INT. J. CONTROL, 2002, VOL. 75, NO. 8, 559–571

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Fixed-structure discrete-time $\mathcal{H}_2/\mathcal{H}_\infty$ controller synthesis using the delta operator

R. SCOTT ERWIN†* and DENNIS S. BERNSTEIN‡

This paper considers the fixed-structure, discrete-time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller synthesis problem in the delta operator (difference operator) framework. The differential operator and shift operator versions of the problem are reviewed for comparison, and necessary conditions are derived for all three formulations. A quasi-Newton/continuation algorithm is then used to obtain approximate solutions to these equations. Controllers are synthesized for two numerical examples, and the performance of the algorithm on the differential, difference and shift operator versions of the problems is compared.

1. Introduction

Control design problems encountered in practice may involve highly coupled multivariable systems with complex dynamics and as a result are modelled by high-dimensional differential equations. While standard optimal control techniques are inherently multivariable, these techniques suffer from the disadvantage that the resulting control equations are the same order (or substantially higher) than those of the system model. This fact often leads to implementation problems in high-bandwidth applications where small sample periods limit the amount of computation that can be performed in real time.

One approach to control design for such problems is through the use of model reduction techniques, that is, algorithms for producing a lower-dimensional approximation of a high-dimensional linear system. Although these strategies can yield acceptable results, they suffer in general from a lack of guarantees about the properties of the closed-loop system; that is, the feedback loop containing the reduced-order controller and the high-dimensional system model. Furthermore, extensions to more specialized controller architectures, such as decentralized controller synthesis, are not available.

As an alternative strategy for addressing control problems subject to architecture constraints, *fixed-structure* technique have been proposed. These techniques provide a direct method for synthesizing high-performance, robust controllers for complex, multivariable systems subject to constraints on signal flow and controller complexity. Fixed-structure methods allow the designer to specify the architecture of the controller while addressing performance objectives and robustness constraints.

Fixed-structure techniques have been applied to both the reduced-order $\mathcal{H}_2$-optimal control and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems. Initial work on the continuous-time (s-domain) mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem (Bernstein and Haddad 1989, Khargonekar and Rotea 1991) utilized a Riccati equation approach that includes a bound on the $\mathcal{H}_2$ performance while enforcing an $\mathcal{H}_\infty$ constraint. More recent research has considered the use of continuation techniques to eliminate the bound on the closed-loop $\mathcal{H}_2$ norm (Haddad and Bernstein 1990, Luke *et al.* 1994), or has utilized non-Riccati methods for enforcing $\mathcal{H}_\infty$ norm constraints (Walker 1994). Discrete-time extensions of these problems have also been proposed in Jacques (1995) and Davis *et al.* (1996) utilizing the standard shift operator (q-domain) framework.

Fixed-structure design problems are, in general, nonconvex optimization problems. In addition, the fundamental problem of determining the existence or uniqueness of stabilizing controllers of a given order or structure is still open (Syrmos *et al.* 1997). Specific fixed-structure problems have been shown to be NP-hard (Toker and Özbay 1995), indicating that there may be no algorithm that computes the solution to the problem such that the time required scales in a polynomial fashion with the dimension of the problem.

Bypassing the question of existence and uniqueness, there has been progress on characterizing and computing optimal fixed-structure controllers using iterative computational algorithms. One such approach is to apply bilinear or alternative linear matrix inequality algorithms (LMI's) (Goh *et al.* 1994, Iwasaka and Skelton 1995, Grigoriadis and Skelton 1996). This technique involves computing the solution to an LMI, the inverse of which is the solution to a second LMI. Another approach is to obtain first order necessary conditions for optimality. These conditions involve nonlinear algebraic equations, which in general have no analytic solution. Various algorithms have been devel-

Received 7 April 1999. Revised 20 October 2001.

* Author for correspondence. e-mail: richard.erwin@kirtland.af.mil

† Air Force Research Laboratory, Space Vehicles Directorate, AFRL/VSSV, 3550 Aberdeen Ave. SE, Kirtland AFB, NM 87117-5776, USA.

‡ Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2140, USA.

International Journal of Control ISSN 0020–7179 print/ISSN 1366–5820 online © 2002 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/00207170210121925

oped for these equations, including homotopic or continuation algorithms (Hyland and Richter 1990, Mercadal 1991, Ge *et al*. 1994, Collins Jr. *et al*. 1995, 1997) and gradient-based algorithms (Dennis Jr. and Schnabel 1983, Ly *et al*. 1985, Toivonen and Mäkilä 1985, Mäkilä and Toivonen 1987, Harn and Kosut 1993).

In the present paper we are concerned with fixed-structure control for discrete-time systems. For discrete-time systems, it has been shown that the standard shift operator formulation can lead to numerical ill-conditioning when high sampling rates are used (Middleton and Goodwin 1990, Goodwin *et al*. 1992, Gevers and Li 1993) in conjunction with finite precision arithmetic. An alternative, but equivalent, formulation of discrete-time systems is based on the delta (difference) operator which has been shown to be less sensitive to numerical round-off errors than the shift operator (Middleton and Goodwin 1990). In particular, Middleton and Goodwin (1990) has shown that shift operator representations of discrete-time systems become numerically ill-conditioned at much lower sampling frequencies than the equivalent difference operator representation of the same system.

The standard LQR/LQG and $\mathcal{H}_\infty$ optimal control problems have been formulated within the difference operator framework in Middleton and Goodwin (1990). These controllers can be computed from the solutions to standard Riccati equations, for which stable, efficient numerical algorithms already exist. The purpose of the present work is to formulate the fixed-structure $\mathcal{H}_2$-optimal and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller synthesis problems within a difference operator framework in order to determine whether the difference operator offers significant advantages over the shift operator for fixed-structure controller synthesis problems. Our results show that this is indeed the case, and we show by example that numerical algorithms may fail to find a solution to the fixed-structure problem when posed in the standard shift operator framework while succeeding in the difference operator framework.

The paper is organized as follows. Section 2 provides a brief review of difference operator theory and definitions. Based on standard results for shift operator systems, § 3 provides state-space techniques for calculating the $\mathcal{H}_2$ norm and $\mathcal{H}_\infty$ norm of difference operator systems. The decentralized static output feedback framework for fixed-structure controller synthesis is summarized, and together with the results of §§ 3 and 4 is used to pose the fixed-structure $\mathcal{H}_2$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problems for difference operator systems. The differential operator and shift operator versions of the problems are stated to provide a unified framework for comparison. Necessary conditions for these problems based on Lagrangian techniques are then derived. To compute approximate solutions to these equations, a hybrid quasi-Newton/continuation algorithm is utilized, which is briefly described in § 7. Finally, $\mathcal{H}_2$-optimal synthesis results for a 10th-order model of a lightly damped flexible beam and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis results for a 4th-order airplane longitudinal dynamics model are presented in §§ 8 and 9, respectively. A comparison of the performance of the quasi-Newton/continuation algorithm on differential operator, shift operator and difference operator formulations of all problems is provided. The paper concludes with a discussion of the results in § 10.

2. Preliminaries

Let $f:[0,\infty)\rightarrow\mathbb{R}^n$ be differentiable, and assume $f(0)=0$. The Laplace transform of $f(t)$, denoted by $\mathbf{f}(s)$ or $\mathbf{L}[f(t)]$, is defined as

$$\mathbf{f}(s)=\mathbf{L}[f(t)]\triangleq\int_0^\infty f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t \tag{1}$$

where $s\in\mathbb{C}$ and satisfies

$$\mathbf{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right]=s\,\mathbf{f}(s) \tag{2}$$

Let $\{g(k)\}_{k=0}^\infty\subset\mathbb{R}^n$, and assume $g(0)=0$. The q-transform (or Z-transform) of this sequence, denoted by $\mathbf{g}(q)$ or $\mathbf{Q}[\{g(k)\}]$, is defined by

$$\mathbf{g}(q)=\mathbf{Q}[\{g(k)\}]\triangleq\sum_{k=0}^\infty g(k)q^{-k} \tag{3}$$

where $q\in\mathbb{C}$ and satisfies

$$\mathbf{Q}[\{g(k+1)\}]=q\,\mathbf{g}(q) \tag{4}$$

Alternatively, a sequence $\{l(k)\}_{k=0}^\infty\subset\mathbb{R}^n$ can also be transformed by the δ-transform (Middleton and Goodwin 1990), which is denoted $\mathbf{l}(\delta)$ or $\mathbf{D}[\{l(k)\}]$ and is defined by

$$\mathbf{l}(\delta)=\mathbf{D}[\{l(k)\}]\triangleq h\sum_{k=0}^\infty l(k)(1+h\delta)^{-k} \tag{5}$$

where $h>0$ is the *sample period* and $\delta\in\mathbb{C}$. The delta transform of a sequence satisfies

$$\mathbf{D}[h^{-1}\{l(k+1)-l(k)\}]=\delta\,\mathbf{l}(\delta) \tag{6}$$

In this paper, we consider linear continuous-time systems of the form

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{7}$$

$$y(t)=Cx(t)+Du(t) \tag{8}$$

and linear discrete-time systems of the form

$$x(k+1) = Ax(k) + Bu(k) \tag{9}$$

$$y(k) = Cx(k) + Du(k) \tag{10}$$

or

$$x(k+1) = x(k) + h[Ax(k) + Bu(k)] \tag{11}$$

$$y(k) = Cx(k) + Du(k) \tag{12}$$

Assuming $x(0) = 0$ and taking the Laplace transform of (7) and (8), the q-transform of (9) and (10) and the δ-transform of (11) and (12) yields

$$\left.\begin{aligned} \zeta\mathbf{x}(\zeta) &= A\mathbf{x}(\zeta) + B\mathbf{u}(\zeta) \\ \mathbf{y}(\zeta) &= C\mathbf{x}(\zeta) + D\mathbf{u}(\zeta) \end{aligned}\right\} \tag{13}$$

where ζ denotes either s, q or δ, where $\mathbf{x}(\zeta)$, $\mathbf{y}(\zeta)$ and $\mathbf{u}(\zeta)$ are the transforms of the corresponding functions or sequences. The transfer function $G(\zeta)$ relating u to y, that is, $\mathbf{y}(\zeta) = G(\zeta)\mathbf{u}(\zeta)$, is then given by

$$G(\zeta) = C(\zeta I - A)^{-1}B + D \tag{14}$$

For convenience we denote (14) by $G(\zeta) \sim (A, B, C, D)$.

Definition 1: Two discrete-time transfer functions are *equivalent* if identical inputs produce identical outputs.

Proposition 1: *If $\hat{G}(q)$ is a q-domain transfer function, then $\hat{G}(q)$ is equivalent to the δ-domain transfer function $\bar{G}(\delta)$ given by*

$$\bar{G}(\delta) \triangleq \hat{G}(1 + h\delta) \tag{15}$$

If $\bar{G}(\delta)$ is a δ-domain transfer function, then $\bar{G}(\delta)$ is equivalent to the q-domain transfer function $\hat{G}(q)$ defined by

$$\hat{G}(q) \triangleq \bar{G}\left(\frac{q-1}{h}\right) \tag{16}$$

Furthermore, if $\hat{G}(q) \sim (\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is a q-domain transfer function, and $\bar{G}(\delta) \sim (\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is an equivalent δ-domain transfer function, then

$$\bar{A} = h^{-1}(\hat{A} - I), \quad \bar{B} = h^{-1}\hat{B}, \quad \bar{C} = \hat{C}, \quad \bar{D} = \hat{D} \tag{17}$$

Proof: See Middleton and Goodwin (1990, p. 46). □

For the following definitions, let spec (A) denote the set of eigenvalues A.

Definition 2: A square matrix A is *ζ-stable* if, for all $\lambda \in \operatorname{spec}(A)$

$$\operatorname{Re}\lambda < 0, \quad \zeta = s$$

$$|\lambda| < 1, \quad \zeta = q$$

$$\frac{h}{2}|\lambda|^2 + \operatorname{Re}\lambda < 0, \quad \zeta = \delta$$

Furthermore, a transfer function $G(\zeta)$ is ζ-stable if there exists a realization (A, B, C, D) of $G(\zeta)$ such that A is ζ-stable.

Lemma 1: *Let $G(\zeta) \sim (A, B, C, D)$ and assume A is ζ-stable. Then there exists a unique, positive-semidefinite matrix P such that*

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C, \qquad \zeta = s \tag{18}$$

$$0 = A^{\mathrm{T}}PA - P + C^{\mathrm{T}}C, \qquad \zeta = q \tag{19}$$

$$0 = A^{\mathrm{T}}P + PA + hA^{\mathrm{T}}PA + C^{\mathrm{T}}C, \quad \zeta = \delta \tag{20}$$

In particular, P is given by

$$P = \int_0^\infty \mathrm{e}^{A^{\mathrm{T}}\tau} C^{\mathrm{T}}C\,\mathrm{e}^{A\tau}\,\mathrm{d}\tau, \qquad \zeta = s \tag{21}$$

$$P = \sum_{k=0}^{\infty}(A^{\mathrm{T}})^k C^{\mathrm{T}}CA^k \qquad \zeta = q \tag{22}$$

$$P = h\sum_{k=0}^{\infty}(I + hA^{\mathrm{T}})^k C^{\mathrm{T}}C(I + hA)^k, \quad \zeta = \delta \tag{23}$$

3. The $\mathcal{H}_2$-norm in the δ-domain

The $\mathcal{H}_2$ norm of a transfer function $G(\zeta) \sim (A, B, C, D)$, where A is ζ-stable, is defined by

$$\|G(\zeta)\|_2 \triangleq \left[\frac{1}{2\pi\jmath}\int_{\operatorname{Re}(s)=0} \operatorname{tr}[G^*(s)G(s)]\,\mathrm{d}s\right]^{1/2}, \qquad \zeta = 5, D = 0 \tag{24}$$

$$\triangleq \left[\frac{1}{2\pi\jmath}\oint_{|q|=1} \operatorname{tr}[G^*(q)G(q)]\frac{1}{q}\,\mathrm{d}q\right]^{1/2}, \qquad \zeta = q \tag{25}$$

$$\triangleq \left[\frac{1}{2\pi\jmath}\oint_{|1+h\delta|=1} \operatorname{tr}[G^*(\delta)G(\delta)]\frac{1}{1+h\delta}\mathrm{d}\delta\right]^{1/2}, \qquad \zeta = \delta \tag{26}$$

Lemma 2: *If $\hat{G}(q)$ and $\bar{G}(\delta)$ are equivalent ζ-stable transfer functions, then*

$$\|\bar{G}(\delta)\|_2 = \frac{1}{\sqrt{h}}\|\hat{G}(q)\|_2 \tag{27}$$

Proof: The proof is immediate from the change of variables $q = 1 + h\delta$ in (25). □

Proposition 2: *Let $G(\zeta) \sim (A, B, C, D)$ and assume A is ζ-stable. Then*

$$\|G(\zeta)\|_2^2 = \operatorname{tr}[B^{\mathrm{T}}PB], \qquad \zeta = s, D = 0 \tag{28}$$

$$= \operatorname{tr}[B^{\mathrm{T}}PB + D^{\mathrm{T}}D], \qquad \zeta = q \tag{29}$$

$$= \operatorname{tr}\left[B^{\mathrm{T}}PB + \frac{1}{h}D^{\mathrm{T}}D\right], \qquad \zeta = \delta \tag{30}$$

where P satisfies (18), (19) or (20).

Proof: See Appendix A. □

4. The $\mathcal{H}_\infty$-norm in the δ-domain

The $\mathcal{H}_\infty$ norm of a matrix transfer function $G(\zeta) \sim (A, B, C, D)$, where A is ζ-stable, is defined as

$$\|G(\zeta)\|_\infty \triangleq \sup_{-\infty<\omega<\infty} \sigma_{\max} G(\jmath\omega), \qquad \zeta = s \tag{31}$$

$$\triangleq \max_{-\pi<\theta<\pi} \sigma_{\max} G(\mathrm{e}^{\jmath\theta}), \qquad \zeta = q \tag{32}$$

$$\triangleq \max_{-\pi<\theta<\pi} \sigma_{\max} G(h^{-1}(\mathrm{e}^{\jmath\theta} - 1)), \qquad \zeta = \delta \tag{33}$$

If $\hat{G}(q)$ is a q-stable transfer function and $\bar{G}(\zeta)$ is the equivalent δ-domain transfer function, then

$$\|\bar{G}(\delta)\|_\infty = \|\hat{G}(q)\|_\infty \tag{34}$$

Proposition 3: *Let $G(\zeta) \sim (A, B, C, D)$, where A is ζ-stable, and define*

$$R \triangleq I - \gamma^{-2}D^{\mathrm{T}}D, \quad S \triangleq \gamma^{-1}C^{\mathrm{T}}D \tag{35}$$

Suppose there exists a positive-semidefinite solution P to the algebraic Riccati equation

$$0 = A^{\mathrm{T}}P + PA + \gamma^{-1}C^{\mathrm{T}}C + \gamma^{-1}(B^{\mathrm{T}}P + S^{\mathrm{T}})^{\mathrm{T}}R^{-1}(B^{\mathrm{T}}P + S^{\mathrm{T}}), \quad \zeta = s \tag{36}$$

$$0 = A^{\mathrm{T}}PA - P + \gamma^{-1}C^{\mathrm{T}}C + \gamma^{-1}(B^{\mathrm{T}}PA + S^{\mathrm{T}})^{\mathrm{T}}(R - \gamma^{-1}B^{\mathrm{T}}PB)^{-1} \times (B^{\mathrm{T}}PA + S^{\mathrm{T}}), \quad \zeta = q \tag{37}$$

$$0 = A^{\mathrm{T}}P + PA + hA^{\mathrm{T}}PA + C^{\mathrm{T}}C + \gamma^{-1}(B^{\mathrm{T}}P(I + hA)S^{\mathrm{T}})^{\mathrm{T}} \times (R - \gamma^{-1}hB^{\mathrm{T}}PB)^{-1} \times (B^{\mathrm{T}}P(I + h\bar{A}) + S^{\mathrm{T}}), \quad \zeta = \delta \tag{38}$$

such that

$$R > 0, \qquad \zeta = s \tag{39}$$

$$R > \gamma^{-1}B^{\mathrm{T}}PB, \qquad \zeta = q \tag{40}$$

$$R > \gamma^{-1}hB^{\mathrm{T}}PB, \qquad \zeta = \delta \tag{41}$$

Then $\|G(\zeta)\|_\infty < \gamma$.

Proof: The proofs for $\zeta = s$ and $\zeta = q$ can be found in Zhou (1996). The proof of the result for $\zeta = \delta$ is analogous to the proof for the case $\zeta = q$. □

The Riccati equations (36)–(38) are equivalent, but not identical, to those given in Ridgely *et al.* (1992 a) and Davis *et al.* (1996). Letting $G(\zeta) \sim (A, B, C, D)$, the results of Ridgely *et al.* (1992 a) and Davis *et al.* (1996) use $\gamma^{-1}G(\zeta) \sim (A, B, \gamma^{-1}C, \gamma^{-1}D)$ or $(A, \gamma^{-1}B, C, \gamma^{-1}D)$. For large values of γ, these realizations are approximately uncontrollable or unobservable, which causes numerical problems when solving the associated Riccati equation. The scaling used in Proposition 3 utilizes $\gamma^{-1}G(\zeta) \sim (A, \gamma^{-1/2}B, \gamma^{-1/2}C, \gamma^{-1}D)$, which provides improved numerical conditioning.

5. Decentralized static output feedback

This section reviews the decentralized static output feedback problem formulation for fixed-structure controller synthesis. As shown in Erwin *et al.* (1998), this formulation captures a large class of centralized and decentralized controller architectures within a common framework so that a common numerical algorithm can be used. Specialization of this formulation to full- and reduced-order, strictly proper, centralized dynamic compensation is given in Appendix B.

Consider the $(m+2)$-vector-input, $(m+2)$-vector-output decentralized system shown in figure 1, where e and d are used to account for model uncertainty, w is the exogenous disturbance input, z is the performance variable, and the signals y_i and u_i, $i = 1, \ldots, m$, are measurement and control signals, respectively. Furthermore, define

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_m \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_m \end{bmatrix} \tag{42}$$

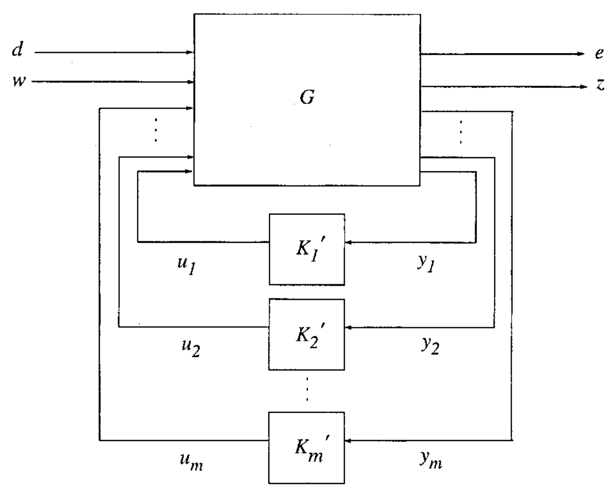

Figure 1. Decentralized static output feedback formulation.

and let $G(\zeta)$ have the realization

$$G(\zeta) \sim \left[\begin{array}{c|c:c:c} \mathcal{A} & \mathcal{B}_u & \mathcal{B}_d & \mathcal{B}_w \\ \hline \mathcal{C}_y & \mathcal{D}_{yu} & \mathcal{D}_{yd} & \mathcal{D}_{yw} \\ \hdashline \mathcal{C}_e & \mathcal{D}_{eu} & \mathcal{D}_{ed} & \mathcal{D}_{ew} \\ \hdashline \mathcal{C}_z & \mathcal{D}_{zu} & \mathcal{D}_{zd} & \mathcal{D}_{zw} \end{array}\right] \tag{43}$$

which represents the linear, time-invariant or shift-invariant dynamic system

$$\zeta \mathbf{x} = \mathcal{A}\mathbf{x} + \mathcal{B}_u\mathbf{u} + \mathcal{B}_d\mathbf{d} + \mathcal{B}_w\mathbf{w} \tag{44}$$

$$\mathbf{y} = \mathcal{C}_y\mathbf{x} + \mathcal{D}_{yu}\mathbf{u} + \mathcal{D}_{yd}\mathbf{d} + \mathcal{D}_{yw}\mathbf{w} \tag{45}$$

$$\mathbf{e} = \mathcal{C}_e\mathbf{x} + \mathcal{D}_{eu}\mathbf{u} + \mathcal{D}_{ed}\mathbf{d} + \mathcal{D}_{ew}\mathbf{w} \tag{46}$$

$$\mathbf{z} = \mathcal{C}_z\mathbf{x} + \mathcal{D}_{zu}\mathbf{u} + \mathcal{D}_{zd}\mathbf{d} + \mathcal{D}_{zw}\mathbf{w} \tag{47}$$

where $\mathbf{x}$, $\mathbf{u}$, $\mathbf{d}$, $\mathbf{w}$, $\mathbf{y}$, $\mathbf{e}$ and $\mathbf{z}$ are the transforms of the corresponding functions or sequences and where the dependence on ζ has been suppressed.

To represent decentralized static output feedback control with possibly repeated gains, let

$$\mathbf{u}_i = \mathcal{K}_i'\mathbf{y}_i, \quad i = 1, \ldots, m \tag{48}$$

where the matrices $\mathcal{K}_i'$ are not necessarily distinct. Reordering the variables in (48) if necessary, (48) can be rewritten as

$$\mathbf{u} = \mathcal{K}\mathbf{y} \tag{49}$$

where $\mathcal{K}$ has the form

$$\mathcal{K} \triangleq \text{block-diag}\,(I_{\phi_1} \otimes \mathcal{K}_1, \ldots, I_{\phi_v} \otimes \mathcal{K}_v) \tag{50}$$

where v is the number of *distinct* gains $\mathcal{K}_i \in \mathbb{R}^{r_i \times c_i}$ and ϕ_i is the number of repetitions of gain $\mathcal{K}_i$. Note that $\mathcal{K}_1, \ldots, \mathcal{K}_v$ are not necessarily square matrices, and that $\sum_{i=1}^{v} \phi_i = m$. We define $\mathcal{U}$ to be the set of all matrices $\mathcal{K}$ that have the structure (50).

For convenience, define the algebraic return difference

$$L_{\mathcal{K}} \triangleq I - \mathcal{D}_{yu}\mathcal{K} \tag{51}$$

Assuming that $L_{\mathcal{K}}$ is non-singular, the closed-loop dynamics, model error, and performance variable are given by

$$\zeta \mathbf{x} = \tilde{A}\mathbf{x} + \tilde{B}_d\mathbf{d} + \tilde{B}_w\mathbf{w} \tag{52}$$

$$\mathbf{e} = \tilde{C}_e\mathbf{x} + \tilde{D}_{ed}\mathbf{d} + \tilde{D}_{ew}\mathbf{w} \tag{53}$$

$$\mathbf{z} = \tilde{C}_z\mathbf{x} + \tilde{D}_{zd}\mathbf{d} + \tilde{D}_{zw}\mathbf{w} \tag{54}$$

where

$$\tilde{A} \triangleq \mathcal{A} + \mathcal{B}_u\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{C}_y, \quad \tilde{B}_d \triangleq \mathcal{B}_d + \mathcal{B}_u\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yd}$$

$$\tilde{B}_w \triangleq \mathcal{B}_w + \mathcal{B}_u\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yw} \quad \tilde{C}_z \triangleq \mathcal{C}_z + \mathcal{D}_{zu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{C}_y$$

$$\tilde{D}_{zd} \triangleq \mathcal{D}_{zd} + \mathcal{D}_{zu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yd}, \quad \tilde{D}_{zw} \triangleq \mathcal{D}_{zw} + \mathcal{D}_{zu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yw}$$

$$\tilde{C}_e \triangleq \mathcal{C}_e + \mathcal{D}_{eu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{C}_y, \quad \tilde{D}_{ed} \triangleq \mathcal{D}_{ed} + \mathcal{D}_{eu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yd}$$

$$\tilde{D}_{ew} \triangleq \mathcal{D}_{ew} + \mathcal{D}_{eu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yw}$$

The closed-loop transfer function $\tilde{G}_{zw}(\zeta)$ from w to z therefore has a realization

$$\tilde{G}_{zw}(\zeta) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B}_w \\ \hline \tilde{C}_z & \tilde{D}_{zw} \end{array}\right] \tag{55}$$

while the closed-loop transfer function $\tilde{G}_{ed}(\zeta)$ from d to e has a realization

$$\tilde{G}_{ed}(\zeta) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B}_d \\ \hline \tilde{C}_e & \tilde{D}_{ed} \end{array}\right] \tag{56}$$

6. $\mathcal{H}_2$ and $\mathcal{H}_2/\mathcal{H}_\infty$ control

The fixed-structure $\mathcal{H}_2$-optimal control problem is defined as

$$\min_{\mathcal{K} \in \mathcal{U}_\zeta} \|\tilde{G}_{zw}(\zeta)\|_2^2 \tag{57}$$

while the fixed-structure mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem is defined as

$$\min_{\mathcal{K} \in \mathcal{U}_\zeta} \|\tilde{G}_{zw}(\zeta)\|_2^2 \quad \text{subject to} \quad \|\tilde{G}_{ed}(\zeta)\|_\infty < \gamma \tag{58}$$

where $\gamma > 0$ and $\mathcal{U}_\zeta$ is the set of all $\mathcal{K} \in \mathcal{U}$ such that $\tilde{A}$ is ζ-stable. If $\mathcal{K} \in \mathcal{U}_\zeta$, then we can evaluate $\|\tilde{G}_{zw}(\zeta)\|_2$ by using Proposition 2 with (A, B, C, D) replaced by $(\tilde{A}, \tilde{B}_w, \tilde{C}_z, \tilde{D}_{zw})$. Necessary conditions for optimality are given in Appendix C, where $P = \tilde{P}_{zw}$ denotes the solution to (18), (19) or (20) with these substitutions.

To enforce the $\mathcal{H}_\infty$ norm constraint (58), we utilize Proposition 3 with (A, B, C, D) replaced by $(\tilde{A}, \tilde{B}_d, \tilde{C}_e, \tilde{D}_{ed})$, where $\tilde{R}$ and $\tilde{S}$ are defined by (35) with these substitutions. The resulting necessary conditions are given in Appendix C, where $P = \tilde{P}_{ed}$ denotes the solution to (36) and (39), (37) and (40), or (38) and (41) with these substitutions. The Lagrangian functions used to generate these necessary conditions include an auxiliary cost $\mathcal{J}_{\text{aux}}$ defined by

$$\mathcal{J}_{\text{aux}} \triangleq \operatorname{tr} \tilde{B}_d^{\mathrm{T}}\tilde{P}_{ed}\tilde{B}_d, \qquad \zeta = s \tag{59}$$

$$\triangleq \operatorname{tr}\,[\tilde{B}_d^{\mathrm{T}}\tilde{P}_{ed}\tilde{B}_d + \tilde{D}_{ed}^{\mathrm{T}}\tilde{D}_{ed}], \qquad \zeta = q \tag{60}$$

$$\triangleq \operatorname{tr}\left[\tilde{B}_d^{\mathrm{T}}\tilde{P}_{ed}\tilde{B}_d + \frac{1}{h}\tilde{D}_{ed}^{\mathrm{T}}\tilde{D}_{ed}\right], \qquad \zeta = \delta \tag{61}$$

The auxiliary cost $\mathcal{J}_{\rm aux}$ and the $\mathcal{H}_2$ cost $\|\tilde{G}_{zw}(\zeta)\|_2$ are weighted by the continuation parameter $\rho \in (0,1)$ in the convex combination

$$\mathcal{J}_\rho \triangleq \rho\|\tilde{G}_{zw}(\zeta)\|_2 + (1-\rho)\mathcal{J}_{\rm aux}$$

so that, as $\rho \to 1$, the effect of $\mathcal{J}_{\rm aux}$ becomes negligible. As discussed in Haddad and Bernstein (1990) and Ridgely *et al.* (1992 a, b), the use of this auxiliary term avoids a degenerate Lyapunov equation in the resulting necessary conditions. It is shown in Ridgely *et al.* (1992 b), however, that the minimizer of (6) is often located at the boundary of the region where the $\mathcal{H}_\infty$ bound holds. In this case the gradient of the performance at the minimizer may point outward from this region for all values of the convexifying parameter. This difficulty is avoided in Ridgely *et al.* (1992 a) where the Lagrangian was modified to include the solution of a Lyapunov equation having the property that its solution becomes unbounded when the $\mathcal{H}_\infty$ bound is approached. Although this modification could be included in a delta-domain formulation of the problem as well, this numerical difficulty did not arise in the course of this investigation.

7. Algorithm description

To compute solutions of the fixed-structure control problems, a general-purpose BFGS quasi-Newton algorithm (Dennis Jr. and Schnabel 1983) is used in conjunction with a continuation technique. For open-loop-stable plants we initialize the algorithm by using a sufficiently low-authority compensator along with an appropriate model-reduction technique (Collins Jr. *et al.* 1996). A series of intermediate problems are solved sequentially to produce the structured, high-authority controller.

The computational procedure is given by the following *two-step continuation algorithm.* Let $\bar{\mathcal{D}}_{zu}$ represent the control effort weighting matrix for the desired high-authority controller problem. Then:

(1) Step 1:
   (a) Choose a scalar $\beta > 1$ such that the optimal controller for the low-authority $\mathcal{H}_2$ problem using $\mathcal{D}_{zu} = \beta\bar{\mathcal{D}}_{zu}$ yields a closed-loop system that satisfies the $\mathcal{H}_\infty$ constraint (36), (37) or (38). This low-authority, full-order controller also has the advantage that it can often be truncated to obtain a reduced-order controller without violating either closed-loop stability or the $\mathcal{H}_\infty$ constraint.
   (b) Define a series of intermediate problems, indexed by the decreasing sequence $\{\alpha_i\}_{i=1}^r$, where $\alpha_1 = 1$ and $\alpha_r = 1/\beta$. Each intermediate problem then utilizes $\mathcal{D}_{zu} = \alpha_i\beta\bar{\mathcal{D}}_{zu}$ and provides an increase in the control authority from the low-authority level used to generate the initializing controller ($\alpha_1 = 1$) to the desired high-authority values ($\alpha_r = 1/\beta$).
   (c) Sequentially solve each of these intermediate problems using the quasi-Newton algorithm, with the solution of each intermediate problem providing the initializing compensator for the next intermediate problem, and so on. The continuation parameter ρ is held at a constant value $\rho_1 \approx 0.9$ for all of these intermediate problems.

(2) Step 2:
   (a) Define a second set of intermediate problems by the increasing sequence $\{\rho_i\}_{i=1}^l$, where ρ_l can be chosen arbitrarily close to 1, implying that the optimality conditions approach those of (58). The problems defined by this sequence approach the $\mathcal{H}_2/\mathcal{H}_\infty$ control problem (58).
   (b) As in (3) above, this second sequence of intermediate problems is solved by sequential application of the quasi-Newton algorithm. The solution of the final intermediate problem is then the solution of a high-authority, near-optimal mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem.

Note that for fixed-structure $\mathcal{H}_2$ synthesis, the two-step continuation algorithm terminates after Step 1(c).

The line search portions of the quasi-Newton algorithm involve a modification of the standard Armijo-type search to include a constraint-checking subroutine. This subroutine decreases the step length along the search direction to ensure that the next iterate satisfies the problem constraints, that is, (i) that $\tilde{A}$ is ζ-stable, implying (18), (19) or (20) has a solution, and (ii) the corresponding Riccati equation (36), (37) or (38) has a solution. This modification ensures that the cost function remains defined at every point the linear-search process.

8. $\mathcal{H}_2$-optimal control example

The 10th-order continuous-time beam example of Hyland and Richter (1990) is considered for s-domain $\mathcal{H}_2$-optimal controller synthesis. A zero-order-hold model of this plant is used for both q-domain and δ-domain synthesis. To ensure that the highest frequency mode of the plant (approximately 160 Hz) is below the Nyquist frequency, a sampling frequency of 400 Hz was chosen. This sampling frequency corresponds to a sampling period of $h = 2.5 \times 10^{-3}$ s. Both full- and reduced-order, strictly proper, centralized dynamic compensators were synthesized within the decentralized static output feedback framework as shown in Appendix B.

8.1. *Full-order $\mathcal{H}_2$-optimal control*

To compare the q-domain and δ-domain formulations for $\mathcal{H}_2$-optimal controller synthesis, full-order $\mathcal{H}_2$-optimal controllers were synthesized in the q-domain and δ-domain using both standard Riccati equation techniques and the quasi-Newton/continuation algorithm technique. Controllers obtained via the ζ-domain Riccati equation approach are called *ζ-domain Riccati solutions*, while controllers obtained via the ζ-domain quasi-Newton/continuation algorithm are called *ζ-domain fixed-structure solutions.*

Figure 2 compares the $\mathcal{H}_2$-optimal costs for the q-domain fixed-structure solution and the q-domain Riccati solution, normalized by the cost of the q-domain Riccati solution. Note that as the controller authority increases with decreasing values of α_i, the q-domain fixed-structure solution diverges from the q-domain Riccati solution. Note that we cannot conclude that the numerical conditioning of the problem is the cause of this behaviour, since the BFGS optimization algorithm is only guaranteed to converge to a local minimum.

Figure 3 presents the $\mathcal{H}_2$-optimal cost for the δ-domain Riccati solution and the δ-domain fixed-structure solution, normalized by the q-domain Riccati solution cost. Note that the δ-domain fixed-structure solution coincides with the δ-domain Riccati solution for each value of the continuation parameter α_i, both of which yield a (marginally) lower cost than the q-domain Riccati solution.

Thus, we have solved identical problems in the q- and δ-domains, using identical algorithms and initializing controllers, obtaining poor performance (i.e. divergence from the global minimizer) in the q-domain and good performance (i.e. convergence to the global minimizer) in the δ-domain. We thus conclude that the numerical conditioning of the problem is the cause of the divergence of the fixed-structure solution from the Riccati solution in the q-domain.

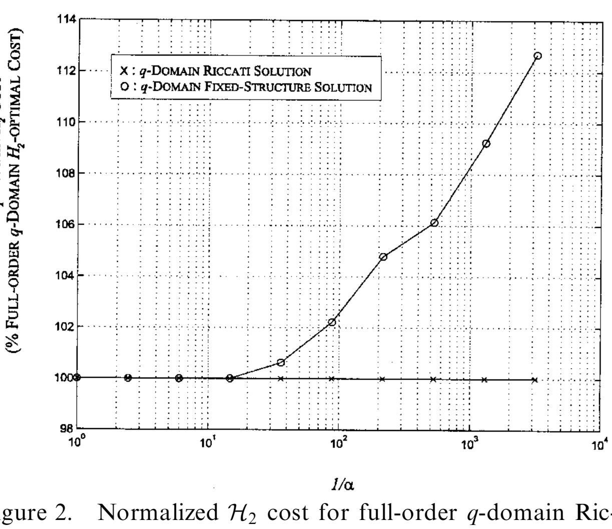

Figure 2. Normalized $\mathcal{H}_2$ cost for full-order q-domain Riccati and fixed-structure solutions versus continuation parameter α.

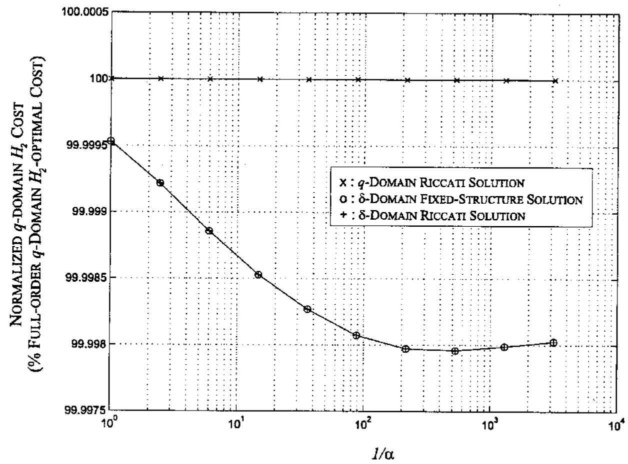

Figure 3. Normalized $\mathcal{H}_2$ cost for full-order δ-domain and q-domain Riccati solutions and δ-domain fixed-structure solution.

The total number of iterations required by the quasi-Newton algorithm summed over the 10 intermediate problems was 3572 for q-domain synthesis and 1207 for δ-domain synthesis.

The optimality of the δ-domain fixed-structure solutions obtained for each value of α_i was tested by transforming them to q-domain realization via (17) and then using the resulting realizations as initializing controllers for the quasi-Newton algorithm in the q-domain formulation of the problem. For each of the 10 intermediate problems, the code was unable to further optimize these initial controllers since they satisfied the small gradient norm condition for termination. However, using the solutions synthesized in the q-domain as initial controllers for δ-domain synthesis, the algorithm was able to further reduce the cost for the intermediate problems for each of the last several values of α_i, indicating that the q-domain fixed-structure solutions were suboptimal.

8.2. *Reduced-order $\mathcal{H}_2$-optimal control*

The quasi-Newton/continuation algorithm was used to solve a reduced-order discrete-time $\mathcal{H}_2$-optimal control problem in the q-domain and the δ-domain. The 10th-order flexible beam model was again used, with the compensator order set to $n_c = 6$. Since no Riccati solution is available for the reduced-order problem, the $\mathcal{H}_2$-optimal cost of the δ-domain Riccati solution for the full-order problem was used to normalize all costs. For the results of this section, the continuation algorithm

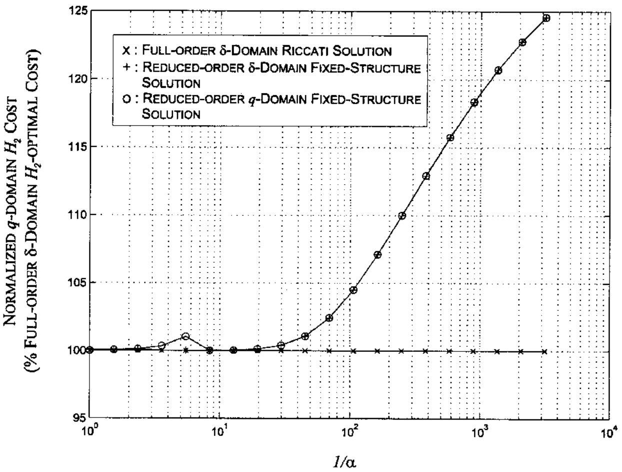

Figure 4. Normalized $\mathcal{H}_2$ cost for reduced-order q-domain and δ-domain fixed-structure solutions.

utilized 20 intermediate problems to generate high-authority controllers from the low-authority initializing controllers.

Figure 4 presents the normalized $\mathcal{H}_2$-optimal costs for the reduced-order q-domain and δ-domain fixed-structure solutions. Note that as the continuation algorithm increases the authority by means of the decreasing sequence $\{\alpha_i\}$, the q-domain fixed-structure solution begins to diverge at $1/\alpha_i = 3.5$, then recovers and continues to track the 'true' solution. As expected, at high authority levels the fixed-structure solutions from both domains yield higher $\mathcal{H}_2$ costs than the full-order solution.

The total number of iterations performed by the quasi-Newton algorithm summed over the 20 intermediate problems of the continuous algorithm was 1791 for the q-domain formulation, and 1433 for the δ-domain formulation. In both formulations, the quasi-Newton algorithm produced a solution that satisfies a small-gradient condition for termination for each value of α_i, including those where, as shown in figure 4, the q-domain fixed structure solution begins to diverge.

9. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control example

Next we consider mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for the 4th-order HIMAT aircraft longitudinal dynamics model given in Design B of Ullauri *et al.* (1994). The transfer function from d to e represents the input weighted complementary sensitivity function, where the weighting function is (Ullauri *et al.* 1994)

$$W_{T_i} = I_{2\times 2} \otimes \frac{50(s+100)}{s+10\,000} \tag{62}$$

For discrete-time control, the sampling rate was chosen based on the highest frequency dynamics of the system. In this case, the weighting function W_{T_i} has poles at 10 000 rad/s (undamped natural frequency), or approximately 1591 Hz. A sampling frequency of $f_s = 4000$ Hz was chosen so that $h = 2.5 \times 10^{-4}$ s. This sampling rate is approximately 2.5 times the frequency of the highest frequency pole. Properties of the full-order $\mathcal{H}_2$-optimal solution properties for this problem are given in table 1.

	s-Domain	δ-Domain	q-Domain
$\lVert\tilde{G}_{zw}(\zeta)\rVert_2$	6.115	6.115	$6.118\sqrt{h}$
$\lVert\tilde{G}_{ed}(\zeta)\rVert_\infty$	55.13	55.31	55.07

Table 1. Full-order $\mathcal{H}_2$-optimal controller properties.

A mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem was considered with $\gamma = 1$. Since the $\mathcal{H}_\infty$ norm of $\tilde{G}_{ed}(\zeta)$ is approximately 55 in all three domains (see table 1), the $\mathcal{H}_2$-optimal solution is not a feasible solution for the mixed problem. The two-step continuation algorithm was used to solve the high-authority, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem. The continuation algorithm used 50 logarithmically spaced values of α_i decreasing from 1 to 0.05. The parameter $\rho = \rho_1$ was held at a constant value of 0.9 during the continuation on α. With $\alpha = \alpha_{50}$ held constant at 0.05, the second step of the continuation algorithm used 50 logarithmically spaced values of ρ_j increasing from 0.9 to 0.9999.

Figures 5 and 6 show $\lVert\tilde{G}_{zw}(\zeta)\rVert_2$ versus the continuation parameters α and ρ for s-domain and δ-domain synthesis solutions, respectively, while figure 7 shows $(1/\sqrt{h})\lVert\tilde{G}_{zw}(q)\rVert_2$ versus the continuation parameters α and ρ for the q-domain synthesis solution. Figures 5 and 6 show similar behaviour of the s-domain and δ-domain solutions during the two-step continuation algorithm, while figure 7 indicates the inability of the algorithm to find a solution for the q-domain formulation of the problem.

Figures 8, 9 and 10 plot $\lVert\tilde{G}_{ed}(\zeta)\rVert_\infty$ versus the continuation parameters α and ρ for s-domain, δ-domain and q-domain synthesis solutions, respectively. Again, figures 8 and 9 illustrate similar properties of the s-domain and δ-domain solutions during the continuation algorithm, while figure 10 shows that in the q-domain formulation, the algorithm effectively gets 'stuck' at an early stage in the continuation, and never recovers. Table 2 gives the solution properties for the final results of the two-step continuation algorithm.

10. Discussion and conclusions

This paper presented the delta operator formulation of the fixed-structure mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem. Using continuation techniques in conjunction with a Lagrangian approach, necessary conditions for sub-optimal controller synthesis were derived for the s-domain, q-domain and δ-domain formulations of the

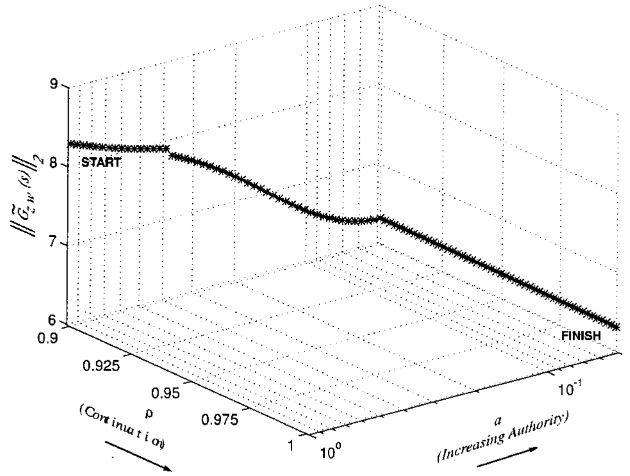

Figure 5. $\|\tilde{G}_{zw}(s)\|_2$ versus continuation parameters α and ρ for s-domain $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis.

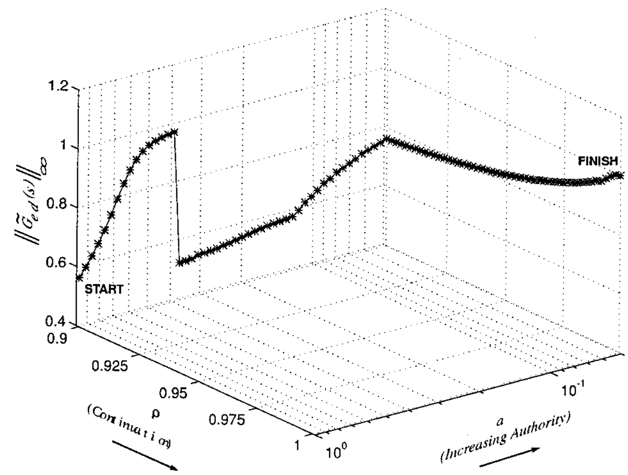

Figure 8. $\|\tilde{G}_{ed}(s)\|_\infty$ versus continuation parameters α and ρ for s-domain $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis.

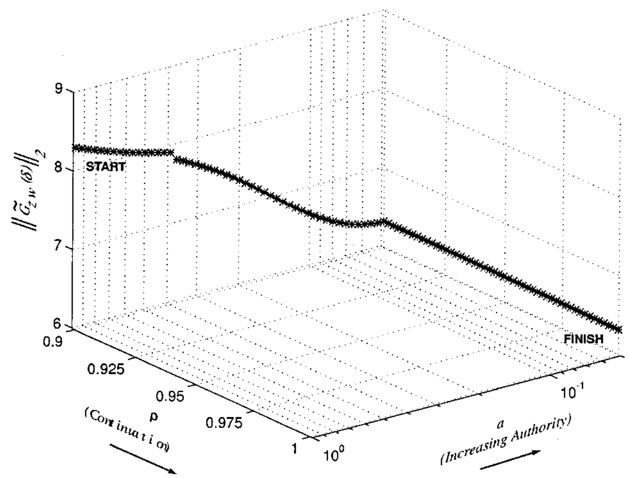

Figure 6. $\|\tilde{G}_{zw}(\delta)\|_2$ versus continuation parameters α and ρ for δ-domain $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis.

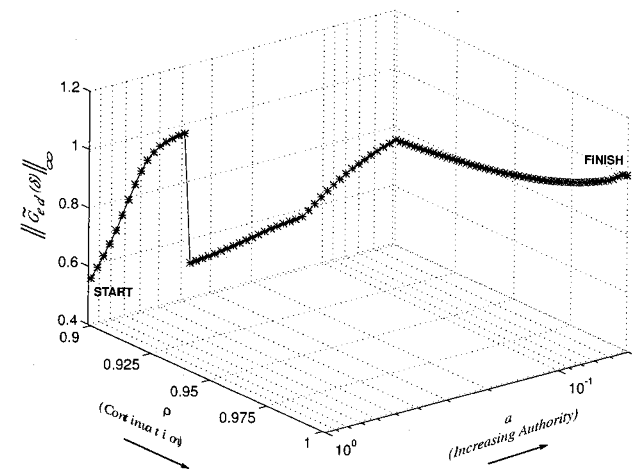

Figure 9. $\|\tilde{G}_{ed}(\delta)\|_\infty$ versus continuation parameters α and ρ for δ-domain $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis.

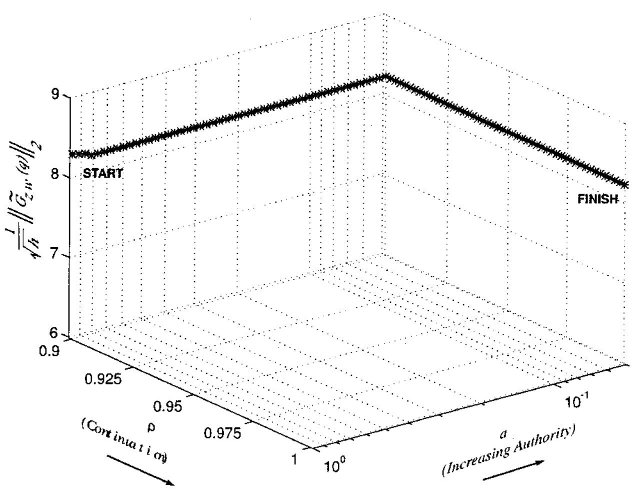

Figure 7. $(1/\sqrt{h})\|\tilde{G}_{zw}(q)\|_2$ versus continuation parameters α and ρ for q-domain $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis.

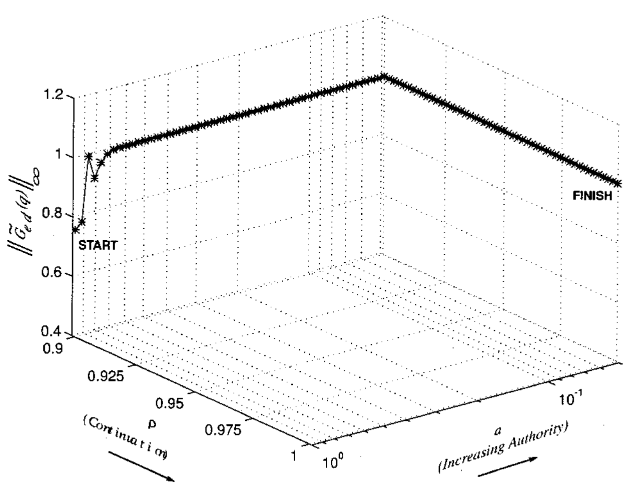

Figure 10. $\|\tilde{G}_{ed}(q)\|_\infty$ versus continuation parameters α and ρ for q-domain $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis.

	s-Domain	δ-Domain	q-Domain
$\|\tilde{G}_{zw}(\zeta)\|_2$	6.303	6.303	$8.291\sqrt{h}$
$\|\tilde{G}_{ed}(\zeta)\|_\infty$	1.000	1.000	1.000

Table 2. Full-order $\mathcal{H}_2$-optimal controller properties.

$\mathcal{H}_2/\mathcal{H}_\infty$ problem. The performance of a quasi-Newton continuation algorithm in computing solutions for the q-domain and δ-domain formulations of the problem was compared numerically. It was found that the δ-domain formulation of the problem yielded significant numerical advantages over the standard q-domain problem formulation. In particular, the algorithm showed increased robustness to large changes in the continuation parameter and required fewer iterations for convergence. The results demonstrate the advantages of the δ-domain formulation over the q-domain formulation in the context of discrete-time fixed-structure controller synthesis.

Numerical experience suggests that the poor numerical conditioning of the q-domain state-space matrices (Middleton and Goodwin 1990, Gevers and Li 1993) and lower numerical accuracy of q-domain Riccati equation solvers relative to their δ-domain counterparts (Middleton and Goodwin 1990, pp. 287, 511–515) lead to inaccurate evaluations of the gradient formulae used by the quasi-Newton algorithm, which is then unable to compute meaningful solutions in the q-domain. Direct confirmation of this is non-trivial, however, as it would require computing the exact (i.e. infinite precision) solution to the Riccati equation at the point where the solutions from the q- and δ-domain formulations diverge and then evaluating the gradient formulae using this exact solution.

The software used to compute the results of this paper is available to interested parties in the form of a Matlab toolbox. Software requests may be directed to the first author.

Acknowledgements

This research was supported by the Air Force Office of Scientific Research under grant F49620-95-1-0019 and the Air Force Research Laboratory.

Appendices

A. *Proof of Proposition 2*

The cases $\zeta = s$ and q are standard; see, for example, Zhou (1996). For $\zeta = \delta$, we note that since $G(\delta) \sim (A, B, C, D)$ is a ζ-stable matrix transfer function, Proposition 1 implies that $\hat{G}(q) \sim (I + hA, hB, C, D)$ is the equivalent q-domain transfer function. Using (27) and (29), it then follows that

$$\|G(\delta)\|_2^2 = h^{-1}\,\mathrm{tr}\,[h^2 B^{\mathrm{T}} \hat{P} B + D^{\mathrm{T}} D] \tag{63}$$

where $\hat{P}$ is the solution of the q-domain Lyapunov equation

$$0 = (I + hA)^{\mathrm{T}} \hat{P} (I + hA) - \hat{P} + C^{\mathrm{T}} C \tag{64}$$

Expanding (64) and defining $P = h\hat{P}$ yields (20). Finally, applying this definition to (63) yields (30). □

B. *Centralized strictly proper dynamic compensation*

Consider a plant of the form

$$\zeta \mathbf{x}_0 = A\mathbf{x}_0 + B\mathbf{u}_0 + M_1\mathbf{d} + D_1\mathbf{w} \tag{65}$$

$$\mathbf{y}_0 = C\mathbf{x}_0 + F\mathbf{u}_0 + M_2\mathbf{d} + D_2\mathbf{w} \tag{66}$$

$$\mathbf{e} = N_1\mathbf{x}_0 + N_2\mathbf{u}_0 + E_4\mathbf{d} + E_5\mathbf{w} \tag{67}$$

$$\mathbf{z} = E_1\mathbf{x}_0 + E_2\mathbf{u}_0 + E_3\mathbf{d} + E_0\mathbf{w} \tag{68}$$

controlled by a full- or reduced-order strictly proper, centralized dynamic compensator having the realization

$$\zeta \mathbf{x}_c = A_c\mathbf{x}_c + B_c\mathbf{y} \tag{69}$$

$$\mathbf{u} = C_c\mathbf{x}_c \tag{70}$$

This system can be written as decentralized static output feedback with $m = v = 3$, $\phi_1 = \phi_2 = \phi_3 = 1$, $G(\zeta)$ given by

$$G(\zeta) \sim \left[\begin{array}{cc|ccc:c:c} A & 0 & 0 & 0 & B & M_1 & D_1 \\ 0 & 0 & I & I & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & F & M_2 & D_2 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ \hdashline N_1 & 0 & 0 & 0 & N_2 & E_4 & E_5 \\ \hdashline E_1 & 0 & 0 & 0 & E_2 & E_3 & E_0 \end{array}\right] \tag{71}$$

and $\mathcal{K}$ denoting the block-diagonal matrix $\mathcal{K} = \text{block-diag}\,(A_c, B_c, C_c)$. This yields

$$L_{\mathcal{K}} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & -DC_c \\ 0 & 0 & I \end{bmatrix} \tag{72}$$

which is non-singular.

Note that in this derivation, every entry in the state-space realization matrices is a parameter to be optimized (a so-called full-matrix parameterization). Thus, any optimal solution in this parameterization, if one exists, is non-unique, as a similarity transformation on the resulting state-space matrices yields another set of parameters yielding the same cost. The decentralized static output feedback framework used in this work can incorporate various other internal parameterizations of the state space realization (canonical forms, etc.). A study on the effect of different internal parameterizations on computational efficency can be found in Erwin *et al.* (1998).

C. *Necessary conditions*

Define $Q_{\mathrm{L}ij}$ and $Q_{\mathrm{R}ij}$ by

$$Q_{\mathrm{L}ij} \triangleq \begin{bmatrix} 0_{r_1\phi_1\times r_i} \\ 0_{r_2\phi_2\times r_i} \\ \vdots \\ 0_{r_{i-1}\phi_{i-1}\times r_i} \\ 0_{r_i(j-1)\times r_i} \\ I_{r_i} \\ 0_{r_i(\phi_i-j)\times r_i} \\ 0_{r_{i+1}\phi_{i+1}\times r_i} \\ \vdots \\ 0_{r_v\phi_v\times r_i} \end{bmatrix}, \quad Q_{\mathrm{R}ij} \triangleq \begin{bmatrix} 0_{c_1\phi_1\times c_i} \\ 0_{c_2\phi_2\times c_i} \\ \vdots \\ 0_{c_{i-1}\phi_{i-1}\times c_i} \\ 0_{c_i(j-1)\times c_i} \\ I_{c_i} \\ 0_{c_i(\phi_i-j)\times c_i} \\ 0_{c_{i+1}\phi_{i+1}\times c_i} \\ \vdots \\ 0_{c_v\phi_v\times c_i} \end{bmatrix}^{\mathrm{T}} \tag{73}$$

The Lagrangian for the s-domain $\mathcal{H}_2/\mathcal{H}_\infty$ control problem is given by

$$\begin{aligned}\mathcal{L}(\tilde P_{zw},\tilde Q_{zw},\tilde P_{ed},\tilde Q_{ed},\mathcal{K}_i) &= \rho\,\mathrm{tr}\,\tilde B_w^{\mathrm{T}}\tilde P_{zw}\tilde B_w + \mathrm{tr}\,\tilde Q_{zw}[\tilde A^{\mathrm{T}}\tilde P_{zw} + \tilde P_{zw}\tilde A + \tilde C_z^{\mathrm{T}}\tilde C_z] \\ &\quad + (1-\rho)\mathrm{tr}\,\tilde B_d^{\mathrm{T}}\tilde P_{ed}\tilde B_d + \mathrm{tr}\,\tilde Q_{ed}[\tilde A^{\mathrm{T}}\tilde P_{ed} \\ &\quad + \tilde P_{ed}\tilde A + \gamma^{-1}\tilde C_e^{\mathrm{T}}\tilde C_e \\ &\quad + \gamma^{-1}(\tilde B_d^{\mathrm{T}}\tilde P_{ed} + \tilde S^{\mathrm{T}})^{\mathrm{T}}\tilde R^{-1}(\tilde B_d^{\mathrm{T}}\tilde P_{ed} + \tilde S^{\mathrm{T}})]\end{aligned} \tag{74}$$

with partial derivatives given by

$$\frac{\partial\mathcal{L}}{\partial\tilde P_{zw}} = \tilde A\tilde Q_{zw} + \tilde Q_{zw}\tilde A^{\mathrm{T}} + \rho\tilde B_w\tilde B_w^{\mathrm{T}} \tag{75}$$

$$\frac{\partial\mathcal{L}}{\partial\tilde Q_{zw}} = \tilde A^{\mathrm{T}}\tilde P_{zw} + \tilde P_{zw}\tilde A + \tilde C_z^{\mathrm{T}}\tilde C_z \tag{76}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\tilde P_{ed}} &= (\tilde A + \gamma^{-1}\tilde B_d\tilde R^{-1}\Sigma)\tilde Q_{ed} \\ &\quad + \tilde Q_{ed}(\tilde A + \gamma^{-1}\tilde B_d\tilde R^{-1}\Sigma)^{\mathrm{T}} + (1-\rho)\tilde B_d\tilde B_d^{\mathrm{T}}\end{aligned} \tag{77}$$

$$\frac{\partial\mathcal{L}}{\partial\tilde Q_{ed}} = \tilde A^{\mathrm{T}}\tilde P_{ed} + \tilde P_{ed}\tilde A + \gamma^{-1}\Sigma^{\mathrm{T}}\tilde R^{-1}\Sigma + \gamma^{-1}\tilde C_e^{\mathrm{T}}\tilde C_e \tag{78}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\mathcal{K}_i} &= 2\sum_{j=1}^{\phi_i} Q_{\mathrm{L}ij}^{\mathrm{T}}(I + \mathcal{D}_{yu}^{\mathrm{T}}L_{\mathcal{K}}^{-\mathrm{T}}\mathcal{K}^{\mathrm{T}}) \\ &\quad \times [\rho\mathcal{B}_u^{\mathrm{T}}\tilde P_{zw}\tilde B_w\mathcal{D}_{yw}^{\mathrm{T}} + (1-\rho)\mathcal{B}_d^{\mathrm{T}}\tilde P_{ed}\tilde B_d\mathcal{D}_{yd}^{\mathrm{T}} \\ &\quad + \mathcal{B}_u^{\mathrm{T}}\tilde P_{zw}\tilde Q_{zw}\mathcal{C}_y^{\mathrm{T}} + \mathcal{D}_{zu}^{\mathrm{T}}\tilde C_z\tilde Q_{zw}\mathcal{C}_y^{\mathrm{T}} \\ &\quad + \mathcal{B}_u^{\mathrm{T}}\tilde P_{ed}\tilde Q_{ed}\mathcal{C}_y^{\mathrm{T}} + \gamma^{-1}\mathcal{D}_{eu}^{\mathrm{T}}\tilde C_e\tilde Q_{ed}\mathcal{C}_y^{\mathrm{T}} \\ &\quad + \gamma^{-1}\mathcal{B}_u^{\mathrm{T}}\tilde P_{ed}\tilde Q_{ed}\Sigma^{\mathrm{T}}\tilde R^{-1}\mathcal{D}_{yd}^{\mathrm{T}} \\ &\quad + \gamma^{-2}\mathcal{D}_{eu}^{\mathrm{T}}\tilde C_e\tilde Q_{ed}\Sigma^{\mathrm{T}}\tilde R^{-1}\mathcal{D}_{yd}^{\mathrm{T}} \\ &\quad + \gamma^{-3}\mathcal{D}_{eu}^{\mathrm{T}}\tilde D_{ed}\tilde R^{-1}\Sigma\tilde Q_{ed}\Sigma^{\mathrm{T}}\tilde R^{-1}\mathcal{D}_{yd}^{\mathrm{T}}]L_{\mathcal{K}}^{-\mathrm{T}}Q_{\mathrm{R}ij}^{\mathrm{T}}\end{aligned} \tag{79}$$

where

$$\Sigma \triangleq \tilde B_d^{\mathrm{T}}\tilde P_{ed} + \tilde S^{\mathrm{T}} \tag{80}$$

The Lagrangian for the q-domain $\mathcal{H}_2/\mathcal{H}_\infty$ control problem is given by

$$\begin{aligned}\mathcal{L}(\tilde P_{zw},\tilde Q_{zw},\tilde P_{ed},\tilde Q_{ed},\mathcal{K}_i) &= \rho\,\mathrm{tr}\,[\tilde B_w^{\mathrm{T}}\tilde P_{zw}\tilde B_w + \tilde D_{zw}^{\mathrm{T}}\tilde D_{zw}] \\ &\quad + \mathrm{tr}\,\tilde Q_{zw}[\tilde A^{\mathrm{T}}\tilde P_{zw}\tilde A + \tilde C_z^{\mathrm{T}}\tilde C_z - \tilde P_{zw}] \\ &\quad + (1-\rho)\,\mathrm{tr}\,[\tilde B_d^{\mathrm{T}}\tilde P_{ed}\tilde B_d + \tilde D_{ed}^{\mathrm{T}}\tilde D_{ed}] \\ &\quad + \mathrm{tr}\,\tilde Q_{ed}\{\tilde A^{\mathrm{T}}\tilde P_{ed}\tilde A + \gamma^{-1}\tilde C_e^{\mathrm{T}}\tilde C_e - \tilde P_{ed} \\ &\quad + \gamma^{-1}[\tilde B_d^{\mathrm{T}}\tilde P_{ed}\tilde A + \tilde S^{\mathrm{T}}]^{\mathrm{T}} \\ &\quad \times [\tilde R - \gamma^{-1}\tilde B_d^{\mathrm{T}}\tilde P_{ed}\tilde B_d]^{-1}[\tilde B_d^{\mathrm{T}}\tilde P_{ed}\tilde A + \tilde S^{\mathrm{T}}]\}\end{aligned} \tag{81}$$

with partial derivatives given by

$$\frac{\partial\mathcal{L}}{\partial\tilde P_{zw}} = \tilde A\tilde Q_{zw}\tilde A^{\mathrm{T}} + \rho\tilde B_w\tilde B_w^{\mathrm{T}} - \tilde Q_{zw} \tag{82}$$

$$\frac{\partial\mathcal{L}}{\partial\tilde Q_{zw}} = \tilde A^{\mathrm{T}}\tilde P_{zw}\tilde A + \tilde C_z^{\mathrm{T}}\tilde C_z - \tilde P_{zw} \tag{83}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\tilde P_{ed}} &= (\tilde A + \gamma^{-1}\tilde B_d\Gamma^{-1}\Sigma)\tilde Q_{ed}(\tilde A + \gamma^{-1}\tilde B_d\Gamma^{-1}\Sigma)^{\mathrm{T}} \\ &\quad + (1-\rho)\tilde B_d\tilde B_d^{\mathrm{T}} - \tilde Q_{ed}\end{aligned} \tag{84}$$

$$\frac{\partial\mathcal{L}}{\partial\tilde Q_{ed}} = \tilde A^{\mathrm{T}}\tilde P_{ed}\tilde A + \gamma^{-1}\Sigma^{\mathrm{T}}\Gamma^{-1}\Sigma + \gamma^{-1}\tilde C_e^{\mathrm{T}}\tilde C_e - \tilde P_{ed} \tag{85}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\mathcal{K}_i} &= 2\sum_{j=1}^{\phi_i} Q_{\mathrm{L}ij}^{\mathrm{T}}(I + \mathcal{D}_{yu}^{\mathrm{T}}L_{\mathcal{K}}^{-\mathrm{T}}\mathcal{K}^{\mathrm{T}}) \\ &\quad \times [\rho(\mathcal{B}_u^{\mathrm{T}}\tilde P_{zw}\tilde B_w\mathcal{D}_{yw}^{\mathrm{T}} + \mathcal{D}_{zu}^{\mathrm{T}}\tilde D_{zw}\mathcal{D}_{zw}^{\mathrm{T}}) \\ &\quad + (1-\rho)(\gamma^{-1}\mathcal{B}_u^{\mathrm{T}}\tilde P_{ed}\tilde B_d\mathcal{D}_{yd}^{\mathrm{T}} + \gamma^{-2}\mathcal{D}_{eu}^{\mathrm{T}}\tilde D_{ed}\mathcal{D}_{yd}^{\mathrm{T}}) \\ &\quad + \mathcal{B}_u^{\mathrm{T}}\tilde P_{zw}\tilde A\tilde Q_{zw}\mathcal{C}_y^{\mathrm{T}} + \mathcal{D}_{zu}^{\mathrm{T}}\tilde C_z\tilde Q_{zw}\mathcal{C}_y^{\mathrm{T}} \\ &\quad + \mathcal{B}_u^{\mathrm{T}}\tilde P_{ed}\tilde A\tilde Q_{ed}\mathcal{C}_y^{\mathrm{T}} + \gamma^{-1}\mathcal{D}_{eu}^{\mathrm{T}}\tilde C_e\tilde Q_{ed}\mathcal{C}_y^{\mathrm{T}} \\ &\quad + \gamma^{-2}\mathcal{D}_{eu}^{\mathrm{T}}\tilde D_{ed}\Gamma^{-1}\Sigma\tilde Q_{ed}\mathcal{C}_y^{\mathrm{T}} \\ &\quad + \gamma^{-1}\mathcal{D}_{eu}^{\mathrm{T}}\tilde C_e\tilde Q_{ed}\Sigma\Gamma^{-1}\mathcal{D}_{yd}^{\mathrm{T}} \\ &\quad + \gamma^{-1}\mathcal{B}_u^{\mathrm{T}}\tilde P_{ed}\tilde B_d\Gamma^{-1}\Sigma^{\mathrm{T}}\tilde Q_{ed}\mathcal{C}_y^{\mathrm{T}} \\ &\quad + \gamma^{-1}\mathcal{B}_u^{\mathrm{T}}\tilde P_{ed}\tilde A\tilde Q_{ed}\Sigma^{\mathrm{T}}\Gamma^{-1}\mathcal{D}_{yd}^{\mathrm{T}} \\ &\quad + \gamma^{-3}\mathcal{D}_{eu}^{\mathrm{T}}\tilde D_{ed}\Gamma^{-1}\Sigma\tilde Q_{ed}\Sigma^{\mathrm{T}}\Gamma^{-1}\mathcal{D}_{yd}^{\mathrm{T}} \\ &\quad + \gamma^{-2}\mathcal{B}_u^{\mathrm{T}}\tilde P_{ed}\tilde B_d\Gamma^{-1}\Sigma\tilde Q_{ed}\Sigma^{\mathrm{T}}\Gamma^{-1}\mathcal{D}_{yd}^{\mathrm{T}}]L_{\mathcal{K}}^{-\mathrm{T}}Q_{\mathrm{R}ij}^{\mathrm{T}}\end{aligned} \tag{86}$$

where

$$\Gamma \triangleq \tilde R + \gamma^{-1}\tilde B_d^{\mathrm{T}}\tilde P_{ed}\tilde B_d, \quad \Sigma \triangleq \tilde B_d^{\mathrm{T}}\tilde P_{ed}\tilde A + \tilde S^{\mathrm{T}} \tag{87}$$

The Lagrangian for the δ-domain $\mathcal{H}_2/\mathcal{H}_\infty$ control problem is given by

$$\mathcal{L}(\tilde{P}_{zw}, \hat{Q}_{zw}, \tilde{P}_{ed}, \tilde{Q}_{ed}, \mathcal{K}_i)$$
$$= \rho\,\mathrm{tr}\left[\tilde{B}_w^{\mathrm{T}}\tilde{P}_{zw}\tilde{B}_w + \frac{1}{h}\tilde{D}_{zw}^{\mathrm{T}}\tilde{D}_{zw}\right]$$
$$+ \mathrm{tr}\,\tilde{Q}_{zw}[\tilde{A}^{\mathrm{T}}\tilde{P}_{zw} + \tilde{P}_{zw}\tilde{A} + h\tilde{A}^{\mathrm{T}}\tilde{P}_{zw}\tilde{A} + \tilde{C}_z^{\mathrm{T}}\tilde{C}_z]$$
$$+ (1-\rho)\,\mathrm{tr}\left[\tilde{B}_d^{\mathrm{T}}\tilde{P}_{ed}\tilde{B}_d + \frac{1}{h}\tilde{D}_{ed}^{\mathrm{T}}\tilde{D}_{ed}\right]$$
$$+ \mathrm{tr}\,\tilde{Q}_{ed}\{\tilde{A}^{\mathrm{T}}\tilde{P}_{ed} + \tilde{P}_{ed}\tilde{A} + h\tilde{A}^{\mathrm{T}}\tilde{P}_{ed}\tilde{A} + \gamma^{-1}\tilde{C}_e^{\mathrm{T}}\tilde{C}_e$$
$$+\gamma^{-1}[\tilde{B}_d^{\mathrm{T}}\tilde{P}_{ed}(I + h\tilde{A}) + \tilde{S}^{\mathrm{T}}]^{\mathrm{T}}[\tilde{R} - h\gamma^{-1}\tilde{B}_d^{\mathrm{T}}\tilde{P}_{ed}\tilde{B}_d]^{-1}$$
$$\times [\tilde{B}_d^{\mathrm{T}}\tilde{P}_{ed}(I + h\tilde{A}) + \tilde{S}^{\mathrm{T}}]\} \qquad (88)$$

with partial derivatives given by

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}_{zw}} = \tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + h\tilde{A}\tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \rho\tilde{B}_w\tilde{B}_w^{\mathrm{T}} \qquad (89)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{Q}_{zw}} = \tilde{A}^{\mathrm{T}}\tilde{P}_{zw} + \tilde{P}_{zw}\tilde{A} + h\tilde{A}^{\mathrm{T}}\tilde{P}_{zw}\tilde{A} + \tilde{C}_z^{\mathrm{T}}\tilde{C}_z \qquad (90)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}_{ed}} = (\tilde{A} + \gamma^{-1}\tilde{B}_d\Gamma^{-1}\Sigma^{\mathrm{T}})\tilde{Q}_{ed} + \tilde{Q}_{ed}(\tilde{A} + \gamma^{-1}\tilde{B}_d\Gamma^{-1}\Sigma^{\mathrm{T}})^{\mathrm{T}}$$
$$+ h(\tilde{A} + \gamma^{-1}\tilde{B}_d\Gamma^{-1}\Sigma^{\mathrm{T}})$$
$$\times \tilde{Q}_{ed}(\tilde{A} + \gamma^{-1}\tilde{B}_d\Gamma^{-1}\Sigma^{\mathrm{T}})^{\mathrm{T}} + (1-\rho)\tilde{B}_d\tilde{B}_d^{\mathrm{T}} \qquad (91)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{Q}_{ed}} = \tilde{A}^{\mathrm{T}}\tilde{P}_{ed} + \tilde{P}_{ed}\tilde{A} + h\tilde{A}^{\mathrm{T}}\tilde{P}_{ed}\tilde{A}$$
$$+ \gamma^{-1}\Sigma^{\mathrm{T}}\Gamma^{-1}\Sigma + \gamma^{-1}\tilde{C}_e^{\mathrm{T}}\tilde{C}_e \qquad (92)$$

$$\frac{\partial\mathcal{L}}{\partial\mathcal{K}_i} = 2\sum_{j=1}^{\phi_i} \mathcal{Q}_{\mathrm{L}ij}^{\mathrm{T}}(I + \mathcal{D}_{yu}^{\mathrm{T}}L_{\mathcal{K}}^{-\mathrm{T}}\mathcal{K}^{\mathrm{T}})$$
$$\times \left[\rho\left(\mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\tilde{B}_w\mathcal{D}_{yw}^{\mathrm{T}} + \frac{1}{h}\mathcal{D}_{zu}^{\mathrm{T}}\tilde{D}_{zw}\mathcal{D}_{yw}^{\mathrm{T}}\right)\right.$$
$$+ (1-\rho)\left(\gamma^{-1}\mathcal{B}_u^{\mathrm{T}}\tilde{P}_{ed}\tilde{B}_d\mathcal{D}_{yd}^{\mathrm{T}} + \frac{1}{h}\gamma^{-2}\mathcal{D}_{eu}^{\mathrm{T}}\tilde{D}_{ed}\mathcal{D}_{yd}^{\mathrm{T}}\right)$$
$$+ \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\tilde{Q}_{zw}\mathcal{C}_y^{\mathrm{T}} + \mathcal{D}_{zu}^{\mathrm{T}}\tilde{C}_z\tilde{Q}_{zw}\mathcal{C}_y^{\mathrm{T}}$$
$$+ h\mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\tilde{A}\tilde{Q}_{zw}\mathcal{C}_y^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{ed}\tilde{Q}_{ed}\mathcal{C}_y^{\mathrm{T}}$$
$$+ \gamma^{-1}\mathcal{D}_{eu}^{\mathrm{T}}\tilde{C}_e\tilde{Q}_{ed}\mathcal{C}_y^{\mathrm{T}} + h\mathcal{B}_u^{\mathrm{T}}\tilde{P}_{ed}\tilde{A}\tilde{Q}_{ed}\mathcal{C}_y^{\mathrm{T}}$$
$$+ \gamma^{-1}\mathcal{B}_u^{\mathrm{T}}\tilde{P}_{ed}(I + h\tilde{A})\tilde{Q}_{ed}\Sigma^{\mathrm{T}}\Gamma^{-1}\mathcal{D}_{yd}^{\mathrm{T}}$$
$$+ h\gamma^{-1}\mathcal{B}_u^{\mathrm{T}}\tilde{P}_{ed}\tilde{B}_d\Gamma^{-1}\Sigma\tilde{Q}_{ed}\mathcal{C}_y^{\mathrm{T}}$$
$$+ \gamma^{-2}\mathcal{D}_{eu}^{\mathrm{T}}\tilde{C}_e\tilde{Q}_{ed}\Sigma^{\mathrm{T}}\Gamma^{-1}\mathcal{D}_{yd}^{\mathrm{T}}$$
$$+ \gamma^{-2}\mathcal{D}_{eu}^{\mathrm{T}}\tilde{D}_{ed}\Gamma^{-1}\Sigma\tilde{Q}_{ed}\mathcal{C}_y^{\mathrm{T}}$$
$$+ \gamma^{-3}\mathcal{D}_{eu}^{\mathrm{T}}\tilde{D}_{ed}\Gamma^{-1}\Sigma\tilde{Q}_{ed}\Sigma^{\mathrm{T}}\Gamma^{-1}\mathcal{D}_{yd}^{\mathrm{T}}$$
$$\left. + h\gamma^{-2}\mathcal{B}_u^{\mathrm{T}}\tilde{P}_{ed}\tilde{B}_d\Gamma^{-1}\Sigma\tilde{Q}_{ed}\Sigma^{\mathrm{T}}\Gamma^{-1}\mathcal{D}_{yd}^{\mathrm{T}}\right] L_{\mathcal{K}}^{-\mathrm{T}}\mathcal{Q}_{\mathrm{R}ij}^{\mathrm{T}} \qquad (93)$$

where

$$\Gamma \triangleq \tilde{R} + \gamma^{-1}\tilde{B}_d^{\mathrm{T}}\tilde{P}_{ed}\tilde{B}_d, \quad \Sigma \triangleq \tilde{B}_d^{\mathrm{T}}\tilde{P}_{ed}(I + h\tilde{A}) + \tilde{S}^{\mathrm{T}} \qquad (94)$$

For s-domain, q-domain and δ-domain $\mathcal{H}_2$-optimal control, the Lagrangians and necessary conditions can be obtained from the corresponding equations for the $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problem, with the substitutions $\tilde{B}_d = 0$, $\tilde{D}_{ed} = 0$, $\tilde{C}_e = 0$, $\tilde{P}_{ed} = 0$, $\tilde{Q}_{ed} = 0$, $\rho = 1$ into the partial derivatives $\partial\mathcal{L}/\partial\tilde{P}_{zw}$, $\partial\mathcal{L}/\partial\tilde{Q}_{zw}$ and $\partial\mathcal{L}/\partial\mathcal{K}_i$.

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