

Robust H_∞ control design for systems with structured parameter uncertainty

Appasaheb N. MADIWALE *

Massachusetts Institute of Technology, Lincoln Laboratory, P.O. Box 73, Lexington, MA 02173, U.S.A.

Wassim M. HADDAD **

Department of Mechanical and Aerospace Engineering, Florida Institute of Technology, Melbourne, FL 32901, U.S.A.

Dennis S. BERNSTEIN **

Government Aerospace Systems Division, Harris Corporation, MS 22/4848, Melbourne, FL 32901, U.S.A.

Received 17 October 1988

Revised 4 February 1989

Abstract: In a recent paper a unification of the H_2 (LQG) and H_∞ control-design problems was obtained in terms of modified algebraic Riccati equations. In the present paper these results are extended to guarantee robust H_2 and H_∞ performance in the presence of structured real-valued parameter variations (ΔA , ΔB , ΔC) in the state space model. For design flexibility the paper considers two distinct types of uncertainty bounds for both full- and reduced-order dynamic compensation. An important special case of these results generates H_2/H_∞ controller designs with guaranteed gain margins.

Keywords: Robust control; parameter uncertainty; H -infinity design.

1. Introduction

It has recently been shown that the solution to the optimal H_∞ disturbance attenuation problem can be expressed in terms of a pair of modified Riccati equations [3,4]. Furthermore, it was shown in [3] that H_2/H_∞ design tradeoffs can be achieved by solving a coupled system consisting of *three* modified Riccati equations. As is well known, the disturbance attenuation problem can be used to guarantee robustness with respect to unstructured plant uncertainties. However, if plant uncertainty is present in the form of structured parametric variations of the state space model, then alternative bounding techniques are required. The goal of the present paper is thus to extend the results of [3] to include bounds on the effects of real-valued structured parameter variations.

In the *absence* of an H_∞ design constraint, robust stability and H_2 performance for dynamic compensator design were guaranteed in [1,2] by incorporating quadratic Lyapunov bounds within LQG design theory. Two distinct bounds were considered. In [1] a quadratic bound was used while in [2] a linear bound was employed. In each case full- and reduced-order dynamic compensators were characterized by means of coupled systems of modified Riccati and Lyapunov equations.

To design H_∞ controllers which are robust with respect to structured real-valued parameter variations we proceed by combining the results of [3] with those of [1,2]. That is, we derive coupled systems of

* This work is sponsored by the Dept. of the Air Force. The views expressed are those of the author and do not reflect the official policy or position of the U.S. Government.

** Supported in part by the Air Force Office of Scientific Research under contracts F49620-86-C-0002 and F49620-89-C0011.

modified Riccati and Lyapunov equations whose solutions yield controllers which are guaranteed to satisfy a prespecified H_∞ attenuation constraint for all variations $(\Delta A, \Delta B, \Delta C)$, belonging to a given uncertainty set. If the uncertainty is absent (i.e., $\Delta A = 0$, etc.), then the results of [3] are recovered, while if the H_∞ constraint is relaxed, then the results of [1,2] are obtained. Thus the results of [3] can be viewed as a specialization of a broader design theory which accounts for structured real-valued parameter uncertainty. Finally, we state all results for the case of a fixed-order (i.e., reduced-order) controller for maximal design flexibility. Extensions to even more general design problems are mentioned in Section 9 but omitted here for lack of space.

Notation

Note: all matrices have real entries.

$\mathbb{R}, \mathbb{R}^{r \times s}, \mathbb{R}^r, \mathbf{E}$: real numbers, $r \times s$ real matrices, $\mathbb{R}^{r \times 1}$, expected value.

$I_r, ()^T, 0_{r \times s}, 0_r$: $r \times r$ identity matrix, transpose, $r \times s$ zero matrix, $0_{r \times r}$.

$\mathbf{S}^r, \mathbf{N}^r, \mathbf{P}^r$: $r \times r$ symmetric, nonnegative-definite, positive-definite matrices.

$Z_1 \leq Z_2, Z_1 < Z_2$: $Z_2 - Z_1 \in \mathbf{N}^r, Z_2 - Z_1 \in \mathbf{P}^r, Z_1, Z_2 \in \mathbf{S}^r$.

n, m, l, n_c : positive integers.

$p, d, d_\infty, q, q_\infty; \tilde{n}$: positive integers; $n + n_c$ ($n_c \leq n$).

x, u, y, x_c, \tilde{x} : n, m, l, n_c, \tilde{n} -dimensional vectors.

$A, \Delta A; B, \Delta B; C, \Delta C$: $n \times n; n \times m; l \times n$ matrices.

A_c, B_c, C_c : $n_c \times n_c, n_c \times l, m \times n_c$ matrices.

$$\tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}, \quad \Delta \tilde{A} = \begin{bmatrix} \Delta A & \Delta BC_c \\ B_c \Delta C & 0_{n_c} \end{bmatrix}.$$

$w(\cdot)$: d -dimensional standard white noise.

D_1, D_2 : $n \times d, l \times d$ matrices; $D_1 D_2^T = 0$.

V_1, V_2 : $D_1 D_1^T, D_2 D_2^T$; $V_2 \in \mathbf{P}^l$.

$$\tilde{D} = \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix}, \quad \tilde{V} = \begin{bmatrix} V_1 & 0_{n \times n_c} \\ 0_{n_c \times n} & B_c V_2 B_c^T \end{bmatrix} = \tilde{D} \tilde{D}^T.$$

E_1, E_2 : $q \times n, q \times m$ matrices; $E_1^T E_2 = 0$.

\tilde{E}, R_1, R_2 : $[E_1 \ E_2 C_c], E_1^T E_1, E_2^T E_2$; $R_2 \in \mathbf{P}^m$.

$$\tilde{R} = \begin{bmatrix} R_1 & 0_{n \times n_c} \\ 0_{n_c \times n} & C_c^T R_2 C_c \end{bmatrix} = \tilde{E}^T \tilde{E}.$$

$E_{1\infty}, E_{2\infty}$: $q_\infty \times n, q_\infty \times m$ matrices; $E_{1\infty}^T E_{2\infty} = 0$.

$\tilde{E}_\infty, R_{1\infty}, R_{2\infty}$: $[E_{1\infty} \ E_{2\infty} C_c], E_{1\infty}^T E_{1\infty}, E_{2\infty}^T E_{2\infty}$.

$$\tilde{R}_\infty = \begin{bmatrix} R_{1\infty} & 0_{n \times n_c} \\ 0_{n_c \times n} & C_c^T R_{2\infty} C_c \end{bmatrix} = \tilde{E}_\infty^T \tilde{E}_\infty.$$

$D_{1\infty}, D_{2\infty}$: $n \times d_\infty, l \times d_\infty$ matrices; $D_{1\infty} D_{2\infty}^T = 0$.

$V_{1\infty}, V_{2\infty}$: $D_{1\infty} D_{1\infty}^T, D_{2\infty} D_{2\infty}^T$.

$$\tilde{D}_\infty = \begin{bmatrix} D_{1\infty} \\ B_c D_{2\infty} \end{bmatrix}, \quad \tilde{V}_\infty = \begin{bmatrix} V_{1\infty} & 0_{n \times n_c} \\ 0_{n_c \times n} & B_c V_{2\infty} B_c^T \end{bmatrix}.$$

$\beta; \gamma, \alpha$: nonnegative constant; positive constants.

$A_\alpha = A + \frac{1}{2}\alpha I_n, A_{c\alpha} = A_c + \frac{1}{2}\alpha I_{n_c}$.

2. Robust stability and H_2 performance with a robust H_∞ constraint

In this section we state the robust stability and H_2 performance problem with an H_∞ disturbance attenuation constraint. Specifically, we consider a fixed-order dynamic output-feedback control-design problem with structured real-valued plant parameter uncertainties and constrained H_∞ disturbance attenuation. This problem involves a set $\mathbf{U} \subset \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{l \times n}$ of uncertain perturbations $(\Delta A, \Delta B, \Delta C)$ of the nominal system matrices (A, B, C) . The goal of the problem is to determine a fixed-order, strictly proper dynamic compensator (A_c, B_c, C_c) which (i) stabilizes the plant for all variations in \mathbf{U} , (ii) satisfies an H_∞ constraint on disturbance rejection for all variations in \mathbf{U} , and (iii) minimizes the worst-case value over the uncertainty set \mathbf{U} of a steady-state H_2 performance criterion. In this and the following section no explicit structure is assumed for the elements of \mathbf{U} . In Sections 4 and 7, two specific structures of variations in \mathbf{U} will be introduced.

H_∞ -constrained robust dynamic compensation problem. Given the n -th-order stabilizable and detectable plant with structured real-valued plant parameter variations

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) + D_1w(t), \tag{2.1}$$

$$y(t) = (C + \Delta C)x(t) + D_2w(t), \tag{2.2}$$

determine an n_c -th-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{2.3}$$

$$u(t) = C_c x_c(t), \tag{2.4}$$

which satisfies the following design criteria:

- (i) the closed-loop system (2.1)–(2.4) is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$, i.e., $\tilde{A} + \Delta \tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$;
- (ii) the $q_\infty \times d$ closed-loop transfer function

$$H_{\Delta \tilde{A}}(s) \triangleq \tilde{E}_\infty [sI_{\tilde{n}} - (\tilde{A} + \Delta \tilde{A})]^{-1} \tilde{D} \tag{2.5}$$

from $w(t)$ to $E_{1\infty}x(t) + E_{2\infty}u(t)$, satisfies the constraint

$$\|H_{\Delta \tilde{A}}(s)\|_\infty \leq \gamma, \quad (\Delta A, \Delta B, \Delta C) \in \mathbf{U}, \tag{2.6}$$

where $\gamma > 0$ is a given constant; and

- (iii) the performance functional

$$J(A_c, B_c, C_c) \triangleq \sup_{(\Delta A, \Delta B, \Delta C) \in \mathbf{U}} \limsup_{t \rightarrow \infty} \mathbb{E} [x^T(t)R_1x(t) + u^T(t)R_2u(t)] \tag{2.7}$$

is minimized.

Note that for each uncertain variation $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$, the closed-loop system can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta \tilde{A})\tilde{x}(t) + \tilde{D}w(t), \quad t \in [0, \infty), \tag{2.8}$$

and that (2.7) becomes

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathbf{U}} \limsup_{t \rightarrow \infty} \mathbb{E} [\tilde{x}^T(t)\tilde{R}\tilde{x}(t)]. \tag{2.9}$$

Furthermore, by defining the transfer function

$$\tilde{H}_{\Delta \tilde{A}}(s) \triangleq \tilde{E} [sI_{\tilde{n}} - (\tilde{A} + \Delta \tilde{A})]^{-1} \tilde{D},$$

it can be shown that when (i) is satisfied, (2.7) is given by

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathbf{U}} \|\tilde{H}_{\Delta\tilde{A}}(s)\|_2^2.$$

Note that the problem statement involves both H_2 and H_∞ performance weights. In particular, the matrices R_1 and R_2 are the H_2 weights for the state and control variables. By introducing the variables

$$z(t) = E_1 x(t), \quad v(t) = E_2 u(t),$$

the cost (2.7) can be written as

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathbf{U}} \limsup_{t \rightarrow \infty} \mathbb{E} [z^T(t)z(t) + v^T(t)v(t)].$$

For convenience we thus define $R_1 \triangleq E_1^T E_1$ and $R_2 \triangleq E_2^T E_2$ which appear in subsequent expressions. Although an H_2 cross-weighting term of the form $2x^T(t)R_{12}u(t)$ can also be included, we shall not do so here to facilitate the presentation.

For the H_∞ performance constraint, the transfer function (2.5) involves weighting matrices $E_{1\infty}$ and $E_{2\infty}$ for the state and control variables. The matrices $R_{1\infty} \triangleq E_{1\infty}^T E_{1\infty}$ and $R_{2\infty} \triangleq E_{2\infty}^T E_{2\infty}$ are thus the H_∞ counterparts of the H_2 weights R_1 and R_2 . Although we do not require that $R_{1\infty}$ and $R_{2\infty}$ be equal to R_1 and R_2 , we shall require for technical reasons that $R_{2\infty} = \beta^2 R_2$, where the nonnegative scalar β is a design variable. We further note that the assumption $E_{1\infty}^T E_{2\infty} = 0$ precludes an H_∞ cross-weighting term which again facilitates the presentation. Finally, similar remarks apply to the disturbance and sensor noise intensities $V_1 \triangleq D_1 D_1^T$, $V_2 \triangleq D_2 D_2^T$, $V_{1\infty} \triangleq D_{1\infty} D_{1\infty}^T$ and $V_{2\infty} \triangleq D_{2\infty} D_{2\infty}^T$ for the H_2 and H_∞ designs respectively. As in [3], $w(t)$ is interpreted as white noise for the H_2 design and as an L_2 signal for the H_∞ design aspect.

Before continuing it is useful to note that if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$ for a given compensator (A_c, B_c, C_c) , then the performance (2.7) is given by

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathbf{U}} \text{tr } \tilde{Q}_{\Delta\tilde{A}} \tilde{R}, \quad (2.10)$$

where the steady-state closed-loop state covariance defined by

$$\tilde{Q}_{\Delta\tilde{A}} \triangleq \lim_{t \rightarrow \infty} \mathbb{E} [\tilde{x}(t) \tilde{x}^T(t)] \quad (2.11)$$

satisfies the $\tilde{n} \times \tilde{n}$ algebraic Lyapunov equation

$$0 = (\tilde{A} + \Delta\tilde{A}) \tilde{Q}_{\Delta\tilde{A}} + \tilde{Q}_{\Delta\tilde{A}} (\tilde{A} + \Delta\tilde{A})^T + \tilde{V}. \quad (2.12)$$

The key step in guaranteeing robust stability and performance is to replace the uncertain terms in the covariance Lyapunov equation (2.12) by a bounding function Ω . Note that since $\Delta\tilde{A}$ is independent of A_c , the bounding function Ω need only depend on B_c and C_c . Furthermore, the H_∞ disturbance attenuation constraint (2.6) is enforced for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$ by replacing the modified algebraic Lyapunov equation (2.12) by an algebraic Riccati equation which overbounds the closed-loop steady-state covariance. Justification for this technique is provided by the following result.

Lemma 2.1. Let $\Omega: \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c} \times \mathbb{N}^{\tilde{n}} \rightarrow \mathbb{S}^{\tilde{n}}$ be such that

$$\Delta\tilde{A}\mathbf{Q} + \mathbf{Q}\Delta\tilde{A}^T \leq \Omega(B_c, C_c, \mathbf{Q}), \quad (\Delta A, \Delta B, \Delta C) \in \mathbf{U}, \quad (B_c, C_c, \mathbf{Q}) \in \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c} \times \mathbb{N}^{\tilde{n}}, \quad (2.13)$$

and, for a given (A_c, B_c, C_c) , suppose there exists $\mathbf{Q} \in \mathbb{N}^{\tilde{n}}$ satisfying

$$0 = \tilde{A}\mathbf{Q} + \mathbf{Q}\tilde{A}^T + \gamma^{-2} \mathbf{Q}\tilde{R}_\infty \mathbf{Q} + \Omega(B_c, C_c, \mathbf{Q}) + \tilde{V}. \quad (2.14)$$

Then

$$(\tilde{A} + \Delta\tilde{A}, \tilde{D}) \text{ is stabilizable, } (\Delta A, \Delta B, \Delta C) \in \mathbf{U}, \tag{2.15}$$

if and only if

$$\tilde{A} + \Delta\tilde{A} \text{ is asymptotically stable, } (\Delta A, \Delta B, \Delta C) \in \mathbf{U}. \tag{2.16}$$

In this case,

$$\|H_{\Delta\tilde{A}}(s)\|_\infty \leq \gamma, \quad (\Delta A, \Delta B, \Delta C) \in \mathbf{U}, \tag{2.17}$$

and

$$\tilde{Q}_{\Delta\tilde{A}} \leq \mathbf{Q}, \quad (\Delta A, \Delta B, \Delta C) \in \mathbf{U}, \tag{2.18}$$

where $\tilde{Q}_{\Delta\tilde{A}}$ is given by (2.12). Consequently,

$$J(A_c, B_c, C_c) \leq \mathbf{J}(A_c, B_c, C_c, \mathbf{Q}), \tag{2.19}$$

where

$$\mathbf{J}(A_c, B_c, C_c, \mathbf{Q}) \triangleq \text{tr } \mathbf{Q}\tilde{\mathbf{R}}. \tag{2.20}$$

Proof. First note for clarity that in (2.13) \mathbf{Q} denotes an arbitrary element of \mathbf{N}^n since (2.13) holds for all $\mathbf{Q} \in \mathbf{N}^n$, while in (2.14) \mathbf{Q} denotes a specific solution to (2.14). Now for $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$, (2.14) is equivalent to

$$0 = (\tilde{A} + \Delta\tilde{A})\mathbf{Q} + \mathbf{Q}(\tilde{A} + \Delta\tilde{A})^T + \gamma^{-2}\mathbf{Q}\tilde{\mathbf{R}}_\infty\mathbf{Q} + \Omega(B_c, C_c, \mathbf{Q}) - (\Delta\tilde{A}\mathbf{Q} + \mathbf{Q}\Delta\tilde{A}^T) + \tilde{\mathbf{V}}. \tag{2.21}$$

Hence, by assumption, (2.21) has a solution $\mathbf{Q} \in \mathbf{N}^n$ for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$ and, by (2.13), $\Omega(B_c, C_c, \mathbf{Q}) - (\Delta\tilde{A}\mathbf{Q} + \mathbf{Q}\Delta\tilde{A}^T)$ is nonnegative definite. Now it follows from Theorem 3.6 of [7] and (2.15) that $(\tilde{A} + \Delta\tilde{A}, [\tilde{\mathbf{V}} + \gamma^{-2}\mathbf{Q}\tilde{\mathbf{R}}_\infty\mathbf{Q} + \Omega(B_c, C_c, \mathbf{Q}) - (\Delta\tilde{A}\mathbf{Q} + \mathbf{Q}\Delta\tilde{A}^T)]^{1/2})$ is stabilizable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$. Thus it follows from (2.21) and Lemma 12.2 of [7] that $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$. Conversely, if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$, then (2.16) holds. The proof of (2.17) follows from a standard manipulation of (2.14). Next, subtracting (2.12) from (2.20) yields

$$\begin{aligned} 0 &= (\tilde{A} + \Delta\tilde{A})(\mathbf{Q} - \tilde{Q}_{\Delta\tilde{A}}) + (\mathbf{Q} - \tilde{Q}_{\Delta\tilde{A}})(\tilde{A} + \Delta\tilde{A})^T \\ &\quad + \gamma^{-2}\mathbf{Q}\tilde{\mathbf{R}}_\infty\mathbf{Q} + \Omega(B_c, C_c, \mathbf{Q}) - (\Delta\tilde{A}\mathbf{Q} + \mathbf{Q}\Delta\tilde{A}^T), \end{aligned}$$

or, equivalently, since $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$,

$$\mathbf{Q} - \tilde{Q}_{\Delta\tilde{A}} = \int_0^\infty e^{(\tilde{A} + \Delta\tilde{A})t} [\gamma^{-2}\mathbf{Q}\tilde{\mathbf{R}}_\infty\mathbf{Q} + \Omega(B_c, C_c, \mathbf{Q}) - (\Delta\tilde{A}\mathbf{Q} + \mathbf{Q}\Delta\tilde{A}^T)] e^{(\tilde{A} + \Delta\tilde{A})^T t} dt \geq 0$$

which implies (2.18). The performance bound (2.19) is now an immediate consequence of (2.18). \square

Remark 2.1. Note that (2.15) is actually a closed-loop ‘disturbability’ condition which is not concerned with control as such. This condition guarantees that the closed-loop system does not possess unstable undisturbed modes.

3. The auxiliary minimization problem

As shown in the previous section, the replacement of (2.12) by (2.14) enforces the H_∞ disturbance attenuation constraint and yields an upper bound for the worst case H_2 performance criterion. That is,

given a compensator (A_c, B_c, C_c) for which there exists a nonnegative-definite solution to (2.14), the actual worst case H_2 performance $J(A_c, B_c, C_c)$ of the compensator is guaranteed to be no worse than the bound given by $\mathbf{J}(A_c, B_c, C_c, \mathbf{Q})$. Hence, $\mathbf{J}(A_c, B_c, C_c, \mathbf{Q})$ can be interpreted as an *auxiliary* cost which leads to the following optimization problem.

Auxiliary Minimization Problem. Determine $(A_c, B_c, C_c, \mathbf{Q})$ which minimizes $\mathbf{J}(A_c, B_c, C_c, \mathbf{Q})$ subject to (2.14) with $\mathbf{Q} \in \mathbb{N}^{\tilde{n}}$.

It follows from Lemma 2.1 that the satisfaction of (2.14) for $\mathbf{Q} \in \mathbb{N}^{\tilde{n}}$ along with the generic condition (2.15) leads to (i) closed-loop stability for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$; (ii) prespecified H_∞ performance attenuation for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$; and (iii) an upper bound for the worst case H_2 performance criterion. Hence, it remains to determine (A_c, B_c, C_c) which *minimizes* $\mathbf{J}(A_c, B_c, C_c, \mathbf{Q})$ and thus provides an optimized bound for the actual worst case H_2 performance $J(A_c, B_c, C_c)$ over all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$.

4. Uncertainty structure: Linear bound

Having established the theoretical basis for our approach, we now assign explicit structure to the set of \mathbf{U} and bounding function Ω . Specifically, the uncertainty set \mathbf{U} is assumed to be of the form

$$\mathbf{U} = \left\{ (\Delta A, \Delta B, \Delta C) : \Delta A = \sum_{i=1}^p \sigma_i A_i, \Delta B = \sum_{i=1}^p \sigma_i B_i, \Delta C = \sum_{i=1}^p \sigma_i C_i, \sum_{i=1}^p \sigma_i^2 / \alpha_i^2 \leq 1 \right\}, \quad (4.1)$$

where, for $i = 1, \dots, p$, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, and $C_i \in \mathbb{R}^{l \times n}$ are fixed matrices denoting the structure of the parametric uncertainty; α_i is a given positive number; and σ_i is an uncertain real parameter. Note that the uncertain parameters σ_i are assumed to lie in a specified ellipsoidal region in \mathbb{R}^p . The closed-loop system (2.8) thus has structured uncertainty of the form

$$\Delta \tilde{A} = \sum_{i=1}^p \sigma_i \tilde{A}_i, \quad \text{where} \quad \tilde{A}_i \triangleq \begin{bmatrix} A_i & B_i C_c \\ B_c C_i & 0_{n_c} \end{bmatrix}, \quad i = 1, \dots, p. \quad (4.2)$$

Note that the symmetry of the uncertainty set entails no loss of generality by requiring only a redefinition of the nominal plant matrices.

In order to obtain explicit gain expressions for (A_c, B_c, C_c) in Sections 5 and 6, we shall require that at most one of the perturbations ΔB and ΔC is nonzero. We thus consider the cases $(\Delta A, \Delta C) \in \mathbf{U}$ or $(\Delta A, \Delta B) \in \mathbf{U}$. If this assumption is not imposed, then optimality conditions can still be derived, but at the expense of closed-form gain expressions. In this section and Section 5 we will assume that $\Delta B = 0$ (i.e., $B_i = 0$, $i = 1, \dots, p$) so that $\Omega(B_c, C_c, \mathbf{Q})$ becomes $\Omega(B_c, \mathbf{Q})$. The dual case $\Delta B \neq 0$, $\Delta C = 0$ (i.e., $C_i = 0$, $i = 1, \dots, p$) will be considered in Section 6.

For the structure of \mathbf{U} specified by (4.1), the bound Ω satisfying (2.13) can now be given a concrete form.

Proposition 4.1. *Let α be an arbitrary positive scalar. Then the function*

$$\Omega(B_c, \mathbf{Q}) \triangleq \alpha \mathbf{Q} + \alpha^{-1} \sum_{i=1}^p \alpha_i^2 \tilde{A}_i \mathbf{Q} \tilde{A}_i^T \quad (4.3)$$

satisfies (2.13) with \mathbf{U} given by (4.1).

Proof. See [2]. \square

Remark 4.1. Note that the bound Ω given by (4.3) consists of two distinct terms. The first term αQ can be thought of as arising from an exponential time weighting of the cost, or, equivalently, from a uniform right shift of the open-loop dynamics. The second term $\alpha^{-1} \sum_{i=1}^p \alpha_i^2 \tilde{A}_i Q \tilde{A}_i^T$ arises naturally from a multiplicative white noise model. Such interpretations have no bearing on the results obtained here since only the bound Ω defined by (4.3) is required. We call (4.3) the *linear bound* since it is linear in Q . For a more detailed discussion on (4.3) see [2].

With Ω defined by (4.3), the modified Riccati equation (2.14) becomes

$$0 = \tilde{A}Q + Q\tilde{A}^T + \gamma^{-2}Q\tilde{R}_\infty Q + \alpha Q + \alpha^{-1} \sum_{i=1}^p \alpha_i^2 \tilde{A}_i Q \tilde{A}_i^T + \tilde{V} \tag{4.4}$$

or, equivalently,

$$0 = \tilde{A}_\alpha Q + Q\tilde{A}_\alpha^T + \gamma^{-2}Q\tilde{R}_\infty Q + \sum_{i=1}^p \delta_i \tilde{A}_i Q \tilde{A}_i^T + \tilde{V}, \tag{4.5}$$

where $\delta_i \triangleq \alpha_i^2/\alpha$ and

$$\tilde{A}_\alpha \triangleq \tilde{A} + \frac{1}{2}\alpha I_n = \begin{bmatrix} A_\alpha & BC_c \\ B_c C & A_{c\alpha} \end{bmatrix}.$$

5. Sufficient conditions for robust stability and performance with robust H_∞ disturbance attenuation: Linear bound

In this section we state sufficient conditions for characterizing fixed-order (i.e., full- and reduced-order) controllers guaranteeing closed-loop stability for all $(\Delta A, \Delta C) \in U$, constrained H_∞ disturbance attenuation for all $(\Delta A, \Delta C) \in U$, and an optimized worst case H_2 performance bound. In order to state the main results we require some additional notation and a factorization lemma.

Lemma 5.1. *Let $\hat{Q}, \hat{P} \in \mathbb{N}^n$ and suppose $\text{rank } \hat{Q}\hat{P} = n_c$. Then there exist $n_c \times n$ G, Γ and $n_c \times n_c$ invertible M , unique except for a change of basis in \mathbb{R}^{n_c} , such that*

$$\hat{Q}\hat{P} = G^T M \Gamma, \quad \Gamma G^T = I_{n_c}. \tag{5.1), (5.2)}$$

Furthermore, the $n \times n$ matrices

$$\tau \triangleq G^T \Gamma, \quad \tau_\perp \triangleq I_n - \tau \tag{5.3), (5.4)}$$

are idempotent and have rank n_c and $n - n_c$, respectively. Finally, if $P \in \mathbb{N}^n$ and $\beta \geq 0$ then the inverse

$$S \triangleq (I_n + \beta^2 \gamma^{-2} \hat{Q}P)^{-1} \tag{5.5}$$

exists.

For arbitrary $Q, \hat{Q} \in \mathbb{R}^{n \times n}$ and $\alpha > 0$ define the following notation:

$$V_{2s} \triangleq V_2 + \sum_{i=1}^p \delta_i C_i (Q + \hat{Q}) C_i^T, \quad Q_s \triangleq QC^T + \sum_{i=1}^p \delta_i A_i (Q + \hat{Q}) C_i^T, \quad \Sigma \triangleq BR_2^{-1} B^T.$$

Theorem 5.1. Suppose there exists $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying

$$0 = A_\alpha Q + QA_\alpha^T + \gamma^{-2}QR_{1\infty}Q + V_1 + \sum_{i=1}^p \delta_i A_i (Q + \hat{Q}) A_i^T - Q_s V_{2s}^{-1} Q_s^T + \tau_\perp Q_s V_{2s}^{-1} Q_s^T \tau_\perp^T, \quad (5.6)$$

$$0 = (A_\alpha + \gamma^{-2}[Q + \hat{Q}]R_{1\infty})^T P + P(A_\alpha + \gamma^{-2}[Q + \hat{Q}]R_{1\infty}) + R_1 \\ + \sum_{i=1}^p \delta_i \left[A_i^T P A_i + (A_i - Q_s V_{2s}^{-1} C_i)^T \hat{P} (A_i - Q_s V_{2s}^{-1} C_i) \right] - S^T P \Sigma P S + \tau_\perp^T S^T P \Sigma P S \tau_\perp, \quad (5.7)$$

$$0 = (A_\alpha - \Sigma P S + \gamma^{-2}QR_{1\infty})\hat{Q} + \hat{Q}(A_\alpha - \Sigma P S + \gamma^{-2}QR_{1\infty})^T \\ + \gamma^{-2}\hat{Q}(R_{1\infty} + \beta^2 S^T P \Sigma P S)\hat{Q} + Q_s V_{2s}^{-1} Q_s^T - \tau_\perp Q_s V_{2s}^{-1} Q_s^T \tau_\perp^T, \quad (5.8)$$

$$0 = (A_\alpha - Q_s V_{2s}^{-1} C + \gamma^{-2}QR_{1\infty})^T \hat{P} + \hat{P}(A_\alpha - Q_s V_{2s}^{-1} C + \gamma^{-2}QR_{1\infty}) + S^T P \Sigma P S - \tau_\perp^T S^T P \Sigma P S \tau_\perp, \quad (5.9)$$

$$\text{rank } \hat{Q} = \text{rank } \hat{P} = \text{rank } \hat{Q}\hat{P} = n_c, \quad (5.10)$$

and let (A_c, B_c, C_c, Q) be given by

$$A_c = \Gamma(A - \Sigma P S - Q_s V_{2s}^{-1} C + \gamma^{-2}QR_{1\infty})G^T, \quad (5.11)$$

$$B_c = \Gamma Q_s V_{2s}^{-1}, \quad (5.12)$$

$$C_c = -R_2^{-1} B^T P S G^T, \quad (5.13)$$

$$Q = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^T \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^T \end{bmatrix}. \quad (5.14)$$

Then $(\tilde{A} + \Delta\tilde{A}, \tilde{D})$ is stabilizable for all $(\Delta A, \Delta C) \in \mathbf{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta C) \in \mathbf{U}$. In this case, the closed-loop transfer function $H_{\Delta\tilde{A}}(s)$ satisfies the H_∞ disturbance attenuation constraint

$$\|H_{\Delta\tilde{A}}(s)\|_\infty \leq \gamma, \quad (\Delta A, \Delta C) \in \mathbf{U}, \quad (5.15)$$

and the worst-case H_2 performance criterion (2.10) satisfies the bound

$$J(A_c, B_c, C_c) \leq \text{tr}[(Q + \hat{Q})R_1 + \hat{Q}S^T P \Sigma P S]. \quad (5.16)$$

Proof. The proof follows from Lemma 2.1 by combining the proofs of Theorem 6.1 of [3] and Theorem 6.1 of [2]. \square

Remark 5.1. Theorem 5.1 presents sufficient conditions for designing controllers yielding robust stability and performance with a constraint on the H_∞ norm of the closed-loop transfer function for a state-space system with structured real-valued plant parameter variations. These sufficient conditions comprise a system of three modified Riccati equations and one modified Lyapunov equation coupled by the optimal projection τ , the linear uncertainty bound, and the H_∞ constraint. If the uncertainty bound is deleted, then the results of [3] are recovered. If, alternatively, the uncertainty terms are retained but the H_∞ constraint is sufficiently relaxed, i.e., $\gamma \rightarrow \infty$, then the results of [2] are recovered for the case $B_i = 0$, $i = 1, \dots, p$.

Remark 5.2. To specialize Theorem 5.1 to the full-order case $n_c = n$, it is only necessary to set $G^T = \Gamma^{-1}$ so that $G = \Gamma = \tau = I_n$ and $\tau_\perp = 0$ without loss of generality. Now the last term in each of (5.6)–(5.9) can be deleted and G and Γ in (5.11)–(5.14) can be taken to be the identity. It is interesting to note that in the full-order case the H_∞ design problem with structured parameter variations is comprised of four coupled

Riccati/Lyapunov equations. This coupling illustrates the breakdown of regulator/estimator separation and shows that the certainty equivalence principle is no longer valid. This is not surprising since separation also breaks down for the full-order H_2 result with parameter uncertainties [2].

Remark 5.3. When solving (5.6)–(5.10) numerically, the uncertainty terms and the H_∞ constraint can be adjusted to examine tradeoffs among performance, robustness, and disturbance rejection. Specifically, the uncertainty range α_i and the structure matrices A_i, C_i appearing in Q_s and V_{2s} along with γ can be varied systematically to determine the region of solvability of the design equations (5.6)–(5.9).

Remark 5.4. We point out that if $\beta = 0$ or, equivalently, $E_{2\infty} = 0$, which corresponds to the ‘cheap’ H_∞ control case (i.e., H_∞ attenuation between disturbances and controls is not constrained), it is possible to obtain closed-form gains (A_c, B_c, C_c) given by a modified set of design equations when all three of $\Delta A, \Delta B$, and ΔC are nonzero. Because of space limitations this result will be given in a future paper.

Remark 5.5. An important special case of the results of Section 5 is obtained by setting $\Delta A = 0, \Delta B = 0, \Delta C = \sigma_1 C_1$, and $C_1 = C$. The resulting H_2/H_∞ design is guaranteed to possess a gain margin of $\pm 100\alpha_1$ percent at the sensor output.

6. The dual case: Linear bound

Unlike the standard LQG result involving a pair of uncoupled Riccati equations, the new result guaranteeing robust stability, robust performance, and H_∞ disturbance rejection involves a coupled system of *four* modified Riccati/Lyapunov equations. The asymmetry of these equations suggests the possibility of a *dual* result in which the modifications to the standard Riccati equations are reversed. One motivation for developing such dual results is that for certain problems the dual bounds may be sharper than the primal bound introduced in Section 4. Furthermore, the dual theory permits distinct H_∞ disturbance weights ($V_{1\infty}$ and $V_{2\infty}$), although we now require $R_{1\infty} = R_1$. Finally, the dual theory allows for uncertainty in the control matrix B (i.e., $\Delta B \neq 0$), although we now require $\Delta C = 0$, (i.e., $C_i = 0, i = 1, \dots, p$) to obtain closed-form gain expressions for (A_c, B_c, C_c) . We begin with the following lemma.

Lemma 6.1. *Suppose the system (2.8) is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in U$ for a given (A_c, B_c, C_c) . Then*

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in U} \text{tr } \tilde{P}_{\Delta \tilde{A}} \tilde{V}, \tag{6.1}$$

where $\tilde{P}_{\Delta \tilde{A}} \in \mathbb{N}^{\tilde{n}}$ is the unique solution to

$$0 = (\tilde{A} + \Delta \tilde{A})^T \tilde{P}_{\Delta \tilde{A}} + \tilde{P}_{\Delta \tilde{A}} (\tilde{A} + \Delta \tilde{A}) + \tilde{R}. \tag{6.2}$$

Proof. See [1]. \square

Utilizing (6.1) in place of (2.10), the H_∞ disturbance attenuation constraint from plant and sensor disturbances to the state and control variables given by

$$\|\hat{H}_{\Delta \tilde{A}}(s)\|_\infty = \|\tilde{E} [sI_{\tilde{n}} - (\tilde{A} + \Delta \tilde{A})]^{-1} \tilde{D}_\infty\|_\infty \leq \gamma \tag{6.3}$$

can now be enforced by replacing (2.14) by the modified Riccati equation

$$0 = \tilde{A}^T \mathbf{P} + \mathbf{P} \tilde{A} + \gamma^{-2} \mathbf{P} \tilde{V}_\infty \mathbf{P} + \hat{\Omega}(C_c, \mathbf{P}) + \tilde{R}, \tag{6.4}$$

where

$$\Delta \tilde{A}^T \mathbf{P} + \mathbf{P} \Delta \tilde{A} \leq \hat{\Omega}(C_c, \mathbf{P}), \quad (\Delta A, \Delta B) \in \mathbf{U}. \quad (6.5)$$

Note that (6.4) is merely the dual of (2.14). We also require the condition dual to (2.15) given by

$$(\tilde{E}, \tilde{A} + \Delta \tilde{A}) \text{ is detectable for all } (\Delta A, \Delta B) \in \mathbf{U}. \quad (6.6)$$

For the structure of \mathbf{U} as specified by (4.1) with $\Delta C = 0$, the bound $\hat{\Omega}$ satisfying (6.5) can now be given a concrete form.

Proposition 6.1. *Let α be an arbitrary positive scalar. Then the function*

$$\hat{\Omega}(C_c, \mathbf{P}) \triangleq \alpha \mathbf{P} + \alpha^{-1} \sum_{i=1}^p \alpha_i^2 \tilde{A}_i^T \mathbf{P} \tilde{A}_i \quad (6.7)$$

satisfies (6.5) with \mathbf{U} given by (4.1) and $\Delta C = 0$. With $\hat{\Omega}$ defined by (6.7), the modified dual Riccati equation (6.4) becomes

$$0 = \tilde{A}_\alpha^T \mathbf{P} + \mathbf{P} \tilde{A}_\alpha + \gamma^{-2} \mathbf{P} \tilde{V}_\infty \mathbf{P} + \sum_{i=1}^p \delta_i \tilde{A}_i^T \mathbf{P} \tilde{A}_i + \tilde{R}. \quad (6.8)$$

We can now state sufficient conditions for robust stability, robust H_2 performance, and robust disturbance attenuation for the dual problem. For arbitrary, $Q, P, \hat{P} \in \mathbb{R}^{n \times n}$ and $\alpha > 0$ define the following notation:

$$R_{2s} \triangleq R_2 + \sum_{i=1}^p \delta_i B_i^T (P + \hat{P}) B_i, \quad P_s \triangleq B^T P + \sum_{i=1}^p \delta_i B_i^T (P + \hat{P}) A_i, \\ \hat{S} \triangleq (I_n + \gamma^{-2} \beta^2 Q \hat{P})^{-1}, \quad \bar{\Sigma} \triangleq C^T V_2^{-1} C.$$

Theorem 6.1. *Suppose there exist $P, Q, \hat{P}, \hat{Q} \in \mathbb{N}^n$ satisfying (5.10) and*

$$0 = A_\alpha^T P + P A_\alpha + \gamma^{-2} P V_{1\infty} P + R_1 + \sum_{i=1}^p \delta_i A_i^T (P + \hat{P}) A_i - P_s^T R_{2s}^{-1} P_s + \tau_\perp^T P_s^T R_{2s}^{-1} P_s \tau_\perp, \quad (6.9)$$

$$0 = (A_\alpha + \gamma^{-2} V_{1\infty} [P + \hat{P}]) Q + Q (A_\alpha + \gamma^{-2} V_{1\infty} [P + \hat{P}])^T + V_1 \\ + \sum_{i=1}^p \delta_i \left[A_i Q A_i^T + (A_i - B_i R_{2s}^{-1} P_s) \hat{Q} (A_i - B_i R_{2s}^{-1} P_s)^T \right] - \hat{S} Q \bar{\Sigma} Q \hat{S}^T + \tau_\perp \hat{S} Q \bar{\Sigma} Q \hat{S}^T \tau_\perp^T, \quad (6.10)$$

$$0 = (A_\alpha - \hat{S} Q \bar{\Sigma} + \gamma^{-2} V_{1\infty} P)^T \hat{P} + \hat{P} (A_\alpha - \hat{S} Q \bar{\Sigma} + \gamma^{-2} V_{1\infty} P) + \gamma^{-2} \hat{P} (V_{1\infty} + \beta^2 \hat{S} Q \bar{\Sigma} Q \hat{S}^T) \hat{P} \\ + P_s^T R_{2s}^{-1} P_s - \tau_\perp^T P_s^T R_{2s}^{-1} P_s \tau_\perp, \quad (6.11)$$

$$0 = (A_\alpha - B R_{2s}^{-1} P_s + \gamma^{-2} V_{1\infty} P) \hat{Q} + \hat{Q} (A_\alpha - B R_{2s}^{-1} P_s + \gamma^{-2} V_{1\infty} P)^T + \hat{S} Q \bar{\Sigma} Q \hat{S}^T - \tau_\perp \hat{S} Q \bar{\Sigma} Q \hat{S}^T \tau_\perp^T, \quad (6.12)$$

and let $(A_c, B_c, C_c, \mathbf{P})$ be given by

$$A_c = \Gamma (A - \hat{S} Q \bar{\Sigma} - B R_{2s}^{-1} P_s + \gamma^{-2} V_{1\infty} P) G^T, \quad (6.13)$$

$$B_c = \Gamma \hat{S} Q C^T V_2^{-1}, \quad (6.14)$$

$$C_c = -R_{2s}^{-1} P_s G^T, \quad (6.15)$$

$$\mathbf{P} = \begin{bmatrix} P + \hat{P} & -\hat{P} G^T \\ -G \hat{P} & G \hat{P} G^T \end{bmatrix}. \quad (6.16)$$

Then $(\tilde{E}, \tilde{A} + \Delta\tilde{A})$ is detectable for all $(\Delta A, \Delta B) \in \mathbf{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B) \in \mathbf{U}$. In this case, the closed-loop transfer function $\hat{H}_{\Delta\tilde{A}}(s)$ satisfies the H_∞ disturbance attenuation constraint

$$\|\hat{H}_{\Delta\tilde{A}}(s)\|_\infty \leq \gamma, \quad (\Delta A, \Delta B) \in \mathbf{U}, \quad (6.17)$$

and the worst-case H_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c) \leq \text{tr}[(P + \hat{P})V_1 + \hat{P}\hat{S}Q\bar{\Sigma}Q\hat{S}^T]. \quad (6.18)$$

Remark 6.1. The dual case of Remark 5.5 is obtained by setting $\Delta A = 0$, $\Delta B = \sigma_1 B_1$, $\Delta C = 0$, and $B_1 = B$. The resulting H_2/H_∞ design is guaranteed to possess a guaranteed gain margin of $\pm 100\alpha_1$ percent at the input.

7. Uncertainty structure and sufficient conditions for robust stability and performance with H_∞ disturbance attenuation: Quadratic bound

We now assign a different structure to the uncertainty set \mathbf{U} and the bounding function Ω . Specifically, the uncertainty set \mathbf{U} is assumed to be of the form

$$\mathbf{U} = \left\{ (\Delta A, \Delta B, \Delta C): \Delta A = \sum_{i=1}^p F_i M_i N_i G_i, \Delta B = \sum_{i=1}^p F_i M_i N_i H_i, \right. \\ \left. \Delta C = \sum_{i=1}^p K_i M_i N_i G_i, M_i M_i^T \leq \bar{M}_i, N_i^T N_i \leq \bar{N}_i, i = 1, \dots, p \right\}, \quad (7.1)$$

where, for $i = 1, \dots, p$, $F_i \in \mathbb{R}^{n \times r_i}$, $G_i \in \mathbb{R}^{t_i \times n}$, $H_i \in \mathbb{R}^{t_i \times m}$, and $K_i \in \mathbb{R}^{l \times r_i}$ are fixed matrices denoting the structure of the uncertainty; $\bar{M}_i \in \mathbb{N}^{r_i}$ and $\bar{N}_i \in \mathbb{N}^{t_i}$ are given uncertainty bounds; and $M_i \in \mathbb{R}^{r_i \times s_i}$ and $N_i \in \mathbb{R}^{s_i \times t_i}$ are uncertain matrices.

In order to obtain explicit gain expressions for (A_c, B_c, C_c) we again consider two cases: (1) $(\Delta A, \Delta C) \in \mathbf{U}$ with $\Delta B = 0$, and (2) $(\Delta A, \Delta B) \in \mathbf{U}$ with $\Delta C = 0$. When $\Delta B = 0$ the closed-loop system has structured uncertainty of the form

$$\Delta\tilde{A} = \sum_{i=1}^p \tilde{F}_i M_i N_i \tilde{G}_i, \quad (7.2)$$

where

$$\tilde{F}_i \triangleq \begin{bmatrix} F_i \\ B_c K_i \end{bmatrix}, \quad \tilde{G}_i \triangleq \begin{bmatrix} G_i & 0_{t_i \times n_c} \end{bmatrix}.$$

In this case the quadratic bound Ω satisfying (2.13) can now be given a concrete form.

Proposition 7.1. *The function*

$$\Omega(B_c, \mathbf{Q}) \triangleq \sum_{i=1}^p \tilde{F}_i \bar{M}_i \tilde{F}_i^T + \mathbf{Q} \tilde{G}_i^T \bar{N}_i \tilde{G}_i \mathbf{Q} \quad (7.3)$$

satisfies (2.13) with \mathbf{U} given by (7.1) and $\Delta B = 0$.

Proof. See [1]. \square

Thus, with Ω defined by (7.3), the modified Riccati equation (2.14) becomes

$$0 = \tilde{A}Q + Q\tilde{A}^T + \gamma^{-2}Q\tilde{R}_\infty Q + \sum_{i=1}^p [\tilde{F}_i \bar{M}_i \tilde{F}_i^T + Q\tilde{G}_i^T \bar{N}_i \tilde{G}_i Q] + \tilde{V}. \quad (7.4)$$

For arbitrary $Q \in \mathbb{R}^{n \times n}$ define:

$$\begin{aligned} Q_a &\triangleq QC^T + \sum_{i=1}^p F_i \bar{M}_i K_i, & D &\triangleq \sum_{i=1}^p F_i \bar{M}_i F_i^T, \\ V_{2a} &\triangleq V_2 + \sum_{i=1}^p K_i \bar{M}_i K_i^T, & E &\triangleq \sum_{i=1}^p G_i^T \bar{N}_i G_i. \end{aligned}$$

Theorem 7.1. Suppose these exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying (5.10) and

$$0 = AQ + QA^T + \gamma^{-2}QR_{1\infty}Q + V_1 + QEQ + D - Q_a V_{2a}^{-1} Q_a^T + \tau_\perp Q_a V_{2a}^{-1} Q_a^T \tau_\perp^T, \quad (7.5)$$

$$\begin{aligned} 0 &= (A + [Q + \hat{Q}][\gamma^{-2}R_{1\infty} + E])^T P + P(A + [Q + \hat{Q}][\gamma^{-2}R_{1\infty} + E]) \\ &\quad + R_1 - S^T P \Sigma P S + \tau_\perp^T S^T P \Sigma P S \tau_\perp, \end{aligned} \quad (7.6)$$

$$\begin{aligned} 0 &= (A - \Sigma P S + Q[\gamma^{-2}R_{1\infty} + E])\hat{Q} + \hat{Q}(A - \Sigma P S + Q[\gamma^{-2}R_{1\infty} + E])^T \\ &\quad + \hat{Q}(\gamma^{-2}[R_{1\infty} + \beta^2 S^T P \Sigma P S] + E)\hat{Q} + Q_a V_{2a}^{-1} Q_a^T - \tau_\perp Q_a V_{2a}^{-1} Q_a^T \tau_\perp^T, \end{aligned} \quad (7.7)$$

$$\begin{aligned} 0 &= (A - Q_a V_{2a}^{-1} C + Q[\gamma^{-2}R_{1\infty} + E])^T \hat{P} + \hat{P}(A - Q_a V_{2a}^{-1} C + Q[\gamma^{-2}R_{1\infty} + E]) + S^T P \Sigma P S \\ &\quad - \tau_\perp^T S^T P \Sigma P S \tau_\perp, \end{aligned} \quad (7.8)$$

and let Q be given by (5.14) and (A_c, B_c, C_c) by

$$A_c = \Gamma(A - \Sigma P S - Q_a V_{2a}^{-1} C + Q[\gamma^{-2}R_{1\infty} + E])G^T, \quad (7.9)$$

$$B_c = \Gamma Q_a V_{2a}^{-1}, \quad (7.10)$$

$$C_c = -R_2^{-1} B^T P S G^T. \quad (7.11)$$

Then $(\tilde{A} + \Delta \tilde{A}, \tilde{D})$ is stabilizable for all $(\Delta A, \Delta C) \in \mathbf{U}$ if and only if $\tilde{A} + \Delta \tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta C) \in \mathbf{U}$. In this case, the closed-loop transfer function $H_{\Delta \tilde{A}}(s)$ satisfies the H_∞ disturbance attenuation constraint

$$\|H_{\Delta \tilde{A}}(s)\|_\infty \leq \gamma, \quad (\Delta A, \Delta C) \in \mathbf{U}, \quad (7.12)$$

and the worst-case H_2 performance criterion (2.10) satisfies the bound

$$J(A_c, B_c, C_c) \leq \text{tr}[(Q + \hat{Q})R_1 + \hat{Q}S^T P \Sigma P S]. \quad (7.13)$$

Proof. The proof follows by combining the proofs of Theorems 6.1 of [3] and Theorem 8.1 of [1]. \square

Remark 7.1. It is interesting to note that in the full-order case $n_c = n$ with $G = \Gamma = \tau = I_n$ and $\tau_\perp = 0$ (see Remark 5.1), \hat{P} plays no role so that (7.8) is superfluous. Thus, unlike the full-order result for the linear bound involving four equations, the full-order quadratic bound involves *three* modified Riccati equations coupled by the quadratic bound and the H_∞ constraint. If, alternatively, the reduced-order constraint is retained, but the uncertainty terms are deleted, then the results of [3] are recovered. If, furthermore, the uncertainty terms are retained, but the H_∞ constraint is sufficiently relaxed, i.e., $\gamma \rightarrow \infty$, then the results of [1] are recovered.

8. The dual case: Quadratic bound

For the structure of U as specified by (7.1) with $\Delta C = 0$, the closed-loop system has structured uncertainty of the form

$$\Delta \tilde{A} = \sum_{i=1}^p \tilde{F}_i M_i N_i \tilde{G}_i, \tag{8.1}$$

where

$$\tilde{F}_i \triangleq \begin{bmatrix} F_i \\ 0_{n_c \times r_i} \end{bmatrix}, \quad \tilde{G}_i \triangleq [G_i \quad H_i C_c].$$

Proposition 8.1. *The function*

$$\hat{\Omega}(C_c, P) \triangleq \sum_{i=1}^p \tilde{G}_i^T \bar{N}_i \tilde{G}_i + P \tilde{F}_i \bar{M}_i \tilde{F}_i^T P \tag{8.2}$$

satisfies (6.5) with U given by (7.1) and $\Delta C = 0$.

With $\hat{\Omega}$ defined by (8.2), the modified dual equation (6.4) becomes

$$0 = \tilde{A}^T P + P \tilde{A} + \gamma^{-2} P \tilde{V}_\infty P + \sum_{i=1}^p [\tilde{G}_i^T \bar{N}_i \tilde{G}_i + P \tilde{F}_i \bar{M}_i \tilde{F}_i^T P] + \tilde{R}. \tag{8.3}$$

For arbitrary $P \in \mathbb{R}^{n \times n}$ define:

$$P_a \triangleq B^T P + \sum_{i=1}^p H_i^T \bar{N}_i G_i, \quad R_{2a} \triangleq R_2 + \sum_{i=1}^p H_i^T \bar{N}_i H_i.$$

Theorem 8.1. *Suppose there exist $P, Q, \hat{P}, \hat{Q} \in \mathbb{N}^n$ satisfying (5.10) and*

$$0 = A^T P + PA + \gamma^{-2} P V_{1\infty} P + R_1 + E + PDP - P_a^T R_{2a}^{-1} P_a + \tau_\perp^T P_a^T R_{2a}^{-1} P_a \tau_\perp, \tag{8.4}$$

$$0 = (A + [\gamma^{-2} V_{1\infty} + D][P + \hat{P}])Q + Q(A + [\gamma^{-2} V_{1\infty} + D][P + \hat{P}])^T + V_1 - \hat{S}Q\bar{\Sigma}Q\hat{S}^T + \tau_\perp \hat{S}Q\bar{\Sigma}Q\hat{S}^T \tau_\perp^T, \tag{8.5}$$

$$0 = (A - \hat{S}Q\bar{\Sigma} + [\gamma^{-2} V_{1\infty} + D]P)^T \hat{P} + \hat{P}(A - \hat{S}Q\bar{\Sigma} + [\gamma^{-2} V_{1\infty} + D]P) + \hat{P}(\gamma^{-2} [V_{1\infty} + \beta^2 \hat{S}Q\bar{\Sigma}Q\hat{S}^T] + D)\hat{P} + P_a^T R_{2a}^{-1} P_a - \tau_\perp^T P_a^T R_{2a}^{-1} P_a \tau_\perp, \tag{8.6}$$

$$0 = (A - BR_{2a}^{-1} P_a + [\gamma^{-2} V_{1\infty} + D]P)\hat{Q} + \hat{Q}(A - BR_{2a}^{-1} P_a + [\gamma^{-2} V_{1\infty} + D]P)^T + \hat{S}Q\bar{\Sigma}Q\hat{S}^T - \tau_\perp \hat{S}Q\bar{\Sigma}Q\hat{S}^T \tau_\perp^T, \tag{8.7}$$

and let P be given by (6.16) and (A_c, B_c, C_c) by

$$A_c = \Gamma(A - \hat{S}Q\bar{\Sigma} - BR_{2a}^{-1} P_a + [\gamma^{-2} V_{1\infty} + D]P)G^T, \tag{8.8}$$

$$B_c = \Gamma \hat{S}Q C^T V_2^{-1}, \tag{8.9}$$

$$C_c = -R_{2a}^{-1} P_a G^T. \tag{8.10}$$

Then $(\tilde{E}, \tilde{A} + \Delta \tilde{A})$ is detectable for all $(\Delta A, \Delta B) \in U$ if and only if $\tilde{A} + \Delta \tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B) \in U$. In this case, the closed-loop transfer function $\hat{H}_{\Delta \tilde{A}}(s)$ satisfies the H_∞ disturbance attenuation constraint

$$\|\hat{H}_{\Delta \tilde{A}}(s)\|_\infty \leq \gamma, \quad (\Delta A, \Delta B) \in U, \tag{8.11}$$

and the worst case L_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c) \leq \text{tr}[(P + \hat{P})V_1 + \hat{P}\hat{S}\hat{Q}\bar{\Sigma}\hat{Q}^T\hat{S}^T]. \quad (8.12)$$

9. Numerical solution of the design equation

One of the principal motivations for the Riccati equation approach is the opportunity it provides for developing efficient computational algorithms for control design. In particular, the goal is to develop numerical methods which exploit the structure of the Riccati equations. It turns out however, that methods for solving standard Riccati equations cannot account for the additional terms which appear in the modified equations such as (5.6)–(5.9). Therefore, a new class of numerical algorithms has been developed based upon homotopic continuation methods. These methods operate by first replacing the original problem by a simpler problem with a known solution. The desired solution is then reached by integrating along a path which connects the starting problem to the original problem. These ideas have been illustrated for the reduced order problem in [5] and the H_∞ constrained problem in [3] where the coupling terms preclude standard Riccati techniques. A complete description of the homotopy algorithm will appear in [6].

10. Further extensions

The results of this paper can readily be extended in several directions:

(1) Mixed bounds, i.e., letting $\Delta A = \Delta A_1 + \Delta A_2$ and bounding ΔA_1 with the linear bound and ΔA_2 with the quadratic bound (this would unify the linear and quadratic bound results).

(2) H_2 and H_∞ cross weighting terms (e.g., $x^T R_{12} u$) as well as correlated plant disturbance and sensor noise.

(3) Nonstrictly proper plant model, i.e., (2.2) replaced by

$$y(t) = (C + \Delta C)x(t) + (D + \Delta D)u(t) + D_2 w(t). \quad (10.1)$$

(4) Nonstrictly proper controller, i.e., (2.4) replaced by

$$u(t) = C_c x_c(t) + D_c y(t) \quad (10.2)$$

and the related problems of singular control weighting ($R_2 \geq 0$) and singular measurement noise ($V_2 \geq 0$).

(5) Discrete-time and sampled-data design.

11. Conclusions

The Riccati equation approach to fixed-order H_∞ constrained LQG design has been extended to account for the presence of parameter uncertainties in the state space plant model. Specifically, by embedding quadratic Lyapunov bounds within the design equations, the resulting controllers are guaranteed to provide robust stability and robust H_2/H_∞ performance over a specified range of parameter uncertainty. Two distinct bounds were considered, namely, a linear bound and a quadratic bound. Among the open problems which remain to be examined are the necessity of the design equations, the conservatism of the bounds, and the existence of solutions.

References

- [1] D.S. Bernstein and W.M. Haddad, the optimal projection equations with Petersen-Hollot bounds: robust stability and performance via fixed-order dynamic compensation for systems with structured real-valued parameter uncertainty, *IEEE Trans. Automat. Control* **33** (1988) 578–582.
- [2] D.S. Bernstein and W.M. Haddad, Robust stability and performance for fixed-order dynamic compensation with guaranteed cost bounds, *Proc. Amer. Control Conf.*, Atlanta, GA (June 1988) 2471–2476; *Math. Control Signal Systems* **2** (1989).
- [3] D.S. Bernstein and W.M. Haddad, LQG Control with an H_∞ performance bound: A Riccati equation approach, *Proc. Amer. Control Conf.*, Atlanta, GA, (June 1988) 796–802; *IEEE Trans. Automat. Control* **34** (1989) 293–305.
- [4] J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis, State-space solutions to standard H_2 and H_∞ control problems, *Proc. Amer. Control Conf.*, Atlanta, GA (June 1988) 1691–1696.
- [5] S. Richter, A homotopy algorithm for solving the optimal projection equations for fixed-order compensation: existence, convergence and global optimality, *Proc. Amer. Control Conf.*, Minneapolis, MN (June 1987) 1527–1531.
- [6] S. Richter and E.G. Collins, Jr., A homotopy algorithm for reduced-order controller design via the optimal projection equations, in preparation.
- [7] W.M. Wonham, *Linear Multivariable Control: A Geometric Approach* (Springer-Verlag, New York, 1979).