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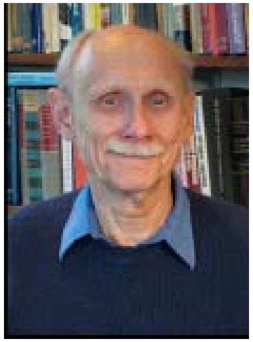
# On the Advantages of Being Periodic

***r**Evolving Horizons in Systems and Control*

*In Honor and Appreciation of*

*Elmer Gilbert*

Evolving Horizons in Systems and Control



# Goal

- 33 years of perspective on what my PhD research was about!

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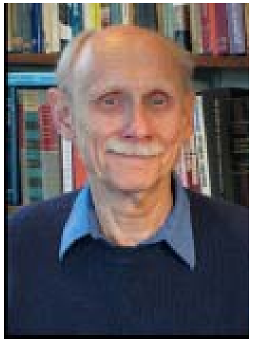
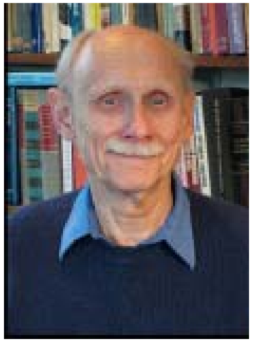


# Periodic History

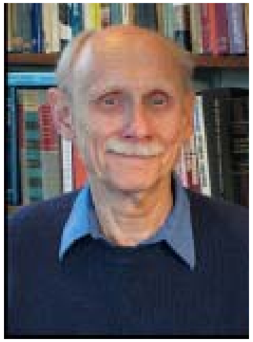
- Marquez—*One Hundred Years of Solitude*
  - “Just like Aureliano,” Ursula exclaimed. “It’s as if the world were repeating itself.”





# Periodic = Monotonous?

- Ecclesiastes (Koheleth)
  - The sun rises and the sun sets,
    and hurries back to where it rises.
  - The wind blows to the south
    and turns to the north;
    round and round it goes,
    ever returning on its course.
  - All streams flow into the sea,
    yet the sea is never full.
    To the place the streams come from,
    there they return again.
  - What has been will be again,
    what has been done will be done again;
    there is nothing new under the sun.



# Periodic Control

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- What is it?
- Why do it?
- Why need it?

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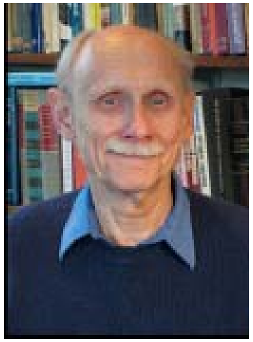

# Control for the Long Term

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- How to best control a system for long-term operation?
  - Ignore transients/startup
    - E.g., ascent, descent
  - Operate sustainably
    - E.g., Cruise
    - Maximize endurance
    - Minimize fuel usage rate
- Constant operation---obvious approach
- Periodic operation---why?

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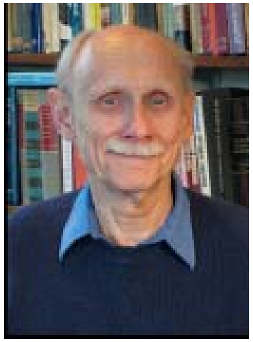

# Optimal Periodic Control

SIAM J. CONTROL AND OPTIMIZATION
Vol. 15, No. 5, August 1977

## OPTIMAL PERIODIC CONTROL: A GENERAL THEORY OF NECESSARY CONDITIONS*

ELMER G. GILBERT†

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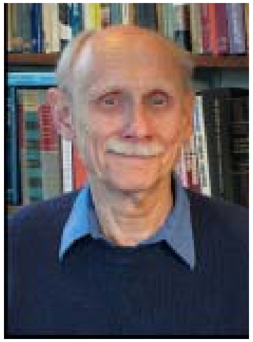


# Optimal Periodic Control

*Optimal periodic control problem* (OPC). Find $u(\cdot)$, $x(\cdot)$ and $\tau$ which minimize $J$ subject to

(2.1-1) $$J = g_0(y, x(0)),$$

(2.1-2) $$g_i(y, x(0)) \leqq 0, \qquad i = -j, \cdots, -1,$$

(2.1-3) $$g_i(y, x(0)) = 0, \qquad i = 1, \cdots, k,$$

(2.1-4) $$y = \frac{1}{\tau}\int_0^\tau \tilde{f}(x(t), u(t))\, dt \in Y,$$

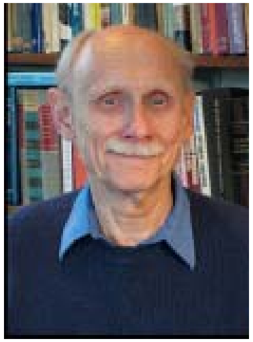
(2.1-5) $$\dot{x}(t) = f(x(t), u(t)) \quad \text{almost all } t \in [0, T], \quad x(0) = x(\tau),$$

(2.1-6) $$u(\cdot) \in \mathscr{U} = \{u(\cdot) : u(\cdot) \text{ measurable and essentially bounded on } [0, T],\ u(t) \in U \text{ for all } t \in [0, T]\},$$

(2.1-7) $$x(\cdot) \in \mathscr{X} = \{x(\cdot) : x(\cdot) \text{ absolutely continuous on } [0, T],\ x(t) \in X \text{ for all } t \in [0, T]\},$$

(2.1-8) $$\tau \in (0, T].$$

Gilbert, SICOPT, 1977



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# Optimal Steady-State Control

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*Optimal steady-state problem* (OSS). Find $u$ and $x$ which minimize $J$ subject to

(2.2-1) $$J = g_0(y, x),$$

(2.2-2) $$g_i(y, x) \leqq 0, \qquad i = -j, \cdots, -1,$$

(2.2-3) $$g_i(y, x) = 0, \qquad i = 1, \cdots, k,$$

(2.2-4) $$y = \tilde{f}(x, u) \in Y,$$

(2.2-5) $$f(x, u) = 0,$$

(2.2-6) $$u \in U,$$

(2.2-7) $$x \in X.$$

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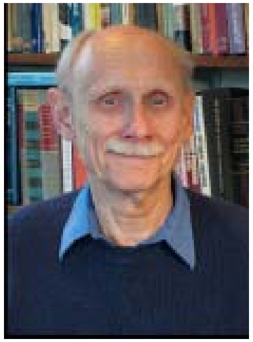

# Solution Sets

(2.3) $$\mathcal{S}(\text{OPC} = \{(u(\cdot), x(\cdot), \tau): (u(\cdot), x(\cdot), \tau \text{ solves OPC}\},$$

(2.4) $$\mathcal{S}(\text{SS}) = \{(u(\cdot), x(\cdot), \tau): \text{(2.1-2)–(2,1-8) are satisfied and } u(\cdot) \text{ and } x(\cdot) \text{ are constant}\},$$

(2.5) $$\mathcal{S}(\text{SSOPC}) = \mathcal{S}(\text{OPC}) \cap \mathcal{S}(\text{SS}),$$

(2.6) $$\mathcal{S}(\text{OSS}) = \{(u(\cdot), x(\cdot), \tau): (u(\cdot), x(\cdot), \tau) \in \mathcal{S}(\text{SS}) \text{ and } (u(0), x(0)) \text{ solves OSS}\}.$$

Of course, $\mathcal{S} = \varnothing$, the null set, is possible in any of the four cases. The particular circumstance $\mathcal{S}(\text{SSOPC}) = \varnothing$, $\mathcal{S}(\text{OSS}) \neq \varnothing$ implies that there exist time-dependent controls which do better than the best steady-state controls. If $\mathcal{S}(\text{SSOPC}) \neq \varnothing$ any $\psi \in \mathcal{S}(\text{SSOPC})$ is also in $\mathcal{S}(\text{OSS})$ since $\psi$ is optimum with respect to choices in $\mathcal{U} \times \mathcal{X} \times (0, T]$ and $\mathcal{S}(\text{SS}) \subset \mathcal{U} \times \mathcal{X} \times (0, T]$. Also, it is clear that all elements of $\mathcal{S}(\text{OSS})$ and $\mathcal{S}(\text{SSOPC})$ yield identical costs $J$. This leads to the following.

*Remark* 2.1. There are three mutually exclusive possibilities:

(i) $\mathcal{S}(\text{SSOPC}) = \mathcal{S}(\text{OSS}) \neq \varnothing$;

(ii) $\mathcal{S}(\text{SSOPC}) = \varnothing$, $\mathcal{S}(\text{OSS}) \neq \varnothing$;

(iii) $\mathcal{S}(\text{SSOPC}) = \mathcal{S}(\text{OSS}) = \varnothing$.

Possibility (iii) is not apt to occur since for well posed problems it is likely that $\mathcal{S}(\text{OSS}) \neq \varnothing$. Possibility (i) implies that OPC has a steady-state solution and consequently, there is no advantage (even though OPC may also have time-dependent solutions) in using time-dependent control. Possibility (ii) implies time-dependent control can do better than steady-state control (a statement which holds true even if $\mathcal{S}(\text{OPC}) = \varnothing$). Because of the importance of possibilities (i) and (ii) the following definitions are introduced.

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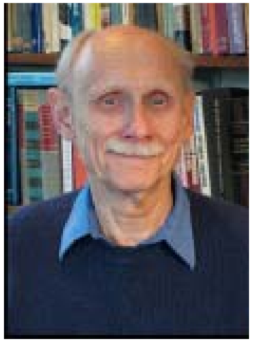


# More Solution Sets

DEFINITION 2.1. If $\mathscr{S}(\text{SSOPC}) = \mathscr{S}(\text{OSS}) \neq \varnothing$ the problem OPC is called *steady-state*.

DEFINITION 2.2. If $\mathscr{S}(\text{SSOPC}) = \varnothing$, $\mathscr{S}(\text{OSS}) \neq \varnothing$ the problem OPC is called *proper* (compare [5]).

The study of relative minima of OPC and OSS will prove to be of value, particularly in the case of steady-state minima.

DEFINITION 2.3. $(u(\cdot), x(\cdot), \tau) \in \mathscr{S}(\text{SS})$ is a *strong* {*weak*} *relative minimum* of OPC if there exists an $\varepsilon > 0$ such that for all $(\hat{u}(\cdot), \hat{x}(\cdot), \hat{\tau})$ which satisfy (2.1-2)–(2.1-8) and $\|\hat{x}(t) - x(0)\| < \varepsilon$ $\{\|\hat{x}(t) - x(0)\| < \varepsilon,\ \|\hat{u}(t) - u(0)\| < \varepsilon\}$, $t \in [0, T]$, it follows that $g_0(y, x(0)) \leqq g_0(\hat{y}, \hat{x}(0))$.

DEFINITION 2.4. $(u(\cdot), x(\cdot), \tau) \in \mathscr{S}(\text{SS})$ is a *strong* {*weak*} *relative minimum* of OSS if there exists an $\varepsilon > 0$ such that for all $(\hat{u}, \hat{x})$ which satisfy (2.2-2)–(2.2-7) and $\|\hat{x} - x(0)\| < \varepsilon$ $\{\|\hat{x} - x(0)\| < \varepsilon,\ \|\hat{u} - u(0)\| < \varepsilon\}$ it follows that $g_0(y, x(0)) \leqq g_0(\hat{y}, \hat{x})$.

In these definitions $\|\cdot\|$ denotes any norm on $R^n$ or $R^l$ and $\hat{y} = y$ for $u = \hat{u}$, $x = \hat{x}$, $\tau = \hat{\tau}$. Corresponding to each of the four types of relative minima, notations for the set of minima are adopted:

$$\mathscr{S}(\text{SRMSSOPC}), \mathscr{S}(\text{WRMSSOPC}), \mathscr{S}(\text{SRMOSS}), \mathscr{S}(\text{WRMOSS}).$$

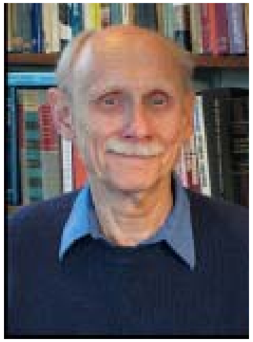
For example,

$$(2.7)\quad \mathscr{S}(\text{SRMSSOPC}) = \{(u(\cdot), x(\cdot), \tau) : (u(\cdot), x(\cdot), \tau) \in \mathscr{S}(\text{SS}) \text{ is a strong relative minimum of OPC}\}.$$

Obviously, $\mathscr{S}(\text{SSOPC}) \subset \mathscr{S}(\text{SRMSSOPC}) \subset \mathscr{S}(\text{WRMSSOPC})$ and $\mathscr{S}(\text{OSS}) \subset \mathscr{S}(\text{SRMOSS}) \subset \mathscr{S}(\text{WRMOSS})$. By using the same reasoning which led to Remark 2.1 it is easy to see that $\mathscr{S}(\text{SRMSSOPC}) \subset \mathscr{S}(\text{SRMOSS})$. However, $\mathscr{S}(\text{SRMSSOPC}) \neq \varnothing$ does not imply $\mathscr{S}(\text{SRMSSOPC}) = \mathscr{S}(\text{SRMOSS})$ because elements of $\mathscr{S}(\text{SRMSSOPC})$ do not necessarily have the same cost as elements of $\mathscr{S}(\text{SRMOSS})$. Similar reasoning applies to the case of weak relative minima. All of this is summarized in

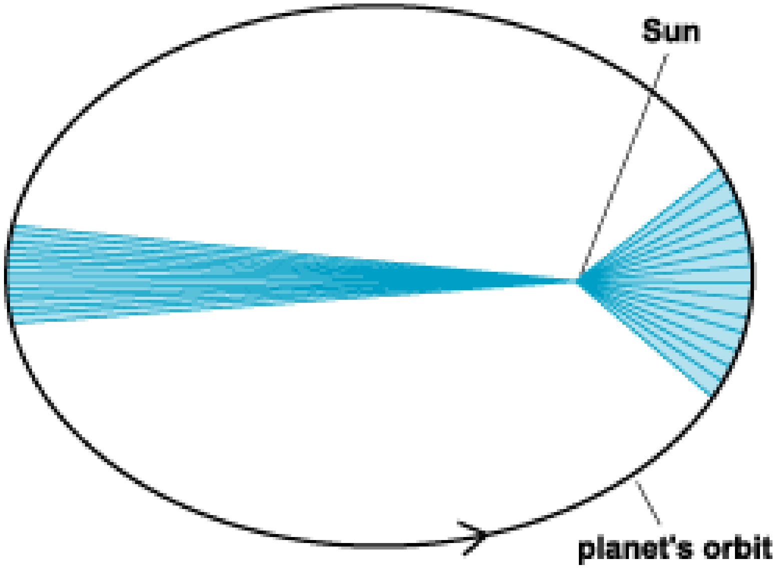
*Remark* 2.2. The following conclusions are valid: $\mathscr{S}(\text{SSOPC}) \subset \mathscr{S}(\text{SRMSSOPC}) \subset \mathscr{S}(\text{WRMSSOPC})$, $\mathscr{S}(\text{OSS}) \subset \mathscr{S}(\text{SRMOSS}) \subset \mathscr{S}(\text{WRMOSS})$, $\mathscr{S}(\text{SSOPC}) \subset \mathscr{S}(\text{OSS})$, $\mathscr{S}(\text{SRMSSOPC}) \subset \mathscr{S}(\text{SRMOSS})$, $\mathscr{S}(\text{WRMSSOPC}) \subset \mathscr{S}(\text{WRMOSS})$.

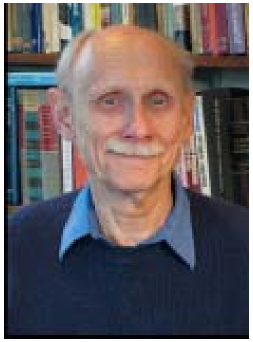
Evolving



# Astrodynamics and Periodicity

- Kepler's laws and elliptical orbits



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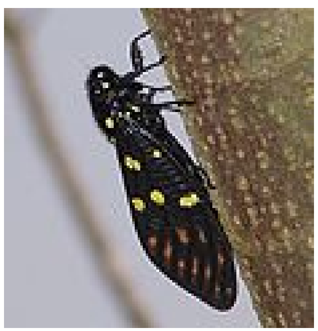
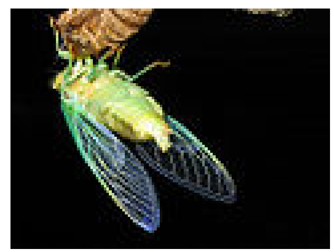
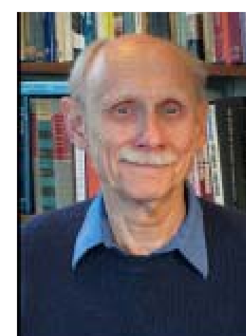
# Prime Periodicity

- Cicada
  - 13 and 17 year cycles
  - Predator resistance







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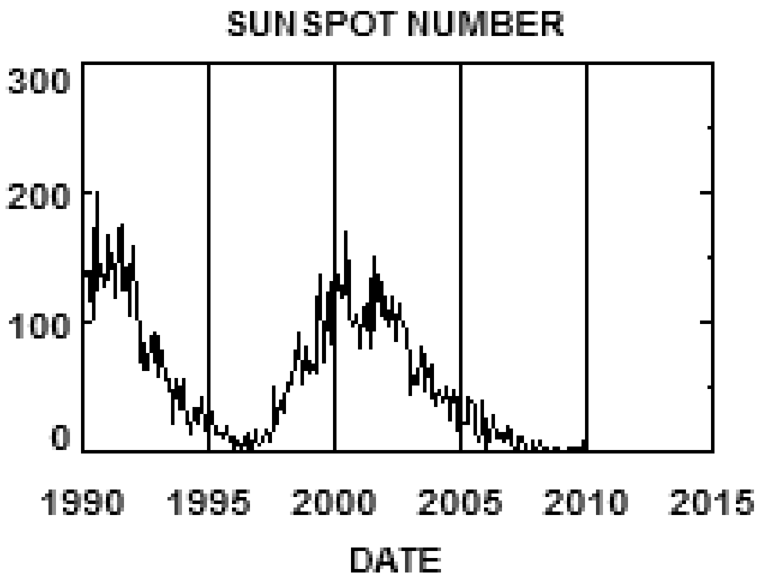
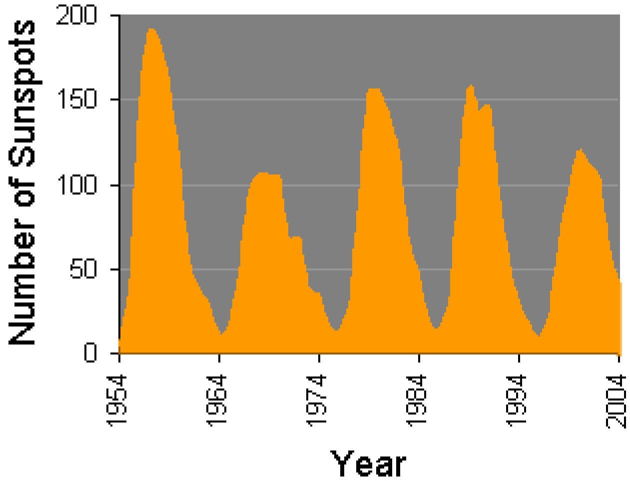
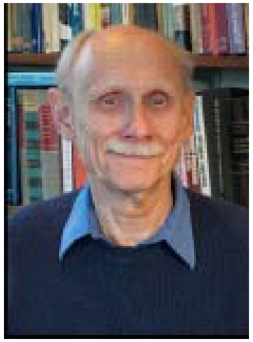
# Sunspot Periodicity

- 9-14 years, 11 years on average



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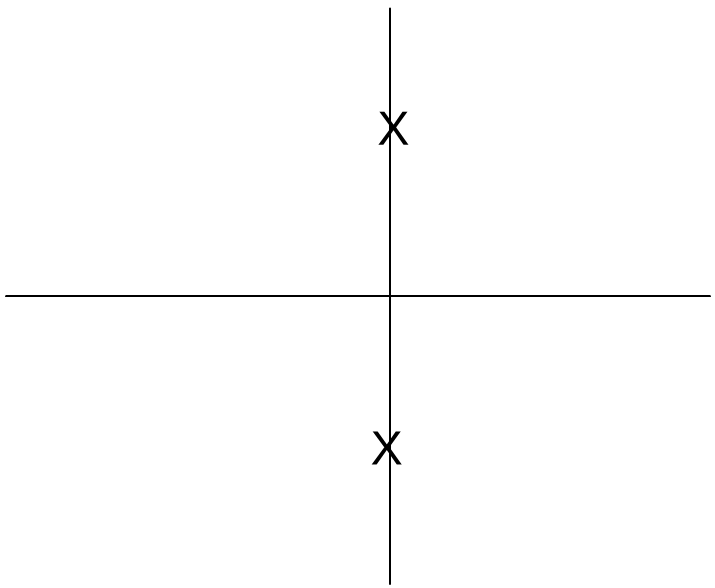
# Linear Systems Periodicity

- Imaginary poles give periodic response



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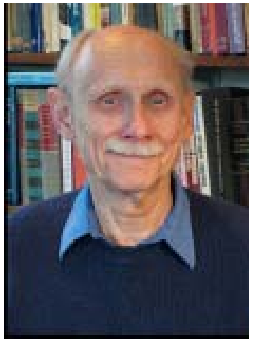
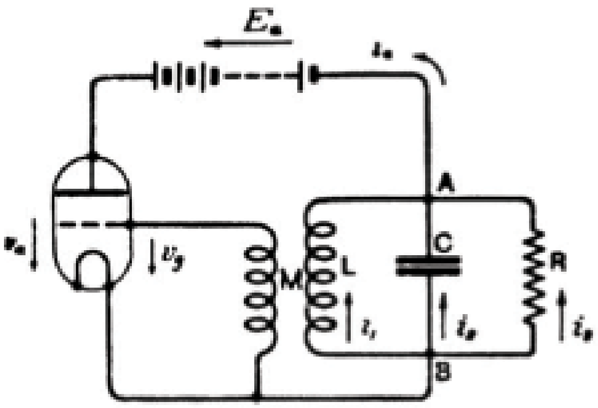
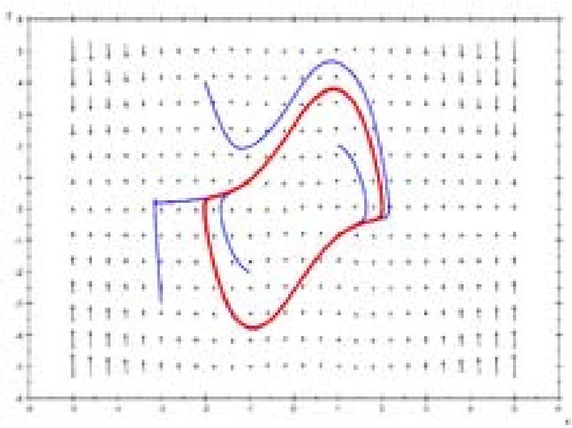


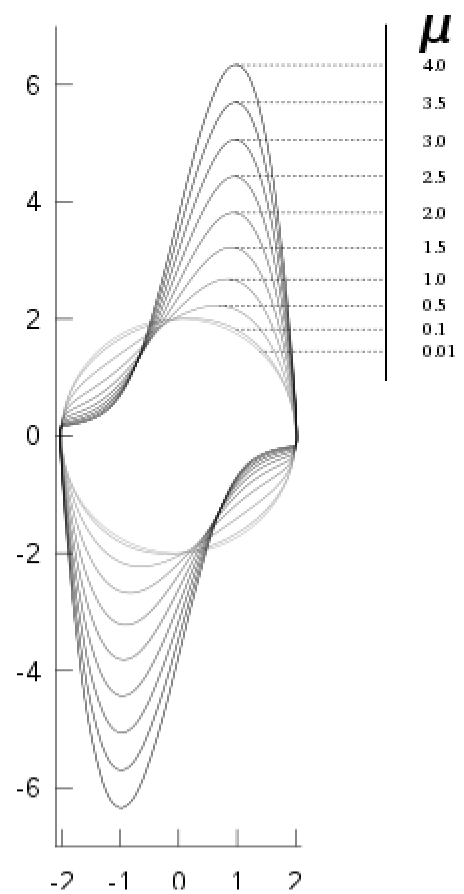
# Nonlinear System Periodicity

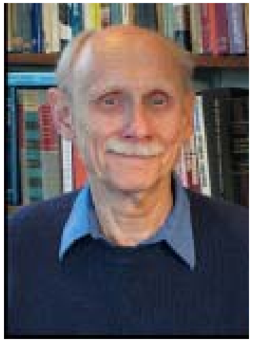
**(Van der Pol 1920)**

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$





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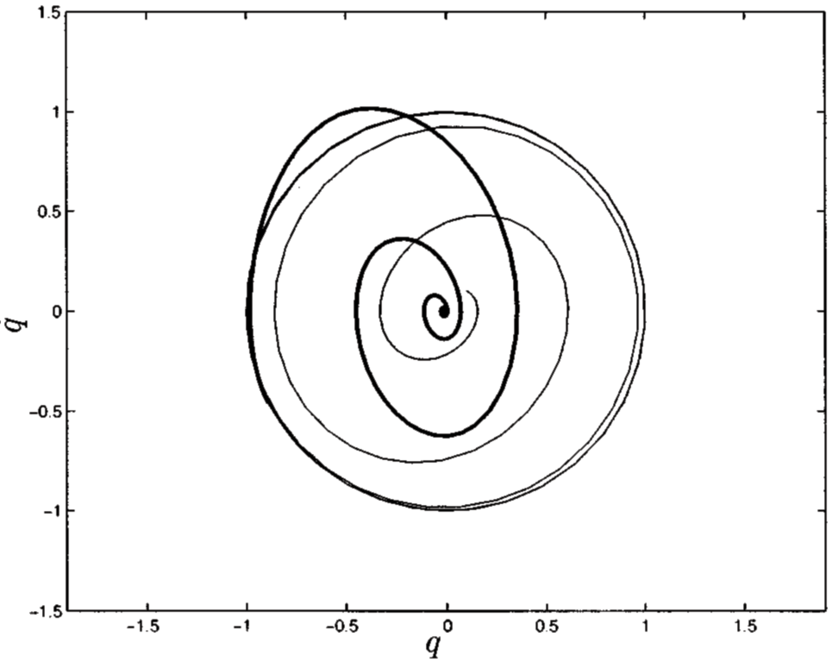
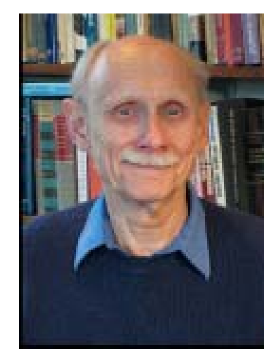
# Roup Oscillator

## 2001

- To obtain a circular/sinusoidal limit cycle with amplitude *a* and frequency $\omega_n$
- Speed of convergence to limit cycle determined by $\lambda$



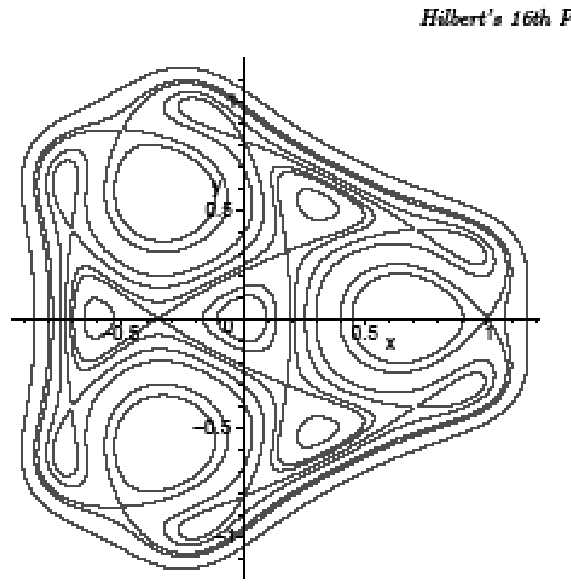

$$\ddot{q} + \lambda(q^2 + \omega_n^{-2}\dot{q}^2 - a^2) + \omega_n^2 q = 0$$

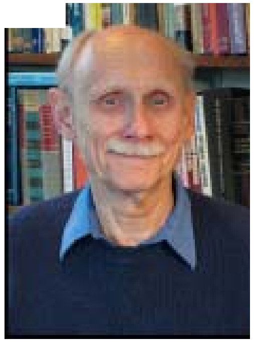


# Hilbert's 16$^{th}$ Problem (1900)

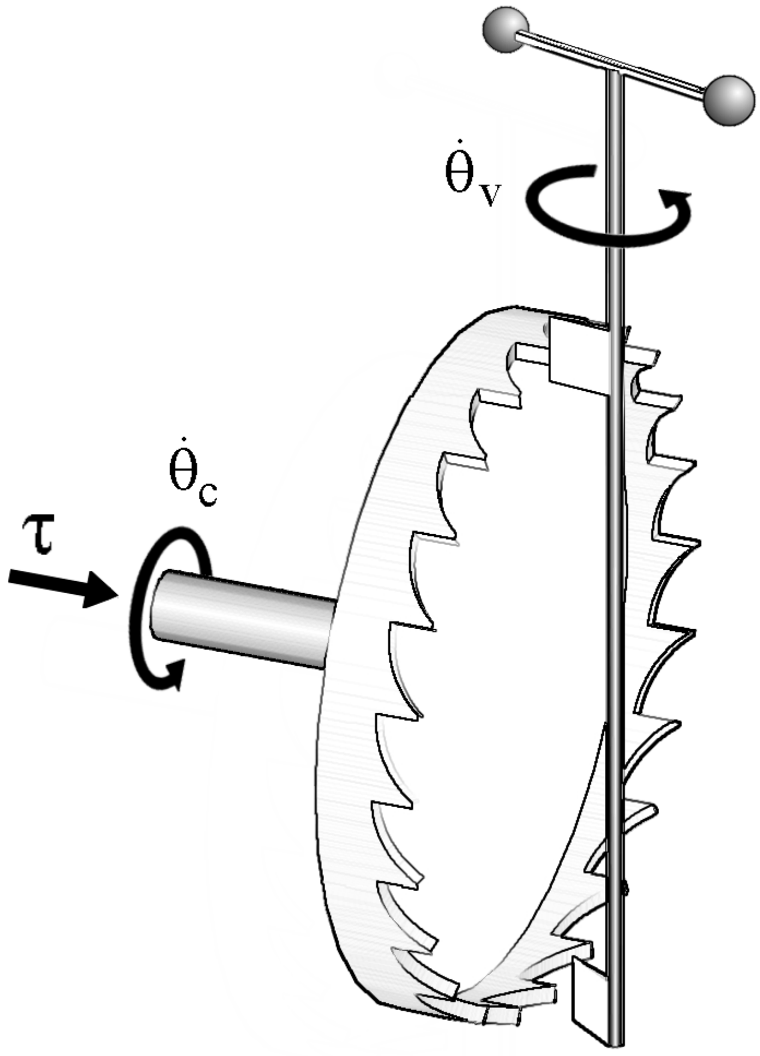
- How many limit cycles does a planar polynomial have?
  - Dulac's theorem 1923: Finite number for each system
    - Incorrect proof stood for 80 years—current status uncertain
  - For n=2, 4 are possible
  - For n=3, 11 are possible
  - For n=5, 24 are possible
  - No upper bound known for ANY n
- Besides the Riemann hypothesis, the most "elusive" of Hilbert's problems

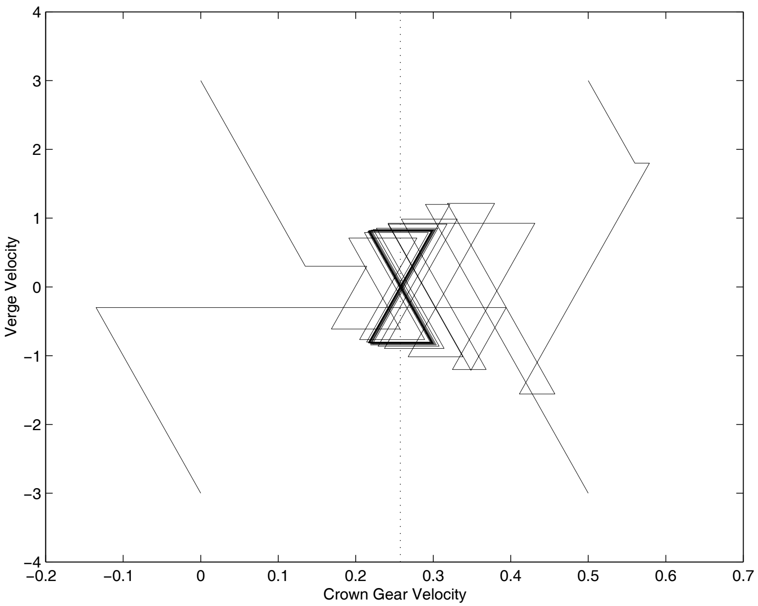
$$
\begin{aligned}
\dot{x} &= P(x,y) \\
\dot{y} &= Q(x,y)
\end{aligned}
$$

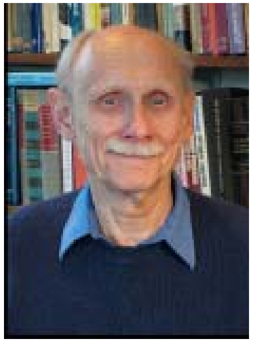


# Clocks



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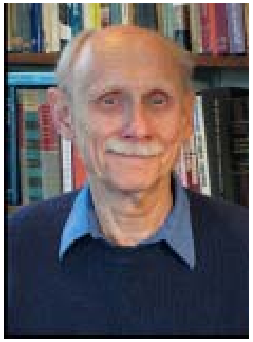
# Self-Recycling Worlds

Shrimp

Ecospheres



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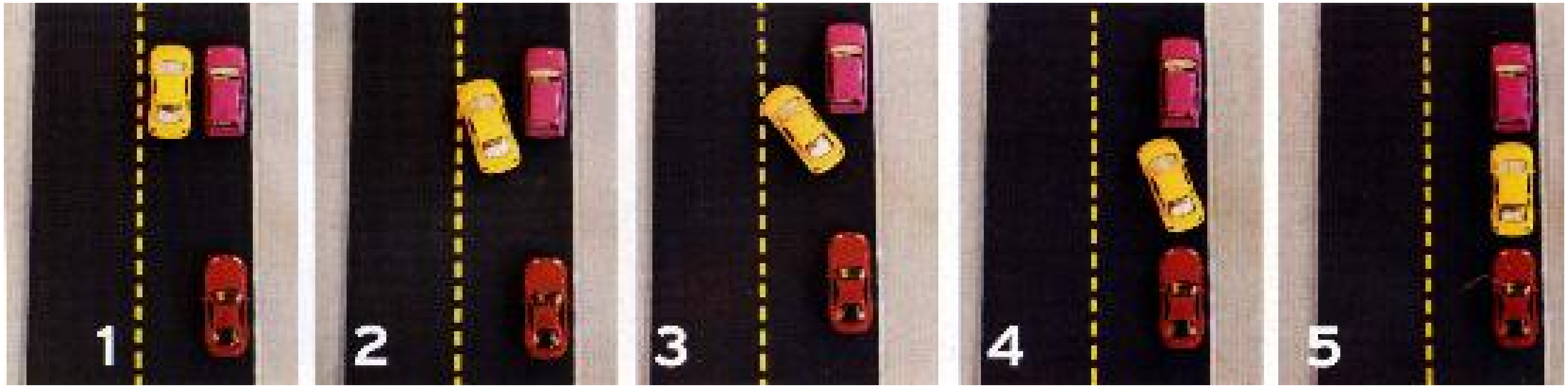
# Nonholonomic Systems

- System with constraint on a velocity..
  - But not a constraint on position




Turning radius is constrained
Multiple passes may be needed
The number of required passes increases as the turning radius decreases

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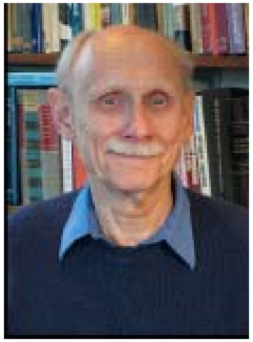
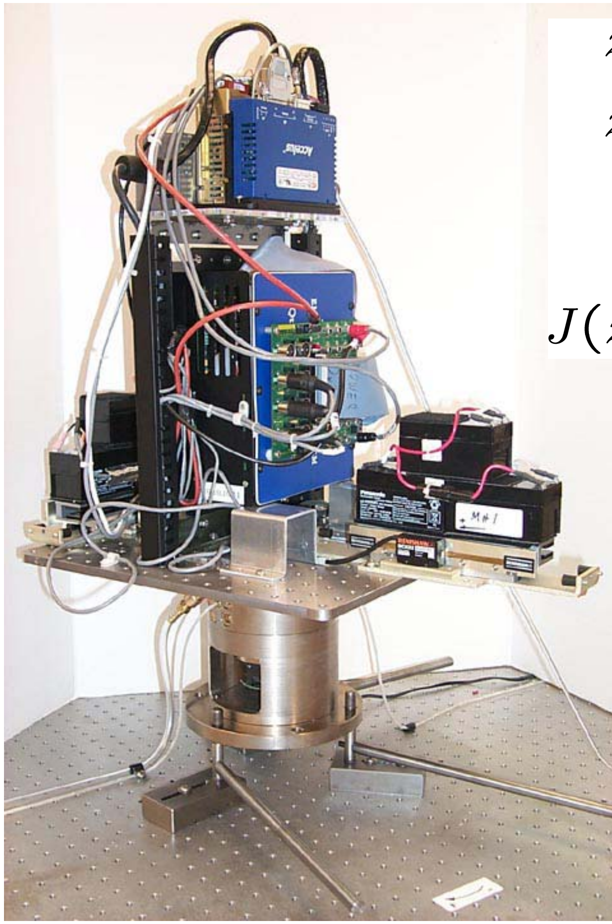


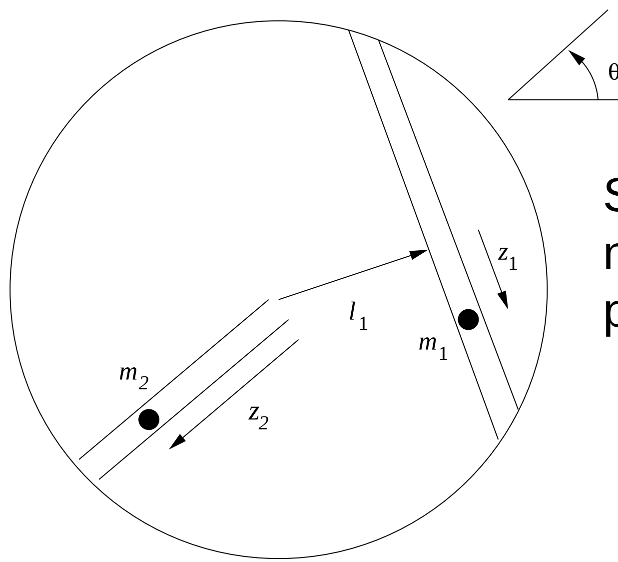
# Shape Change Actuation

**(Shen and McClamroch)**

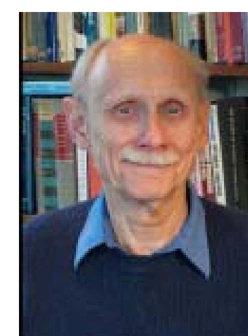
Angular momentum is conserved
…..but attitude is not constrained



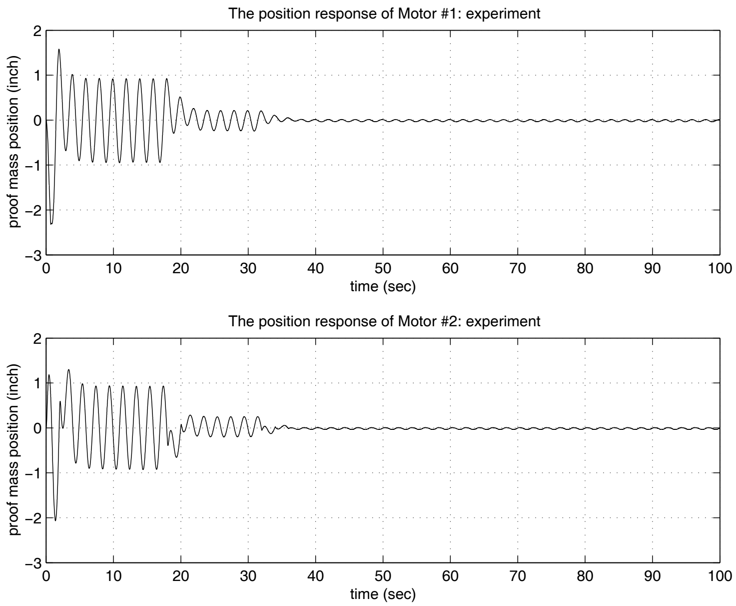
$$
\begin{aligned}
\dot{z}_1 &= v_1 \\
\dot{z}_2 &= v_2 \\
\dot{\theta} &= \frac{m_1 l_1}{J(z)} v_1 + \frac{m_2 l_2}{J(z)} v_2 \\
J(z) &= I + m_1(l_1^2 + z_1^2) + m_2(l_2^2 + z_2^2)
\end{aligned}
$$

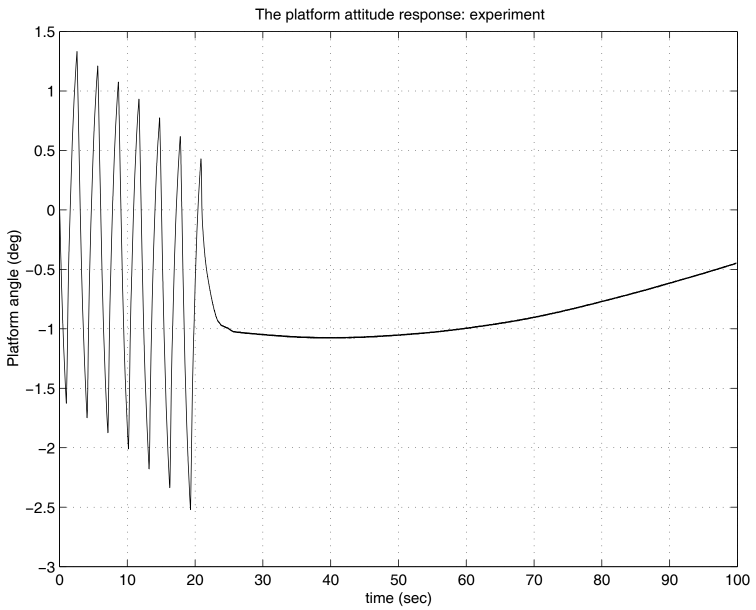


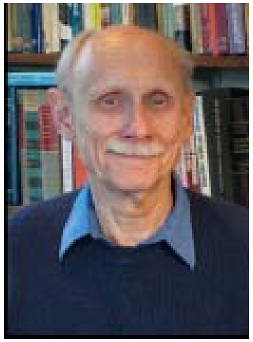

Stroke constraints necessitate multiple passes



# Platform Reorientation



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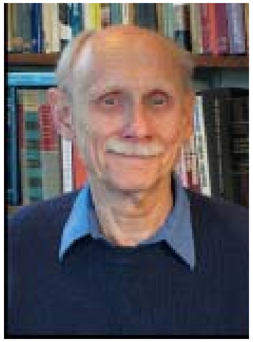
# Periodic Flight

- How do birds stay aloft and cover huge distances with minimal energy?

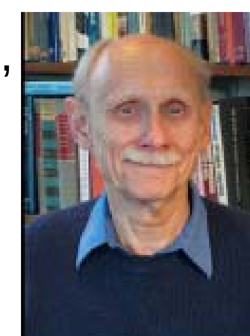
- How can we keep an aircraft aloft indefinitely?





# Dynamic Soaring

- **Harvesting energy from wind gradients—from special terrain**
  - **<u>Not</u> from vertical components/requires special conditions**
  - **Conjectured by Lord Rayleigh 1883 for albatrosses**
  - **Accomplished in 1974 by glider pilot:**
    - **"By repeating this manoeuvre he successfully maintained his height for around 20 minutes <u>without the existence of ascending air</u>…"**
  - 392 mph RC glider record from 45 mph winds in 2009
  - UAV strategy to reduce fuel consumption
    - Zhao/Qi 2004
    - "All problem formulations are subject to UAV equations of motion, UAV operational constraints, proper initial conditions, and terminal conditions that enforce a <u>periodic flight</u>."



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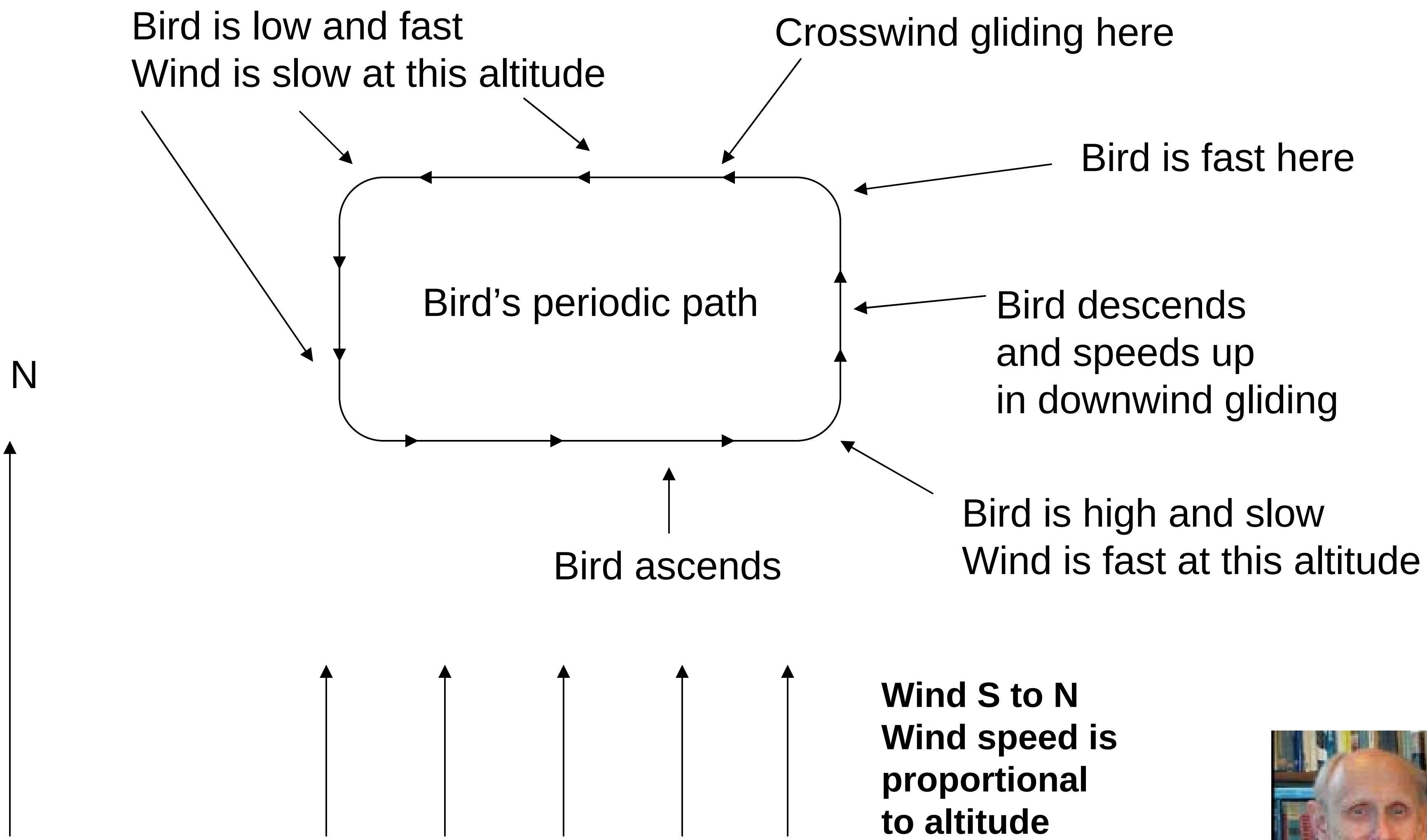
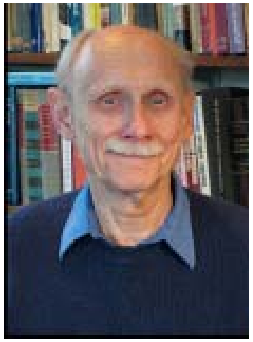
# How the Albatross Loiters



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# Assigning Equilibria

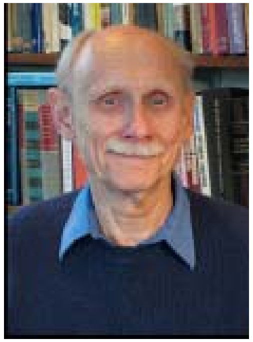
Can we maintain arbitrary equilibria?

$$\dot{x} = Ax + Bu \Rightarrow 0 = A\bar{x} + B\bar{u}$$

- Why not?
  - Range of B is too small---need $m \geq n$
  - Unattainable equilibria

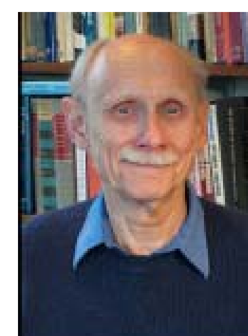
- Contradicts controllability??





# Vibration versus Shape Control

- Vibration suppression
  - Bring motion to rest (origin) and stay there
  - E.g., vibrating membrane
- Assign shape
  - Bring motion to rest at desired equilibrium
  - Vibrating membrane with desired aperture shape



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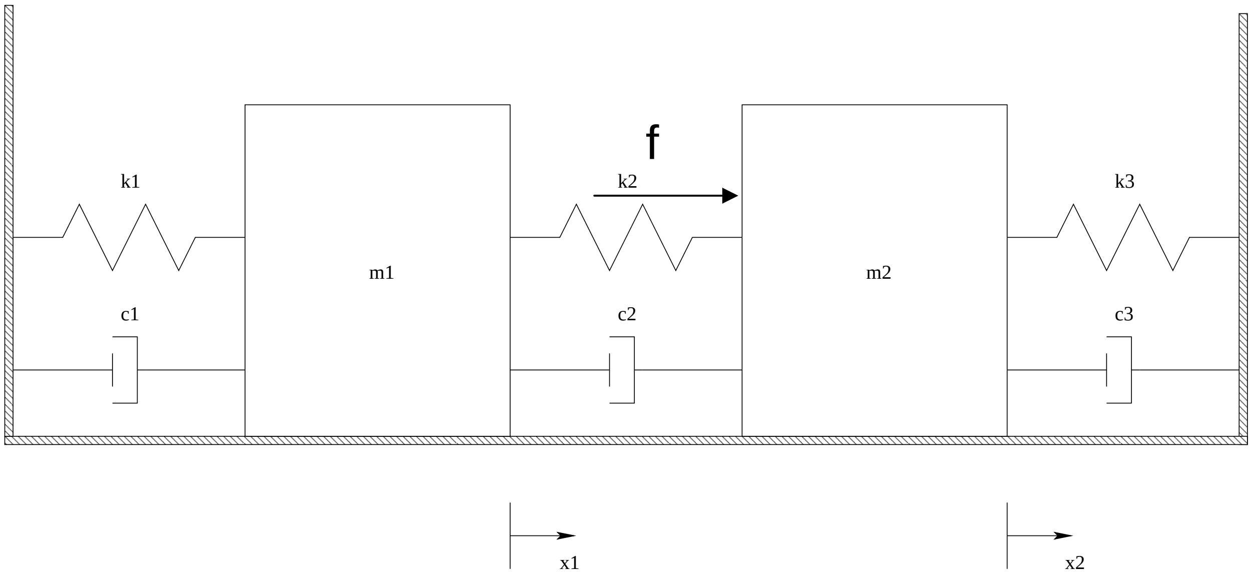
# Vibration versus Shape Control

- 4-state structure is <u>controllable</u> with one force input
  - Can bring masses to arbitrary configuration at arbitrary time
  - Cannot stay there!
  - Desired equilibrium is <u>unattainable</u>
- Idea:
  - Reach, leave, and return quickly to desired “equilibrium”
- Do this periodically

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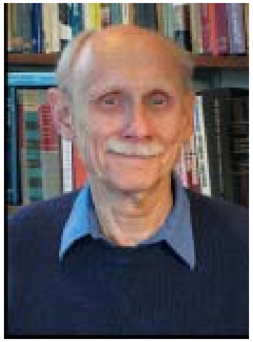
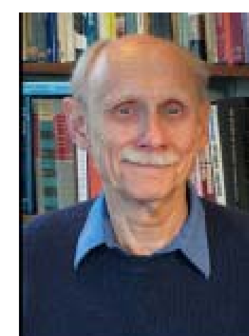


# Hovering/Loitering

- All airplanes are controllable in position and velocity
  - Can reach desired position with zero velocity
    - Not a good idea for most airplanes
- But most airplanes cannot hover
  - Not enough actuation
  - Works for helicopters

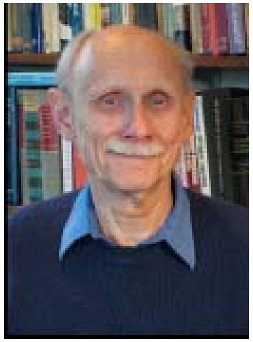
- Hummingbirds control periodically by flapping
  - Flapping induces small periodic motion
- Research problem:  What is the best way to maintain operation near an unattainable equilibrium?
  - Loitering limited by actuation constraints
  - Barabanov "Non-assignable Equilibria," Automatica, 2007



# Dealing with Unattainable Equilibria

- Slow switching
  - Quasi steady state (QSS)
  - Convexify SSs to meet constraints
- Fast switching
  - Relaxed steady state (RSS)
  - Convexifies velocity set
- Slow switching between RSS's
  - Quasi-relaxed steady state (QRSS)
  - Convexify RSS's





# Aircraft Cruise

The vehicle cruise problem is formulated as follows. The performance function

$$J(T(\cdot), V(\cdot), \tau) = V_{\mathrm{avg}}(F_{\mathrm{avg}})^{-1}$$
$$= \text{specific range} \qquad (1)$$

depends on the thrust $T(\cdot)$ (measurable on $[0, \tau)$), the speed $V(\cdot)$, and the period $\tau > 0$ which satisfy the following constraints:

$$\dot{V} = -D(V) + T(t), \quad V(0) = V(\tau) \geqslant 0, \qquad (2)$$

$$0 \leqslant T(t) \leqslant 1, \quad a.a.\ t \in [0, \tau], \qquad (3)$$

$$F_{\mathrm{avg}} = \frac{1}{\tau}\int_0^\tau F(T(t))\,\mathrm{d}t = \text{average fuel rate}, \qquad (4)$$

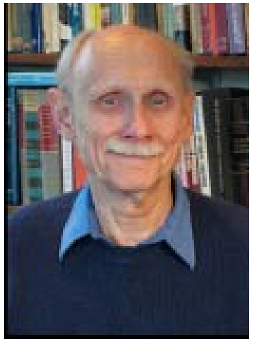
$$V_{\mathrm{avg}} = \frac{1}{\tau}\int_0^\tau V(t)\,\mathrm{d}t = \text{average speed}, \qquad (5)$$

$$V_{\mathrm{avg}} \geqslant V_{\mathrm{min}} \geqslant 0. \qquad (6)$$

The condition $V(0) = V(\tau)$ assures that both $V(\cdot)$ and $T(\cdot)$ are periodic when the domain of these functions is extended to $(-\infty, +\infty)$ by

$$V(t+\tau) = V(t) \quad \text{and} \quad T(t+\tau) = T(t).$$

Gilbert, Automatica, 1976



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# SS, RSS,QSS, QRSS

- SS—equilibrium solution
- Relaxed steady state—fast switching between SS's
- QSS—slow switching between SS's
- QRSS—slow switching between RSS's

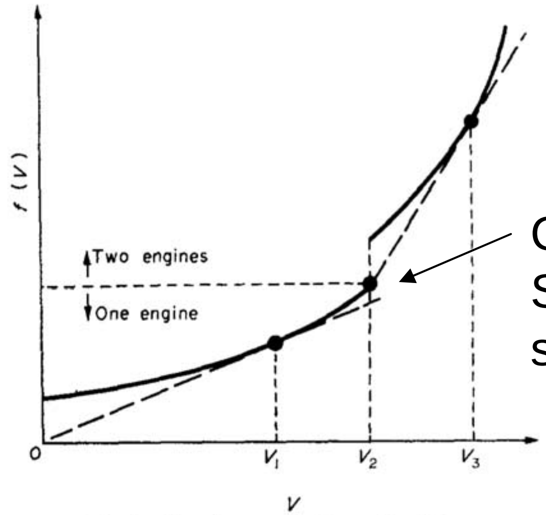



FIG. 3. Steady-state fuel rate for ship.

QSS
Slow
switching

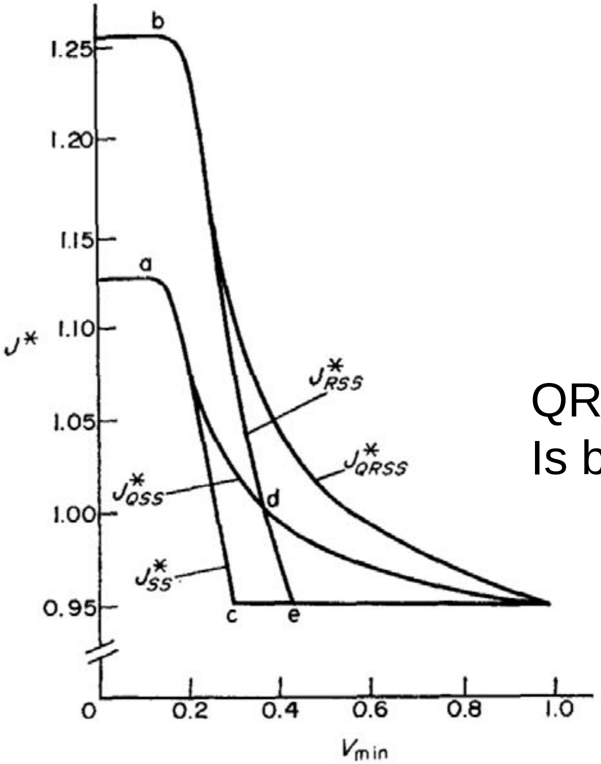
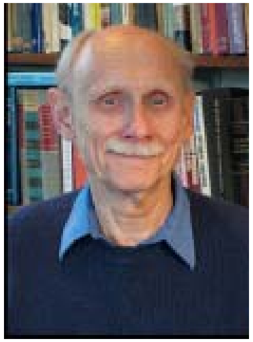



FIG. 2. Optimal cost functions for Example 1.

QRSS
Is best



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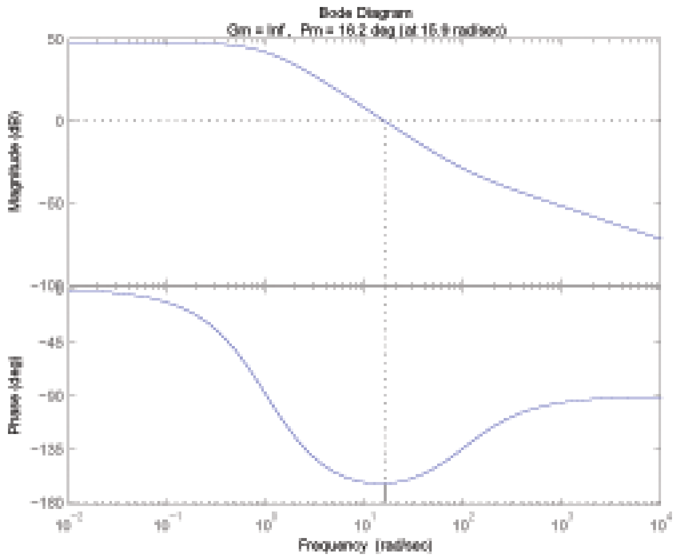
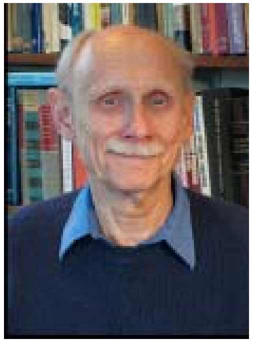
# Forced Periodicity: Linear Case

- Fundamental theorem of linear systems
  - Sinusoidal forcing of an AS LTI systems is eventually periodic
  - Gain and phase captured by Bode plot



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# Forced Periodicity: Nonlinear Case

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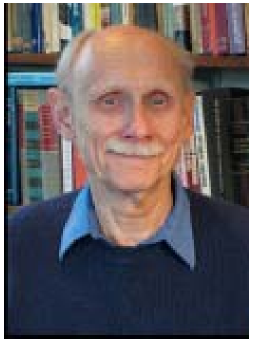
- Periodic forcing is a special case of periodically time-varying dynamics
- When does a nonlinear system have the property “periodic-input/eventually-periodic-output”?
- Subharmonic, superharmonic, and nonperiodic solutions may exist
- Extremely complex problem

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{x} &= f(x, \sin \omega t) \\ \dot{x} &= f(x,t) \end{aligned}$$



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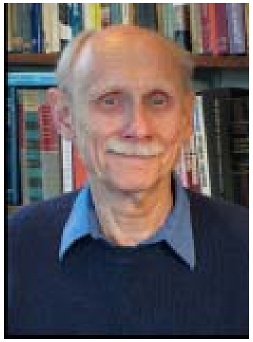
# Local Improvement

- Dynamics and cost $\dot{x} = f(x,u) \quad J = \frac{1}{\tau}\int_0^\tau g(x,u)dt$
- Find optimal steady state solution $(\bar{x}, \bar{u})$
- Linearize the cost and dynamics $\delta \dot{x} = A\delta x + B\delta u$
- Note $\delta J = 0$
- Evaluate $\delta^2 J$ with $\delta u = \sin \omega t$
- Find $\omega$ such that $\Pi(\omega) < 0$

$$\Pi(\omega) = G^*(\omega)L_{xx}G(\omega) + G^*(\omega)L_{xu} + L_{ux}G(\omega) + L_{uu}$$

  – Pi Test----Guardabassi
- Then periodic control can locally do better than constant control



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# Why Periodic Control?

- Periodic control is necessitated by
  - Unassignable equilibria
  - Constraints
  - Nonconvex velocity set
- Periodic control can do dramatically better than constant control
  - Dynamic soaring
- Periodic control ensures sustainability
- Nature and humans have discovered these advantages and benefits

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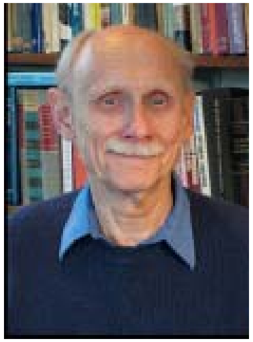

# Evolving Horizons

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- Thank you, Elmer, for your constant guidance and wisdom, and for always setting the highest example of scholarship and integrity

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