The analysis of this algorithm is very difficult and is beyond the scope of this paper. Extensive simulations conducted show very good results, some of which are presented here.

IV. Simulation Results

Extensive simulations have been conducted, some to verify the analysis of Section II and some to test the algorithm for the problem in Section III. In all our simulations, the following first-order system has been used

$$G(s) = \frac{1.5e^{-1.5s}}{s+1}. \quad (4.1)$$

In Fig. 1, we see the result of using the algorithm with a unit step input and employed as in Theorem 2.1. The convergence is fast and seems to be exponential, as predicted by the proof of Theorem 2.1. In Figs. 2–4, we repeated the experiment, each time with a different modification proposed in Remark 2.2. In each case, the convergence is quite similar to the one in Fig. 1, as predicted in Remark 2.2. To test for robustness, we have tried to use the algorithm without any modification when $b_1 \neq b$, and the algorithm diverged. In Fig. 5, we see the behavior of the algorithm when we use a square wave as the input. Again, the behavior verifies our discussion and (2.25). In Fig. 5(a), we take $b_1 < b$, and in Fig. 5(b), $b_1 > b$, both with similar results.

Finally, we have used the algorithm proposed in (3.5) for the case when a and b are unknown, with a square wave input. The results are very encouraging and are given in Fig. 6. We see that all three parameters converge to the correct values.

V. Conclusion

An algorithm for direct identification of an unknown time delay in an LTI system was presented. It is based on the commonly used RLS algorithm. The convergence of the proposed algorithm for minimal phase and stable systems, where only the time delay is unknown, is analyzed and proven. The robustness of the proposed algorithm to the knowledge of other parameters is also discussed. It is shown that, with an oscillating input such as the square wave or with integrators resetting, the algorithm is robust to inaccuracies in system parameters.

The algorithm is extended to the case where all parameters of the system are unknown. For this, there is no analytical support; the simulations conducted ,however, show very encouraging results.

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Dissipative H_2/H_∞ Controller Synthesis

Wassim M. Haddad, Dennis S. Bernstein, and Y. William Wang

Abstract— **In certain applications, such as the colocated control of flexible structures, the plant is known to be positive real. Hence, closed-loop stability is unconditionally guaranteed as long as the controller is also positive real. One approach to designing positive real controllers is the LQG-based positive real synthesis technique of Lozano-Leal and Joshi. The contribution of this paper is the extension of this positive real synthesis technique to include an H_∞–norm constraint on closed-loop performance.**

I. Introduction

In certain applications, such as the control of flexible structures, the plant transfer function is known to be positive real. This property arises if the sensor and actuator are colocated and also dual, for example, force actuator and velocity sensor or torque actuator and angular rate sensor. In practice, the prospects for controlling such systems is quite good since, if sensor and actuator dynamics are negligible, stability is unconditionally guaranteed as long as the controller is strictly positive real [1]–[3]. Although there is no general theory yet available for designing positive real controllers, a variety of techniques have been proposed based on H_2 theory [4]–[10] and H_∞ theory [11], [12].

In this paper, we focus on the H_2-based positive real controller synthesis method of Lozano-Leal and Joshi [7]. In [7], it is shown that if the plant is positive real and if the error and disturbance matrices satisfy certain constraints, then the LQG controller is also positive real. This approach is appealing in practice since it requires only standard LQG synthesis techniques. Our goal in this note is to extend the synthesis technique of [7] to include an H_∞-norm bound on the closed-loop transfer function [13]. This extension thus provides the control designer with more flexibility in specifying closed-loop system performance.

II. Preliminaries

In this section, we establish definitions and notation. Let $\mathcal{R}$ and $\mathcal{C}$ denote the real and complex numbers, let $(\ )^T$ and $(\ )^*$ denote transpose and complex conjugate transpose, respectively, and let I_n or I denote the $n \times n$ identity matrix. Furthermore, we write $\|\cdot\|_2$ for the Euclidean norm, $\|\cdot\|_F$ for the Frobenius matrix norm, $\sigma(\cdot)$ for the maximum singular value, tr for the trace operator, and

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$M \geq 0$ ($M > 0$) to denote the fact that the Hermitian matrix M is nonnegative (positive) definite. In this paper a *real-rational matrix function* is a matrix whose elements are rational functions with real coefficients. Furthermore, a *transfer function* is a real-rational matrix function each of whose elements is *proper*, i.e., finite at $s = \infty$. A *strictly proper transfer function* is a transfer function that is zero at infinity. Finally, an *asymptotically stable transfer function* is a transfer function each of whose poles is in the open left-half plane. The space of asymptotically stable transfer functions is denoted by RH_∞, i.e., the real-rational subset of H_∞. Let

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

denote a state-space realization of a transfer function $G(s)$, that is, $G(s) = C(sI - A)^{-1}B + D$. The notation "$\overset{\min}{\sim}$" is used to denote a minimal realization. The H_2 and H_∞ norms of an asymptotically stable transfer function $G(s)$ are defined as

$$\|G(s)\|_2 \triangleq \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|G(j\omega)\|_F^2 \, d\omega\right)^{1/2}, \tag{1}$$

$$\|G(s)\|_\infty \triangleq \sup_{\omega \in \mathcal{R}} \sigma_{\max}(G(j\omega)). \tag{2}$$

A square transfer function $G(s)$ is called *positive real* [14, p. 216] if: 1) all poles of $G(s)$ are in the closed left-half plane, and 2) $G(s) + G^*(s)$ is nonnegative definite for Re$[s] > 0$. A square transfer function $G(s)$ is called *strictly positive real* or *dissipative* [15], [16] if: 1) $G(s)$ is asymptotically stable, and 2) $G(j\omega) + G^*(j\omega)$ is positive definite for all real ω. Recall that a minimal realization of a positive real transfer function is stable in the sense of Lyapunov [17], while a strictly positive real transfer function is asymptotically stable [15].

For notational convenience in this paper, G will denote an $l \times m$ transfer function with input $u \in \mathcal{R}^m$, output $y \in \mathcal{R}^l$, and internal state $x \in \mathcal{R}^n$. We will omit all matrix dimensions throughout and assume that all quantities have compatible dimensions. Note that if the plant is positive real, then $l = m$ and the resulting compensator is square. Next, we state the well-known positive real lemma [17], [18].

Lemma 2.1: The strictly proper transfer function $G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ is positive real if and only if there exist matrices Q_0 and L with Q_0 positive definite such that

$$AQ_0 + Q_0A^T = -LL^T \tag{3}$$

$$Q_0C^T = B. \tag{4}$$

This form of the positive real lemma is the dual of that given in [17], and the derivation is similarly dual. See [18] for further details on the dual positive real lemma.

The dual version of Lemma 2.1 can be obtained by replacing A by A^T and B by C^T. In this case, $G(s)$ is positive real if and only if there exist matrices P_0 and $\hat{L}$ with P_0 positive definite such that

$$A^TP_0 + P_0A = -\hat{L}^T\hat{L}, \tag{5}$$

$$P_0B = C^T. \tag{6}$$

Recall that in the case in which $G(s)$ is strictly positive real, it follows that $(A, \tilde{L})$ is observable [15]. Finally, we give a key definition and a lemma involving self-dual realizations.

Definition 2.1: Let $G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ be a positive real transfer function. Then $\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ is a *self-dual realization* of $G(s)$ if $A + A^T \leq 0$ and $B = C^T$.

Self-dual realizations are convenient since conditions (3)–(6) are satisfied by $Q_0 = P_0 = I$ and $LL^T = \hat{L}^T\hat{L} = -(A + A^T)$. The next result due to [7] shows that positive real transfer functions always have self-dual realizations.

Lemma 2.2: Let $G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ be positive real, and let positive-definite Q_0 and L satisfy (3), (4). Then $\left[\begin{array}{c|c} Q_0^{-1/2}AQ_0^{1/2} & Q_0^{-1/2}B \\ \hline CQ_0^{1/2} & 0 \end{array}\right]$ is a self-dual realization of $G(s)$.

III. Problem Statement and Main Results

In this section, we begin by obtaining H_2 dynamic output-feedback controllers with constrained H_∞ disturbance attenuation. We then use this result to derive dissipative H_2/H_∞ controllers for a given positive real plant.

H_2/H_∞ Control Problem: Given the n-th order stabilizable and detectable plant

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t), \tag{7}$$

$$y(t) = Cx(t) + D_2w(t) \tag{8}$$

determine an n-th order dynamic compensator $G_c(s) \sim \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$ of the form

$$\dot{x}_c(t) = A_cx_c(t) + B_cy(t), \tag{9}$$

$$u(t) = C_cx_c(t) \tag{10}$$

that satisfies the following design criteria:

1) the closed-loop system (7)–(10) given by $\tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}$ is asymptotically stable,

2) the closed-loop transfer function $\tilde{G}(s) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{D} \\ \hline \tilde{E} & 0 \end{array}\right]$ from the disturbance $w(t)$ to the performance variables $z(t) = E_1x(t) + E_2u(t)$ satisfies the constraint

$$\left\|\tilde{G}(s)\right\|_\infty \leq \gamma \tag{11}$$

where $\gamma > 0$ is a given constant, $\tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}$, and $\tilde{E} \triangleq [E_1 \quad E_2C_c]$, and

3) the H_2 performance measure

$$J(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \frac{1}{t}\int_0^t [x^T(s)R_1x(s) + u^T(s)R_2u(s)]\, ds \tag{12}$$

$$= \left\|\tilde{G}(s)\right\|_2^2 \tag{13}$$

is minimized, where $R_1 \triangleq E_1^TE_1$, $R_2 \triangleq E_2^TE_2 > 0$, and $E_1^TE_2 = 0$.

The basis for our approach is the mixed-norm H_2/H_∞ framework developed in [13]. For the case of equalized H_2/H_∞ weights, a full-order dynamic compensator satisfying design constraints 1), 2), and providing a bound for 3) is given by the following theorem. For convenience, define $V_1 \triangleq D_1D_1^T$, $V_2 \triangleq D_2D_2^T > 0$, and assume $D_1D_2^T = 0$.

Theorem 3.1: Suppose there exist $n \times n$ nonnegative-definite matrices Q and P satisfying

$$0 = AQ + QA^T + V_1 + \gamma^{-2}QR_1Q - QC^TV_2^{-1}CQ \quad (14)$$

$$0 = (A + \gamma^{-2}QR_1)^TP + P(A + \gamma^{-2}QR_1) + R_1 - PBR_2^{-1}B^TP + \gamma^{-2}PQC^TV_2^{-1}CQP \quad (15)$$

and let (A_c, B_c, C_c) be given by

$$A_c = A - QC^TV_2^{-1}C - BR_2^{-1}B^TP + \gamma^{-2}QR_1 \quad (16)$$

$$B_c = QC^TV_2^{-1} \quad (17)$$

$$C_c = -R_2^{-1}B^TP. \quad (18)$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $\tilde{G}(s)$ satisfies the H_∞ disturbance attenuation constraint (11) and the H_2 performance criterion (13) satisfies the bound

$$J(A_c, B_c, C_c) \leq \operatorname{tr}[QR_1 + QC^TV_2^{-1}CQP]. \quad (19)$$

Proof: See [13, Proposition 5.6]. □

Note that using (16)–(18), the dynamic compensator (9), (10) is given by

$$\dot{x}_c(t) = (A - QC^TV_2^{-1}C - BR_2^{-1}B^TP + \gamma^{-2}QR_1)x_c(t) + QC^TV_2^{-1}y(t), \quad (20)$$

$$u(t) = -R_2^{-1}B^TPx_c(t). \quad (21)$$

We now assume that the plant (7), (8) is positive real and seek a strictly positive real controller within a negative feedback configuration.

Dissipative H_2/H_∞ Control Problem: Given the n-th order minimal positive real plant (7), (8), determine an n-th order compensator $G_c(s) \sim \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$ that satisfies the design criteria 2) and 3) with the additional property that $-G_c(s) \sim \left[\begin{array}{c|c} A_c & B_c \\ \hline -C_c & 0 \end{array}\right]$ is strictly positive real.

Note that in this case, since the plant is positive real and the negative feedback compensator is strictly positive real, condition 1) is automatically satisfied [1]. We now present our main result, which shows that if the design weights are chosen in a specific manner, then the controller is positive real. This choice of design weights is a direct generalization to the H_2/H_∞ problem of the H_2 design weights that were originally proposed in [7].

Theorem 3.2: Assume $G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ is positive real, and let Q_0 and L satisfy (3), (4), where Q_0 is positive definite. Furthermore, assume that there exist $n \times n$ nonnegative-definite matrices Q and P satisfying (14), (15), where R_1, R_2, V_1, V_2 satisfy

$$V_1 = LL^T + BR_2^{-1}B^T - \gamma^{-2}Q_0R_1Q_0 > 0, \quad (22)$$

$$V_2 = R_2, \quad (23)$$

$$R_1 > C^TR_2^{-1}C. \quad (24)$$

Then the negative feedback dynamic compensator $-G_c(s) \sim \left[\begin{array}{c|c} A_c & B_c \\ \hline -C_c & 0 \end{array}\right]$ given by (16)–(18) is strictly positive real and satisfies the design criteria 1), 2). Furthermore, the H_2 performance criterion satisfies the bound

$$J(A_c, B_c, C_c) \leq \operatorname{tr}[QR_1 + QC^TV_2^{-1}CQP]. \quad (25)$$

Proof: The H_2 performance bound (25) and the closed-loop H_∞ disturbance attenuation constraint are direct consequences of Theorem 3.1. Next, we show that $-G_c(s) \sim \left[\begin{array}{c|c} A_c & B_c \\ \hline -C_c & 0 \end{array}\right]$ is strictly positive real. Using (22), (14) can be written as

$$0 = AQ + QA^T + LL^T + BR_2^{-1}B^T - \gamma^{-2}Q_0R_1Q_0 + \gamma^{-2}QR_1Q - QC^TV_2^{-1}CQ. \quad (26)$$

Since the open-loop plant is positive real, it follows from Lemma 2.1 that $LL^T = -(AQ_0 + Q_0A^T)$ and $C = B^TQ_0^{-1}$. Hence, $Q = Q_0$ is a solution to (26). Next, adding and subtracting $PBR_2^{-1}B^TQ_0^{-1}$, $Q_0^{-1}BR_2^{-1}B^TP$, $PBR_2^{-1}B^TP$ to and from (15) yields

$$0 = (A - Q_0C^TV_2^{-1}C - BR_2^{-1}B^TP + \gamma^{-2}QR_1)^TP + P(A - Q_0C^TV_2^{-1}C - BR_2^{-1}B^TP + \gamma^{-2}QR_1) + R_1 - Q_0^{-1}BR_2^{-1}B^TQ_0^{-1} + (Q_0^{-1} + P) \cdot BR_2^{-1}B^T(Q_0^{-1} + P) + \gamma^{-2}PBR_2^{-1}B^TP. \quad (27)$$

Using (16), (27) can be written as

$$A_c^TP + PA_c = -[R_1 - Q_0^{-1}BR_2^{-1}B^TQ_0^{-1} + (Q_0^{-1} + P) \cdot BR_2^{-1}B^T(Q_0^{-1} + P) + \gamma^{-2}PBR_2^{-1}B^TP] \quad (28)$$

or, equivalently, since by (4) and (24) $R_1 - Q_0^{-1}BR_2^{-1}B^TQ_0^{-1} > 0$,

$$A_c^TP + PA_c = -L_c^TL_c = -L_c^2 < 0 \quad (29)$$

where L_c is the positive-definite square root of the positive-definite matrix on the right-hand side of (28). Furthermore, using (17), (18), (23), and $B^T = QC$, it follows that $PB_c = -C_c^T$. Finally, since $G(s)$ is positive real and $-G_c(s)$ is strictly positive real, it follows from [3, Theorem 7.2] that $\tilde{A}$ is asymptotically stable. □

Remark 3.1: Inequality (24) assures that $A_c^TP + PA_c < 0$. Nevertheless, if (24) does not hold, but rather the weaker condition

$$R_1 - Q_0^{-1}BR_2^{-1}B^TQ_0^{-1} + (Q_0^{-1} + P)BR_2^{-1}B^T(Q_0^{-1} + P) + \gamma^{-2}PBR_2^{-1}B^TP > 0 \quad (30)$$

is satisfied, then $A_c^TP + PA_c < 0$. Thus, when (24) fails, one can use (30) to guarantee that the controller is positive real. Note, however, that (30) cannot be verified *a priori* since it involves the matrix P which satisfies (15).

Remark 3.2: Note that if the H_∞ disturbance attenuation constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, then (14) and (15) approach the standard LQG observer and regulator Riccati equations. In this case, Theorem 3.2 can be applied with (22) replaced by

$$V_1 = LL^T + BR_2^{-1}B^T \quad (31)$$

to yield dissipative LQG controllers. If the plant realization is self-dual, then the dissipative LQG controller given by (20), (21) is equivalent to the dissipative controller obtained in [7].

To apply Theorems 3.1 and 3.2, it is necessary to satisfy the positive real conditions (3), (4). For the case of a flexible structure with m force inputs and m velocity measurements, the colocated admittance, or driving point mobility, is characterized by

$$M\ddot{q} + C\dot{q} + Kq = B_0u \quad (32)$$

$$y = B_0^T\dot{q} \quad (33)$$

where M, C, and K are mass, damping, and stiffness matrices, respectively, and B_0 is determined by the sensor/actuator locations.

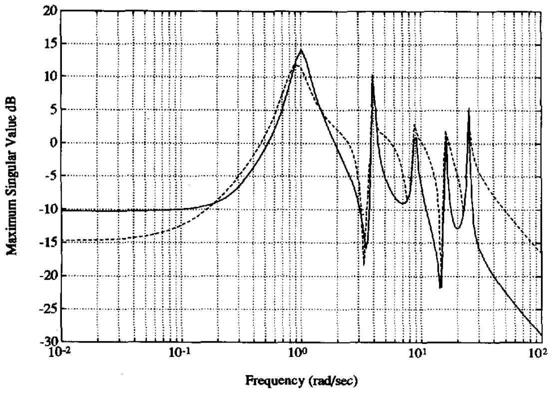

Fig. 1. Comparison of $\|\tilde{G}(s)\|_\infty$ for H_2 positive real (solid line) and H_2/H_∞ positive real (dashed line) controllers.

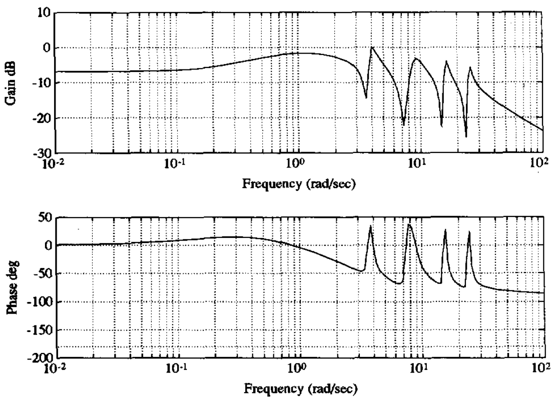

Fig. 2. Frequency response of the H_2/H_∞ positive real controller.

Choosing a realization for the system (32), (33) by

$$G(s) \sim \left[\begin{array}{c|c} \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} & \begin{bmatrix} 0 \\ M^{-1}B_0 \end{bmatrix} \\ \hline [0 \quad B_0^T] & 0 \end{array}\right], \tag{34}$$

it follows that (3), (4) are satisfied by

$$Q_0 = \begin{bmatrix} K^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix}, \qquad L = \begin{bmatrix} 0 \\ \sqrt{2}M^{-1}C^{1/2} \end{bmatrix}, \tag{35}$$

while (5), (6) hold with

$$P_0 = \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}, \qquad \hat{L} = [0 \quad \sqrt{2}C^{1/2}]. \tag{36}$$

Thus, $G(s)$ has the self-dual realization

$$G(s) \sim \left[\begin{array}{c|c} \begin{bmatrix} 0 & K^{1/2}M^{-1/2} \\ -M^{-1/2}K^{1/2} & -M^{-1/2}CM^{-1/2} \end{bmatrix} & \begin{bmatrix} 0 \\ M^{-1/2}B_0 \end{bmatrix} \\ \hline [0 \quad B_0^T M^{-1/2}] & 0 \end{array}\right]. \tag{37}$$

Similar expressions appear in [8].

IV. Illustrative Numerical Example

For illustrative purposes, consider a simply supported Euler–Bernouli beam. The partial differential equation for the transverse deflection $w(x, t)$ is given by

$$m(x)\frac{\partial^2 w(x, t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x, t)}{\partial x^2}\right] = f(x, t) \tag{38}$$

with boundary conditions

$$w(x, t)|_{x=0,\, L} = 0, \qquad EI\left.\frac{\partial^2 w(x, t)}{\partial x^2}\right|_{x=0,\, L} = 0, \tag{39}$$

where $m(x)$ is the mass per unit length and $EI(x)$ is the flexural rigidity, with E denoting Young's modulus of elasticity and $I(x)$ denoting the cross-sectional area moment of inertia about an axis normal to the plane of vibration and passing through the center of the cross-sectional area. Finally, $f(x, t)$ is the force distribution due to a single control actuator. Assuming uniform beam properties with $m = m(x)$, the modal decomposition of this system has the form

$$w(x, t) = \sum_{r=1}^{\infty} W_r(x)q_r(t), \tag{40}$$

$$\int_0^L mW_r^2(x)\,dx = 1, \qquad W_r(x) = \sqrt{\frac{2}{mL}}\sin\frac{r\pi x}{L}, \quad r = 1, 2, \cdots \tag{41}$$

where, assuming uniform proportional damping, the modal coordinates q_r satisfy

$$\ddot{q}_r(t) + 2\zeta\omega_r\dot{q}_r(t) + \omega_r^2 q_r(t) = \int_0^L f(x, t)W_r(x)\,dx, \quad r = 1, 2, \cdots. \tag{42}$$

For simplicity, assume $L = \pi$ and $m = EI = 2/\pi$ so that $\sqrt{2/mL} = 1$. Furthermore, assume that $f(x, t)$ arises from a point force actuator and a velocity sensor both located at $x = 0.55L$. Finally, modeling the first five modes and defining the plant state as $x = [q_1 \; \dot{q}_1 \; \cdots \; q_5 \; \dot{q}_5]^T$ and defining the performance of the beam in terms of the velocity at $x = 0.7L$, the resulting state space model and problem data are

$$A = \text{block-diag}_{i=1,\cdots,5}\begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix}, \quad \omega_i = i^2, \quad i = 1, \cdots, 5, \quad \zeta = 0.01,$$

$$B = C^T = [0 \; 0.09877 \; 0 \; -0.309 \; 0 \; -0.891 \; 0 \; 0.5878 \; 0 \; 0.7071]^T$$

$$E_1 = \begin{bmatrix} 0 & 0.809 & 0 & -0.951 & 0 & 0.309 & 0 & 0.5878 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_2 = [0 \quad 1.9]^T, \qquad R_1 = E_1^T E_1,$$
$$D_1 = [B \quad 0_{10\times 1}], \qquad D_2 = [0 \quad 1.9]$$

$$V_2 = R_2 = D_2 D_2^T = E_2^T E_2 = 3.61.$$

Note that with the above data, conditions (22) and (24) are not satisfied with strict inequality. Nevertheless, for $\gamma = \infty$, the H_2 controller was found to be positive real and yielded a closed-loop H_∞ performance of 14.13 dB (i.e, 14.13 dB above unity gain). Furthermore, for $\gamma = 12.02$ dB, the H_2/H_∞ positive real controller yielded a net H_∞ performance improvement of 2.65 dB (see Fig. 1). This result is consistent with [19, Theorem 1], which implies that the maximum ratio of the H_∞ performance of the optimal H_2 controller to the H_∞ performance of the optimal H_∞ controller can be no more than twice the number of right-half plane transmission zeros for the transfer function between disturbances and measurements, and between control signals and performance variables. For the present problem with one nonminimum phase zero for the second transfer function, this bound corresponds to a factor of 2 (i.e., 6 dB). Finally, Fig. 2 shows the gain and phase plots of the H_2/H_∞ positive real controller.

V. Conclusion

In this note, we extended the positive real synthesis technique of Lozano-Leal and Joshi to include an H_∞ constraint on closed-loop performance. The result involves constraining the allowable H_2/H_∞ weights to guarantee that the controller is positive real. Current research is focusing on more general choices of the design weights as well as applications to passive absorber synthesis.

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An Algorithm for Robust Pole Assignment Via Polynomial Approach

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***Abstract*— A computational method for designing controllers which attempt to place the roots of the characteristic polynomial of an uncertain system inside some prescribed regions is presented. The analysis is based on transfer functions of characteristic polynomials, and the problem is formulated as one of semi-infinite programming. An example of an application is given to illustrate this approach.**

I. Introduction

In the selection of a suitable control scheme and associated controller parameters, the designer must decide what can be considered as an acceptable closed-loop response. When the plant parameters are liable to vary, the designer is faced with the question of whether the closed-loop system will remain stable, and if so, whether the perturbed closed-loop dynamics will continue to resemble the nominal response. From the point of view of robust performance subject to plant parameter variations, it is desirable that the closed-loop transfer function poles remain constrained within certain closed regions surrounding their nominal locations.

Pole assignment is a common approach for designing closed-loop controllers in order to meet desired control specifications (see, for instance, Kučera [1] or Åström and Wittenmark [2]). However, this problem has received some criticism due to the assumption of complete state observation in earlier works, and the implicit assumption that the models used in design are accurate. It has been shown that if the condition of complete state observation is relaxed, then a dynamic output compensator can assign almost arbitrary poles for the closed-loop system.

In a previous work, Soh *et al.* [3], the objective of assigning closed-loop poles was replaced by that of assigning characteristic polynomials, and a solution was given for transfer functions with coefficients varying on an interval region; after characterizing the set of admissible controllers, either the distance from a nominal controller or a robustness measure is optimized (see also Soh [4]). Rotstein *et al.* [5], [6] considered a more general plant uncertainty description, but the control objective is still that of assigning characteristic polynomials. This substitution (assigning characteristic polynomials instead assigning closed-loop poles), which is certainly not trivial, constitute the main drawback of those approaches. This limitation is overcome in the present paper by reformulating the problem, incorporating additional constraints. These constraints relate the pole position in the open left-half plane to the real variation of the coefficients in the characteristic polynomial, thus defining a region in

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