stable, whereas (20) is exponentially unstable. Figure 3(c) and 3(d) shows the poles and zeros of G and $G_{0,n}$ for $n = 5$ and $n = 10$, respectively.

Next, note that $\mathbb{S}(0, 1) \cap \mathbb{P}(0.8, \rho(G, 0.8)) \neq \varnothing$, where $\rho(G, 0.8) = 0.4$. In fact, $\{e^{j\theta} : \theta \in (-\pi, -0.14\pi] \cup [0.14\pi, \pi]\} \subset \mathbb{P}(0.8, 0.4)$, as shown in Figure 4(a). Thus, $\theta \mapsto G_{0.8,n}(e^{j\theta})$ converges uniformly to $\theta \mapsto G(e^{j\theta})$ on $(-\pi,$

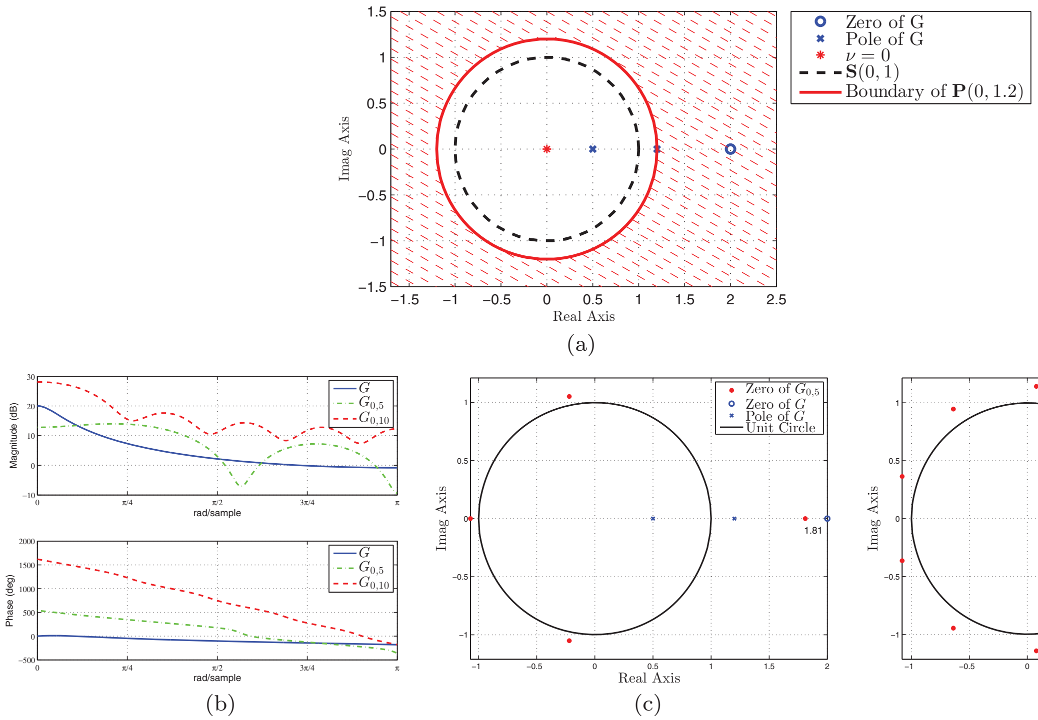

Figure 3. Example 5.2 with G given by (20) and $\nu = 0$. Since $\mathbb{S}(0,1) \cap \mathbb{P}(0,1.2) = \varnothing$, as shown in (a), for all $\theta \in (-\pi, \pi]$, $G_{0,n}(e^{j\theta})$ does not converge to $G(e^{j\theta})$ as $n \to \infty$, as shown in (b). However, for large n, $G_{0,n}$ has a zero that is close to the NMP zero 2 of G. In particular, $G_{0,5}$ has a zero at 1.81, as shown in (c), and $G_{0,10}$ has a zero at 1.99, as shown in (d).

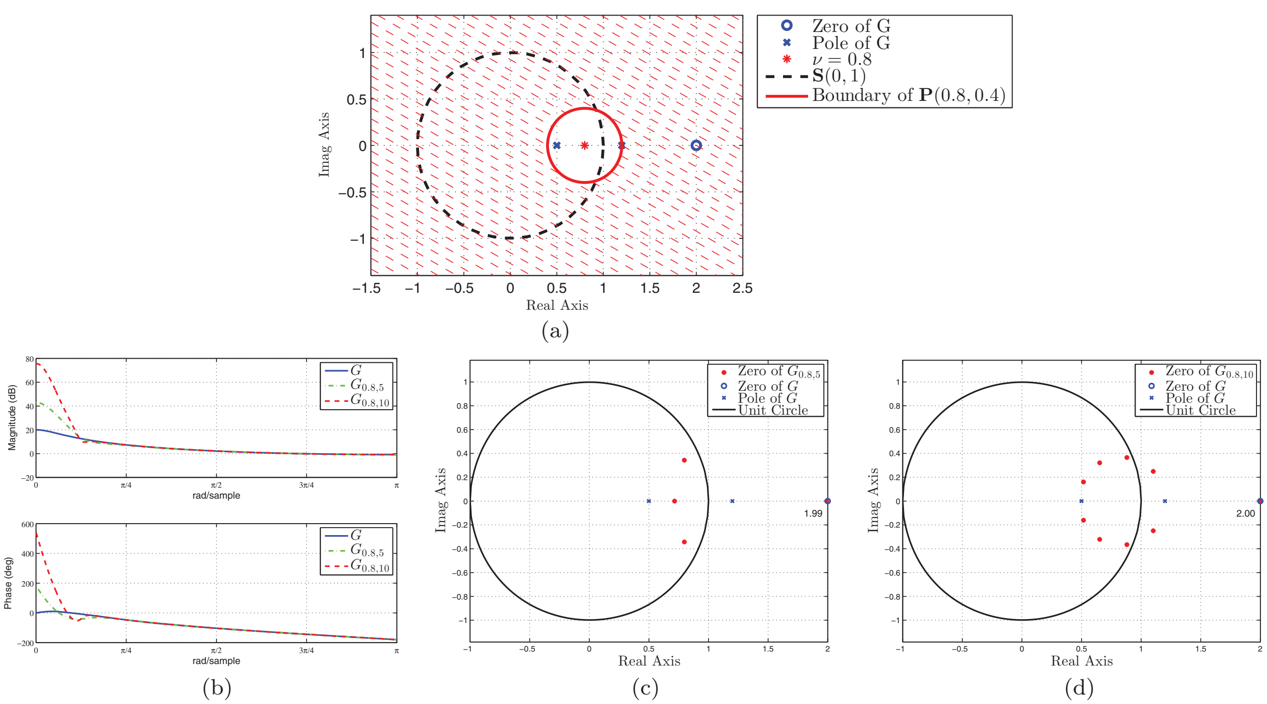

Figure 4. Example 5.2 with G given by (20) and $\nu = 0.8$. Note that $\mathbb{S}(0,1) \cap \mathbb{P}(0.8, 0.4) \neq \varnothing$. In fact, $\{e^{j\theta} : \theta \in (-\pi, -0.14\pi] \cup [0.14\pi, \pi]\} \subset \mathbb{P}(0.8, 0.4)$, as shown in (a). Thus, $\theta \mapsto G_{0.8,n}(e^{j\theta})$ converges uniformly to $\theta \mapsto G(e^{j\theta})$ on $(-\pi, -0.14\pi] \cup [0.14\pi, \pi]$ as $n \to \infty$, as shown in (b). Furthermore, for large n, $G_{0.8,n}$ has a zero that is close to the NMP zero 2 of G. In particular, $G_{0.8,5}$ has a zero at 1.99, as shown in (c), and $G_{0.8,10}$ has a zero close to 2.00, as shown in (d).

$-0.14\pi]\cup[0.14\pi, \pi]$ as $n\to\infty$, as shown in Figure 4(b). Furthermore, for large n, $G_{0.8,\,n}$ also has a zero that is close to the NMP zero of G as in Example 5.1. Figure 4(c) and 4(d) shows the poles and zeros of G and $G_{0.8,\,n}$ for $n=5$ and $n=10$, respectively.

Figures 3(b) and 4(b) show that, for $\theta\in[0.14\pi, \pi]$, $G_{0.8,\,5}(e^{j\theta})$ is closer to $G(e^{j\theta})$ than $G_{0,\,5}(e^{j\theta})$.

For a proper rational function G, let $\mathrm{rd}(G)$ denote the relative degree of G. The following result is immediate.

Lemma 5.1: *Let G be a proper rational function, $\nu\in\mathbb{R}$, and $\rho\in[0,\infty)$, assume that G is analytic in $\mathbb{P}(\nu,\rho)$, let d be a positive integer, and consider the coefficients $\{L_i\}_{i=0}^{\infty}$ of the Laurent series (15) in $\mathbb{P}(\nu,\rho)$. Then, $rd(G)=d$ if and only if $L_0=L_1=\cdots=L_{d-1}=0$ and $L_d\neq 0$. Thus, if $1\le n\le rd(G)-1$, then $G_{\nu,\,n}=0$. Furthermore, if $n\ge rd(G)$, then $rd(G_{\nu,\,n})=rd(G)$.*

Proposition 5.1: *Let G be a proper rational function, $\nu_1,\nu_2\in\mathbb{R}$, and $n\ge m\ge 1$. Then $\mathcal{L}_{\nu_1,m}\{G_{\nu_2,n}\}=\mathcal{L}_{\nu_1,m}\{G\}$.*

Proof: In the case $m\le n\le \mathrm{rd}(G)-1$, Lemma 5.1 implies that $G_{\nu_2,n}=0$ and thus, for all $\mathbf{z}\neq\nu_1$, $\mathcal{L}_{\nu_1,m}\{G_{\nu_2,n}(\mathbf{z})\}=\mathcal{L}_{\nu_1,m}\{G(\mathbf{z})\}=0$. In the case $m+1\le\mathrm{rd}(G)\le n$, Lemma 5.1 implies that $m+1\le\mathrm{rd}(G)=\mathrm{rd}(G_{\nu_2,n})$ and thus, for all $\mathbf{z}\neq\nu_1$, $\mathcal{L}_{\nu_1,m}\{G_{\nu_2,n}(\mathbf{z})\}=\mathcal{L}_{\nu_1,m}\{G(\mathbf{z})\}=0$. Next consider the case $\mathrm{rd}(G)\le m\le n$. Then there exists $\{L_i\}_{i=1}^{\infty}$ such that, for all $\mathbf{z}\in\mathbb{P}(\nu_2,\rho(G,\nu_2))$,

$$G(\mathbf{z})=\sum_{i=0}^{\infty}\frac{L_i}{(\mathbf{z}-\nu_2)^i}=G_{\nu_2,n}(\mathbf{z})+\sum_{i=n+1}^{\infty}\frac{L_i}{(\mathbf{z}-\nu_2)^i}.$$

Since $\sum_{i=n+1}^{\infty}\frac{L_i}{(\mathbf{z}-\nu_2)^i}$ is the Laurent series of the rational function $G-G_{\nu_2,n}$ in $\mathbb{P}(\nu_2,\rho(G,\nu_2))$, Lemma 5.1 implies that $\mathrm{rd}(G-G_{\nu_2,n})\ge n+1\ge m+1$. Note that G is analytic in $\mathbb{P}(\nu_1,\rho(G,\nu_1))$. Furthermore, all of the poles of $G_{\nu_2,n}$ are ν_2 and thus $G_{\nu_2,n}$ is analytic in $\mathbb{P}(\nu_1,|\nu_2-\nu_1|)$. Thus $G-G_{\nu_2,n}$ is analytic in $\mathbb{P}(\nu_1,\rho(G,\nu_1))\cap\mathbb{P}(\nu_1,|\nu_2-\nu_1|)$. Since $m\le\mathrm{rd}(G-G_{\nu_2,n})-1$, Lemma 5.1 implies that, for all $\mathbf{z}\in\mathbb{P}(\nu_2,\rho(G,\nu_2))\cap\mathbb{P}(\nu_1,\rho(G,\nu_1))\cap\mathbb{P}(\nu_1,|\nu_2-\nu_1|)$, $\mathcal{L}_{\nu_1,m}\left\{G(\mathbf{z})-G_{\nu_2,n}(\mathbf{z})\right\}=0$. Therefore, for all $\mathbf{z}\in\mathbb{P}(\nu_2,\rho(G,\nu_2))\cap\mathbb{P}(\nu_1,\rho(G,\nu_1))\cap\mathbb{P}(\nu_1,|\nu_2-\nu_1|)$,

$$\begin{aligned}\mathcal{L}_{\nu_1,m}\{G(\mathbf{z})\}&=\mathcal{L}_{\nu_1,m}\{G_{\nu_2,n}(\mathbf{z})\}+\mathcal{L}_{\nu_1,m}\left\{G(\mathbf{z})-G_{\nu_2,n}(\mathbf{z})\right\}\\&=\mathcal{L}_{\nu_1,m}\{G_{\nu_2,n}(\mathbf{z})\}.\end{aligned}$$

Since $\mathbb{P}(\nu_2,\rho(G,\nu_2))\cap\mathbb{P}(\nu_1,\rho(G,\nu_1))\cap\mathbb{P}(\nu_1,|\nu_2-\nu_1|)$ is a nonempty open set, it follows that the rational functions $\mathcal{L}_{\nu_1,m}\{G_{\nu_2,n}\}$ and $\mathcal{L}_{\nu_1,m}\{G\}$ are equal. ■

Let G be a proper rational function, $\nu\in\mathbb{R}$, and $n\ge 1$. Then, Proposition 5.1 with $\nu_1=0$, $\nu_2=\nu$, and $m=n$ implies that $\mathcal{L}_{0,n}\{G_{\nu,n}\}=\mathcal{L}_{0,n}\{G\}$. Therefore, for all $\nu\in\mathbb{R}$, the first n components of the impulse responses of $G_{\nu,\,n}$ and G are the same, as illustrated by the next example.

Example 5.3: The Laurent series of $G(\mathbf{z})=1/(\mathbf{z}-2)$ in $\mathbb{P}(1,1)$ is given by

$$G(\mathbf{z})=\sum_{i=1}^{\infty}\frac{1}{(\mathbf{z}-1)^i}.$$

Thus, $G_{1,3}(\mathbf{z})=\sum_{i=1}^{3}\frac{1}{(\mathbf{z}-1)^i}$, and the Laurent series of $G_{1,\,3}$ in $\mathbb{P}(0,1)$ is given by

$$\begin{aligned}G_{1,3}(\mathbf{z})&=\sum_{i=1}^{\infty}\frac{1}{\mathbf{z}^i}+\sum_{i=2}^{\infty}\frac{i-1}{\mathbf{z}^i}+\sum_{i=3}^{\infty}\frac{(i-1)(i-2)}{2\mathbf{z}^i}\\&=\frac{1}{\mathbf{z}}+\frac{2}{\mathbf{z}^2}+\frac{4}{\mathbf{z}^3}+\frac{7}{\mathbf{z}^4}+\cdots.\end{aligned}$$

Therefore, $\mathcal{L}_{0,3}\{G_{1,3}(\mathbf{z})\}=1/\mathbf{z}+2/\mathbf{z}^2+4/\mathbf{z}^3$. Furthermore, the Laurent series of G in $\mathbb{P}(0,2)$ is given by

$$G(\mathbf{z})=\sum_{i=1}^{\infty}\frac{2^{i-1}}{\mathbf{z}^i}=\frac{1}{\mathbf{z}}+\frac{2}{\mathbf{z}^2}+\frac{4}{\mathbf{z}^3}+\frac{8}{\mathbf{z}^4}+\cdots,$$

and thus $\mathcal{L}_{0,3}\{G(\mathbf{z})\}=1/\mathbf{z}+2/\mathbf{z}^2+4/\mathbf{z}^3$. Hence, in accordance with Proposition 5.1, $\mathcal{L}_{0,3}\{G_{1,3}\}=\mathcal{L}_{0,3}\{G\}$, which shows that the first three components of the impulse responses of $G_{1,\,3}$ and G are the same.

6. Filter construction

In this section, we present a method for constructing the FIR filters α and β in (9) such that $\beta/\alpha=G_{\nu,\,n}$. Theorem 6.1 shows that α and β can be constructed using ν and the first n Markov parameters (i.e. $\{H_i\}_{i=0}^{n}$ in (16)) of G, which can be obtained from system identification methods. This technique thus circumvents the need to determine the coefficients $\{L_i\}_{i=0}^{n}$ of the Laurent expansion of G in $\mathbb{P}(\nu,\rho(G,\nu))$ in (15), for which there is no known identification method.

Theorem 6.1: *Let G be a proper rational function, $\nu\in\mathbb{R}$, and $n\ge 1$, and define the rational functions*

$$\alpha(\mathbf{z})\triangleq\frac{(\mathbf{z}-\nu)^n}{\mathbf{z}^n}\tag{21}$$

and

$$\beta\triangleq\mathcal{L}_{0,n}\{\alpha G_{0,n}\}.\tag{22}$$

Then

$$\beta=\alpha G_{\nu,n}=\mathcal{L}_{0,n}\{\alpha G\}.\tag{23}$$

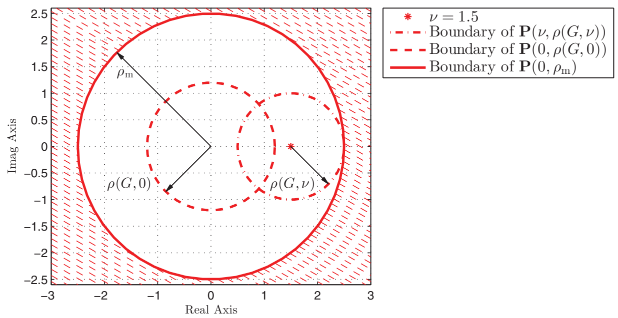

Figure 5. The shaded region is $\mathbb{P}(0, \rho_m)$, where $\nu = 1.5$, $\rho(G, 0) = 1.2$, $\rho(G, \nu) = 1$, and $\rho_m = \max\{\rho(G, 0), |\nu| + \rho(G, \nu)\} = \max\{1.2, 1.5 + 1\} = 2.5$.

Proof: Using (16), there exist $H_0', \ldots, H_{2n}' \in \mathbb{R}$ such that, for all $\mathbf{z} \neq 0$,

$$\alpha(\mathbf{z})G_{0,n}(\mathbf{z}) = \frac{(\mathbf{z}-\nu)^n}{\mathbf{z}^n}\sum_{i=0}^{n}\frac{H_i}{\mathbf{z}^i} = \sum_{i=0}^{2n}\frac{H_i'}{\mathbf{z}^i}, \tag{24}$$

and thus

$$\mathcal{L}_{0,n}\{\alpha(\mathbf{z})G_{0,n}(\mathbf{z})\} = \sum_{i=0}^{n}\frac{H_i'}{\mathbf{z}^i}. \tag{25}$$

On the other hand, (15) implies that there exist $H_0'', \ldots, H_n''$ such that, for all $\mathbf{z} \in \{\mathbf{z} \in \mathbb{C} : \mathbf{z} \neq 0, \mathbf{z} \neq \nu\}$,

$$\begin{aligned}\alpha(\mathbf{z})G_{\nu,n}(\mathbf{z}) &= \frac{(\mathbf{z}-\nu)^n}{\mathbf{z}^n}\sum_{i=0}^{n}\frac{L_i}{(\mathbf{z}-\nu)^i}\\ &= \sum_{i=0}^{n}\frac{L_i(\mathbf{z}-\nu)^{n-i}}{\mathbf{z}^n} = \sum_{i=0}^{n}\frac{H_i''}{\mathbf{z}^i}.\end{aligned} \tag{26}$$

Define $\rho_m \overset{\triangle}{=} \max\{\rho(G, 0), |\nu| + \rho(G, \nu)\} \in \mathbb{R}$. Then $\mathbb{P}(0, \rho_m) \subseteq \mathbb{P}(0, \rho(G, 0))$ and $\mathbb{P}(0, \rho_m) \subseteq \mathbb{P}(\nu, \rho(G, \nu))$, as shown in Figure 5. Note that $0 \notin \mathbb{P}(0, \rho_m)$ and $\nu \notin \mathbb{P}(0, \rho_m)$. Therefore, for all $\mathbf{z} \in \mathbb{P}(0, \rho_m)$, (15), (16), and (24)–(26) hold.

Using (15), (16), (24), and (26) yields, for all $\mathbf{z} \in \mathbb{P}(0, \rho_m)$,

$$\begin{aligned}\alpha(\mathbf{z})G(\mathbf{z}) &= \sum_{i=0}^{n}\frac{H_i''}{\mathbf{z}^i} + f_\nu(\mathbf{z})\\ &= \sum_{i=0}^{n}\frac{H_i'}{\mathbf{z}^i} + \sum_{i=n+1}^{2n}\frac{H_i'}{\mathbf{z}^i} + f_0(\mathbf{z}),\end{aligned} \tag{27}$$

where the rational functions f_ν and f_0 are defined by

$$f_\nu(\mathbf{z}) \overset{\triangle}{=} \alpha(\mathbf{z})(G(\mathbf{z}) - G_{\nu,n}(\mathbf{z})) = \sum_{i=n+1}^{\infty}\frac{L_i}{\mathbf{z}^n(\mathbf{z}-\nu)^{i-n}},$$

$$f_0(\mathbf{z}) \overset{\triangle}{=} \alpha(\mathbf{z})(G(\mathbf{z}) - G_{0,n}(\mathbf{z})) = \sum_{i=n+1}^{\infty}\frac{H_i(\mathbf{z}-\nu)^n}{\mathbf{z}^{i+n}},$$

which are analytic in $\mathbb{P}(0, \rho_m)$. Note that $\mathrm{rd}(f_\nu) \geq n + 1$ and $\mathrm{rd}(f_0) \geq n + 1$, and thus, by Lemma 5.1, $\mathcal{L}_{0,n}\{f_\nu\} = \mathcal{L}_{0,n}\{f_0\} = 0$. Therefore, it follows from (27) that, for all $\mathbf{z} \in \mathbb{P}(0, \rho_m)$,

$$\sum_{i=0}^{n}\frac{H_i''}{\mathbf{z}^i} = \mathcal{L}_{0,n}\left\{\sum_{i=0}^{n}\frac{H_i''}{\mathbf{z}^i}\right\} = \mathcal{L}_{0,n}\left\{\sum_{i=0}^{n}\frac{H_i'}{\mathbf{z}^i}\right\} = \sum_{i=0}^{n}\frac{H_i'}{\mathbf{z}^i}. \tag{28}$$

Thus, (22), (25), (26), and (28) imply that, for all $\mathbf{z} \in \mathbb{P}(0, \rho_m)$,

$$\begin{aligned}\beta(\mathbf{z}) &= \mathcal{L}_{0,n}\{\alpha(\mathbf{z})G_{0,n}(\mathbf{z})\}\\ &= \sum_{i=0}^{n}\frac{H_i'}{\mathbf{z}^i} = \sum_{i=0}^{n}\frac{H_i''}{\mathbf{z}^i} = \alpha(\mathbf{z})G_{\nu,n}(\mathbf{z}).\end{aligned}$$

Furthermore, $\mathcal{L}_{0,n}\{f_0\} = 0$ implies that, for all $\mathbf{z} \in \mathbb{P}(0, \rho_m)$,

$$\begin{aligned}\mathcal{L}_{0,n}\{\alpha(\mathbf{z})G_{0,n}(\mathbf{z})\} &= \mathcal{L}_{0,n}\{\alpha(\mathbf{z})G(\mathbf{z})\} - \mathcal{L}_{0,n}\{f_0(\mathbf{z})\}\\ &= \mathcal{L}_{0,n}\{\alpha(\mathbf{z})G(\mathbf{z})\}.\end{aligned}$$

Since $\mathbb{P}(0, \rho_m)$ is a nonempty open set, it follows that the rational functions β, $\alpha G_{\nu,n}$, and $\mathcal{L}_{0,n}\{\alpha G\}$ are equal. ■

The next example illustrates Theorem 6.1.

Example 6.1: We reconsider Example 5.3, where $G(\mathbf{z}) = 1/(\mathbf{z} - 2)$, $G_{1,3}(\mathbf{z}) = \sum_{i=1}^{3}\frac{1}{(\mathbf{z}-1)^i}$, and $G_{0,3}(\mathbf{z}) = 1/\mathbf{z} + 2/\mathbf{z}^2 + 4/\mathbf{z}^3$. Let $n = 3$ and $\nu = 1$ so that $\alpha(\mathbf{z}) = (\mathbf{z} - 1)^3/\mathbf{z}^3$. Thus, for all $\mathbf{z} \neq 0$,

$$\begin{aligned}\beta(\mathbf{z}) &= \mathcal{L}_{0,3}\left\{\alpha(\mathbf{z})G_{0,3}(\mathbf{z})\right\}\\ &= \mathcal{L}_{0,3}\left\{\frac{(\mathbf{z}-1)^3}{\mathbf{z}^3}\left[\frac{1}{\mathbf{z}} + \frac{2}{\mathbf{z}^2} + \frac{4}{\mathbf{z}^3}\right]\right\}\\ &= \mathcal{L}_{0,3}\left\{\frac{1}{\mathbf{z}} + \frac{-1}{\mathbf{z}^2} + \frac{1}{\mathbf{z}^3} + \frac{-7}{\mathbf{z}^4} + \frac{10}{\mathbf{z}^5} + \frac{-4}{\mathbf{z}^6}\right\}\\ &= \frac{1}{\mathbf{z}} + \frac{-1}{\mathbf{z}^2} + \frac{1}{\mathbf{z}^3}.\end{aligned}$$

Furthermore, for all $\mathbf{z} \in \{\mathbf{z} \in \mathbb{C} : \mathbf{z} \neq 0, \mathbf{z} \neq 1\}$,

$$\alpha(\mathbf{z})G_{1,3}(\mathbf{z}) = \frac{(\mathbf{z}-1)^3}{\mathbf{z}^3}\left[\frac{1}{\mathbf{z}-1} + \frac{1}{(\mathbf{z}-1)^2} + \frac{1}{(\mathbf{z}-1)^3}\right] = \frac{1}{\mathbf{z}} + \frac{-1}{\mathbf{z}^2} + \frac{1}{\mathbf{z}^3}.$$

Moreover, for all $\mathbf{z} \in \mathbb{P}(0, \rho(\alpha G, 0))$,

$$\begin{aligned}\mathcal{L}_{0,3}&\{\alpha(\mathbf{z})G(\mathbf{z})\}\\ &= \mathcal{L}_{0,3}\left\{\frac{(\mathbf{z}-1)^3}{\mathbf{z}^3(\mathbf{z}-2)}\right\}\\ &= \mathcal{L}_{0,3}\left\{\frac{1}{\mathbf{z}} + \frac{-1}{\mathbf{z}^2} + \frac{1}{\mathbf{z}^3} + \frac{1}{\mathbf{z}^4} + \frac{2}{\mathbf{z}^5} + \frac{4}{\mathbf{z}^6} + \cdots\right\}\\ &= \frac{1}{\mathbf{z}} + \frac{-1}{\mathbf{z}^2} + \frac{1}{\mathbf{z}^3},\end{aligned}$$

where $\rho(\alpha G, 0) = 2$. Since $0 \notin \mathbb{P}(0, 2)$ and $1 \notin \mathbb{P}(0, 2)$, it follows that, for all $\mathbf{z} \in \mathbb{P}(0, 2)$, $\beta(\mathbf{z}) = \alpha(\mathbf{z})G_{1,3}(\mathbf{z}) = \mathcal{L}_{0,3}\{\alpha(\mathbf{z})G(\mathbf{z})\}$. Since $\mathbb{P}(0, 2)$ is a nonempty open set, it follows that $\beta = \alpha G_{1,3} = \mathcal{L}_{0,3}\{\alpha G\}$, in accordance with (23).

It follows from (23) that, for all $\nu \in \mathbb{R}$ and $n \geq 1$, $G_{\nu,n} = \beta/\alpha$. Note that α and β are not constructed using the transfer function G. Instead, α is constructed using ν, and β is constructed using α and $H_0, H_1, \ldots, H_n$ of G. By viewing α and β as FIR filters, β can be constructed by using the α-filtered impulse response of G. In practice, $H_0, H_1, \ldots, H_n$ can be obtained either by impulsing the plant from zero initial conditions or from system identification. Guidelines for choosing ν are given in the next section.

7. Application of shifted Laurent series to RCAC

In this section, we apply Theorem 6.1 to the RCAC algorithm given by (4), (9), (13), and (14). The filters α and β in (9) are chosen as in (21) and (22), respectively, where $\mathbf{z}$ is replaced by $\mathbf{q}$ in order to obtain the time-domain operators

$$\alpha(\mathbf{q}) = (1 - \nu\mathbf{q}^{-1})^{n_\mathrm{f}} \tag{29}$$

and

$$\beta(\mathbf{q}) = \mathcal{L}_{0,n_\mathrm{f}}\{\alpha(\mathbf{q})G_{0,n_\mathrm{f}}(\mathbf{q})\}. \tag{30}$$

The following assumption about the plant transfer function $G_{zu}(\mathbf{z}) \overset{\triangle}{=} E_1(\mathbf{z}I - A)^{-1}B$ facilitates the application of Theorem 6.1 to RCAC. This assumption provides conditions under which a limited number of Markov parameters can be used to capture the modelling data required for RCAC. Complete assumptions are given in Hoagg and Bernstein (2012) along with a proof of convergence. Roughly speaking, the following assumption requires that none of the NMP zeros of G_{zu} be contained in a disk centred at a point in the interval $(-1, 1)$ that contains all of the poles of G_{zu}.

Assumption 7.1: *If G_{zu} is NMP, then assume there exists $\nu \in (-1, 1)$ such that $\rho(G_{zu}, \nu) < 1$ and $\{\lambda \in \mathbb{C} : G_{zu}(\lambda) = 0$ and $|\lambda| \geq 1\} \subset \mathbb{P}(\nu, \rho(G_{zu}, \nu))$.*

If G_{zu} is NMP, then Assumption 7.1 requires that all of the NMP zeros of G_{zu} are contained in $\mathbb{P}(\nu, \rho(G_{zu}, \nu))$ and that $\rho(G_{zu}, \nu) < 1$. It can be seen that, if Assumption 7.1 is satisfied, then there exists a unit disk that contains all the poles of G_{zu}, $\rho(G_{zu}, 0) < 2$, and no NMP zero of G_{zu} can be contained in a line segment connecting a pair of poles of G_{zu}. Figure 6 shows examples of NMP transfer functions for which Assumption 7.1 is or is not satisfied.

Note that Assumption 7.1 applies only to the case where G_{zu} is NMP. In fact, if G_{zu} is minimum phase, then it is shown in Hoagg et al. (2008) that Theorem 6.1 can be applied to RCAC with $\nu = 0$ and $n_\mathrm{f} = \mathrm{rd}(G_{zu})$ without requiring $\rho(G_{zu}, \nu) < 1$. In addition, if G_{zu} is minimum phase, then $\{\lambda \in \mathbb{C} : G_{zu}(\lambda) = 0 \text{ and } |\lambda| \geq 1\} = \varnothing \subset \mathbb{P}(\nu, \rho(G_{zu}, \nu))$. Thus, Assumption 7.1 is not needed in the case where G_{zu} is minimum phase.

It is shown in Hoagg and Bernstein (2012) that the zeros of α are the poles of the ideal closed-loop associated with RCAC. Thus, it is assumed in Hoagg and Bernstein (2012) that the absolute value of the zeros of α are smaller than 1. Therefore, since the zeros of α are located at ν, we require $\nu \in (-1, 1)$. On the other hand, $\lim_{i\to\infty} L_i = 0$ is a necessary condition for $G_{\nu,n}$ to approximate G_{zu}. Furthermore, Proposition 4.1 implies that $\lim_{i\to\infty} L_i = 0$ if and only if $\rho(G_{zu}, \nu) < 1$. Thus $\rho(G_{zu}, \nu) < 1$ is required in order for $G_{\nu,n}$ to approximate G_{zu}. In addition, every NMP zero of G_{zu} is required to be contained in $\mathbb{P}(\nu, \rho(G_{zu}, \nu))$ so that it can be approximated by one of the zeros of $G_{\nu,n}$ for sufficiently large n.

Note that, if G_{zu} is not asymptotically stable, then $\rho(G_{zu}, 0) \geq 1$, and thus Assumption 7.1 cannot be satisfied with $\nu = 0$. Thus, if G_{zu} is NMP and not asymptotically stable, then ν must be nonzero in order to satisfy Assumption 7.1. If G_{zu} is NMP and asymptotically stable, then $\rho(G_{zu}, \nu) < 1$ can be achieved by $\nu = 0$. However, $\nu \neq 0$ can still be beneficial if $\rho(G_{zu}, \nu) < \rho(G_{zu}, 0)$, as shown in Example 8.1 in the next section.

8. Numerical examples

In this section, we consider command following problems, where, in (1)–(3), $w(k) = r(k) \in \mathbb{R}$ is the

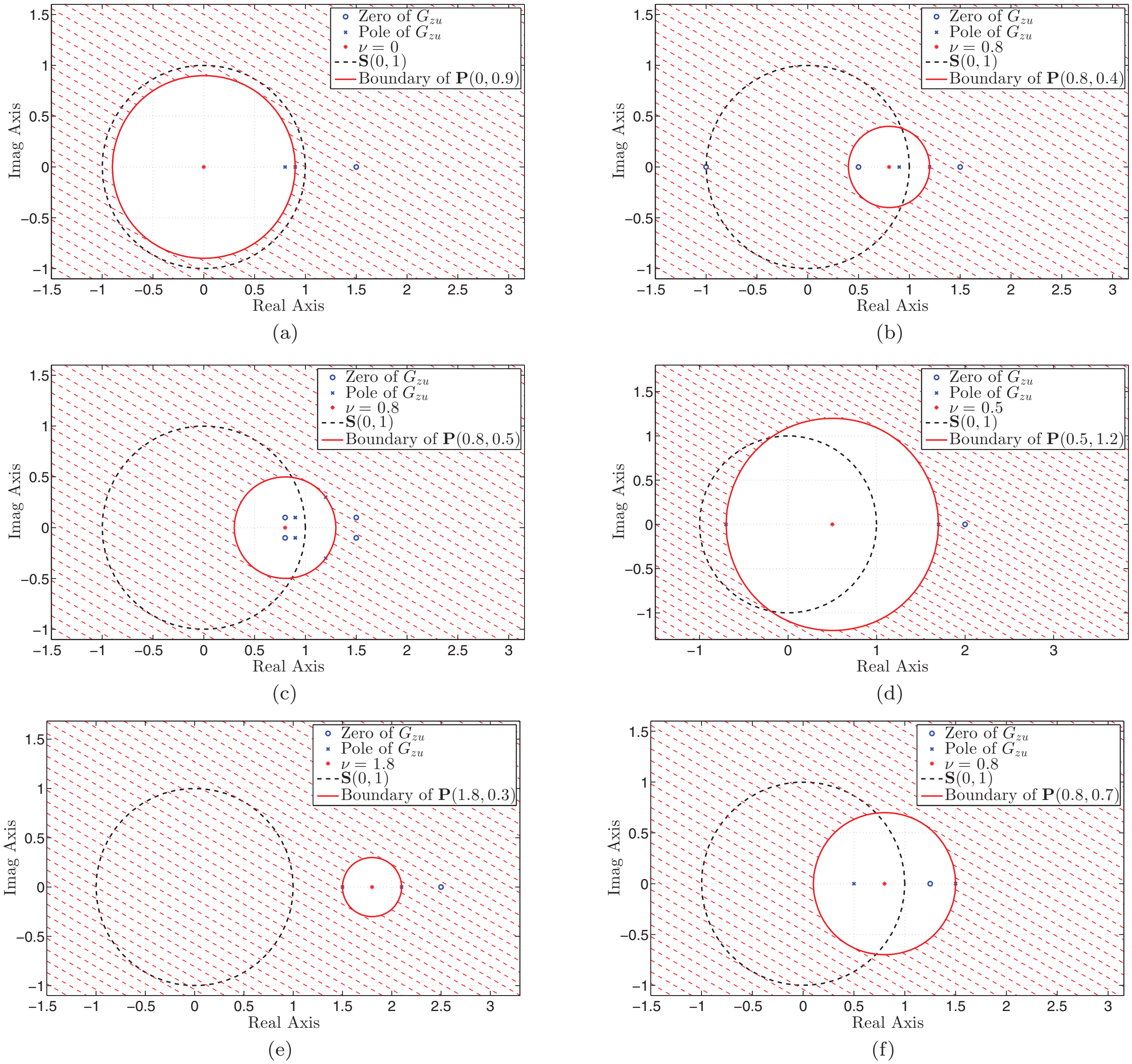

Figure 6. Examples to illustrate Assumption 7.1. (a) Let $G_{zu}(\mathbf{z}) = \frac{\mathbf{z}-1.5}{(\mathbf{z}-0.8)(\mathbf{z}-0.9)}$ and $\nu = 0$, and thus $\rho(G_{zu}, 0) = 0.9$. The shaded region is $\mathbb{P}(0, \rho(G_{zu}, 0))$. Note that $\rho(G_{zu}, 0) = 0.9 < 1$ and all of the NMP zeros of G_{zu} are contained in $\mathbb{P}(0, \rho(G_{zu}, 0))$. Thus Assumption 7.1 is satisfied with $\nu = 0$. (b) Let $G_{zu}(\mathbf{z}) = \frac{(\mathbf{z}-1.5)(\mathbf{z}-0.5)(\mathbf{z}+1)}{(\mathbf{z}-1.2)^3(\mathbf{z}-0.9)}$ and $\nu = 0.8$, and thus $\rho(G_{zu}, 0.8) = 0.4$. The shaded region is $\mathbb{P}(0.8, \rho(G_{zu}, 0.8))$. Note that $\rho(G_{zu}, 0) = 0.4 < 1$ and all of the NMP zeros of G_{zu} are contained in $\mathbb{P}(0.8, \rho(G_{zu}, 0.8))$. Thus Assumption 7.1 is satisfied with $\nu = 0.8$. Note that Assumption 7.1 does not require that the minimum-phase zeros of G_{zu} be contained in $\mathbb{P}(\nu, \rho(G_{zu}, \nu))$. (c) Let $G_{zu}(\mathbf{z}) = \frac{(\mathbf{z}-0.8-0.1\jmath)(\mathbf{z}-0.8+0.1\jmath)(\mathbf{z}-1.5+0.1\jmath)(\mathbf{z}-1.5-0.1\jmath)}{(\mathbf{z}-1.2+0.3j)(\mathbf{z}-1.2-0.3j)(\mathbf{z}-0.9-0.1\jmath)^2(\mathbf{z}-0.9+0.1\jmath)^2}$ and $\nu = 0.8$, and thus $\rho(G_{zu}, 0.8) = 0.5$. The shaded region is $\mathbb{P}(0.8, \rho(G_{zu}, 0.8))$. Note that $\rho(G_{zu}, 0) = 0.5 < 1$ and all of the NMP zeros of G_{zu} are contained in $\mathbb{P}(0.8, \rho(G_{zu}, 0.8))$. Thus Assumption 7.1 is satisfied with $\nu = 0.8$. Note that all of the poles and zeros of G_{zu} are complex. (d) Let $G_{zu}(\mathbf{z}) = \frac{(\mathbf{z}-2)}{(\mathbf{z}+0.7)(\mathbf{z}-1.7)}$. Note that there does not exist a unit disk that contains all of the poles of G_{zu}. Thus Assumption 7.1 cannot be satisfied. For example, let $\nu = 0.5$, which is the midpoint between -0.7 and 1.7. The shaded region is $\mathbb{P}(0.5, \rho(G_{zu}, 0.5))$. Since $\rho(G_{zu}, 0.5) = 1.2 > 1$, Assumption 7.1 is not satisfied with $\nu = 0.5$. (e) Let $G_{zu}(\mathbf{z}) = \frac{(\mathbf{z}-2.5)}{(\mathbf{z}-1.5)(\mathbf{z}-2.1)}$. Note that $\rho(G_{zu}, 0) = 2.1 > 2$. Thus Assumption 7.1 cannot be satisfied. Different from the case in (d), there exists a unit disk that contains all of the poles of G_{zu}. For example, let $\nu = 1.8$. The shaded region is $\mathbb{P}(1.8, \rho(G_{zu}, 1.8))$. Note that $\rho(G_{zu}, 1.8) = 0.3 < 1$ and all of the NMP zeros of G_{zu} are contained in $\mathbb{P}(1.8, \rho(G_{zu}, 1.8))$. However, since Assumption 7.1 requires $\nu \in (-1, 1)$, Assumption 7.1 is not satisfied with $\nu = 1.8$. (f) Let $G_{zu}(\mathbf{z}) = \frac{(\mathbf{z}-1.25)^2}{(\mathbf{z}-0.5)^2(\mathbf{z}-1.5)}$. Note that the NMP zero 1.25 is contained in the line segment connecting the poles 0.5 and 1.5. Thus Assumption 7.1 cannot be satisfied. For example, let $\nu = 0.8$. The shaded region is $\mathbb{P}(0.8, \rho(G_{zu}, 0.8))$. Note that $\rho(G_{zu}, 0.8) = 0.7 < 1$. However, since the NMP zero 1.25 is not contained in $\mathbb{P}(0.8, \rho(G_{zu}, 0.8))$, Assumption 7.1 is not satisfied with $\nu = 0.8$.

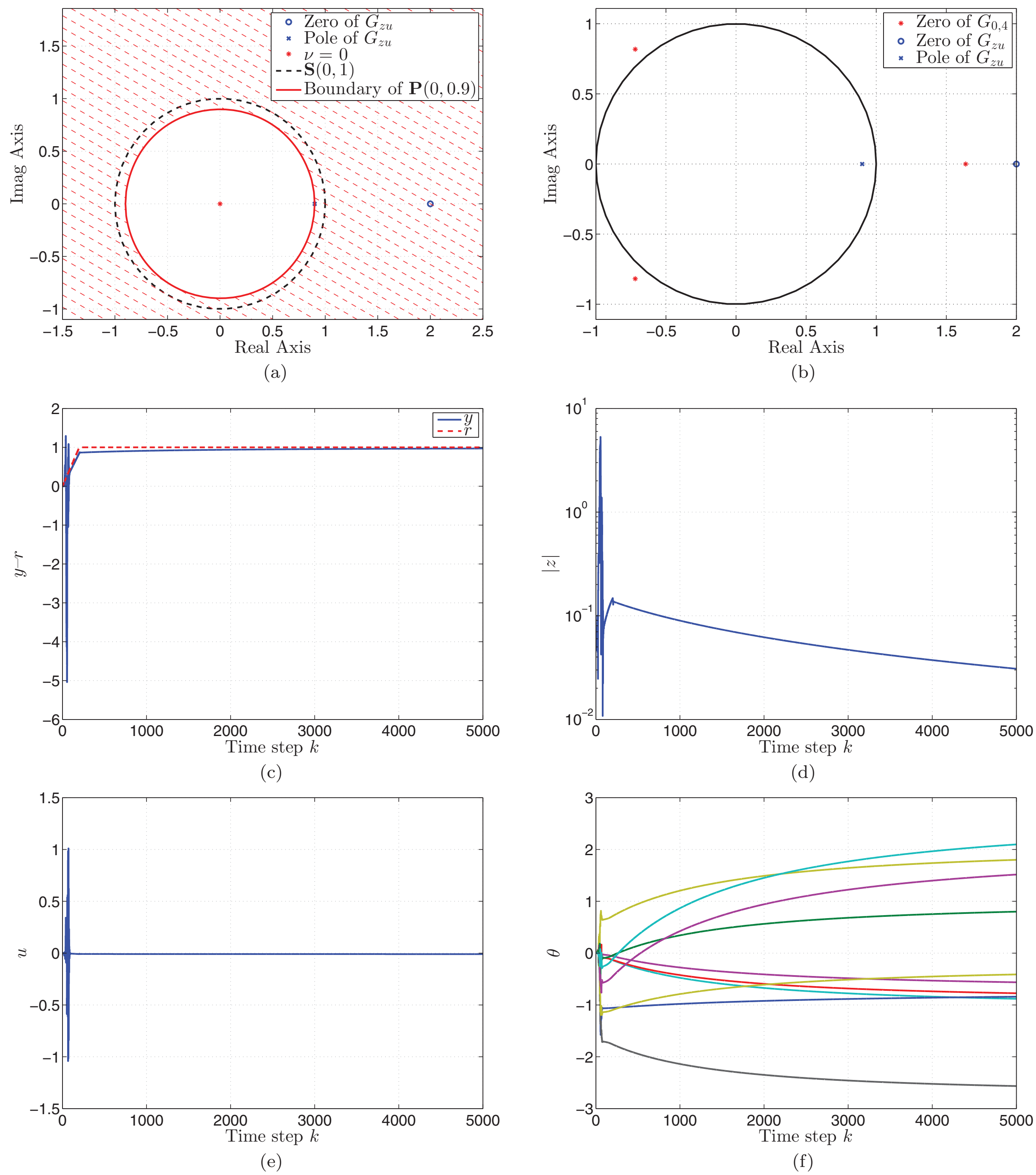

Figure 7. Example 8.1 [Trapezoidal-command following for the asymptotically stable, NMP plant G_{zu} given by (31)]. (a) shows that Assumption 7.1 is satisfied with $\nu = 0$. H_1, H_2, H_3, H_4 are used. (d) shows that the command-following error z is about 0.03 for $k = 5000$.

command to be followed, $D_1 = D_2 = 0$, $E_0 = 1$, and $E_1 = C$. The goal is to investigate the ability of RCAC to achieve asymptotic command following for NMP plants that satisfy Assumption 7.1 without using direct knowledge of the NMP zeros of G_{zu}. All of the examples in this section satisfy Assumption 7.1.

The tuning parameters required by the RCAC controller (4), (9), (13), (14), (29), and (30) are n_c, R_z, R_θ, ν, n_f. Once these parameters are chosen, the only modelling information needed by RCAC is the Markov parameters $H_1, H_2, \ldots, H_{n_f}$. Since G_{zu} with McMillian degree n can be reconstructed from $H_1, H_2, \ldots, H_{2n+1}$ using

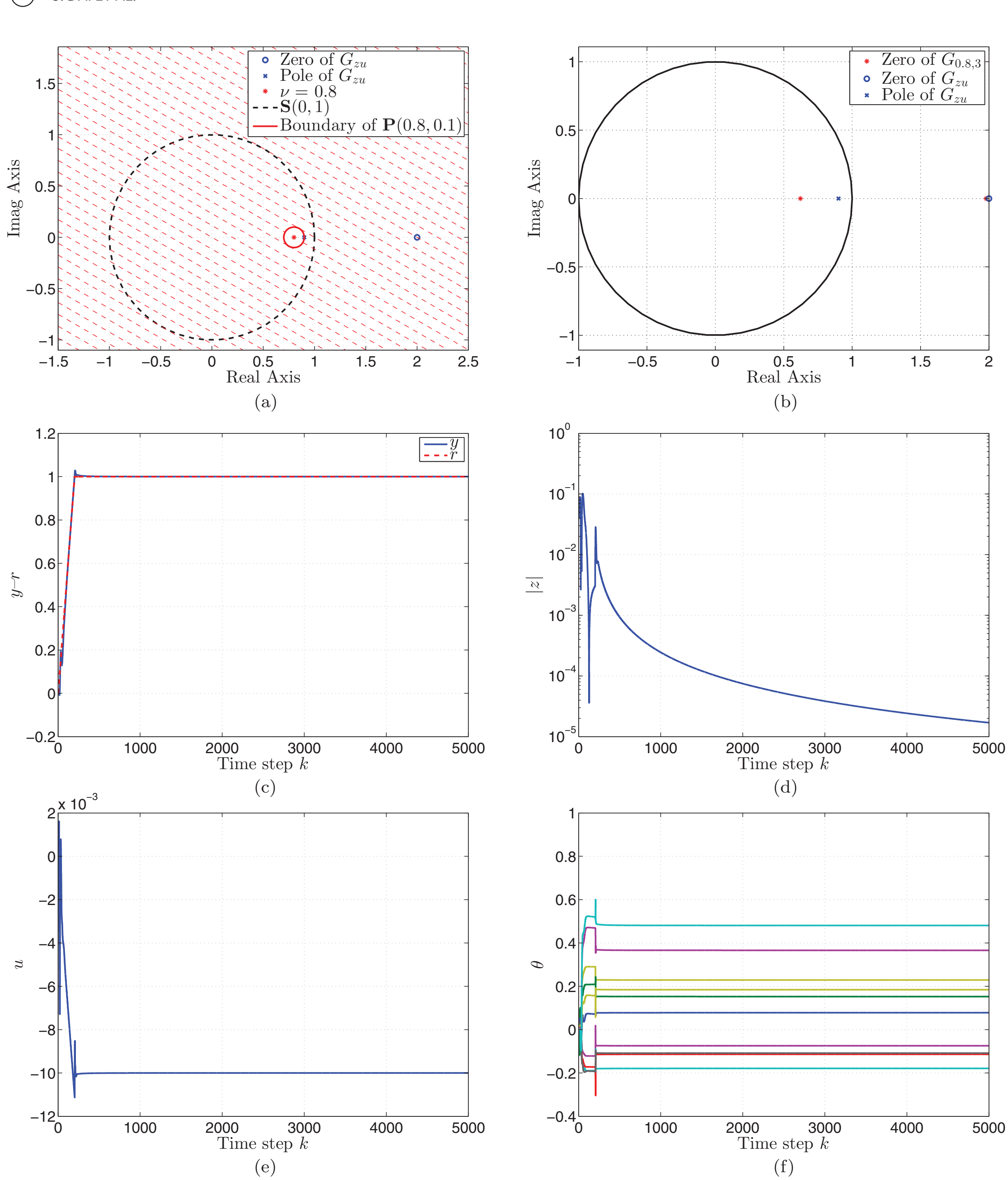

Figure 8. Example 8.1 [Trapezoidal-command following for the asymptotically stable, NMP plant G_{zu} given by (31)]. (a) shows that Assumption 7.1 is satisfied with $\nu = 0.8$. H_1, H_2, H_3 are used.(d) shows that the command-following error z is about 1.7×10^{-6} for $k = 5000$.

Ho-Kalman realisation theory, we choose $n_f \leq 2n$ for all of the examples in this section. Moreover, we do not require knowledge of the characteristics of the reference command r. In particular, the height of a step, the offset and slope of a ramp, the frequency of a harmonic, and the type of the reference command need not be known. Since feedforward control is not used, the signal r is not used directly within RCAC. In addition, we choose the initial controller coefficients $\theta(0) = 0_{2n_c \times 1}$ for all of the examples in this section.

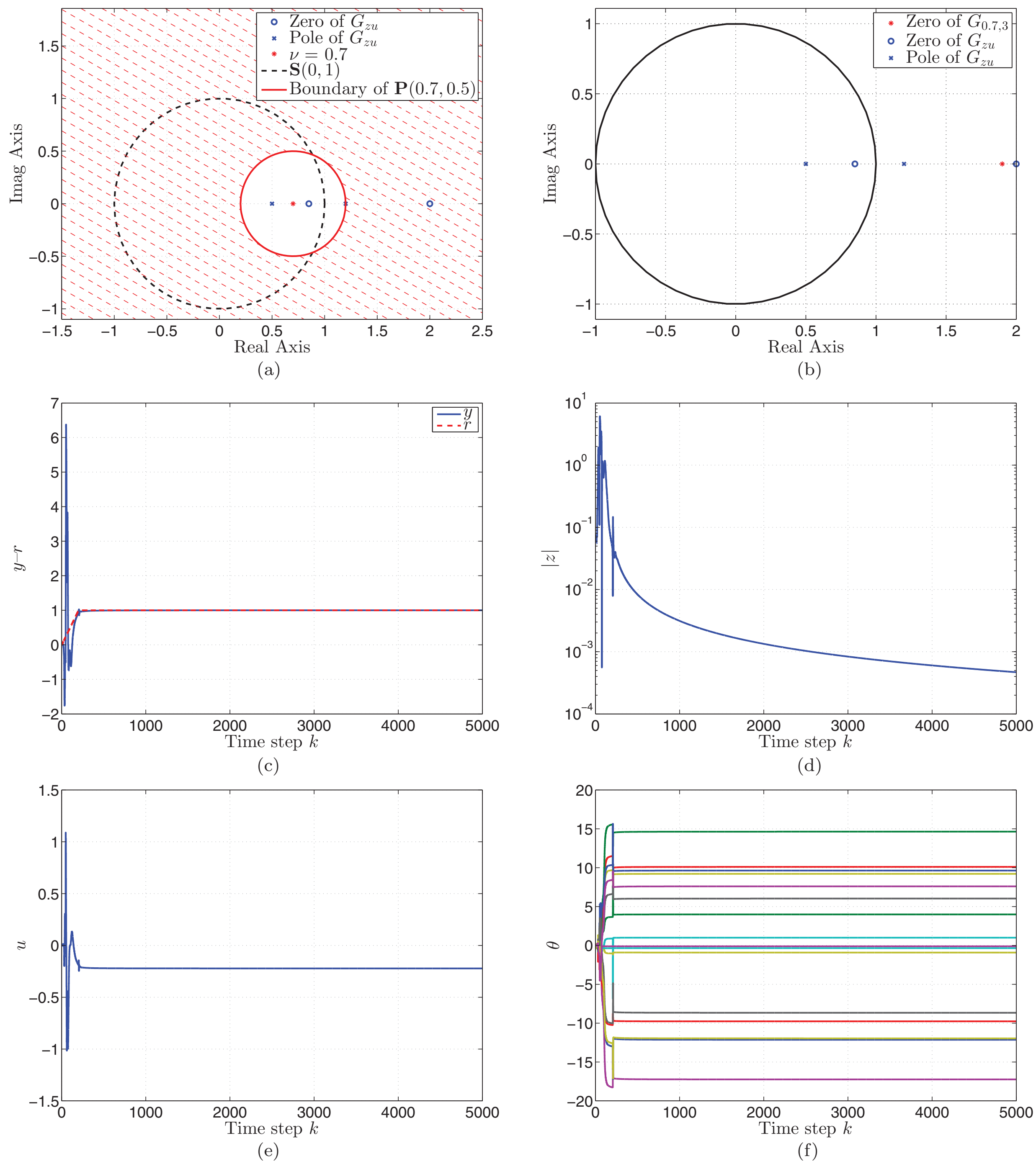

Figure 9. Example 8.2 [Trapezoidal-command following for the exponentially unstable, NMP plant G_{zu} given by (32)]. (a) shows that Assumption 7.1 is satisfied with $\nu = 0.7$. H_1, H_2, H_3 are used. (d) shows that the command-following error z is about 0.0005 for $k = 5000$.

Example 8.1: Consider the asymptotically stable, NMP transfer function

$$G_{zu}(\mathbf{z}) = \frac{\mathbf{z} - 2}{(\mathbf{z} - 0.9)^2}, \tag{31}$$

which satisfies Assumption 7.1 for all $\nu \in (-0.1, 1)$. Let r be the trapezoidal command $r(k) = \min(1, 0.005k)$.

Figure 7(a) illustrates Assumption 7.1 with $\nu = 0$. Hence, we choose $n_c = 5$, $R_z = 1$, $R_\theta = I_{10}$, $\nu = 0$, and $n_f = 4$. Then, (29) and (30) yield

$$\alpha(\mathbf{q}) = 1, \ \beta(\mathbf{q}) = \mathbf{q}^{-1} - 0.2\mathbf{q}^{-2} - 1.17\mathbf{q}^{-3} - 1.944\mathbf{q}^{-4}.$$

Figure 7 shows the performance in this case. Figure 7(d) shows that the command-following error z is about 0.03

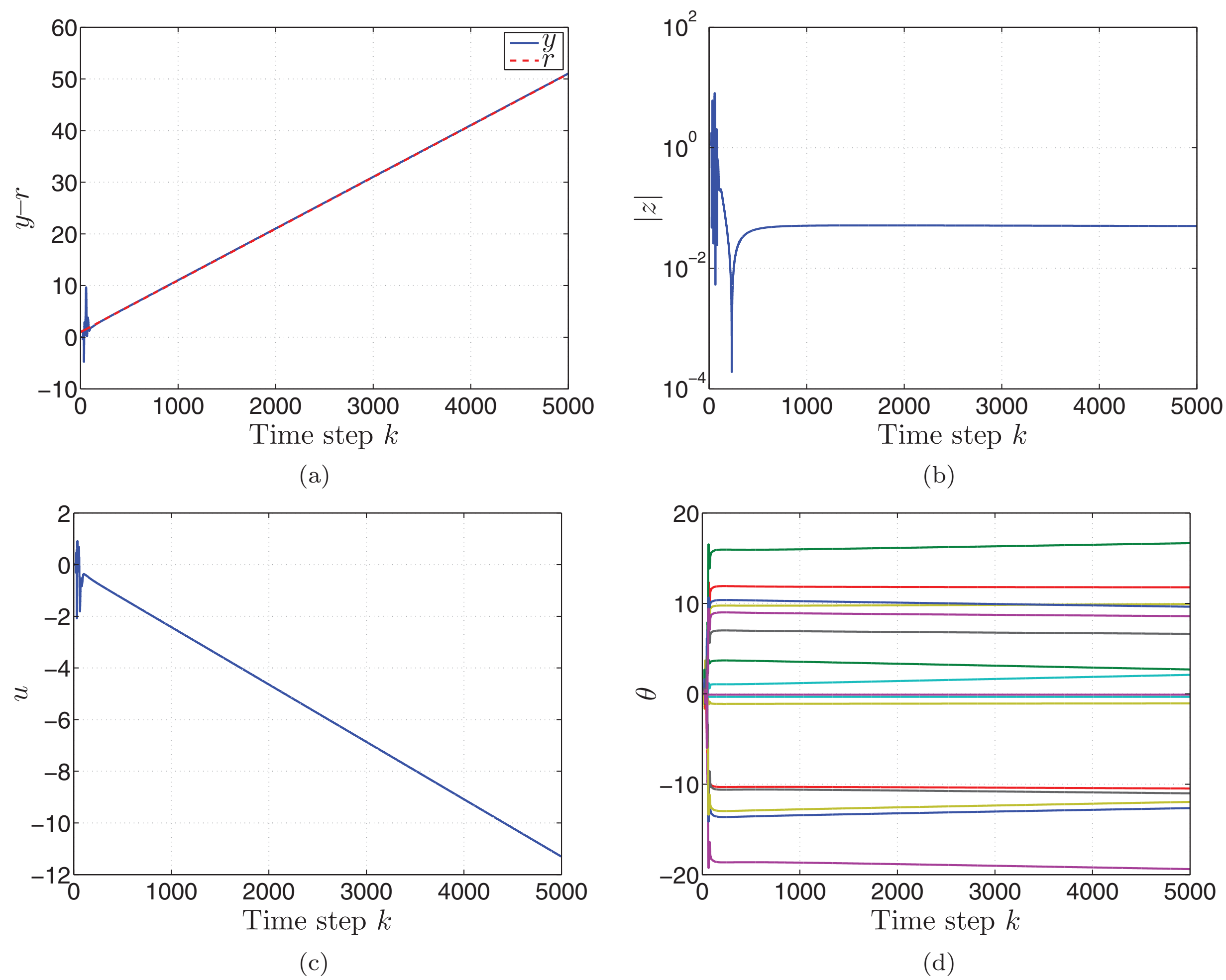

Figure 10. Example 8.2 [Ramp-command following for the exponentially unstable, NMP plant G_{zu} given by (32)]. We choose the same tuning parameters as in Figure 9. H_1, H_2, H_3 are used.(b) shows that the command-following error z is about 0.05 for $k = 5000$.

for $k = 5000$. Smaller value of n_{f} do not yield error convergence. Figure 8(a) illustrates Assumption 7.1 with $\nu = 0.8$. Hence, we choose $n_{\mathrm{c}} = 5$, $R_z = 1$, $R_\theta = 0.00001 I_{10}$, $\nu = 0.8$, and $n_{\mathrm{f}} = 3$. Then, (29) and (30) yield

$$\alpha(\mathbf{q}) = 1 - 2.4\mathbf{q}^{-1} + 1.92\mathbf{q}^{-2} - 0.512\mathbf{q}^{-3},$$
$$\beta(\mathbf{q}) = \mathbf{q}^{-1} - 2.6\mathbf{q}^{-2} + 1.23\mathbf{q}^{-3}.$$

Figure 8 shows the performance in this case. Figure 8(d) shows that the command-following error z is about 1.7×10^{-6} for $k = 5000$.

Note that $n_{\mathrm{f}} = 3$ suffices for $\nu = 0.8$, but $n_{\mathrm{f}} = 4$ is required for $\nu = 0$. Hence less modelling information is required for $\nu = 0.8$ than for $\nu = 0$. Moreover, comparing Figures 7 and 8, it can be seen that it is more beneficial to choose $\nu = 0.8$ than $\nu = 0$ in terms of control performance. An intuitive reason is that $\rho(G_{zu}, 0.8) = 0.1 < \rho(G_{zu}, 0) = 0.9$. In other words, the spectral radius of G_{zu} relative to $\nu = 0.8$ is smaller than the spectral radius of G_{zu} relative to $\nu = 0$, which implies that G_{zu} looks more 'stable' relative to $\nu = 0.8$ compared with $\nu = 0$.

Example 8.2: Consider the exponentially unstable, NMP transfer function

$$G_{zu}(\mathbf{z}) = \frac{(\mathbf{z} - 2)(\mathbf{z} - 0.85)^2}{(\mathbf{z} - 1.2)^2(\mathbf{z} - 0.5)^3}, \tag{32}$$

which satisfies Assumption 7.1 for all $\nu \in (0.2, 1)$. Figure 9 (a) illustrates Assumption 7.1 with $\nu = 0.7$. Hence, we choose $n_{\mathrm{c}} = 8$, $R_z = 1$, $R_\theta = 0.00001 I_{16}$, $\nu = 0.7$, and $n_{\mathrm{f}} = 3$. Then, (29) and (30) yield

$$\alpha(\mathbf{q}) = 1 - 2.1\mathbf{q}^{-1} + 1.47\mathbf{q}^{-2} - 0.343\mathbf{q}^{-3},$$
$$\beta(\mathbf{q}) = \mathbf{q}^{-2} - 1.9\mathbf{q}^{-3}. \tag{33}$$

Let r be the trapezoidal command $r(k) = \min(1, 0.005k)$. Figure 9(d) shows that the command-following error z is about 0.0005 for $k = 5000$. Next, let r be the ramp

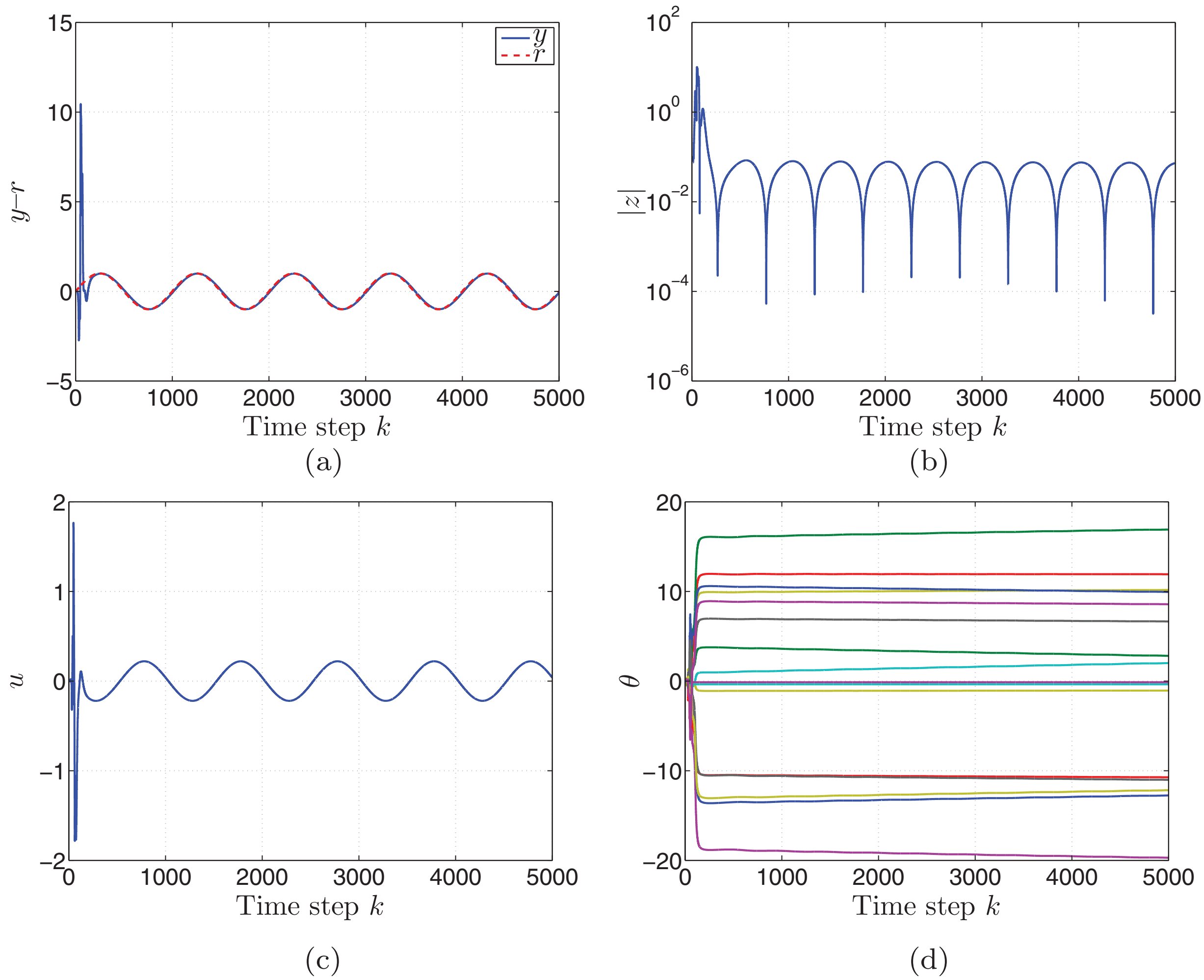

Figure 11. Example 8.2 [Harmonic-command following for the exponentially unstable, NMP plant G_{zu} given by (32)]. We choose the same tuning parameters as in Figure 9. H_1, H_2, H_3 are used. (b) shows that the largest absolute value of the command-following error z in the last period is about 0.07.

$r(k) = 1 + 0.01k$. Figure 10 (b) shows that the command-following error z for $k = 5000$ is about 0.05. Finally, let $r(k) = \sin(0.002\pi k)$. Figure 11 (b) shows that the largest absolute value of the command-following error z in the last period is about 0.07.

Example 8.3: Alternatively, we apply the method given in Hoagg and Bernstein (2012) to (32), where the NMP zeros are assumed to be known. In this case, $\alpha(\mathbf{q})$ and $\beta(\mathbf{q})$ are constructed as

$$\alpha(\mathbf{q}) = 1,\ \beta(\mathbf{q}) = H_d \mathbf{q}^{-d} \prod_{i=1}^{n_\zeta} (1 - \zeta_i \mathbf{q}^{-1}), \tag{34}$$

Table 1. Transient performance and steady-state performance using (33), (35), and (36) for G_{zu} given by (32). The steady-state performance is given by $|z(5000)|$ for trapezoidal and ramp commands, and by the maximum value of $|z|$ in the last period for the harmonic command. The transient performance is given by the maximum value of $|z|$ over the entire simulation, which occurs in the first 100 steps for all cases.

	Transient performance			Steady-state performance		
$r(k)$	Using (33)	Using (35)	Using (36)	Using (33)	Using (35)	Using (36)
min (1, 0.005k)	7.1	14.1	45.3	0.02	0.01	0.08
$1 + 0.01k$	24.2	132.5	1537	0.02	0.02	0.02
sin (0.002πk)	7.9	20.3	127.5	0.08	0.07	0.35

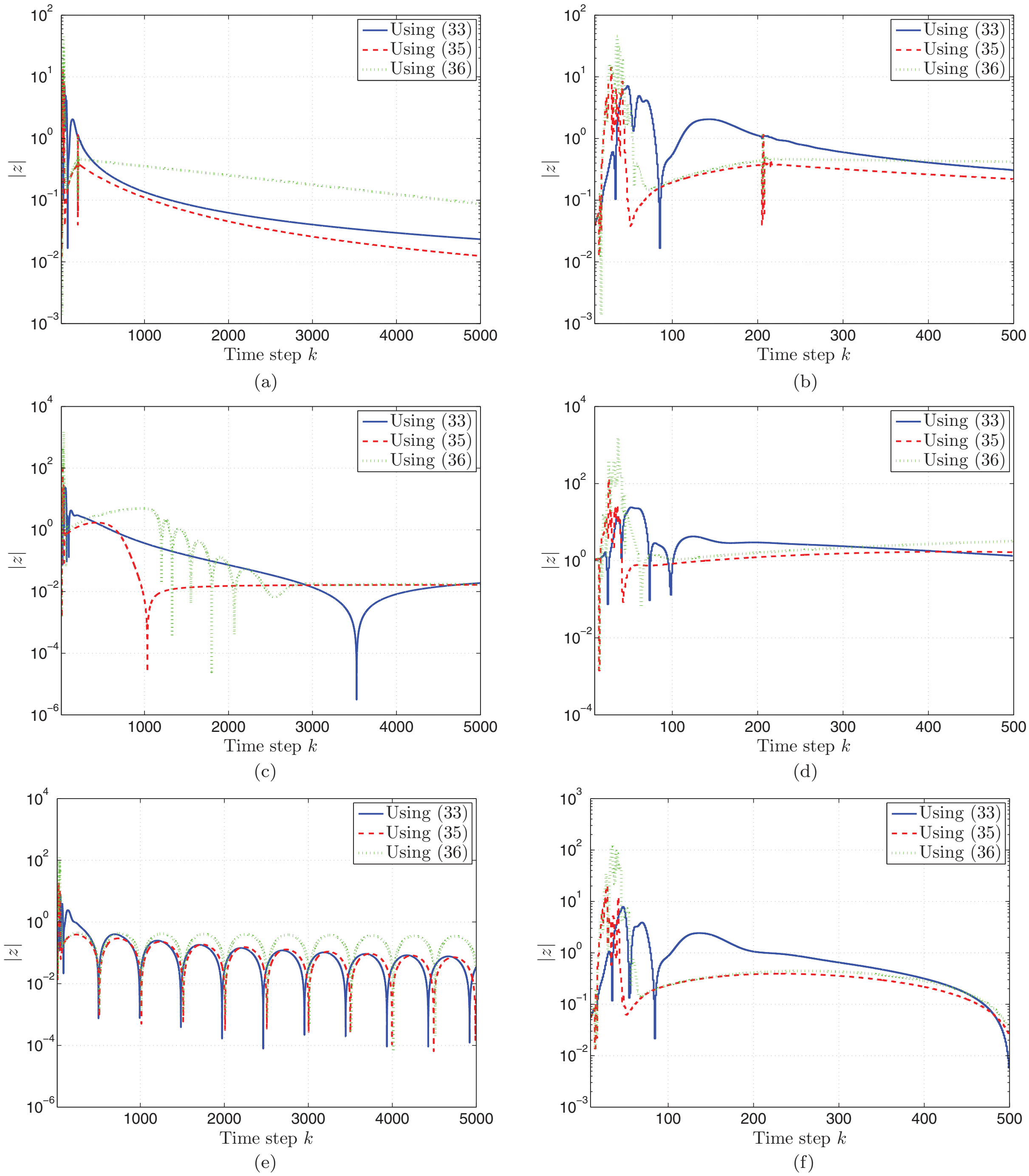

Figure 12. Example 8.3 [Comparison of command-following performance using (33), (35), and (36) for the exponentially unstable, NMP plant G_{zu} given by (32)]. (a), (c), and (e) show that the steady-state performance using (35) is better than using (33). However, (b), (d), and (f) show that the transient of z using (35) is larger than using (33). In addition, (33) and (35) have better transient and steady-state performance than (36) for all three types of commands.

where $d \stackrel{\triangle}{=} \mathrm{rd}(G_{zu})$, n_ζ is the number of NMP zeros of G_{zu}, and $\zeta_1, \ldots, \zeta_{n_\zeta}$ are the NMP zeros of G_{zu}. We use (34) with the actual NMP zero 2 of (32), which implies

$$\alpha(\mathbf{q}) = 1,\ \beta(\mathbf{q}) = \mathbf{q}^{-2}(1 - 2\mathbf{q}^{-1}) = \mathbf{q}^{-2} - 2\mathbf{q}^{-3}. \tag{35}$$

In addition, to compare with (33) and (35), we use (34) with the NMP zero 1.9 of $G_{0.7,3}$ (the NMP zero of $\beta(\mathbf{q})$ in (33)), which implies

$$\alpha(\mathbf{q}) = 1,\ \beta(\mathbf{q}) = \mathbf{q}^{-2}(1 - 1.9\mathbf{q}^{-1}) = \mathbf{q}^{-2} - 1.9\mathbf{q}^{-3}. \tag{36}$$

To compare (33), (35), and (36), we choose the same tuning parameters for all three methods, specifically, $n_c = 5$, $R_z = 1$, and $R_\theta = 0.00001 I_{10}$, which differ from the tuning parameters in Example 8.2, where $n_c = 8$, $R_z = 1$, and $R_\theta = 0.00001 I_{16}$. Different tuning parameters are chosen due to the fact that (36) does not yield error convergence for a ramp command with the latter tuning parameters.

Figure 12 compares the command-following performance using (33), (35), and (36) for trapezoidal, ramp, and harmonic commands. Table 1 summarises the transient performance and steady-state performance. Figure 12 and Table 1 show that (35) achieves better steady-state performance than (33). However, it can be seen that (35) causes larger transient responses than (33). In addition, (33) and (35) have better transient and steady-state performance than (36) for all three types of commands.

9. Conclusion

A method is developed for constructing an approximate transfer function using two FIR filters based on Markov parameters, where the ratio of the FIR filters is equal to a truncated Laurent expansion of G_{zu} centred at $\nu \in \mathbb{R}$. This technique avoids the need for direct knowledge of the NMP zeros of G_{zu} as required in Hoagg and Bernstein (2012). This method can be considered as an extension of Santillo and Bernstein (2010), where $\nu = 0$. With this extension, RCAC is shown to be effective for command following for a class of unstable, NMP systems.

Numerical examples show that this method can achieve asymptotic command following for unstable, NMP systems without direct knowledge of the NMP zeros but using Markov parameters instead. The number of Markov parameters used in each example is insufficient reconstructing G_{zu} using Ho-Kalman realisation theory. In addition, knowledge of the properties of the command signal is not needed. Future work includes improvements on the optimisation methods, proof of closed-loop convergence, and the quantification of robustness to uncertainty in the Markov parameters as well as sensor noise.

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Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendices

Appendix 1. Definition and lemmas for the proof of Proposition 4.1

Definition A1: The sequence $\{p_n\}_{n=1}^{\infty} \subset \mathbb{C}$ is *almost periodic* if, for all $\varepsilon > 0$, there exists $l_\varepsilon \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, there exists $\tau_{\varepsilon,n} \in \{1, \ldots, l_\varepsilon\}$ such that $|p_n - p_{n+\tau_{\varepsilon,n}}| < \varepsilon$.

Lemma A1: *Let $c_1, \ldots, c_r$ be nonzero complex numbers, let $\theta_1, \ldots, \theta_r \in (-\pi, \pi]$ be distinct, and, for all $n \in \mathbb{N}$, define*

$$E_n \triangleq \sum_{i=1}^{r} c_i e^{\jmath n\theta_i}. \tag{A1}$$

Then, the following statements hold:

1. *There exists $n_0 \in \{1, \ldots, r\}$ such that $E_{n_0} \neq 0$.*
2. *$\{E_n\}_{n=1}^{\infty}$ is almost periodic.*
3. *$\limsup_{n\to\infty} |E_n|$ is a positive number.*

Proof: To prove 1), suppose that $E_1 = \cdots = E_r = 0$. Thus

$$\begin{bmatrix} e^{\jmath\theta_1} & \cdots & e^{\jmath\theta_r} \\ e^{\jmath 2\theta_1} & \cdots & e^{\jmath 2\theta_r} \\ \vdots & \cdots & \vdots \\ e^{\jmath r\theta_1} & \cdots & e^{\jmath r\theta_r} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{A2}$$

Fact 5.16.3 in Bernstein (2009, p. 387) implies that the $r \times r$ matrix M in (A2) satisfies

$$\det M = \left(\prod_{i=1}^{r} e^{\jmath\theta_i}\right) \prod_{1\le i<j\le r} \left(e^{\jmath\theta_i} - e^{\jmath\theta_j}\right). \tag{A3}$$

Since $\theta_1, \ldots, \theta_r \in (-\pi, \pi]$ are distinct, it follows that $e^{\jmath\theta_1}, \ldots, e^{\jmath\theta_r}$ are distinct, and thus M is nonsingular. Hence $c_1 = \cdots = c_r = 0$, which contradicts the assumption that $c_1, \ldots, c_r$ are nonzero. Hence, there exists $n_0 \in \{1, \ldots, r\}$ such that $E_{n_0} \neq 0$.

To prove 2), note that Remark 2 of Proposition 3.1 of Corduneanu (2006, p. 39) and Proposition 7.2 of Corduneanu (2006, p. 116) imply that $\{E_n\}_{n=1}^{\infty}$ is almost periodic.

To prove 3), note that 2) and Definition A1 imply that, for all $k \in \mathbb{N}$, there exists $l_k \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, there exists $\tau_{n,k} \in \{1, \ldots, l_k\}$ such that $|E_n - E_{n+\tau_{n,k}}| < \frac{1}{k^2}$, and thus

$$\begin{aligned}|E_n - E_{n+k\tau_{n,k}}| &\le |E_n - E_{n+\tau_{n,k}}| + |E_{n+\tau_{n,k}} - E_{n+2\tau_{n,k}}| \\ &+ \cdots + |E_{n+(k-1)\tau_{n,k}} - E_{n+k\tau_{n,k}}| \le k\frac{1}{k^2} = \frac{1}{k}.\end{aligned}$$

Thus, for all $n \in \mathbb{N}$,

$$\lim_{k\to\infty} E_{n+k\tau_{n,k}} = E_n. \tag{A4}$$

By 1), there exists $n_0 \in \{1, \ldots, r\}$ such that $E_{n_0} \neq 0$. Setting $n = n_0$ in (A4) yields $\lim_{k\to\infty} E_{n_0+k\tau_{n_0,k}} = E_{n_0}$. Thus,

$$\begin{aligned}\limsup_{n\to\infty} |E_n| &\ge \limsup_{k\to\infty} |E_{n_0+k\tau_{n_0,k}}| \\ &= \lim_{k\to\infty} |E_{n_0+k\tau_{n_0,k}}| = |E_{n_0}| > 0.\end{aligned}$$

Furthermore, (A1) implies that

$$\limsup_{n\to\infty} |E_n| = \limsup_{n\to\infty} \left|\sum_{i=1}^{r} c_i e^{\jmath n\theta_i}\right| \le \sum_{i=1}^{r} |c_i| < \infty.$$

Hence $0 < \limsup_{n\to\infty} |E_n| < \infty$, and thus 3) is proved. ■

Lemma A2: *Let G be the rational function*

$$G(z) = \sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{c_{i,j}}{(z - p_i)^j}, \tag{A5}$$

where r is the number of distinct poles of G, k_i is the order of the pole $z = p_i$, and, for all $i = 1, \ldots, r$ and $j = 1, \ldots, k_i$, $c_{i,j} \in \mathbb{C}$ and $c_{i,k_i} \neq 0$. Let $1 \le r_0 \le r$ be such that, $\rho \triangleq \rho(G, 0) = |p_1| = \cdots = |p_{r_0}| > |p_{r_0+1}| \ge \cdots \ge |p_r|$. Let $1 \le r_0' \le r_0$ be such that $k_{\max} \triangleq |k_1| = \cdots = |k_{r_0'}| > |k_{r_0'+1}| \ge \cdots \ge |k_{r_0}|$. Then, defining $\{L_n\}_{n=0}^{\infty}$ as in (15) with $\nu = 0$ and assuming $\rho \neq 0$, for all $n \ge \max\{k_1, \ldots, k_r\}$,

$$\begin{aligned}L_n &= \sum_{i=1}^{r} \sum_{j=1}^{k_i} c_{i,j} p_i^{n-j} \frac{(n-1)!}{(j-1)!(n-j)!} \\ &= \left(\sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{c_{i,j}}{p_i^j (j-1)!} \frac{p_i^n}{\rho^n} \frac{(n-k_{\max})!}{(n-j)!}\right) \rho^n \frac{(n-1)!}{(n-k_{\max})!}\end{aligned} \tag{A6}$$

and

$$\begin{aligned}&\limsup_{n\to\infty} \left|\sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{c_{i,j}}{p_i^j (j-1)!} \frac{p_i^n}{\rho^n} \frac{(n-k_{\max})!}{(n-j)!}\right| \\ &= \limsup_{n\to\infty} \left|\sum_{i=1}^{r_0'} \frac{c_{i,k_{\max}}}{p_i^{k_{\max}} (k_{\max}-1)!} \frac{p_i^n}{\rho^n}\right|.\end{aligned} \tag{A7}$$

Proof: To prove (A6), we first give the Laurent series of $1/(z-p)^k$ in $\mathbb{P}(0, |p|)$, where $p \in \mathbb{C}$. Note that, for all $z \in \mathbb{P}(0, |p|)$,

$$\frac{1}{z-p} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{p}{z}\right)^n. \tag{A8}$$

Differentiating both side of (A8) yields, for all $z \in \mathbb{P}(0, |p|)$ and $k \geq 1$,

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\frac{1}{z-p} = (-1)^{k-1}(k-1)!\frac{1}{(z-p)^k} = (-1)^{k-1}\sum_{n=0}^{\infty} p^n \frac{(n+k-1)!}{n!} z^{-n+k}.$$

Hence, for all $z \in \mathbb{P}(0, |p|)$ and $k \geq 1$,

$$\frac{1}{(z-p)^k} = \sum_{n=0}^{\infty} p^n \frac{(n+k-1)!}{(k-1)!n!} z^{-n+k} = \sum_{n=k}^{\infty} p^{n-k} \frac{(n-1)!}{(k-1)!(n-k)!} z^{-n}. \tag{A9}$$

Next, substituting (A9) into (A5) and comparing coefficients with (15) with $\nu = 0$ yields, for all $n \geq \max\{k_1, \dots, k_r\}$,

$$L_n = \sum_{i=1}^{r}\sum_{j=1}^{k_i} c_{i,j} p_i^{n-j} \frac{(n-1)!}{(j-1)!(n-j)!} = \left(\sum_{i=1}^{r}\sum_{j=1}^{k_i} \frac{c_{i,j}}{p_i^j (j-1)!} \frac{p_i^n}{\rho^n} \frac{(n-k_{\max})!}{(n-j)!}\right) \rho^n \frac{(n-1)!}{(n-k_{\max})!},$$

which proves (A6).

To prove (A7), note that, for all $n \geq \max\{k_1, \dots, k_r\}$,

$$\sum_{i=1}^{r}\sum_{j=1}^{k_i} \frac{c_{i,j}}{p_i^j (j-1)!} \frac{p_i^n}{\rho^n} \frac{(n-k_{\max})!}{(n-j)!} = \sum_{i=1}^{r_0'} \frac{c_{i,k_{\max}}}{p_i^{k_{\max}}(k_{\max}-1)!} \frac{p_i^n}{\rho^n} + f_n + f_n' + f_n'', \tag{A10}$$

where

$$f_n \triangleq \sum_{i=1}^{r_0'}\sum_{j=1}^{k_{\max}-1} \frac{c_{i,j}}{p_i^j (j-1)!} \frac{p_i^n}{\rho^n} \frac{(n-k_{\max})!}{(n-j)!},$$

$$f_n' \triangleq \sum_{i=r_0'+1}^{r_0}\sum_{j=1}^{k_i} \frac{c_{i,j}}{p_i^j (j-1)!} \frac{p_i^n}{\rho^n} \frac{(n-k_{\max})!}{(n-j)!},$$

$$f_n'' \triangleq \sum_{i=r_0+1}^{r}\sum_{j=1}^{k_i} \frac{c_{i,j}}{p_i^j (j-1)!} \frac{p_i^n}{\rho^n} \frac{(n-k_{\max})!}{(n-j)!}.$$

Note that $\rho = |p_1| = \cdots = |p_{r_0'}|$ implies that

$$\lim_{n\to\infty} |f_n| = \lim_{n\to\infty} \left| \sum_{i=1}^{r_0'}\sum_{j=1}^{k_{\max}-1} \frac{c_{i,j}}{p_i^j (j-1)!} \frac{p_i^n}{\rho^n} \times \frac{1}{(n-j)(n-j-1)\cdots(n-k_{\max}+1)} \right| = 0; \tag{A11}$$

$\rho = |p_{r_0'+1}| = \cdots = |p_{r_0}|$ and $k_i < k_{\max}$ for all $i = r_0' + 1, \dots, r_0$ implies that

$$\lim_{n\to\infty} |f_n'| = \lim_{n\to\infty} \left| \sum_{i=r_0'+1}^{r_0}\sum_{j=1}^{k_i} \frac{c_{i,j}}{p_i^j (j-1)!} \times \frac{p_i^n}{\rho^n} \frac{1}{(n-j)(n-j-1)\cdots(n-k_{\max}+1)} \right| = 0; \tag{A12}$$

and $\rho > |p_{r_0+1}| \geq \cdots \geq |p_r|$ implies that

$$\lim_{n\to\infty} |f_n''| = \lim_{n\to\infty} \left| \sum_{i=r_0+1}^{r}\sum_{j=1}^{k_i} \frac{c_{i,j}}{p_i^j (j-1)!} \frac{p_i^n}{\rho^n} \frac{(n-k_{\max})!}{(n-j)!} \right| = 0. \tag{A13}$$

Since, by (A11)–(A13), $\lim_{n\to\infty} \left|f_n + f_n' + f_n''\right| = 0$, (110) and 2.4.19 in Kaczor and Nowak (2000, p. 45) imply

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \sum_{i=1}^{r_0'} \frac{c_{i,k_{\max}}}{p_i^{k_{\max}}(k_{\max}-1)!} \frac{p_i^n}{\rho^n} \right| \\
&= \limsup_{n\to\infty} \left(\left| \sum_{i=1}^{r_0'} \frac{c_{i,k_{\max}}}{p_i^{k_{\max}}(k_{\max}-1)!} \frac{p_i^n}{\rho^n} \right| - \left|f_n + f_n' + f_n''\right| \right) \\
&\leq \limsup_{n\to\infty} \left| \sum_{i=1}^{r}\sum_{j=1}^{k_i} \frac{c_{i,j}}{p_i^j (j-1)!} \frac{p_i^n}{\rho^n} \frac{(n-k_{\max})!}{(n-j)!} \right| \\
&\leq \limsup_{n\to\infty} \left(\left| \sum_{i=1}^{r_0'} \frac{c_{i,k_{\max}}}{p_i^{k_{\max}}(k_{\max}-1)!} \frac{p_i^n}{\rho^n} \right| + \left|f_n + f_n' + f_n''\right| \right) \\
&= \limsup_{n\to\infty} \left| \sum_{i=1}^{r_0'} \frac{c_{i,k_{\max}}}{p_i^{k_{\max}}(k_{\max}-1)!} \frac{p_i^n}{\rho^n} \right|.
\end{aligned}$$

Hence, (A7) is proved. ■

Appendix 2. Proof of Proposition 4.1

Proof: For convenience and without loss of generality, we consider on the case where $\nu = 0$. If $\nu \neq 0$, then z can be replaced by $z - \nu$.

To prove 1), first suppose that $\rho(G,0)=0$. Then all of the poles of G are 0, and thus (15) is a finite sum. Hence $\rho(G,0)=\limsup_{i\to\infty}|L_i|^{1/i}=0$. Alternatively, suppose that $\rho(G,0)>0$ and define

$$\hat{G}(\hat{z}) \triangleq \begin{cases} G(\frac{1}{\hat{z}}), & \hat{z}\in\mathbb{C}\backslash\{0\},\\ 0, & \hat{z}=0.\end{cases}$$

Then (15) implies that, for all $\hat{z}\in\mathbb{D}(0,\rho_{\rm i}(\hat{G},0))$, $\hat{G}(\hat{z})=\sum_{i=0}^{\infty}L_i\hat{z}^i$. Thus, the Cauchy-Hadamard formula (Gamelin, 2001, p. 142) implies that $\rho_{\rm i}(\hat{G},0)=1/\limsup_{i\to\infty}|L_i|^{1/i}$. Note that

$$\begin{aligned}\rho_{\rm i}(\hat{G},0) &= \min\{|\hat{z}| : \hat{z}\text{ is a pole of }\hat{G}\}\\ &= \min\{|\hat{z}| : 1/\hat{z}\text{ is a pole of }G\}\\ &= \min\{1/|z| : z\text{ is a pole of }G\}\\ &= 1/\max\{|z| : z\text{ is a pole of }G\} = 1/\rho(G,0).\end{aligned}$$

Thus, $\rho(G,0)=1/\rho_{\rm i}(\hat{G},0)=\limsup_{i\to\infty}|L_i|^{1/i}$, and thus 1) is proved.

To prove 2), 3), and 4), we first prove sufficiency in each statement and then prove necessity using contradiction.

For the following development we write G in the form (A5). Thus, assuming $\rho\neq 0$, (A6) implies that, for all $n\geq\max\{k_1,\ldots,k_r\}$,

$$L_n = g_n\rho^n\frac{(n-1)!}{(n-k_{\max})!}, \tag{B1}$$

where, for all $n\geq\max\{k_1,\ldots,k_r\}$,

$$g_n \triangleq \sum_{i=1}^{r}\sum_{j=1}^{k_i}\frac{c_{i,j}}{p_i^j(j-1)!}\frac{(n-k_{\max})!}{(n-j)!}\frac{p_i^n}{\rho^n}.$$

Note that (A7) implies that

$$\limsup_{n\to\infty}|g_n| = \limsup_{n\to\infty}\left|\sum_{i=1}^{r_0'}\frac{c_{i,k_{\max}}}{p_i^{k_{\max}}(k_{\max}-1)!}\frac{p_i^n}{\rho^n}\right|. \tag{B2}$$

For all $i=1,\ldots,r_0'$, since $|p_i|=\rho$, it follows that there exists $\theta_i\in(-\pi,\pi]$ such that $e^{\jmath n\theta_i}=p_i^n/\rho^n$. Thus,

$$\limsup_{n\to\infty}|g_n| = \limsup_{n\to\infty}\left|\sum_{i=1}^{r_0'}\frac{c_{i,k_{\max}}}{p_i^{k_{\max}}(k_{\max}-1)!}e^{\jmath n\theta_i}\right|. \tag{B3}$$

Since $p_1,\ldots,p_{r_0'}$ are distinct complex numbers, it follows that $\theta_1,\ldots,\theta_r\in(-\pi,\pi]$ are distinct, thus 3) of Lemma A1 implies that $g_\infty\triangleq\limsup_{n\to\infty}|g_n|$ is a positive number.

To prove sufficiency in 2), we first consider the case where $\rho\triangleq\rho(G,0)\in(0,1)$. It follows that

$$\lim_{n\to\infty}\rho^n\frac{(n-1)!}{(n-k_{\max})!}=0.$$

Since $\{g_n\}_{n=\max\{k_1,\ldots,k_r\}}^{\infty}$ is bounded, (B1) implies that

$$\lim_{n\to\infty}L_n = \lim_{n\to\infty}g_n\rho^n\frac{(n-1)!}{(n-k_{\max})!}=0.$$

Next, in the case where $\rho=0$, it follows that the number of nonzero components of $\{L_n\}_{n=0}^{\infty}$ is finite, and thus $\lim_{n\to\infty}L_n=0$. Hence, sufficiency in 2) is proved.

To prove sufficiency in 3), note that, since $\rho\triangleq\rho(G,0)=1$ and G has no repeated poles in $\mathbb{S}(0,1)$, it follows that $k_{\max}=1$. Substituting $k_{\max}=1$ and $\rho=1$ into (B1) yields, for all $n\geq\max\{k_1,\ldots,k_r\}$, $L_n=g_n$, and thus $0<g_\infty=\limsup_{n\to\infty}|L_n|<\infty$. Hence, sufficiency in 3) is proved.

To prove sufficiency in 4), we first consider the case where $\rho\triangleq\rho(G,0)=1$ and G has at least one repeated pole in $\mathbb{S}(0,1)$, that is $k_{\max}\geq 2$. Hence $\lim_{n\to\infty}|\rho^n\frac{(n-1)!}{(n-k_{\max})!}|=\infty$. Next, in the case where $\rho\in(1,\infty]$, it follows that $\lim_{n\to\infty}|\rho^n\frac{(n-1)!}{(n-k_{\max})!}|=\infty$. Hence, in the case where either $\rho\in(1,\infty]$ or both $\rho=1$ and G has at least one repeated pole in $\mathbb{S}(0,1)$, it follows that

$$\lim_{n\to\infty}\left|\rho^n\frac{(n-1)!}{(n-k_{\max})!}\right|=\infty. \tag{B4}$$

Next, $\limsup_{n\to\infty}|g_n|=g_\infty>0$ implies that there exists a subsequence $\{g_{n_j}\}_{j=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ such that, for all $j\geq 1$, $|g_{n_j}|\geq\frac{1}{2}g_\infty>0$. Replacing n in (B1) by n_j yields

$$\begin{aligned}\limsup_{j\to\infty}|L_{n_j}| &= \limsup_{j\to\infty}\left|g_{n_j}\rho^{n_j}\frac{(n_j-1)!}{(n_j-k_{\max})!}\right|\\ &\geq \frac{1}{2}g_\infty\limsup_{j\to\infty}\left|\rho^{n_j}\frac{(n_j-1)!}{(n_j-k_{\max})!}\right|.\end{aligned} \tag{B5}$$

Since, by (B4), $\limsup_{j\to\infty}|\rho^{n_j}\frac{(n_j-1)!}{(n_j-k_{\max})!}|=\infty$, (B5) implies that

$$\limsup_{n\to\infty}|L_n|\geq\limsup_{j\to\infty}|L_{n_j}|=\infty.$$

Hence sufficiency in 4) is proved.

To prove necessity in 2), suppose that $\rho(G,0)=1$ and every pole of G in $\mathbb{S}(0,1)$ is not repeated. Then, sufficiency in 3) implies that $0<\limsup_{n\to\infty}|L_n|<\infty$, which contradicts $\lim_{n\to\infty}L_n=0$. Next, suppose that either $\rho(G,0)\in(1,\infty]$ or both $\rho(G,0)=$

1 and G has at least one repeated pole in $\mathbb{S}(0, 1)$. Then, sufficiency in 4) implies that $\limsup_{n\to\infty} |L_n| = \infty$, which contradicts $\lim_{n\to\infty} L_n = 0$. Thus, $\lim_{n\to\infty} L_n = 0$ implies $\rho(G, 0) \in [0, 1)$. Hence, necessity in 2) is proved.

To prove necessity in 3), suppose that $\rho(G, 0) \in [0, 1)$. Then, sufficiency in 2) implies that $\lim_{n\to\infty} L_n = 0$, which contradicts $0 < \limsup_{n\to\infty} |L_n| < \infty$. Next, suppose that either $\rho(G, 0) \in (1, \infty]$ or both $\rho(G, 0) = 1$ and G has at least one repeated pole in $\mathbb{S}(0, 1)$. Then, sufficiency in 4) implies that $\limsup_{n\to\infty} |L_n| = \infty$, which contradicts $0 < \limsup_{n\to\infty} |L_n| < \infty$. Thus, $0 < \limsup_{n\to\infty} |L_n| < \infty$ implies that $\rho(G, 0) = 1$ and every pole of G in $\mathbb{S}(0, 1)$ is not repeated. Hence, necessity in 3) is proved.

To prove necessity in 4), suppose that $\rho(G, 0) \in [0, 1)$. Then, sufficiency in 2) implies that $\lim_{n\to\infty} L_n = 0$, which contradicts $\limsup_{n\to\infty} |L_n| = \infty$. Next, suppose that $\rho(G, 0) = 1$ and every pole of G in $\mathbb{S}(0, 1)$ is not repeated. Then, sufficiency in 3) implies that $0 < \limsup_{n\to\infty} |L_n| < \infty$, which contradicts $\limsup_{n\to\infty} |L_n| = \infty$. Thus, $\limsup_{n\to\infty} |L_n| = \infty$ implies that either $\rho(G, 0) \in (1, \infty]$ or both $\rho(G, 0) = 1$ and G has at least one repeated pole in $\mathbb{S}(0, 1)$. Hence, necessity in 4) is proved. ■