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Generalized Riccati equations for the full- and reduced-order mixed-norm H_2/H_∞ standard problem *

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Abstract: This paper considers the mixed-norm H_2/H_∞ standard problem. Specifically, an LQG control design problem involving a constraint on H_∞ disturbance attenuation is addressed. It is shown that the H_2/H_∞ dynamic compensator gains are completely characterized via coupled Riccati/Lyapunov equations. The principal result involves sufficient conditions for characterizing full- and reduced-order controllers that satisfy bounds on both H_2 and H_∞ performance costs. As a special case of this unified result we obtain the full-order H_∞ solution to the standard control problem and the pure reduced-order H_∞ solution with no H_2 contribution. Further extensions include nonstrictly proper dynamics, a direct transmission term from disturbances to H_∞ performance variables, cross-weighting and sensor noise/plant disturbance correlation, and a treatment of the pure *reduced-order* H_∞ control problem.

Keywords: H_2/H_∞ design; mixed norm; H_∞ reduced-order controllers.

1. Introduction

In a recent paper [1] a unification of the H_2 (LQG) and H_∞ control-design problems was obtained in terms of modified coupled algebraic Riccati equations. Specifically, the results of [1] address a unified solution of the H_2/H_∞ standard problem for full- and reduced-order controllers. This mixed-norm problem thus permits design tradeoffs between H_2 performance and H_∞ disturbance rejection.

The goal of the H_2/H_∞ problem is to minimize an H_2 performance criterion subject to a prespecified H_∞ constraint on the closed-loop transfer function. The H_∞ constraint is embedded within the optimization process by replacing the closed-loop covariance Lyapunov equation by a Riccati equation whose solution leads to an upper bound on the H_2 performance. The key idea to this approach is to view this upper bound as an auxiliary cost and, for a fixed controller structure, seek compensator gains that minimize the H_2 bound and guarantee that the disturbance attenuation constraint is enforced. The principal result is a sufficient condition involving coupled modified Riccati equations whose solutions, when they exist, are used to explicitly construct feedback gains for characterizing full- and reduced-order controllers with bounded H_2 and H_∞ costs. Note that, strictly speaking, the problem addressed is suboptimal in both the H_2 sense and the H_∞ sense. However, solving the design equations for progressively smaller H_∞ disturbance attenuation constraints should, in the limit, yield an H_∞-optimal controller over the class of fixed-structure stabilizing controllers. Although our main result gives sufficient conditions, these conditions will also be necessary as long as the mixed-norm optimization problem possesses at least one extremal over the class of fixed-structure controllers (see Lemma 2.2 and [2]).

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The solution given in [1] however, was restricted to the case in which the plant was strictly proper and there was no direct transmission from disturbances to H_∞ performance variables. The main contribution of the present paper is to extend the results of [1] to remove these restrictions and to allow further generalizations. First, a direct transmission term in the state space plant dynamics is included within the problem formulation along with a direct feedthrough term from exogenous disturbances to H_∞ performance variables. Next, to allow for greater design flexibility we permit correlated plant and measurement noise. And, finally, we consider the dual design feature of cross weighting in both the H_2 and H_∞ performance criteria. These generalizations have been studied in [14] for full-state feedback and in [4,5,11] for dynamic compensation. However, the results of [4,5,11] are limited to the 'pure' full-order H_∞ standard problem without the H_2/H_∞ unification. Furthermore, the results given in [4,5,11] are obtained by indirect transformation methods. In the present paper we derive the solution a to mixed-norm H_2/H_∞ fixed-order (i.e., full-, and reduced-order) dynamic compensation problem without employing such transformations.

It should be noted that the approach developed in [4,5] is quite different from our fixed-structure optimization design approach. Specifically, the authors in [4,5] consider a general H_∞ optimization problem of the form $\| T - UQV \|_\infty$, where Q is a parameterization of all stabilizing controllers that give infinity norm better than γ. It is shown that the central member of this set minimizes an entropy functional at infinity and yields a set of decoupled Riccati equations that characterize *full-order* compensators satisfying an H_∞ norm bound [5,8]. Furthermore, the results of [4,5,11] are necessary as well as sufficient. In contrast, the approach of [1] and the present paper is based upon Lagrange multiplier methods which permit the fixed-order-constraints as well as different H_2 and H_∞ performance weights.

Finally, as a special case of the results given in the present paper we obtain the full-order H_2 solution (LQG), reduced-order H_2 solution [6], full-order H_∞ solution [3,4,5,11], and the 'pure' reduced-order H_∞ solution with no H_2 contribution. It is interesting to note that in the full-order H_∞ controller case with no H_2 contribution our results specialize to [3,4,5,11]. Since the results of [3,4,5,11] are necessary as well as sufficient, these connections show that our sufficient conditions (at least in this special case) are also necessary.

Notation. Note: All matrices have real entries.

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$, $\mathbb{E}$	real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value.
I_r, $(\ )^{\mathrm{T}}$, $(\ )^*$	$r \times r$ identity matrix, transpose, complex conjugate transpose.
$\rho(\ )$	spectral radius.
$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices.
x, u, y, x_c, $\tilde{x}$	n, m, l, n_c, $\tilde{n}$-dimensional vectors.
A, B, C, D	$n \times n$, $n \times m$, $l \times n$, $l \times m$ matrices.
A_c, B_c, C_c	$n_c \times n_c$, $n_c \times l$, $m \times n_c$ matrices.
$\tilde{x}$, $\tilde{A}$	$\begin{bmatrix} x \\ x_c \end{bmatrix}$, $\begin{bmatrix} A & BC_c \\ B_cC & A_c + B_cDC_c \end{bmatrix}$.
γ	positive constant.
E_∞	$q_\infty \times d$ matrix.
M	$I_{q_\infty} - \gamma^{-2}E_\infty E_\infty^{\mathrm{T}}$, $M \in \mathbb{P}^{q_\infty}$.
N	$I_d - \gamma^{-2}E_\infty^{\mathrm{T}}E_\infty$, $N \in \mathbb{P}^d$.
$w(\cdot)$	d-dimensional standard white noise or L_2 signal.
D_1, D_2	$n \times d$, $l \times d$ matrices.
V_1, V_2, V_{12}	$D_1D_1^{\mathrm{T}}$, $D_2D_2^{\mathrm{T}}$, $D_1D_2^{\mathrm{T}}$; $V_2 \in \mathbb{P}^l$.
$V_{1\infty}$, $V_{2\infty}$, $V_{12\infty}$	$D_1N^{-1}D_1^{\mathrm{T}}$, $D_2N^{-1}D_2^{\mathrm{T}}$, $D_1N^{-1}D_2^{\mathrm{T}}$; $V_{2\infty} \in \mathbb{P}^l$.
$\tilde{D}$, $\tilde{V}$	$\begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}$, $\begin{bmatrix} V_1 & V_{12}B_c^{\mathrm{T}} \\ B_cV_{12}^{\mathrm{T}} & B_cV_2B_c^{\mathrm{T}} \end{bmatrix} = \tilde{D}\tilde{D}^{\mathrm{T}}$.
E_1, E_2	$q \times n$, $q \times m$ matrices.
$\tilde{E}$, R_1, R_2	$[E_1 \ \ E_2C_c]$, $E_1^{\mathrm{T}}E_1$, $E_2^{\mathrm{T}}E_2$; $R_2 \in \mathbb{P}^m$.
R_{12}, $\tilde{R}$	$E_1^{\mathrm{T}}E_2$, $\tilde{E}^{\mathrm{T}}\tilde{E}$.

$E_{1\infty}$, $E_{2\infty}$	$q_\infty \times n$, $q_\infty \times m$ matrices.
$\tilde{E}_\infty$, $R_{1\infty}$, $R_{2\infty}$	$[E_{1\infty}\ \ E_{2\infty}C_c]$, $E_{1\infty}^{\mathrm{T}}M^{-1}E_{1\infty}$, $E_{2\infty}^{\mathrm{T}}M^{-1}E_{2\infty}$.
$R_{12\infty}$, $\tilde{R}_\infty$	$E_{1\infty}^{\mathrm{T}}M^{-1}E_{2\infty}$, $\tilde{E}_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty$.
$R_{01\infty}$, $R_{02\infty}$	$E_\infty^{\mathrm{T}}M^{-1}E_{1\infty}$, $E_\infty^{\mathrm{T}}M^{-1}E_{2\infty}$.
α, β	nonnegative constants.

2. Statement of the problem

In this section we introduce the LQG dynamic output-feedback control problem with constrained H_∞ disturbance attenuation. Without the H_2 performance criterion the problem considered here is the standard H_∞ control problem [3,4,5]. For simplicity, the first part of the paper addresses controllers of order $n_c = n$ only, i.e., controllers whose order is equal to the dimension of the plant. This constraint is removed in Section 6 where controllers of reduced order are considered. Hence, throughout Sections 2–5 the controller dimension n_c and closed-loop plant dimension $\tilde{n} \triangleq n + n_c$ should be interpreted as n and $2n$, respectively.

***H_∞*-Constrained LQG Control Problem.** Given the nth-order stabilizable and detectable plant

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t), \tag{2.1}$$

$$y(t) = Cx(t) + Du(t) + D_2 w(t), \tag{2.2}$$

determine an nth-order dynamic compensator

$$\dot{x}(t) = A_c x_c(t) + B_c y(t), \tag{2.3}$$

$$u(t) = C_c x_c(t), \tag{2.4}$$

that satisfies the following design criteria:

(i) the closed-loop system (2.1)–(2.4) is asymptotically stable, i.e., $\tilde{A}$ is asymptotically stable;

(ii) the $q_\infty \times p$ nonstrictly proper transfer function

$$H(s) \triangleq \tilde{E}_\infty\left(sI_{\tilde{n}} - \tilde{A}\right)^{-1}\tilde{D} + E_\infty \tag{2.5}$$

from $w(t)$ to $z_\infty(t) = E_{1\infty}x(t) + E_{2\infty}u(t) + E_\infty w(t)$ satisfies the constraint

$$\| H(s) \|_\infty \le \gamma, \tag{2.6}$$

where $\gamma > 0$ is a given constant; and

(iii) the performance functional

$$J(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\left\{\int_0^t \left[x^{\mathrm{T}}(t)R_1 x(t) + 2x^{\mathrm{T}}(t)R_{12}u(t) + u^{\mathrm{T}}(t)R_2 u(t)\right]\,\mathrm{d}t\right\} \tag{2.7}$$

is minimized.

Note that the closed-loop system (2.1)–(2.4) can be written as

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t)$$

and that (2.7) becomes

$$J(A_c, B_c, C_c) = \lim_{t\to\infty} \mathbb{E}\left\{\left[\tilde{E}\tilde{x}(t)\right]^{\mathrm{T}}\left[\tilde{E}\tilde{x}(t)\right]\right\} = \lim_{t\to\infty} \mathbb{E}\left[\tilde{x}^{\mathrm{T}}(t)\tilde{R}\tilde{x}(t)\right]. \tag{2.8}$$

Furthermore, by defining the transfer function

$$\tilde{H}(s) \triangleq \tilde{E}\left(sI_{\tilde{n}} - \tilde{A}\right)^{-1}\tilde{D}, \tag{2.9}$$

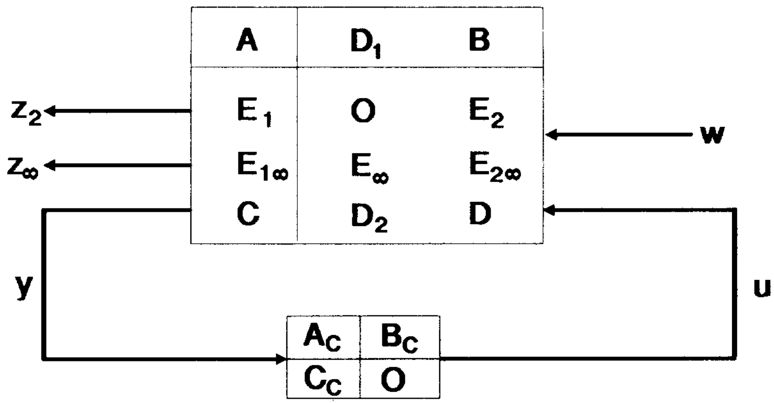

Fig. 1. The mixed-norm H_2/H_∞ standard problem includes the H_∞ standard problem and the LQG problem as special cases.

it can be shown that when $\tilde{A}$ is asymptotically stable, (2.8) is given by

$$J(A_c, B_c, C_c) = \| \tilde{H}(s) \|_2^2. \tag{2.10}$$

Note that the problem statement involves both H_2 and H_∞ performance weights. In particular, the matrices R_1 and R_2 are the H_2 weights for the state and control variables. By introducing the variables

$$z(t) = E_1 x(t), \qquad v(t) = E_2 u(t), \tag{2.11}$$

the H_2 cost (2.7) can be written as

$$J(A_c, B_c, C_c) = \lim_{t\to\infty} \mathbb{E}\left[z^{\mathrm{T}}(t)z(t) + 2z^{\mathrm{T}}(t)v(t) + v^{\mathrm{T}}(t)v(t)\right]. \tag{2.12}$$

For convenience we thus define $R_1 \triangleq E_1^{\mathrm{T}}E_1$ and $R_2 \triangleq E_2^{\mathrm{T}}E_2$ which appear in subsequent expressions. Note that $R_{12} \triangleq E_1^{\mathrm{T}}E_2$ is an H_2 cross-weighting term which is included for greater design flexibility.

For the H_∞ performance constraint, the transfer function (2.5) involves weighting matrices $E_{1\infty}$, $E_{2\infty}$, and E_∞ for the state, control, and disturbance variables. The matrices $R_{1\infty} \triangleq E_{1\infty}^T M^{-1} E_{1\infty}$ and $R_{2\infty} \triangleq E_{2\infty}^{\mathrm{T}} M^{-1} E_{2\infty}$ are thus the H_∞ counterparts of the H_∞ weights R_1 and R_2. Here $M \triangleq I_{q_\infty} - \gamma^{-2} E_\infty E_\infty^{\mathrm{T}}$ arises due to the feedthrough term to the H_∞ performance variables. Although we do not require that $R_{1\infty}$ and $R_{2\infty}$ be equal to R_1 and R_2, we shall assume for simplicity that $R_2 = \alpha^2 \hat{R}_2$ and $R_{2\infty} = \beta^2 \hat{R}_2$, where the nonnegative scalars α, β are design variables such that $\alpha^2 + \beta^2 \neq 0$. As in the H_2 case we allow an H_∞ cross-weighting term $R_{12\infty} \triangleq E_{1\infty}^{\mathrm{T}} M^{-1} E_{2\infty}$. Finally, the dual design feature of plant disturbance and sensor noise correlation is also permitted. As in [1], $w(t)$ is interpreted as white noise for the H_2 design aspect and as an L_2 signal for the H_∞ design aspect. Note that without the H_2 performance criterion, i.e., $R_1 = 0$ and $\alpha = 0$, the problem considered here reduces to the 'pure' H_∞ standard problem (see Figure 1).

Before continuing, it is useful to note that if $\tilde{A}$ is asymptotically stable for a given compensator (A_c, B_c, C_c) then the H_2 performance (2.8) is given by

$$J(A_c, B_c, C_c) = \operatorname{tr} \tilde{Q}\tilde{R}, \tag{2.13}$$

where the steady-state closed-loop state covariance defined by

$$\tilde{Q} \triangleq \lim_{t\to\infty} \mathbb{E}\left[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)\right] \tag{2.14}$$

satisfies the $\tilde{n} \times \tilde{n}$ algebraic Lyapunov equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V}. \tag{2.15}$$

The key step in enforcing the disturbance attenuation constraint (2.6) is to replace the algebraic Lyapunov equation (2.15) by an algebraic Riccati equation that overbounds the closed-loop steady-state covariance. Justification for this technique is provided by the following result.

Lemma 2.1. *Let* (A_c, B_c, C_c) *be given and assume there exists* $\mathscr{Q} \in \mathbb{R}^{n \times n}$ *satisfying*

$$\mathscr{Q} \in \mathbb{N}^{\tilde{n}} \tag{2.16}$$

and

$$0 = \tilde{A}\mathscr{Q} + \mathscr{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\left(\tilde{D}E_\infty^{\mathrm{T}} + \mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}\right)M^{-1}\left(\tilde{D}E_\infty^{\mathrm{T}} + \mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}\right)^{\mathrm{T}} + \tilde{V}. \tag{2.17}$$

Then

$$(\tilde{A},\ \tilde{D}) \text{ is stabilizable} \tag{2.18}$$

if and only if

$$\tilde{A} \text{ is asymptotically stable.} \tag{2.19}$$

In this case,

$$\| H(s) \|_\infty \leq \gamma \tag{2.20}$$

and

$$\tilde{Q} \leq \mathscr{Q}. \tag{2.21}$$

Consequently,

$$J(A_c,\ B_c,\ C_c) \leq \mathscr{J}(A_c,\ B_c,\ C_c,\ \mathscr{Q}), \tag{2.22}$$

where

$$\mathscr{J}(A_c,\ B_c,\ C_c,\ \mathscr{Q}) \triangleq \operatorname{tr} \mathscr{Q}\tilde{R}. \tag{2.23}$$

Proof. It follows from [13, Theorem 3.6] that (2.18) implies that

$$\left(\tilde{A},\ \left[\gamma^{-2}\left(\tilde{D}E_\infty^{\mathrm{T}} + \mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}\right)M^{-1}\left(\tilde{D}E_\infty + \mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}\right)^{\mathrm{T}} + \tilde{V}\right]^{1/2}\right)$$

is also stabilizable. Using the assumed existence of a nonnegative-definite solution to (2.17) and [13, Lemma 12.2] it now follows that $\tilde{A}$ is asymptotically stable. The converse is immediate. To prove (2.20), replace $\tilde{V}$ by $\tilde{D}\tilde{D}^{\mathrm{T}}$ and add and subtract $\mathrm{j}\omega I_{\tilde{n}}\mathscr{Q}$ to (2.17) so that (2.17) becomes

$$0 = \left(-\mathrm{j}\omega I_{\tilde{n}} + \tilde{A}\right)\mathscr{Q} + \mathscr{Q}\left(\mathrm{j}\omega I_{\tilde{n}} + \tilde{A}\right)^{\mathrm{T}} + \gamma^{-2}\left(\tilde{D}E_\infty^{\mathrm{T}} + \mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}\right)M^{-1}\left(\tilde{D}E_\infty^{\mathrm{T}} + \mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}\right)^{\mathrm{T}} + \tilde{D}\tilde{D}^{\mathrm{T}} \tag{2.24}$$

or, equivalently,

$$\tilde{D}\tilde{D}^{\mathrm{T}} = \left(\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\right)\mathscr{Q} + \mathscr{Q}\left(-\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\right)^{\mathrm{T}} - \gamma^{-2}\left(\tilde{D}E_\infty^{\mathrm{T}} + \mathscr{Q}E_\infty^{\mathrm{T}}\right)M^{-1}\left(\tilde{D}E_\infty^{\mathrm{T}} + \mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}\right)^{\mathrm{T}}. \tag{2.25}$$

Next, forming

$$\tilde{E}_\infty\left(\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\right)^{-1}(2.25)\left(-\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\right)^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}}$$

yields

$$\begin{aligned}
&\tilde{E}_\infty\left(\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\right)^{-1}\tilde{D}\tilde{D}^{\mathrm{T}}\left(-\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\right)^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}} \\
&\quad = \tilde{E}_\infty\left(\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\right)^{-1}\mathscr{Q}\tilde{E}_\infty^{\mathrm{T}} + \tilde{E}_\infty\mathscr{Q}\left(-\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\right)^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}} \\
&\qquad - \gamma^{-2}\tilde{E}_\infty\left(\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\right)^{-1}\left[\left(\tilde{D}E_\infty^{\mathrm{T}} + \mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}\right)M^{-1}\left(\tilde{D}E_\infty^{\mathrm{T}} + \mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}\right)^{\mathrm{T}}\right]\left(-\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\right)^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}}.
\end{aligned} \tag{2.26}$$

Now adding $\tilde{E}_\infty(j\omega I_{\tilde{n}}-\tilde{A})^{-1}\tilde{D}E_\infty^{\mathrm{T}}+E_\infty\tilde{D}^{\mathrm{T}}(-j\omega I_{\tilde{n}}-\tilde{A})^{-\mathrm{T}}+E_\infty E_\infty^{\mathrm{T}}$ to both sides of (2.26) yields

$$\begin{aligned}&\tilde{E}_\infty\left(j\omega I_{\tilde{n}}-\tilde{A}\right)^{-1}\tilde{D}\tilde{D}^{\mathrm{T}}\left(-j\omega I_{\tilde{n}}-\tilde{A}\right)^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}}+\tilde{E}_\infty\left(j\omega I_{\tilde{n}}-\tilde{A}\right)^{-1}\tilde{D}E_\infty^{\mathrm{T}}\\&\quad+E_\infty\tilde{D}^{\mathrm{T}}\left(-j\omega I_{\tilde{n}}-\tilde{A}\right)^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}}+E_\infty E_\infty^{\mathrm{T}}\\&=\tilde{E}_\infty\left(j\omega I_{\tilde{n}}-\tilde{A}\right)^{-1}\left[\tilde{D}E_\infty^{\mathrm{T}}+\mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\right]+\left[\tilde{D}E_\infty^{\mathrm{T}}+\mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\right]^{\mathrm{T}}\left(-j\omega I_{\tilde{n}}-\tilde{A}\right)^{-\mathrm{T}}+E_\infty E_\infty^{\mathrm{T}}\\&\quad+\gamma^{-2}\tilde{E}_\infty\left(j\omega I_{\tilde{n}}-\tilde{A}\right)^{-1}\left[\left(\tilde{D}E_\infty^{\mathrm{T}}+\mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\right)M^{-1}\left(\tilde{D}E_\infty^{\mathrm{T}}+\mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\right)^{\mathrm{T}}\right]\left(-j\omega I_{\tilde{n}}-\tilde{A}\right)^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}}.\end{aligned}\tag{2.27}$$

Note that the left hand side of (2.27) is equal $H(j\omega)H^*(j\omega)$ and the right hand side of (2.27) can be written as

$$S+S^*-\gamma^{-2}SM^{-1}S^*+\gamma^2\left(I_{q_\infty}-M\right)\tag{2.28}$$

where

$$S\triangleq\tilde{E}_\infty\left(j\omega I_{\tilde{n}}-\tilde{A}\right)^{-1}\left[\tilde{D}E_\infty^{\mathrm{T}}+\mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\right]$$

and $E_\infty E_\infty^{\mathrm{T}}$ is replaced by $\gamma^2(I_{q_\infty}-M)$. Hence, it follows from (2.27) and (2.28) that

$$H(j\omega)H^*(j\omega)=-\left[\left(\gamma M^{1/2}-\gamma^{-1}SM^{1/2}\right)\left(\gamma M^{1/2}-\gamma^{-1}SM^{1/2}\right)^*\right]+\gamma^2 I_{q_\infty}\geq 0,\tag{2.29}$$

which implies $H(j\omega)H^*(j\omega)\leq\gamma^2 I_{q_\infty}$. This proves (2.20). To prove (2.21), subtract (2.15) from (2.17) to obtain

$$0=\tilde{A}(\mathcal{Q}-\tilde{Q})+(\mathcal{Q}-\tilde{Q})\tilde{A}^{\mathrm{T}}+\gamma^{-2}\left(\tilde{D}E_\infty^{\mathrm{T}}+\mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\right)M^{-1}\left(\tilde{D}E_\infty^{\mathrm{T}}+\mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\right)^{\mathrm{T}}\tag{2.30}$$

which, since $\tilde{A}$ is asymptotically stable, is equivalent to

$$\mathcal{Q}-\tilde{Q}=\int_0^\infty e^{\tilde{A}t}\left[\gamma^{-2}\left(\tilde{D}E_\infty^{\mathrm{T}}+\mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\right)M^{-1}\left(\tilde{D}E_\infty^{\mathrm{T}}+\mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\right)^{\mathrm{T}}\right]e^{\tilde{A}^{\mathrm{T}}t}\,dt\geq 0.$$

Finally, (2.22) follows immediately from (2.21). □

Remark 2.1. An equivalent form of (2.17) is given by

$$0=\left(\tilde{A}+\gamma^{-2}\tilde{D}E_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty\right)\mathcal{Q}+\mathcal{Q}\left(\tilde{A}+\gamma^{-2}\tilde{D}E_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty\right)^{\mathrm{T}}+\gamma^{-2}\mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty\mathcal{Q}+\tilde{D}N^{-1}\tilde{D}^{\mathrm{T}}.\tag{2.31}$$

The equivalence of (2.17) and (2.31) is easily shown by noting that (2.17) can be rewritten as

$$\begin{aligned}0&=\left(\tilde{A}+\gamma^{-2}\tilde{D}E_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty\right)\mathcal{Q}+\mathcal{Q}\left(\tilde{A}+\gamma^{-2}\tilde{D}E_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty\right)^{\mathrm{T}}\\&\quad+\gamma^{-2}\mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty\mathcal{Q}+\gamma^{-2}\tilde{D}E_\infty^{\mathrm{T}}M^{-1}E_\infty\tilde{D}^{\mathrm{T}}+\tilde{D}\tilde{D}^{\mathrm{T}}\end{aligned}\tag{2.32}$$

and noting that $\tilde{D}[\gamma^{-2}E_\infty^{\mathrm{T}}M^{-1}E_\infty+I_d]\tilde{D}^{\mathrm{T}}$ is equal to $\tilde{D}N^{-1}\tilde{D}^{\mathrm{T}}$ since $E_\infty^{\mathrm{T}}M^{-1}=N^{-1}E_\infty^{\mathrm{T}}$ and $N^{-1}(\gamma^{-2}E_\infty^{\mathrm{T}}E_\infty+N)=N^{-1}$.

Lemma 2.1 shows that H_∞ disturbance attenuation is automatically enforced when a nonnegative-definite solution to (2.17) is known to exist and $\tilde{A}$ is asymptotically stable. Furthermore, all such solutions provide upper bounds for the actual closed-loop state covariance $\tilde{Q}$ along with a bound on the H_2 performance criterion. Next, we present a partial converse of Lemma 2.1 that guarantees the existence of a unique minimal nonnegative-definite solution to (2.17) when (2.20) is satisfied. The minimal solution is desirable since it yields the tightest performance bound in (2.22). This was first pointed out in [7].

Lemma 2.2. *Let* (A_c, B_c, C_c) *be given, suppose* $\tilde{A}$ *is asymptotically stable, and assume the disturbance attenuation constraint* (2.20) *is satisfied. Then there exists a unique nonnegative-definite solution* $\mathcal{Q}$ *satisfying* (2.17) *and such that the eigenvalues of* $\tilde{A}+\gamma^{-2}\tilde{D}E_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty+\gamma^{-2}\mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty$ *lie in the closed left half plane. Furthermore, this solution is also minimal.*

Proof. The result is an immediate extension of [2, pp. 150 and 167], using Theorems 3 and 2. The proof of minimality of given in [12]. □

3. The Auxiliary Minimization Problem

As shown in the previous section, replacing (2.15) by (2.17) enforces the H_∞ disturbance attenuation constraint and yields an upper bound for the H_2 performance criterion. That is, given a compensator (A_c, B_c, C_c) for which there exists a nonnegative-definite solution to (2.17), the *actual* H_2 performance $J(A_c, B_c, C_c)$ of the compensator is guaranteed to be no worse than the bound given by $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$. Hence, $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$ can be interpreted as an *auxiliary* cost which leads to the following optimization problem.

Auxiliary Minimization Problem. Determine $(A_c, B_c, C_c, \mathcal{Q})$ which minimizes $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$ subject to (2.17) with $\mathcal{Q} \in \mathbb{N}^{\tilde{n}}$.

It follows from Lemma 2.1 that the satisfaction of (2.16) and (2.17) along with the generic condition (2.18) lead to: (1) closed-loop stability; (2) prespecified H_∞ performance attenuation; (3) an upper bound for the H_2 performance criterion. Hence, it remains to determine $(A_c, B_c, C_c, \mathcal{Q})$ that minimizes $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$, and thus provides an optimized bound for the actual H_2 performance $J(A_c, B_c, C_c)$.

4. Sufficient conditions for H_∞ disturbance attenuation

In this section we state sufficient conditions for characterizing full-order controllers guaranteeing closed-loop stability, constrained H_∞ disturbance attenuation, and an optimized H_2 performance bound. For arbitrary $Q, P, \hat{Q} \in \mathbb{R}^{n\times n}$ and $\alpha, \beta \geq 0$ define the notation

$$Q_a \triangleq QC^{\mathrm{T}} + V_{12\infty}, \qquad P_a \triangleq \left[B^{\mathrm{T}} + \gamma^{-2}R_{02\infty}^{\mathrm{T}}D_1^{\mathrm{T}} + \gamma^{-2}R_{12\infty}^{\mathrm{T}}(Q+\hat{Q})\right]P + R_{12}^{\mathrm{T}},$$
$$S \triangleq \left(\alpha^2 I_n + \beta^2\gamma^{-2}\hat{Q}P\right)^{-1}$$

when the indicated inverse exists.

Theorem 4.1. *Suppose there exist* $Q, P, \hat{Q} \in \mathbb{N}^n$ *satisfying*

$$0 = \left(A + \gamma^{-2}D_1R_{01\infty}\right)Q + Q\left(A + \gamma^{-2}D_1R_{01\infty}\right)^{\mathrm{T}} + \gamma^{-2}QR_{1\infty}Q + V_{1\infty} - Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}}, \tag{4.1}$$

$$\begin{aligned}0 &= \left(A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty} + \gamma^{-2}D_1R_{01\infty} - \gamma^{-2}\hat{Q}S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}R_{12\infty}^{\mathrm{T}}\right)^{\mathrm{T}}P \\ &\quad + P\left(A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty} + \gamma^{-2}D_1R_{01\infty} - \gamma^{-2}\hat{Q}S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}R_{12\infty}^{\mathrm{T}}\right) + R_1 - S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}P_aS,\end{aligned} \tag{4.2}$$

$$\begin{aligned}0 &= \left(A - B\hat{R}_2^{-1}P_aS + \gamma^{-2}Q\left[R_{1\infty} - R_{12\infty}\hat{R}_2^{-1}P_aS\right] + \gamma^{-2}\left[D_1R_{01\infty} - D_1R_{02\infty}\hat{R}_2^{-1}P_aS\right]\right)\hat{Q} \\ &\quad + \hat{Q}\left(A - B\hat{R}_2^{-1}P_aS + \gamma^{-2}Q\left[R_{1\infty} - R_{12\infty}\hat{R}_2^{-1}P_aS\right] + \gamma^{-2}\left[D_1R_{01\infty} - D_1R_{02\infty}\hat{R}_2^{-1}P_aS\right]\right)^{\mathrm{T}} \\ &\quad + \gamma^{-2}\hat{Q}\left(R_{1\infty} - R_{12\infty}\hat{R}_2^{-1}P_aS - S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}R_{12\infty}^{\mathrm{T}} + \beta^2S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}P_aS\right)\hat{Q} + Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}},\end{aligned} \tag{4.3}$$

and let $(A_c, B_c, C_c, \mathcal{Q})$ *be given by*

$$\begin{aligned}A_c &= A - B\hat{R}_2^{-1}P_aS - Q_aV_{2\infty}^{-1}C - Q_aV_{2\infty}^{-1}D\hat{R}_2^{-1}P_aS \\ &\quad + \gamma^{-2}\big(QR_{1\infty} + D_1R_{01\infty} - D_1R_{02\infty}\hat{R}_2^{-1}P_aS - QR_{12\infty}\hat{R}_2^{-1}P_aS \\ &\qquad - Q_aV_{2\infty}^{-1}D_2R_{01\infty} + Q_aV_{2\infty}^{-1}D_2R_{02\infty}\hat{R}_2^{-1}P_aS\big),\end{aligned} \tag{4.4}$$

$$B_c = Q_a V_{2\infty}^{-1}, \qquad C_c = -\hat{R}_2^{-1} P_a S, \tag{4.5), (4.6}$$

$$\mathcal{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q} \\ \hat{Q} & \hat{Q} \end{bmatrix}. \tag{4.7}$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ disturbance attenuation constraint (2.20) and the H_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c) \le \operatorname{tr}\left[(Q + \hat{Q})R_1 - 2R_{12}\hat{R}_2^{-1}P_a S\hat{Q} + S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}R_2\hat{R}_2^{-1}P_a S\hat{Q}\right]. \tag{4.8}$$

Proof. The proof follows as in the proof given in [1]. □

Remark 4.1. Theorem 4.1 presents sufficient conditions for designing controllers with a prespecified H_∞ constraint on the closed-loop transfer function. These sufficient conditions comprise a system of three modified algebraic Riccati equations in variables Q, P, and $\hat{Q}$. The Q and P equations are similar to the estimator and regulator Riccati equations of LQG theory, while the $\hat{Q}$ equation has no counterpart in the standard theory. Note that the Q equation is decoupled from the P and $\hat{Q}$ equations and thus can be solved independently. The P equation, however depends on Q. Thus, regulator/estimator separation holds in only one direction which clearly shows that the certainty equivalence principle is no longer valid for the mixed H_2/H_∞ design problem. Finally, note that if the H_∞ disturbance attenuation constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, then the P equation becomes decoupled from the $\hat{Q}$ equation and thus the $\hat{Q}$ equation becomes superfluous. Furthermore, the remaining Q and P equations separate and coincide with the standard LQG result. Alternatively, note that if both $\beta = 0$ and $R_{1\infty} = 0$, then Theorem 4.1 also specializes to the standard LQG result.

Remark 4.2. The results of [1] are a special case of Theorem 4.1. To see this set the plant/measurement noise correlation to zero ($V_{12} = 0$), set both the H_2 and H_∞ cross weighting terms to zero (R_{12}, $R_{12\infty} = 0$), set the direct transmission term in the plant dynamics to zero ($D = 0$) and set the feedthrough term from disturbances to H_∞ performance variables to zero ($E_\infty = 0$). This yields Theorem 3.1 of [1].

Remark 4.3. When solving (4.1)–(4.3) numerically, the H_∞ constraint can be adjusted to examine tradeoffs between H_2 performance and disturbance rejection. Specifically, γ can be varied systematically to determine the region of solvability of the design equations (4.1)–(4.3) and to study tradeoffs between the H_2/H_∞ performance criteria (see [1]).

5. The pure H_∞ standard problem

As shown in Theorem 4.1, the Riccati equations (4.1)–(4.3) provide sufficient conditions for explicitly synthesizing controllers (A_c, B_c, C_c) satisfying both H_2 and H_∞ performance bounds. The main purpose of this section is to completely eliminate the H_2 aspect in the design problem. This section also provides connections between our approach and the recent results obtained in [3,4,6]. In [1] it was shown that by equalizing the H_2/H_∞ weights the three coupled Riccati equation form could be transformed into two decoupled Riccati equations as in [3,7]. Furthermore, it was shown in [7] that the auxiliary cost (2.23) is equivalent to an entropy integral. However, it is important to note that, as noticed in Remark 2.1, the results of [7] cannot consider a general direct transmission term from disturbances to H_∞ performance variables in order to guarantee that the minimum value of the entropy evaluated at infinity is finite. In the present paper we utilize a simpler approach wherein we eliminate the H_2 contribution by letting R_1, R_{12}, α (and thus R_2) approach zero. By eliminating the H_2 contribution to the problem, the resulting setting corresponds to the H_∞ standard problem. In order to state the main result we require some additional notation. For arbitrary $Y_\infty \in \mathbb{R}^{n\times n}$ define the notation

$$Y_{\infty a} \triangleq B^{\mathrm{T}} Y_\infty + \gamma^{-2} R_{02\infty}^{\mathrm{T}} D_1^{\mathrm{T}} Y_\infty + R_{12\infty}^{\mathrm{T}}.$$

Theorem 5.1. *Suppose there exist $Q \in \mathbb{N}^n$ and $Y_\infty \in \mathbb{P}^n$ satisfying*

$$0 = (A + \gamma^{-2} D_1 R_{01\infty}) Q + Q(A + \gamma^{-2} D_1 R_{01\infty})^{\mathrm{T}} + V_{1\infty} + \gamma^{-2} Q R_{1\infty} Q - Q_a V_{2\infty}^{-1} Q_a^{\mathrm{T}}, \tag{5.1}$$

$$0 = (A + \gamma^{-2} D_1 R_{01\infty})^{\mathrm{T}} Y_\infty + Y_\infty (A + \gamma^{-2} D_1 R_{01\infty}) + R_{1\infty} + \gamma^{-2} Y_\infty V_{1\infty} Y_\infty - Y_{\infty a}^{\mathrm{T}} R_{2\infty}^{-1} Y_{\infty a}, \tag{5.2}$$

$$\rho(Q Y_\infty) < \gamma^2, \tag{5.3}$$

and let (A_c, B_c, C_c) be given by

$$\begin{aligned} A_c = A &- B R_{2\infty}^{-1} Y_{\infty a} (I_n - \gamma^{-2} Q Y_\infty)^{-1} - Q_a V_{2\infty}^{-1} C + Q_a V_{2\infty}^{-1} D R_{2\infty} Y_{\infty a} (I_n - \gamma^{-2} Q Y_\infty)^{-1} \\ &+ \gamma^{-2} \Big[Q R_{1\infty} + D_1 R_{01\infty} - D_1 R_{02\infty} R_{2\infty}^{-1} Y_{\infty a} (I_n - \gamma^{-2} Q Y_\infty)^{-1} \\ &\quad - Q R_{12\infty} R_{2\infty}^{-1} Y_{\infty a} (I_n - \gamma^{-2} Q Y_\infty)^{-1} - Q_a V_{2\infty}^{-1} D_2 R_{01\infty} \\ &\quad + Q_a V_{2\infty}^{-1} D_2 R_{02\infty} R_{2\infty}^{-1} Y_{\infty a} (I_n - \gamma^{-2} Q Y_\infty)^{-1} \Big], \end{aligned} \tag{5.4}$$

$$B_c = Q_a V_{2\infty}^{-1}, \qquad C_c = -R_{2\infty}^{-1} Y_{\infty a} (I_n - \gamma^{-2} Q Y_\infty)^{-1}. \tag{5.5), (5.6}$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ disturbance attenuation constraint (2.20).

Proof. First let R_1, R_{12}, $\alpha \to 0$ in equations (4.1)–(4.3) so that $S \to \beta^{-2}\gamma^2 P^{-1} \hat{Q}^{-1}$. Next, note that $P_a S = \beta^{-2} \gamma^2 \Sigma \hat{Q}^{-1}$, where

$$\Sigma \triangleq B^{\mathrm{T}} + \gamma^{-2} R_{02\infty}^{\mathrm{T}} D_1^{\mathrm{T}} + \gamma^{-2} R_{12\infty}^{\mathrm{T}} (Q + \hat{Q}).$$

Now define $Z_\infty \triangleq \gamma^2 \hat{Q}^{-1}$ and substitute into (4.3) to obtain

$$\begin{aligned} 0 = {} & (A + \gamma^{-2} Q R_{1\infty} + \gamma^{-2} D_1 R_{01\infty})^{\mathrm{T}} Z_\infty + Z_\infty (A + \gamma^{-2} Q R_{1\infty} + \gamma^{-2} D_1 R_{01\infty}) + R_{1\infty} \\ & + \gamma^2 Z_\infty Q_a V_{2\infty}^{-1} Q_a^{\mathrm{T}} Z_\infty - Z_{\infty a}^{\mathrm{T}} R_{2\infty}^{-1} Z_{\infty a}, \end{aligned}$$

where $Z_{\infty a} \triangleq \Sigma Z_\infty$. Now note that (5.2) follows by forming $Y_\infty \triangleq (Z_\infty^{-1} + \gamma^2 Q)^{-1}$. The gain expressions (5.4)–(5.6) follow as a direct consequence. □

Remark 5.1. The solutions Q and Y_∞ of (5.1) and (5.2) are analogous to the matrices S and P of [5] and Y_∞ and X_∞ of [4], while (5.3) corresponds to condition 5.2 (iii) of [4].

Remark 5.2. By setting $R_{12\infty}$, E_∞, and D to zero, the results of Theorem 5.1 specialize to Theorem 6 of [3] and Proposition 5.7 of [1] without the H_2 performance bound.

6. Mixed-norm reduced-order dynamic compensation

In this section we extend Theorem 4.1 by expanding the formulation of Sections 2 and 3 to allow the compensator to be of fixed dimension n_c which may be less than the plant order n. Hence, in this section define $\tilde{n} = n + n_c$, where $n_c \le n$. As in [1,6] this additional constraint leads to an oblique projection that introduces additional coupling in the design equations along with an additional equation. The following lemma is required for the statement of the main theorem (see [1].)

Lemma 6.1. *Let $\hat{Q}$, $\hat{P} \in \mathbb{N}^n$ and suppose* rank $\hat{Q}\hat{P} = n_c$. *Then there exist $n_c \times n$ G, Γ, and $n_c \times n_c$ invertible M, unique except for a change of basis in $\mathbb{R}^{n_c}$, such that*

$$\hat{Q}\hat{P} = G^{T} M \Gamma, \qquad \Gamma G^{\mathrm{T}} = I_{n_c}. \tag{6.1), (6.2}$$

Furthermore, the $n \times n$ matrices

$$\tau = G^{\mathrm{T}}\Gamma, \qquad \tau_{\perp} = I_n - \tau \tag{6.3), (6.4}$$

are idempotent and have rank n_c and $n - n_c$.

Theorem 6.1. *Let $n_c \leq n$, suppose there exist Q, P, $\hat{Q}$, $\hat{P} \in \mathbb{N}^n$ satisfying*

$$0 = (A + \gamma^{-2}D_1R_{01\infty})Q + Q(A + \gamma^{-2}D_1R_{01\infty})^{\mathrm{T}} + \gamma^{-2}QR_{1\infty}Q + V_{1\infty} - Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}} + \tau_{\perp}Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}}\tau_{\perp}^{\mathrm{T}}, \tag{6.5}$$

$$0 = (A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty} + \gamma^{-2}D_1R_{01\infty} - \gamma^{-2}\hat{Q}S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}R_{12\infty}^{\mathrm{T}})^{\mathrm{T}}P + P(A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty} + \gamma^{-2}D_1R_{01\infty} - \gamma^{-2}\hat{Q}S^{\mathrm{T}}P_a\hat{R}_2^{-1}R_{12\infty}^{\mathrm{T}}) + R_1 - S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}P_aS + \tau_{\perp}^{\mathrm{T}}S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}P_aS\tau_{\perp}, \tag{6.6}$$

$$0 = (A - B\hat{R}_2^{-1}P_aS + \gamma^{-2}Q[R_{1\infty} - R_{12\infty}\hat{R}_2^{-1}P_aS] + \gamma^{-2}D_1[R_{01\infty} - R_{02\infty}\hat{R}_2^{-1}P_aS])\hat{Q} + \hat{Q}(A - B\hat{R}_2^{-1}P_aS + \gamma^{-2}Q[R_{1\infty} - R_{12\infty}\hat{R}_2^{-1}P_aS] + \gamma^{-2}D_1[R_{01\infty} - R_{02\infty}\hat{R}_2^{-1}P_aS])^{\mathrm{T}} + \gamma^{-2}\hat{Q}(R_{1\infty} - R_{12\infty}\hat{R}_2^{-1}P_aS - S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}R_{12\infty}^{\mathrm{T}} + \beta^2S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}P_aS)\hat{Q} + Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}} - \tau_{\perp}Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}}\tau_{\perp}^{\mathrm{T}}, \tag{6.7}$$

$$0 = (A - Q_aV_{2\infty}^{-1}C + \gamma^{-2}D_1R_{01\infty} + \gamma^{-2}QR_{1\infty} - \gamma^{-2}Q_aV_{2\infty}^{-1}D_2R_{01\infty})^{\mathrm{T}}\hat{P} + \hat{P}(A - Q_aV_{2\infty}^{-1}C + \gamma^{-2}D_1R_{01\infty} + \gamma^{-2}QR_{1\infty} - \gamma^{-2}Q_aV_{2\infty}^{-1}D_2R_{01\infty}) + S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}P_aS - \tau_{\perp}^{\mathrm{T}}S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}P_aS\tau_{\perp}, \tag{6.8}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c, \tag{6.9}$$

and let $(A_c, B_c, C_c, \mathcal{Q})$ be given by

$$A_c = \Gamma[A - B\hat{R}_2^{-1}P_aS - Q_aV_{2\infty}^{-1}C + Q_aV_{2\infty}^{-1}D\hat{R}_2^{-1}P_aS + \gamma^{-2}(QR_{1\infty} + D_1R_{01\infty} - D_1R_{02\infty}\hat{R}_2^{-1}P_aS - QR_{12\infty}\hat{R}_2^{-1}P_aS - Q_aV_{2\infty}^{-1}D_2R_{01\infty} + Q_aV_{2\infty}^{-1}D_2R_{02\infty}\hat{R}_2^{-1}P_aS)]G^{\mathrm{T}}, \tag{6.10}$$

$$B_c = \Gamma Q_aV_{2\infty}^{-1}, \qquad C_c = -\hat{R}_2^{-1}P_aSG^{\mathrm{T}}, \tag{6.11), (6.12}$$

$$\mathcal{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^{\mathrm{T}} \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}} \end{bmatrix}. \tag{6.13}$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ disturbance attenuation constraint (2.20) *and the H_2 performance criterion* (2.7) *satisfies the bound*

$$J(A_c, B_c, C_c) \leq \operatorname{tr}[(Q + \hat{Q})R_1 - 2R_{12}\hat{R}_2^{-1}P_aS\hat{Q} + S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}P_aS\hat{Q}]. \tag{6.14}$$

Proof. The proof follows as in [1] with the additional terms arising due to cross weighting, disturbance/measurement noise correlation, and direct feedthrough terms. □

Remark 6.1. It is easy to see that Theorem 6.1 is a direct generalization of Theorem 4.1. To recover Theorem 4.1, set $n_c = n$ so that $\tau = G = \Gamma = I_n$ and $\tau_\perp = 0$. In this case the last term in each of (6.5)–(6.8) can be deleted and (6.8) becomes superfluous. Furthermore, (6.5)–(6.7) now reduce to (4.1)–(4.3), as expected. Alternatively, setting $\gamma = \infty$ and retaining the reduced-order constraint $n_c < n$ yields the result of [6]. Finally, to recover Theorem 6.1 of [1] set $V_{12} = 0$, $R_{12} = 0$, $R_{12\infty} = 0$, $D = 0$, and $E_\infty = 0$.

Remark 6.2. As was noted earlier, the assumption that $R_2 = \alpha^2 \hat{R}_2$ and $R_{2\infty} = \beta^2 \hat{R}_2$ was made for simplicity. If it is desired that R_2 and $R_{2\infty}$ be independent then (6.12) is given by

$$C_c = -\mathrm{vec}^{-1}\left[\Omega \mathrm{vec}\left(P_a G^{\mathrm{T}}\right)\right],$$

where

$$\Omega \triangleq R_2 \otimes I_{n_c} + \gamma^{-2} R_{2\infty} \otimes \Gamma \hat{Q} P G^{\mathrm{T}},$$

'vec' is the column stacking operation, and $\otimes$ denotes Kronecker product. In this case, the compensator dynamics (6.10) along with the design equations (6.5)–(6.8) have to be changed accordingly. However, due to lack of space this result is not given. Similar remarks apply to the full-order mixed-norm problem given by Theorem 4.1.

7. The pure H_∞ reduced-order dynamic compensation problem

In this section we eliminate the H_2 aspect of the reduced-order design problem to obtain reduced-order controllers for the pure H_∞ standard problem. As in the full-order controller case (Section 5) we eliminate the H_2 contribution by letting R_1, R_{12}, α (and thus R_2) approach zero. In order to state the main result we require some additional notation. For arbitrary Q, $\hat{Q}$, $P \in \mathbb{N}^n$, and G, $\Gamma \in \mathbb{R}^{n_c \times n}$ define

$$P_{a\infty} \triangleq B^{\mathrm{T}} P + \gamma^{-2} R_{02\infty}^{\mathrm{T}} D_1^{\mathrm{T}} P + \gamma^{-2} R_{12\infty}^{\mathrm{T}} (Q + \hat{Q}) P, \tag{7.1}$$

$$M_\infty \triangleq (\Gamma \hat{Q} \Gamma^{\mathrm{T}})^{-1}, \qquad N_\infty \triangleq (G P G^{\mathrm{T}})^{-1}, \qquad (7.2), (7.3)$$

$$S_\infty \triangleq \gamma^2 N_\infty M_\infty, \qquad W_\infty \triangleq \gamma^4 \Gamma^{\mathrm{T}} S_\infty^{\mathrm{T}} G P_{a\infty}^{\mathrm{T}} R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty \Gamma. \qquad (7.4), (7.5)$$

Theorem 7.1. *Suppose there exist* Q, P, $\hat{Q}$, $\hat{P} \in \mathbb{N}^n$ *satisfying* (6.9), $GPG^{\mathrm{T}} > 0$, *and*

$$0 = \left(A + \gamma^{-2} D_1 R_{01\infty}\right) Q + Q\left(A + \gamma^{-2} D_1 R_{01\infty}\right)^{\mathrm{T}} + V_{1\infty} + \gamma^{-2} Q R_{1\infty} Q - Q_a V_{2\infty}^{-1} Q_a^{\mathrm{T}} + \tau_\perp Q_a V_{2\infty}^{-1} Q_a^{\mathrm{T}} \tau_\perp^{\mathrm{T}}, \tag{7.6}$$

$$\begin{aligned} 0 = {} & \left(A + \gamma^{-2}\left[Q + \hat{Q}\right] R_{1\infty} + \gamma^{-2} D_1 R_{01\infty} - \gamma^{-2} \hat{Q} \Gamma^{\mathrm{T}} S_\infty^{\mathrm{T}} G P_{a\infty}^{\mathrm{T}} R_{2\infty}^{-1} R_{12\infty}^{\mathrm{T}} + \gamma^{-2} \hat{Q} W_\infty\right)^{\mathrm{T}} P \\ & + P\left(A + \gamma^{-2}\left[Q + \hat{Q}\right] R_{1\infty} + \gamma^{-2} D_1 R_{01\infty} - \gamma^{-2} Q \Gamma^{\mathrm{T}} S_\infty^{\mathrm{T}} G P_{a\infty}^{\mathrm{T}} R_{2\infty}^{-1} R_{12\infty}^{\mathrm{T}} + \gamma^{-2} \hat{Q} W_\infty\right) \\ & + R_{1\infty} - P_{a\infty} R_{2\infty}^{-1} P_{a\infty} + \left(I_n - G^{\mathrm{T}} S_\infty \Gamma\right)^{\mathrm{T}} P_{a\infty}^{\mathrm{T}} R_{2\infty}^{-1} P_{a\infty}\left(I_n - G^{\mathrm{T}} S_\infty \Gamma\right), \end{aligned} \tag{7.7}$$

$$\begin{aligned} 0 = {} & \left(A - B R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty \Gamma + \gamma^{-2} Q\left[R_{1\infty} - R_{12\infty} R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty \Gamma\right]\right. \\ & \qquad \left. + \gamma^{-2}\left[D_1 R_{01\infty} - R_{02\infty} R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty \Gamma\right]\right) \hat{Q} \\ & + \hat{Q}\left(A - B R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty \Gamma + \gamma^{-2} Q\left[R_{1\infty} - R_{12\infty} R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty \Gamma\right]\right. \\ & \qquad \left. + \gamma^{-2}\left[D_1 R_{01\infty} - R_{02\infty} R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty \Gamma\right]\right) \\ & + \gamma^{-2} \hat{Q}\left(R_{1\infty} - R_{12\infty} R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty \Gamma - \Gamma^{\mathrm{T}} S_\infty^{\mathrm{T}} P_{a\infty}^{\mathrm{T}} R_{2\infty}^{-1} R_{12\infty}^{\mathrm{T}} + W_\infty\right) \hat{Q} \\ & + Q_a V_{2\infty}^{-1} Q_a^{\mathrm{T}} - \tau_\perp Q_a V_{2\infty}^{-1} Q_a^{\mathrm{T}} \tau_\perp^{\mathrm{T}}, \end{aligned} \tag{7.8}$$

$$0=\left(A-Q_aV_{2\infty}^{-1}C+\gamma^{-2}D_1R_{01\infty}+\gamma^{-2}QR_{1\infty}-\gamma^{-2}Q_aV_{2\infty}^{-1}D_2R_{01\infty}\right)^{\mathrm{T}}\hat{P}$$
$$+\hat{P}\left(A-Q_aV_{2\infty}^{-1}C+\gamma^{-2}D_1R_{01\infty}+\gamma^{-2}QR_{1\infty}-\gamma^{-2}Q_aV_{2\infty}^{-1}D_2R_{01\infty}\right)$$
$$+P_{a\infty}^{T}R_{2\infty}^{-1}P_{a\infty}-\left(I_n-G^{\mathrm{T}}S_\infty\Gamma\right)^{\mathrm{T}}P_{a\infty}^{\mathrm{T}}R_{2\infty}^{-1}P_{a\infty}\left(I_n-G^{\mathrm{T}}S_\infty\Gamma\right)-\gamma^{-2}\left(W_\infty\hat{Q}P+P\hat{Q}W_\infty\right), \tag{7.9}$$

and let (A_c, B_c, C_c) *be given by*

$$A_c=\Gamma\Big[A-R_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty\Gamma-Q_aV_{2\infty}^{-1}C+Q_aV_{2\infty}^{-1}DR_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty\Gamma$$
$$+\gamma^{-2}\big(QR_{1\infty}+D_1R_{01\infty}-D_1R_{02\infty}R_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty\Gamma-QR_{12\infty}R_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty\Gamma$$
$$-Q_aV_{2\infty}^{-1}D_2R_{01\infty}+Q_aV_{2\infty}^{-1}D_2R_{02\infty}R_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty\Gamma\big)\Big]G^{\mathrm{T}}, \tag{7.10}$$

$$B_c=\Gamma Q_aV_{2\infty}^{-1},\qquad C_c=-R_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty. \qquad (7.11),(7.12)$$

Then $(\tilde{A}, \tilde{D})$ *is stabilizable if and only if* $\tilde{A}$ *is asymptotically stable. In this case, the closed-loop transfer function* $H(s)$ *satisfies the* H_∞ *disturbance attenuation constraint* (2.20).

Proof. The proof follows from Theorem 6.1 by using the relation $G^{\mathrm{T}}\hat{S}\Gamma=S\tau$, where $\hat{S}\triangleq(\alpha^2 I_{n_c}+\gamma^{-2}\beta^2\Gamma\hat{Q}PG^{\mathrm{T}})^{-1}$ and letting R_1, R_{12}, $\alpha\to 0$. □

Remark 7.1. Theorem 7.1 presents sufficient conditions for designing *reduced-order* controllers with a prespecified H_∞ constraint on the closed-loop transfer function with no H_2 contribution. Thus, Theorem 7.1 addresses the pure reduced-order H_∞-standard problem. Note that considerable simplification can be achieved in the design equations by setting $R_{12\infty}$, E_∞, and D to zero.

8. Numerical solution of the design equations

Although the design equations appearing in Theorems 4.1, 6.1 and 7.1 appear formidable, they are, in fact, quite numerically tractable. One of the principal motivations of the Riccati equation approach to the mixed norm problem is the opportunity it provides for developing efficient computational algorithms for control design. In particular, the goal is to develop numerical methods that exploit the structure of these modified Riccati equations. It should be noted, however, that existing methods for solving standard Riccati equations cannot account for the additional terms that appear in the modified equations such as (6.5)–(6.8). Therefore, a new class of numerical algorithms has been developed based upon homotopic continuation methods. These methods operate by first replacing the original problem by a simpler problem with a known solution. The desired solution is then reached by integrating along a path (homotopy path) that connects the starting problem to the original problem. The advantage of such algorithms is that they are based on theories which are global in nature. In particular, homotopy methods facilitate the finding of (multiple) solutions to a problem, and the convergence of the homotopy algorithms is generally not dependent upon having initial conditions which are in some sense close to the actual solution. These ideas have been illustrated for the H_2 reduced-order problem in [9] and the H_∞ constrained problem in [1] where the additional coupling terms preclude standard solution techniques. A complete description of the homotopy algorithm is given in [10].

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