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Lyapunov-stable adaptive stabilization of non-linear time-varying systems with matched uncertainty

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This paper considers direct adaptive stabilization of multi-input non-linear time-varying systems with full-state feedback and state-and-time-dependent uncertainty. The novel result of this paper is a single-parameter adaptive controller that yields a Lyapunov-stable closed-loop system. We demonstrate the controller on a non-linear spring-mass-damper, a three-degree-of-freedom Mathieu equation, and a 4th-order non-linear time-varying system.

1. Introduction

Adaptive stabilization of linear time-invariant plants under full-state feedback has been considered in Narendra and Annaswamy (1989), Åström and Wittenmark (1995), Ioannou and Sun (1996) and Hong and Bernstein (2001) using Lyapunov-based gradient update laws. These methods require that the state-space parameterization of the plant (A, B) have matched uncertainty. In the case of linear time-invariant plants, matched uncertainty implies that there exists K_s such that $A_s \triangleq A + BK_s$ is known and asymptotically stable.

Lyapunov-based adaptive stabilization has been extended to linear time-varying systems and non-linear time-invariant systems. In Roup and Bernstein (2004), a variation of the controller presented in Hong and Bernstein (2001) is shown to stabilize a class of scalar second-order linear time-varying systems. In particular, the adaptive controller of Roup and Bernstein (2004) can stabilize the scalar time-varying system $m\ddot{q}(t) + g(t)\dot{q}(t) + f(t)q(t) = bu(t)$, where $f(\cdot)$ and $g(\cdot)$ are piecewise continuous and bounded but otherwise unknown.

In Roup and Bernstein (2001), an alternative version of the controller presented in Hong and Bernstein (2001) is shown to stabilize a class of scalar second-order non-linear systems with partial-state-dependent uncertainty. In particular, the adaptive controller of Roup and Bernstein (2001) can stabilize the scalar non-linear system $m\ddot{q}(t) + g(q(t))\dot{q}(t) + f(q(t))q(t) = bu(t)$, where $f(\cdot)$ and $g(\cdot)$ are lower bounded but otherwise unknown. The results of Roup and Bernstein (2001) are extended in Chellaboina *et al.* (2003) and Haddad *et al.* (submitted 2005) to stabilize vector second-order non-linear time-varying systems with partial-state-and-time-dependent uncertainty.

Parameter-monotonic (or high-gain) adaptive control of non-linear time-varying systems with matched uncertainty has been considered in Ryan (1991) Corless and Ryan (1993) and Ilchmann and Ryan (2003). Specifically, Corless and Ryan (1993) and Ilchmann and Ryan (2003) consider output feedback adaptive control for classes of non-linear time-varying systems where known functions bound the unknown non-linear dynamics. In Ryan (1991), full-state feed-back adaptive control is considered for single-input nth-order non-linear time-varying systems where the system is modelled as a differential inclusion. In particular, the adaptive controller of Ryan (1991) guarantees asymptotic convergence for the scalar non-linear time-varying system $q^{(n)}(t) + g(t, x(t)) = bu(t)$ where $x \triangleq [q^{(n-1)}\ q^{(n-2)}\ \dots\ \dot{q}\ q]$, and there exists $a > 0$ and a known function $\hat{g} : \mathbb{R}^n \to [0, \infty)$ such that, for almost all $t \geq 0$ and for all $x \in \mathbb{R}^n$, $|g(t, x(t))| \leq a\hat{g}(x(t))$.

The results of Ryan (1991), Corless and Ryan (1993) and Ilchmann and Ryan (2003), guarantee that the state (or output) of the closed-loop adaptive system asymptotically converges to zero. However, they do not

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prove Lyapunov stability for the closed-loop system. Lyapunov stability of a closed-loop adaptive system provides information about the system's transient performance and prevents small perturbations (such as noise) from driving the system far away from the equilibrium before the asymptotic behaviour returns the system to the equilibrium. In fact, without verification of Lyapunov stability, the nomenclature "adaptive stabilization" is arguably a misnomer.

In the present paper, a full-state-feedback adaptive controller is used to stabilize multi-input nth-order non-linear time-varying systems with matched state-and-time-dependent uncertainty. In the case of bounded state-and-time-dependent uncertainty, the adaptive controller requires no additional information concerning the system non-linearities or time dependence. If the state-and-time-dependent uncertainty is unbounded, then additional bounding functions are required. Whereas the results of Roup and Bernstein (2001, 2004), Chellaboina *et al.* (2003), and Haddad *et al.* (submitted 2005) stabilize 2nd-order systems with matched partial-state-dependent and time-varying uncertainty, the adaptive controller presented herein stabilizes nth-order systems with full-state-and-time-dependent uncertainty. In contrast to the results of Ryan (1991), Corless and Ryan (1993) and Ilchmann and Ryan (2003), the results of this paper guarantee that a continuum of equilibria of the closed-loop system are Lyapunov stable. In addition, Ryan (1991) considers only single-input systems whereas this paper considers multi-input systems. Notably, the present paper differs from Ryan (1991), Corless and Ryan (1993), Roup and Bernstein (2001, 2004), Ilchmann and Ryan (2003), Chellaboina *et al.* (2003) and Haddad *et al.* (inpress 2007), in both method of proof and resulting parameter-monotonic adaptive law. More specifically, the current paper's proofs utilize new tools presented in appendix A, and the current paper's parameter-monotonic adaptive law incorporates an exponentially decaying factor, which has no counterpart in Ryan (1991), Corless and Ryan (1993), Roup and Bernstein (2001, 2004), Chellaboina *et al.* (2003), Ilchmann and Ryan (2003), and Haddad *et al.* (submitted 2005). Nevertheless, the present paper and the previous work (Ryan 1991, Corless and Ryan 1993, Roup and Bernstein 2001, Chellaboina *et al.* 2003, Ilchmann and Ryan 2003, Roup and Bemstein 2004, Haddad *et al.* (inpress 2007), both require the assumption of matched uncertainty. The problem of adaptive stabilization of non-linear time-varying systems with unmatched uncertainty is open.

In § 2, we introduce the notation used in this paper. In § 3, an adaptive controller is derived for nth-order non-linear time-varying systems with bounded uncertainty. In § 4, an adaptive controller is provided for non-linear time-varying systems where the state-and-time-dependent uncertainty is unbounded. Examples are given in §§ 5–7, and conclusions are given in § 8.

2. Notation

$A \otimes B$	Kronecker product of A and B
$A \oplus B$	Kronecker sum of A and B
vec A	vector formed by stacking the columns of A
$\lambda_{\max}(A)$	maximum eigenvalue of A
$\operatorname{diag}(A_1, \ldots, A_l)$	block-diagonal

$$\text{matrix} \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{bmatrix}$$

$$E_1 \triangleq \begin{bmatrix} I_m \\ 0_{m(n-1)\times m} \end{bmatrix}$$

3. Parameter-monotonic adaptive stabilization for non-linear time-varying systems

In this section, we consider parameter-monotonic adaptive stabilization for the nth-order vector non-linear time-varying system

$$\begin{aligned} q^{(n)}(t) &+ M_{n-1}\big(t, q^{(n-1)}, \ldots, \dot{q}, q\big) q^{(n-1)}(t) \\ &+ M_{n-2}\big(t, q^{(n-1)}, \ldots, \dot{q}, q\big) q^{(n-2)}(t) \\ &+ \cdots + M_1\big(t, q^{(n-1)}, \ldots, \dot{q}, q\big)\dot{q}(t) \\ &+ M_0\big(t, q^{(n-1)}, \ldots, \dot{q}, q\big) q(t) \\ &= B_0\big(t, q^{(n-1)}, \ldots, \dot{q}, q\big) u(t), \end{aligned} \tag{1}$$

where $q \in \mathbb{R}^m$, $u \in \mathbb{R}^m$, $B_0 : [0, \infty) \times \mathbb{R}^{mn} \to \mathbb{R}^{m\times m}$, and for $i = 0, \ldots, n-1$, $M_i : [0, \infty) \times \mathbb{R}^{mn} \to \mathbb{R}^{m\times m}$. We make the following assumptions.

(A1) $M_0(\cdot), \ldots, M_{n-1}(\cdot)$, and $B_0(\cdot)$ are locally Lipschitz in $q^{(n-1)}, \ldots, \dot{q}, q$ and piecewise continuous in t.

(A2) $M_0(\cdot), \ldots, M_{n-1}(\cdot)$ are bounded. That is, there exists $\mu > 0$ such that, for all $i = 0, \ldots, n-1$, for all $q^{(n-1)}, \ldots, \dot{q}$, $q \in \mathbb{R}^m$ and for all $t \geq 0$, $||M_i(t, q^{(n-1)}, \ldots, \dot{q}, q)|| \leq \mu$. The bound μ need not be known.

(A3) $B_0(\cdot)$ is globally invertible.

(A4) There exists a positive-definite matrix $H \in \mathbb{R}^{m\times m}$ such that $F(t, q^{(n-1)}, \ldots, \dot{q}, q) \triangleq B_0^{-1}(t, q^{(n-1)}, \ldots, \dot{q}, q) H$ is known.

(A5) The full state $q, \dot{q}, \ldots, q^{(n-1)}$ is available for feedback.

Assumption (A3) restricts our attention to systems with matched uncertainty. Assumption (A4) requires limited knowledge of the mapping $B_0(\cdot)$. Specifically, in the single-input linear time-invariant case, B_0 is a constant scalar and Assumption (A4) is equivalent to the assumption that the sign of the high-frequency gain is known.

If $M_0(\cdot),\dots,M_{n-1}(\cdot)$ and $B_0(\cdot)$ are constant maps, then (1) is a multi-input linear time-invariant system. Furthermore, Assumptions (A1)–(A5) are satisfied if B_0 has full rank and there exists a positive-definite matrix $H\in\mathbb{R}^{m\times m}$ such that $F\triangleq B_0^{-1}H$ is known.

The system (1) can be written in the state-and-time-dependent block controllable canonical form

$$\dot{x}(t)=A(t,x(t))x(t)+B(t,x(t))u(t), \tag{2}$$

where

$$A(t,x)\triangleq\begin{bmatrix}-M_{n-1}(t,x) & -M_{n-2}(t,x) & \cdots & -M_1(t,x) & -M_0(t,x)\\ I_m & 0 & & 0 & 0\\ 0 & I_m & & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & & I_m & 0\end{bmatrix}, \tag{3}$$

$$B(t,x)\triangleq\begin{bmatrix}B_0(t,x)\\ 0_{m(n-1)\times m}\end{bmatrix}, \tag{4}$$

$$x^{\mathrm{T}}\triangleq\begin{bmatrix}q^{(n-1)} & q^{(n-2)} & \dots & \dot{q} & q\end{bmatrix}. \tag{5}$$

We now present a high-gain-stabilizing controller for the non-linear time-varying system (2)–(4).

Lemma 1: *Consider the non-linear time-varying system (2)–(4). Let $g(s)$ be the Hurwitz polynomial*

$$g(s)\triangleq g_{n-1}s^{n-1}+g_{n-2}s^{n-2}+\cdots+g_0, \tag{6}$$

where $g_{n-1}>0$. Define $G(t,x)\triangleq[g_{n-1}\ g_{n-2}\ \dots\ g_0]\otimes F(t,x)$, and consider the feedback

$$u(t)=-kG(t,x(t))x(t), \tag{7}$$

where $k\in\mathbb{R}$. Then, there exists $k_s>0$ such that, for all $k\ge k_s$, the origin of the closed-loop system is globally exponentially stable.

Proof: The system (2)–(4) with the feedback (7) is

$$\dot{x}(t)=[A_s(k)+E_1\Delta(t,x(t))]x(t),$$

where

$$A_s(k)\triangleq\begin{bmatrix}-kg_{n-1}H & -kg_{n-2}H & \cdots & -kg_1H & -kg_0H\\ I_m & 0 & & 0 & 0\\ 0 & I_m & & 0 & 0\\ \vdots & & \ddots & & \vdots\\ 0 & 0 & \cdots & I_m & 0\end{bmatrix}, \tag{8}$$

and $\Delta(t,x)\triangleq[-M_{n-1}(t,x)\cdots-M_0(t,x)]\in\mathbb{R}^{m\times mn}$.

Lemma A.3 implies that there exists $k_1>0$ such that, for all $k\ge k_1$, the matrix $A_s(k)$ is asymptotically stable. Let $Q>0$ and, for all $k\ge k_1$, let $P(k)$ be the positive-definite solution to the Lyapunov equation

$$A_s^{\mathrm{T}}(k)P(k)+P(k)A_s(k)=-(Q+I_n)\otimes I_m.$$

Furthermore, let $P_1(k)$ denote the first m columns of $P(k)$. Next, for $k\ge k_1$, consider the Lyapunov candidate

$$V(x)\triangleq x^{\mathrm{T}}P(k)x.$$

Taking the derivative along a closed-loop trajectory yields

$$\begin{aligned}\dot{V}(x)&=x^{\mathrm{T}}\big[A_s^{\mathrm{T}}(k)P(k)+P(k)A_s(k)+\Delta^{\mathrm{T}}(t,x)E_1^{\mathrm{T}}P(k)\\&\quad+P(k)E_1\Delta(t,x)\big]x\\&=-x^{\mathrm{T}}[(Q+I_n)\otimes I_m]x\\&\quad+x^{\mathrm{T}}\big[\Delta^{\mathrm{T}}(t,x)E_1^{\mathrm{T}}P(k)+P(k)E_1\Delta(t,x)\big]x.\end{aligned} \tag{9}$$

Note that

$$0\le\left(\sqrt{2}P(k)E_1\Delta(t,x)-\frac{1}{\sqrt{2}}I\right)^{\mathrm{T}}\times\left(\sqrt{2}P(k)E_1\Delta(t,x)-\frac{1}{\sqrt{2}}I\right),$$

and thus

$$\begin{aligned}&\Delta^{\mathrm{T}}(t,x)E_1^{\mathrm{T}}P(k)+P(k)E_1\Delta(t,x)\\&\quad\le\tfrac{1}{2}I+2\Delta^{\mathrm{T}}(t,x)E_1^{\mathrm{T}}P^2(k)E_1\Delta(t,x).\end{aligned} \tag{10}$$

Combining (9) and (10) yields

$$\begin{aligned}\dot{V}(x)&\le-x^{\mathrm{T}}\big[(Q+I_n)\otimes I_m\big]x\\&\quad+x^{\mathrm{T}}\left[\frac{1}{2}I+2\Delta^{\mathrm{T}}(t,x)E_1^{\mathrm{T}}P^2(k)E_1\Delta(t,x)\right]x\\&\le-x^{\mathrm{T}}(Q\otimes I_m)x-\frac{1}{2}x^{\mathrm{T}}x\\&\quad+2x^{\mathrm{T}}\big[\Delta^{\mathrm{T}}(t,x)P_1^{\mathrm{T}}(k)P_1(k)\Delta(t,x)\big]x.\end{aligned} \tag{11}$$

Since Lemma A.3 implies that $P_1(k)\to0$ as $k\to\infty$, let $k_s\ge k_1$ be such that, for all $k\ge k_s$,

$$P_1^{\mathrm{T}}(k)P_1(k)\le\frac{1}{4\nu}I_m, \tag{12}$$

where $\nu\triangleq\sup_{t\ge0,\,x\in\mathbb{R}}\lambda_{\max}(\Delta^{\mathrm{T}}(t,x)\Delta(t,x))$. Note that ν exists since $\Delta(t,\ x)$ is bounded by assumption (A2). Therefore, for all $k\ge k_s$, it follows from (11) and (12) that

$$\dot{V}(x)\le-x^{\mathrm{T}}(Q\otimes I_m)x-\frac{1}{2}x^{\mathrm{T}}x+\frac{1}{2}x^{\mathrm{T}}\left[\frac{\Delta^{\mathrm{T}}(t,x)\Delta(t,x)}{\nu}\right]x\le-x^{\mathrm{T}}(Q\otimes I_m)x.$$

Hence, for all $k \geq k_s$, the origin is globally exponentially stable. □

Now we present the main result of this paper, namely, Lyapunov-stable adaptive stabilization of a class of nth-order vector non-linear time-varying systems.

Theorem 1: *Consider the non-linear time-varying system* (2)–(4). *Let $g(s)$ be the Hurwitz polynomial* (6) *where $g_{n-1} > 0$. Define $G(t,x) \triangleq [g_{n-1}\ g_{n-2}\ \dots\ g_0] \otimes F(t,x)$, and consider the adaptive feedback controller*

$$u(t) = -k(t)G(t,x(t))x(t), \tag{13}$$

$$\dot{k}(t) = e^{-\alpha k(t)} x^{\mathrm{T}}(t)(R \otimes I_m)x(t), \tag{14}$$

where $R \in \mathbb{R}^{n\times n}$ is positive definite and $\alpha > 0$. Then, there exists $k_s > 0$ such that, for all $k_e \geq k_s$, the equilibrium solution $(0, k_e)$ of the closed-loop system (2)–(4) *and* (13) and (14) *is uniformly Lyapunov stable. Furthermore, for all initial conditions $x(0)$ and $k(0)$, $\lim_{t\to\infty} k(t)$ exists and $\lim_{t\to\infty} x(t) = 0$.*

Proof: The dynamics (2)–(4) with the feedback (13) is

$$\dot{x}(t) = [A_s(k) + E_1\Delta(t,x)]x(t), \tag{15}$$

where $A_s(k)$ is given by (8) and $\Delta(t,x) \triangleq [-M_{n-1}(t,x) \cdots -M_0(t,x)] \in \mathbb{R}^{m\times mn}$.

Lemma A.3 implies that there exists k_1 such that, for all $k \geq k_1$, $A_s(k)$ is asymptotically stable. Let $Q > 0$ and, for all $k \geq k_1$, let $P_s(k)$ be the positive definite solution to the Lyapunov equation

$$A_s^{\mathrm{T}}(k)P_s(k) + P_s(k)A_s(k) = -(Q + I_n)\otimes I_m.$$

Let $P_{s,1}(k) = P_s(k)E_1$ denote the first m columns of $P_s(k)$. Since Lemma A.3 implies that $P_{s,1}(k) \to 0$ as $k \to \infty$, let $k_s \geq k_1$ be such that, for all $k \geq k_s$,

$$P_{s,1}^{\mathrm{T}}(k)P_{s,1}(k) \leq \tfrac{1}{4\nu} I, \tag{16}$$

where $\nu \triangleq \sup_{t\geq 0,\, x\in\mathbb{R}} \lambda_{\max}(\Delta^{\mathrm{T}}(t,x)\Delta(t,x))$. Assumption (A2) implies that ν is finite.

Let $k_e \geq k_s$, define $A_e \triangleq A_s(k_e)$, and define $\tilde{k}(t) \triangleq k_e - k(t)$ so that (15) can be written as

$$\dot{x} = A_e x + \tilde{k}E_1\hat{H}x + E_1\Delta(t,x)x,$$

where $\hat{H} \triangleq [g_{n-1}\ \dots\ g_0] \otimes H$. Define $P_e \triangleq P_s(k_e)$, and consider the Lyapunov candidate

$$V(x,\tilde{k}) \triangleq x^T P_e x + \tilde{k}^2,$$

where $V : \mathbb{R}^{mn} \times \mathcal{D} \to [0,\infty)$ and the domain $\mathcal{D} \subset \mathbb{R}$ will be specified later. The derivative of $V(x,\tilde{k})$ along a trajectory of the closed-loop system is

$$\begin{aligned}
\dot{V}(x,\tilde{k}) &= x^{\mathrm{T}}\big[A_e^{\mathrm{T}}P_e + P_eA_e\big]x \\
&\quad + x^{\mathrm{T}}\big[\Delta^{\mathrm{T}}(t,x)E_1^{\mathrm{T}}P_e + P_eE_1\Delta(t,x)\big]x \\
&\quad + \tilde{k}x^{\mathrm{T}}\Big[\hat{H}^{\mathrm{T}}E_1^{\mathrm{T}}P_e + P_eE_1\hat{H}\Big]x - 2\tilde{k}\dot{\tilde{k}} \\
&\leq -x^{\mathrm{T}}[(Q+I_n)\otimes I_m]x \\
&\quad + x^{\mathrm{T}}\Big[\frac{1}{2}I + 2\Delta^{\mathrm{T}}(t,x)E_1^{\mathrm{T}}P_e^2E_1\Delta(t,x)\Big]x \\
&\quad + \tilde{k}x^{\mathrm{T}}\Big[\hat{H}^{\mathrm{T}}E_1^{\mathrm{T}}P_e + P_eE_1\hat{H} - 2e^{-\alpha k}(R\otimes I_m)\Big]x \\
&\leq -x^{\mathrm{T}}(Q\otimes I_m)x - \frac{1}{2}x^{\mathrm{T}}x \\
&\quad + 2x^{\mathrm{T}}\big[\Delta^{\mathrm{T}}(t,x)P_{s,1}(k_e)^{\mathrm{T}}P_{s,1}(k_e)\Delta(t,x)\big]x \\
&\quad + \tilde{k}x^{\mathrm{T}}\Big[\hat{H}^{\mathrm{T}}E_1^{\mathrm{T}}P_e + P_eE_1\hat{H} - 2e^{-\alpha k}(R\otimes I_m)\Big]x.
\end{aligned} \tag{17}$$

Since $k_e \geq k_s$, it follows from (16) and (17) that

$$\begin{aligned}
\dot{V}(x,\tilde{k}) &\leq -x^{\mathrm{T}}(Q\otimes I_m)x - \frac{1}{2}x^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}\Big[\frac{\Delta^{\mathrm{T}}(t,x)\Delta(t,x)}{\nu}\Big]x \\
&\quad + \tilde{k}x^{\mathrm{T}}\Big[\hat{H}^{\mathrm{T}}E_1^{\mathrm{T}}P_e + P_eE_1\hat{H} - 2e^{-\alpha k}(R\otimes I_m)\Big]x \\
&\leq -x^{\mathrm{T}}(Q\otimes I_m)x \\
&\quad + \tilde{k}x^{\mathrm{T}}\Big[\hat{H}^{\mathrm{T}}E_1^{\mathrm{T}}P_e + P_eE_1\hat{H} - 2e^{-\alpha k}(R\otimes I_m)\Big]x.
\end{aligned}$$

To show that $\dot{V}$ is negative semi-definite, we first consider the case $\tilde{k} \geq 0$, in which

$$\begin{aligned}
\dot{V}(x,\tilde{k}) &\leq -x^{\mathrm{T}}(Q\otimes I_m)x + \tilde{k}x^{\mathrm{T}}\Big[\hat{H}^{\mathrm{T}}E_1^{\mathrm{T}}P_e + P_eE_1\hat{H}\Big]x \\
&\leq -x^{\mathrm{T}}(Q\otimes I_m)x + \tilde{k}x^{\mathrm{T}}\Big[I + \hat{H}^{\mathrm{T}}E_1^{\mathrm{T}}P_e^2E_1\hat{H}\Big]x \\
&\leq -x^{\mathrm{T}}(Q\otimes I_m)x + \tilde{k}\sigma_1x^{\mathrm{T}}x,
\end{aligned}$$

where $\sigma_1 \triangleq \lambda_{\max}(I + \hat{H}^{\mathrm{T}}E_1^{\mathrm{T}}P_e^2E_1\hat{H})$. Let ε_1 satisfy $0 < \varepsilon_1 < \lambda_{\min}(Q\otimes I_m)$. Then, for all $\tilde{k}$ such that $0 \leq \tilde{k} \leq (\lambda_{\min}(Q\otimes I_m) - \varepsilon_1/\sigma_1)$,

$$\begin{aligned}
\dot{V}(x,\tilde{k}) &\leq -x^{\mathrm{T}}(Q\otimes I_m)x \\
&\quad + \left(\frac{\lambda_{\min}(Q\otimes I_m) - \varepsilon_1}{\sigma_1}\right)\sigma_1x^{\mathrm{T}}x \leq -\varepsilon_1x^{\mathrm{T}}x.
\end{aligned}$$

Now, consider the case $\tilde{k} \leq 0$, in which

$$\begin{aligned}
\dot{V}(x,\tilde{k}) &\leq -x^{\mathrm{T}}(Q\otimes I_m)x \\
&\quad - \tilde{k}x^{\mathrm{T}}\Big[-\hat{H}^{\mathrm{T}}E_1^{\mathrm{T}}P_e - P_eE_1\hat{H} + 2(R\otimes I_m)\Big]x \\
&\leq -x^{\mathrm{T}}(Q\otimes I_m)x \\
&\quad - \tilde{k}x^{\mathrm{T}}\Big[I + \hat{H}^{\mathrm{T}}E_1^{\mathrm{T}}P_e^2E_1\hat{H} + 2(R\otimes I_m)\Big]x \\
&\leq -x^{\mathrm{T}}(Q\otimes I_m)x - \tilde{k}\sigma_2x^{\mathrm{T}}x,
\end{aligned}$$

where $\sigma_2 \triangleq \lambda_{\max}(I + \hat{H}^{\mathrm{T}} E_1^{\mathrm{T}} P_e^2 E_1 \hat{H} + 2(R \otimes I_m))$. Let ε_2 satisfy $0 < \varepsilon_2 < \lambda_{\min}(Q \otimes I_m)$. Then, for all $\tilde{k}$ such that $-((\lambda_{\min}(Q \otimes I_m) - \varepsilon_2)/\sigma_2) \le \tilde{k} \le 0$,

$$\dot{V}(x, \tilde{k}) \le -x^{\mathrm{T}}(Q \otimes I_m)x + \left(\frac{\lambda_{\min}(Q \otimes I_m) - \varepsilon_2}{\sigma_2}\right)\sigma_2 x^{\mathrm{T}} x \le -\varepsilon_2 x^{\mathrm{T}} x.$$

Define the domain

$$\mathcal{D} \triangleq \left\{\tilde{k} \in \mathbb{R} : -\frac{\lambda_{\min}(Q \otimes I_m) - \varepsilon_2}{\sigma_2} < \tilde{k} < \frac{\lambda_{\min}(Q \otimes I_m) - \varepsilon_1}{\sigma_1}\right\}.$$

Thus, for all $x \in \mathbb{R}^{mn}$ and all $\tilde{k} \in \mathcal{D}$, $\dot{V}(x, \tilde{k}) \le -\min(\varepsilon_1, \varepsilon_2)x^{\mathrm{T}}x$, and thus the solution $(0, k_e)$ is uniformly Lyapunov stable.

Next, we show that $k(t)$ converges. Lemma A.3 implies that there exists $k_1 > 0$ such that, for all $k \ge k_1$, $A_{\mathrm{s}}(k)$ is asymptotically stable. For $k \ge k_1$, define

$$V_0(x, k) \triangleq e^{-\alpha k} x^T P(k) x,$$

where, for $k \ge k_1$, $P(k)$ is the positive definite solution to the equation

$$A_{\mathrm{s}}^{\mathrm{T}}(k)P(k) + P(k)A_{\mathrm{s}}(k) = -(R \otimes I_m).$$

Taking the derivative of $V_0(x, k)$ along a trajectory of (14) and (15) yields

$$\dot{V}_0(x, k) = -e^{-\alpha k} x^{\mathrm{T}}(R \otimes I_m)x + e^{-\alpha k} x^{\mathrm{T}}\left[\Delta^{\mathrm{T}}(t, x)E_1^{\mathrm{T}} P(k) + P(k)E_1\Delta(t, x)\right]x - \dot{k}e^{-\alpha k} x^{\mathrm{T}}\left[\alpha P(k) - \frac{\partial P(k)}{\partial k}\right]x.$$

Lemma A.3 implies that there exists $k_2 \ge k_1$ such that, for all $k \ge k_2$, $\alpha P(k) - (\partial P(k)/\partial k) > 0$. Thus, for all $k \ge k_2$,

$$\dot{V}_0(x, k) \le -e^{-\alpha k} x^{\mathrm{T}}(R \otimes I_m)x + e^{-\alpha k} x^{\mathrm{T}}\left[\Delta^{\mathrm{T}}(t, x)E_1^{\mathrm{T}} P(k) + P(k)E_1\Delta(t, x)\right]x. \tag{18}$$

Since

$$0 \le \left[\sqrt{\frac{1}{2}}(R \otimes I_m)^{1/2} - \sqrt{2}(R \otimes I_m)^{-1/2} P(k)E_1\Delta(t, x)\right]^{\mathrm{T}} \times \left[\sqrt{\frac{1}{2}}(R \otimes I_m)^{1/2} - \sqrt{2}(R \otimes I_m)^{-1/2} P(k)E_1\Delta(t, x)\right],$$

it follows that

$$\Delta^{\mathrm{T}}(t, x)E_1^{\mathrm{T}} P(k) + P(k)E_1\Delta(t, x) \le \frac{1}{2}(R \otimes I_m) + 2\Delta^{\mathrm{T}}(t, x)E_1^{\mathrm{T}} \times P(k)(R \otimes I_m)^{-1} P(k)E_1\Delta(t, x). \tag{19}$$

Combining (18) and (19) yields

$$\begin{aligned}\dot{V}_0(x, k) &\le -e^{-\alpha k} x^{\mathrm{T}}(R \otimes I_m)x + \frac{1}{2}e^{-\alpha k} x^{\mathrm{T}}(R \otimes I_m)x + 2e^{-\alpha k} x^{\mathrm{T}} \\ &\quad \times \left[\Delta^{\mathrm{T}}(t, x)E_1^{\mathrm{T}} P(k)(R \otimes I_m)^{-1} P(k)E_1\Delta(t, x)\right]x \\ &\le -\frac{1}{2}e^{-\alpha k} x^{\mathrm{T}}(R \otimes I_m)x + 2e^{-\alpha k} x^{\mathrm{T}} \\ &\quad \times \left[\Delta^{\mathrm{T}}(t, x)P_1^{\mathrm{T}}(x)(R \otimes I_m)^{-1} P_1(k)\Delta(t, x)\right]x, \end{aligned} \tag{20}$$

where $P_1(k)$ denotes the first m columns of $P(k)$. Since, by Lemma A.3, $\lim_{k\to\infty} P_1(k) = 0$, it follows that there exists $k_3 \ge k_2$ such that, for all $k \ge k_3$, $P_1^{\mathrm{T}}(k)(R \otimes I_m)^{-1} P_1(k) \le (\lambda_{\min}(R \otimes I_m)/8\nu)I_m$. Then, it follows from (20) that, for all $k \ge k_3$,

$$\dot{V}_0(x, k) \le -\frac{1}{4}e^{-\alpha k} x^{\mathrm{T}}(R \otimes I_m)x = -\frac{1}{4}\dot{k}. \tag{21}$$

Since (2)–(4) and (13)–(14) are locally Lipschitz in (x, k) and piecewise continuous in t, it follows that the solution to (2)–(4) and (13)–(14) exists and is unique locally, that is, there exists $t_e > 0$ such that $(x(t), k(t))$ exists on the interval $[0, t_e)$. Note that it follows from (14) that if at least one component of $x(t)$ diverges to infinity at t_e, then $k(t)$ diverges to infinity at t_e. To prove that $(x(\cdot), k(\cdot))$ exists and is unique on all finite intervals, suppose that $(x(t), k(t))$ diverges to infinity at t_e. Then $k(t)$ diverges to infinite at t_e, and there exists $t_3 < t_e$ such that $k(t_3) = k_3$. Let $t \in [t_3, t_e)$. Integrating (21) from t_3 to t and solving for $k(t)$ yields

$$\begin{aligned} k(t) &\le k_3 + 4V_0(x(t_3), k_3) - 4V_0(x(t), k(t)) \\ &\le k_3 + 4V_0(x(t_3), k_3), \end{aligned} \tag{22}$$

for $t \in [t_3, t_e)$. Hence $k(t)$ is bounded on $[0, t_e)$, which is a contradiction. Therefore, the solution to (2)–(4) and (13)–(14) exists and is unique on all finite intervals. Now integrating (21) from t_3 to t yields (22) for all $t \in [t_3, \infty)$. Therefore, $k(\cdot)$ is bounded on $[0, \infty)$. Since $k(t)$ is non-decreasing, $k_\infty \triangleq \lim_{t\to\infty} k(t)$ exists.

Next, we show that $x(\cdot)$ is bounded. Taking the derivative of

$$V_1(x) \triangleq x^{\mathrm{T}} x$$

along a trajectory of (15) yields

$$\dot{V}_1(x, k) = x^{\mathrm{T}}\left[A_{\mathrm{s}}^{\mathrm{T}}(k) + \Delta^{\mathrm{T}}(t, x)E_1^{\mathrm{T}} + A_{\mathrm{s}}(k) + E_1\Delta(t, x)\right]x.$$

Since $k(t)$ converges and $\Delta(\cdot, \cdot)$ is bounded, there exists $\eta > 0$ such that

$$\dot{V}_1(x, k) \le \eta x^{\mathrm{T}}(R \otimes I_m)x = \eta e^{\alpha k}\dot{k}.$$

Integrating the above from 0 to t and solving for $V_1(x(t))$ yields

$$V_1(x(t)) \le \frac{\eta}{\alpha}e^{\alpha k(t)} + V_1(x(0)) - \frac{\eta}{\alpha}e^{\alpha k(0)}.$$

Since $k(\cdot)$ is bounded, we conclude that $V_1(\cdot)$ is bounded. Thus, $x(\cdot)$ is bounded.

Next, we show that $\lim_{t\to\infty} x(t) = 0$. The dynamics (15) implies

$$||\dot{x}(t)|| \le (||A_s(k(t))|| + ||E_1||\,||\Delta(t, x(t))||)\,||x(t)||. \quad (23)$$

Since $\lim_{t\to\infty} k(t)$ exists, $A_s(\cdot)$ is bounded. Furthermore, $\Delta(\cdot,\cdot)$ is bounded. Since $A_s(\cdot)$, $\Delta(\cdot,\cdot)$, and $x(\cdot)$ are bounded, it follows from (23) that $\dot{x}(\cdot)$ is bounded. Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\dot{k}(t)\Big] &= \frac{\mathrm{d}}{\mathrm{d}t}\big[e^{-\alpha k(t)}x^{\mathrm{T}}(t)(R\otimes I_m)x(t)\big]\\ &= e^{-\alpha k(t)}\Big(-\alpha\dot{k}(t)x^{\mathrm{T}}(t)(R\otimes I_m)x(t)\\ &\quad + 2x^{\mathrm{T}}(t)(R\otimes I_m)\dot{x}(t)\Big),\end{aligned}$$

is bounded, and thus $\dot{k}(\cdot)$ is uniformly continuous. Since $\dot{k}(\cdot)$ is uniformly continuous and $\lim_{t\to\infty}\int_0^t \dot{k}(\tau)\mathrm{d}\tau = k_\infty - k(0)$ exists, Barabalat's lemma implies that $\lim_{t\to\infty}\dot{k}(t) = 0$. Therefore, $\lim_{t\to\infty} x^{\mathrm{T}}(t)(R\otimes I_m)x(t) = \lim_{t\to\infty}\dot{k}(t)e^{\alpha k(t)} = (\lim_{t\to\infty}\dot{k}(t))(\lim_{t\to\infty} e^{\alpha k(t)}) = 0$, and thus $\lim_{t\to\infty} x(t) = 0$. □

4. Parameter-monotonic adaptive stabilization for non-linear time-varying systems with unbounded non-linearities

Assumption (A2) in §4 requires that $M_0(\cdot),\ldots,M_{n-1}(\cdot)$ be bounded. In this section, we weaken this assumption for systems where additional information about the non-linearity is available. Specifically, we make the following alternative assumption.

(A2b) For $i = 0,\ldots,n-1$, there exists a known map $\hat{M}:[0,\infty)\times\mathbb{R}^{mn}\to\mathbb{R}^{m\times m}$ such that $M_i(\cdot) - B_0(\cdot)\hat{M}_i(\cdot)$ is bounded. That is, there exists $\bar{\mu} > 0$ such that, for $i = 0,\ldots,n-1$, for all $q^{(n-1)},\ldots,\dot{q},q\in\mathbb{R}^m$ and for all $t\ge 0$,

$$\Big\| M_i\big(t, q^{(n-1)},\ldots,\dot{q},q\big) - B_0\big(t, q^{(n-1)},\ldots,\dot{q},q\big)\hat{M}_i(t, q^{(n-1)},\ldots,\dot{q},q)\Big\| \le \bar{\mu}.$$

The bound $\bar{\mu}$ need not be known.

For example, consider the Duffing equation with control

$$\ddot{q}(t) + c_1\dot{q}(t) + \big[c_0 + dq^2(t)\big]q(t) = u(t), \quad (24)$$

which is also given by (1) where $M_1 = c_1$, $M_0(q) = c_0 + \mathrm{d}q^2$, and $B_0 = 1$. The controlled Duffing equation (1) does not satisfy Assumption (A2) because the function $M_0(\cdot)$ is not bounded. However, if we let $\hat{M}_1 = 0$ and define the known function $\hat{M}_0(q) \triangleq \mathrm{d}q^2$, then (24) does satisfy Assumption (A2b).

The following corollary to Theorem 1 addresses non-linear time-varying systems with unbounded non-linearities.

Corollary 1: *Consider the non-linear time-varying system* (2)–(4) *satisfying assumptions* (A1), (A2b), *and* (A3)–(A5). *Let $g(s)$ be the Hurwitz polynomial* (6), *where* $g_{n-1} > 0$. *Define*

$$G(t,x) \triangleq \big[\, g_{n-1} \quad g_{n-2} \quad \cdots \quad g_0 \,\big]\otimes F(t,x),$$

$$\hat{M}(t,x) \triangleq \big[\, \hat{M}_{n-1}(t,x) \quad \hat{M}_{n-2}(t,x) \quad \cdots \quad \hat{M}_0(t,x)\,\big],$$

and consider the adaptive feedback controller

$$u(t) = -k(t)G(t,x(t))x(t) - \hat{M}(t,x(t))x(t), \quad (25)$$

$$\dot{k}(t) = e^{-\alpha k(t)}x^{\mathrm{T}}(t)(R\otimes I_m)x(t), \quad (26)$$

where $R\in\mathbb{R}^{n\times n}$ is positive definite and $\alpha > 0$. Then, there exists $k_s > 0$, such that, for all $k_e \ge k_s$, the equilibrium solution $(0, k_e)$ of the closed-loop system (2)–(4) *and* (25)–(26) *is uniformly Lyapunov stable. Furthermore, for all initial conditions $x(0)$ and $k(0)$, $\lim_{t\to\infty} k(t)$ exists and* $\lim_{t\to\infty} x(t) = 0$.

Proof: The dynamics (2)–(4) and (25) can be written as

$$\dot{x}(t) = \bar{A}(t,x(t))x(t) + B\bar{u}(t), \quad (27)$$

where

$$\bar{A}(t,x) \triangleq \begin{bmatrix} -\bar{M}_{n-1}(t,x) & -\bar{M}_{n-2}(t,x) & \cdots & -\bar{M}_1(t,x) & -M_0(t,x)\\ I_m & 0 & & 0 & 0\\ 0 & I_m & & 0 & 0\\ \vdots & & \ddots & \vdots & \\ 0 & 0 & & I_m & 0\end{bmatrix}, \quad (28)$$

$$\bar{u}(t) \triangleq -k(t)G(t,x(t))x(t), \quad (29)$$

and, for $i = 0,\ldots,n-1$, $\bar{M}_i(t,x) \triangleq M_i(t,x) - B_0(t,x)\hat{M}_i(t,x)$. Since $\bar{M}_1(\cdot),\ldots,\bar{M}_{n-1}(\cdot)$ are bounded, the system (27)–(29) satisfies Assumptions (A1)–(A5), and the result follows immediately from Theorem 1. □

5. Example: non-linear spring-mass-damper

In this section, we consider the non-linear spring-mass-damper

$$m\ddot{q}(t) + \hat{c}(\dot{q}(t)) + \hat{k}(q(t)) = u(t), \quad (30)$$

where

$$\hat{c}(\dot{q}(t)) \triangleq \begin{cases} \left(c + \dfrac{d}{\delta}\right)\dot{q}(t), & |\dot{q}(t)| < \delta, \\ c\dot{q}(t) + \operatorname{sgn}(\dot{q}(t))d, & |\dot{q}(t)| \geq \delta, \end{cases}$$

$$\hat{k}(q(t)) \triangleq \begin{cases} k_0\left(q(t) + \dfrac{h}{2}\right), & q(t) \leq \dfrac{-h}{2}, \\ 0, & |q(t)| < \dfrac{h}{2}, \\ k_0\left(q(t) - \dfrac{h}{2}\right), & q(t) \geq \dfrac{h}{2}, \end{cases}$$

and $\delta, c, d, h, k_0 > 0$. The function $\hat{c}(\cdot)$ is a continuous approximation of Coulomb friction plus linear damping and satisfies assumption (*i*). The function $\hat{k}(\cdot)$ is a linear spring with a deadzone. This non-linear system is shown in figure 1. Note that the uncontrolled system has a continuum of equilibria, each of which is Lyapunov stable and semistable, but not asymptotically stable; for the definition of semistability, see Bhat and Bernstein (2003). For this example, the mass $m = 3\,\mathrm{kg}$, the viscous friction $c = 2\,\mathrm{kg/s}$, the Coulomb friction $d = 20\,\mathrm{N}$, the spring stiffness $k_0 = 2\,\mathrm{kg/s^2}$, the deadzone gap $h = 10\,\mathrm{m}$, and $\delta = 0.1\,\mathrm{m/s}$.

The system (30) can be written as

$$\ddot{q}(t) + \left[\frac{\hat{c}(\dot{q}(t))}{m\dot{q}(t)}\right]\dot{q}(t) + \left[\frac{\hat{k}(q(t))}{mq(t)}\right]q(t) = \frac{1}{m}u(t),$$

where the functions $[\hat{c}(\dot{q})/m\dot{q}]$ and $[\hat{k}(q)/mq]$ are locally Lipschitz and bounded. Thus the system (30) satisfies Assumptions (A1)–(A5) and the adaptive controller presented in Theorem 1 can be used to stabilize the origin. This controller is given by

$$u(t) = -k(t)\begin{bmatrix} g_1 & g_0 \end{bmatrix}\begin{bmatrix} \dot{q}(t) \\ q(t) \end{bmatrix}, \tag{31}$$

$$k(t) = e^{-\alpha k(t)}\begin{bmatrix} \dot{q}(t) \\ q(t) \end{bmatrix}^{\mathrm{T}} R \begin{bmatrix} \dot{q}(t) \\ q(t) \end{bmatrix}, \tag{32}$$

where $g_0 > 0, g_1 > 0, \alpha > 0$, and R is positive definite. We choose the controller parameters $g_0 = 11$, $g_1 = 7$, $\alpha = 0.1$, and $R = I$. The system (30) with the adaptive controller (31) and (32) is simulated with the initial conditions $k(0) = 0$, $q(0) = -25\,\mathrm{m}$, and $\dot{q}(0) = 10\,\mathrm{m/s}$. The time histories of the position $q(t)$ and velocity $\dot{q}(t)$ for the open-loop and closed-loop systems are shown in figure 2. Since the open-loop system has a continuum of semistable equilibria, the velocity converges to zero and the position converges, with a limiting value of approximately $-15.6\,\mathrm{m}$. The adaptive controller stabilizes the $(0, 0)$ equilibrium so that both the velocity and position converge to zero. Time histories of the adaptive parameter $k(t)$ and the control signal $u(t)$ are shown in figure 3. The adaptive parameter converges to approximately 42.3.

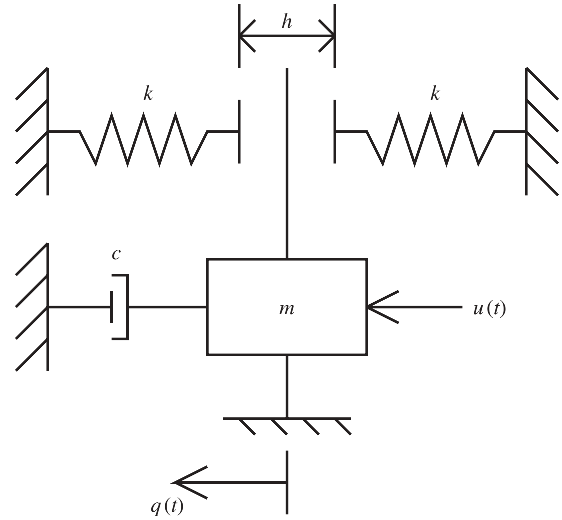

Figure 1. Non-linear spring-mass-damper with Coulomb friction plus linear damping and a deadzone in the spring stiffness.

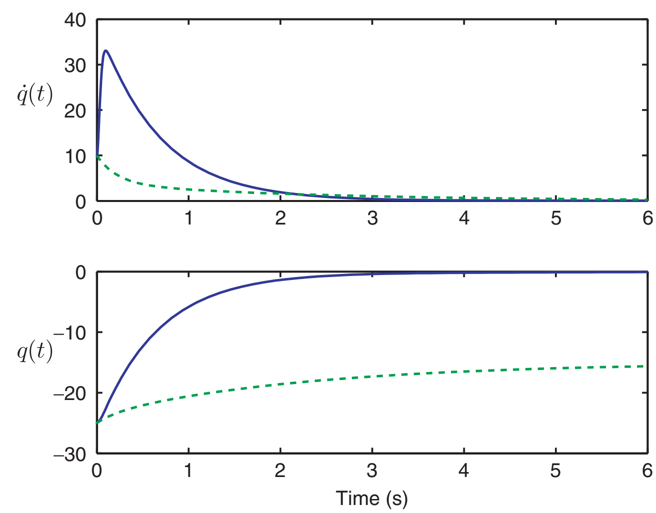

Figure 2. The open-loop (dashed) and closed-loop (solid) time histories for the position and velocity of the mass. The open-loop system has a continuum of semistable equilibria, and the adaptive controller stabilizes the origin.

6. Example: Mathieu equation

Consider the forced three-degree-of-freedom Hill equation

$$\ddot{y}(t) + [C - 2Dp(t)]y(t) = u(t), \tag{33}$$

where $y \in \mathbb{R}^3, u \in \mathbb{R}^3, C \in \mathbb{R}^{3\times 3}, D \in \mathbb{R}^{3\times 3}$ and $p(t)$ is a periodic function. The forced three-degree-of-freedom Mathieu equation is a special case of the Hill equation (33) with

$$p(t) = \cos 2t. \tag{34}$$

In this section, we use the forcing term $u(t)$ to adaptively stabilize the Mathieu equation (33) and (34). The unforced Mathieu equation (33) and (34) with $u(t) \equiv 0$ can exhibit either stable or unstable behaviour depending on the coefficients C and D. For more

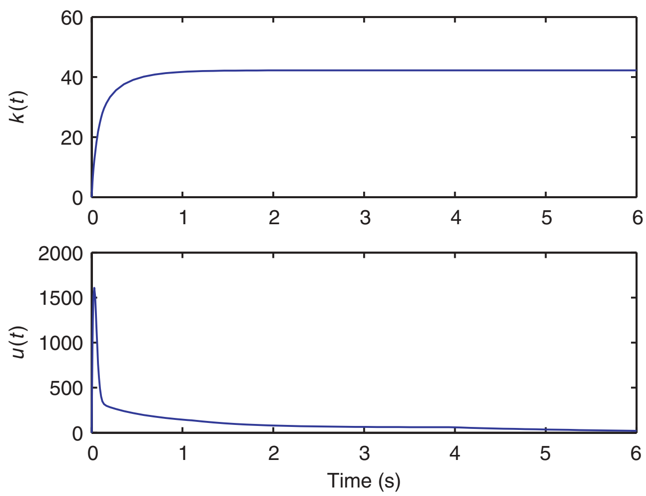

Figure 3. Time histories of the adaptive parameter $k(t)$ and the control signal $u(t)$. The adaptive parameter converges to approximately 42.3.

information on the unforced behaviour of a Mathieu equation, see D'Angelo (1970) and Tondl *et al*. (2000). For this example, we choose

$$C = D = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 3 & -4 & 5 \end{bmatrix}$$

so that the origin of the unforced system is unstable.

Since $C - 2Dp(t)$ is piecewise continuous in t and bounded, the linear time-varying system (33) and (34) satisfies Assumptions (A1)–(A5) and the adaptive controller presented in Theorem 1 can be used to stabilize the origin. This controller is given by

$$u(t) = -k(t)\left(\begin{bmatrix} g_1 & g_0 \end{bmatrix} \otimes F\right)\begin{bmatrix} \dot{y}(t) \\ y(t) \end{bmatrix}, \tag{35}$$

$$k(t) = e^{-\alpha k(t)}\begin{bmatrix} \dot{y}(t) \\ y(t) \end{bmatrix}^{\mathrm{T}} (R \otimes I_3)\begin{bmatrix} \dot{y}(t) \\ y(t) \end{bmatrix}, \tag{36}$$

where $g_0 > 0$, $g_1 > 0$, $\alpha > 0$, and F and R are positive definite. We choose the controller parameters $g_0 = 1$, $g_1 = 1$, $\alpha = 1.0$, $F = I_3$, and $R = I_2$. The system (33) and (34) with the adaptive controller (35) and (36) connected in feedback is simulated with the initial conditions $k(0) = 0$, $y(0) = [-5 \quad -10 \quad 1]^{\mathrm{T}}$, and $\dot{y}(0) = [-7 \quad 8 \quad 2]$. The time histories of $y(t)$ and $\dot{y}(t)$ for the open-loop and closed-loop systems are shown in figures 4 and 5. The origin of the open-loop system is unstable, and the adaptive controller stabilizes this equilibrium. Time histories of the adaptive parameter $k(t)$ and the control signal $u(t)$ are shown in figure 6. The adaptive parameter converges to approximately 6.2.

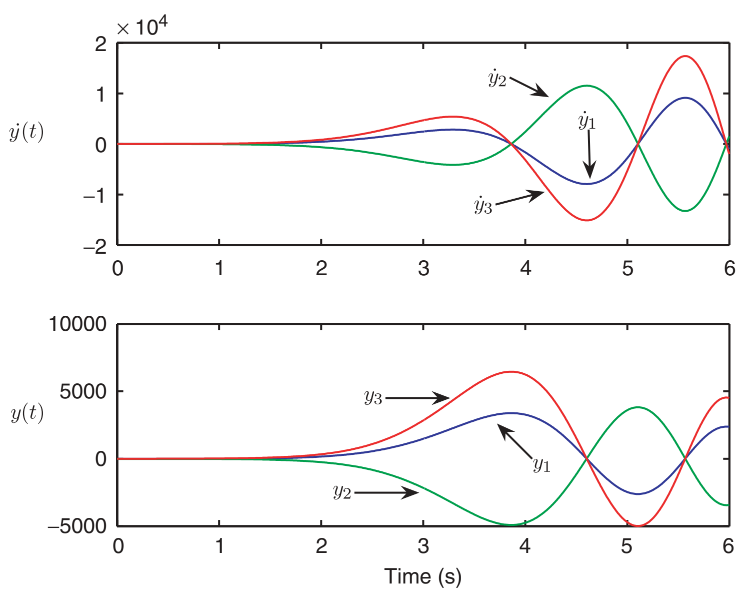

Figure 4. The open-loop time histories for the Mathieu equation. The origin of the open-loop system is unstable.

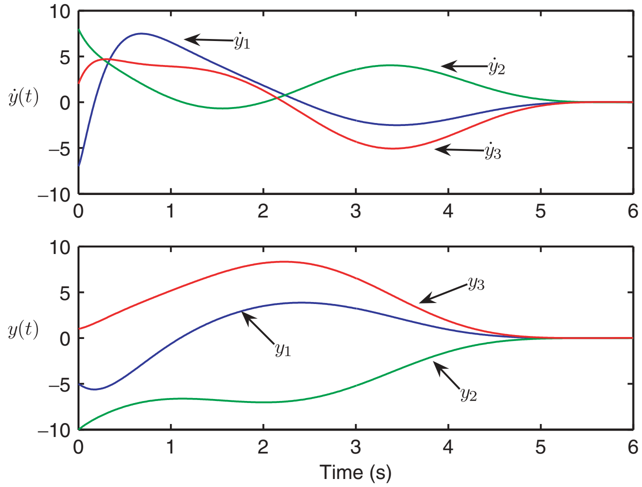

Figure 5. The closed-loop time histories for the Mathieu equation. The adaptive controller stabilizes the origin.

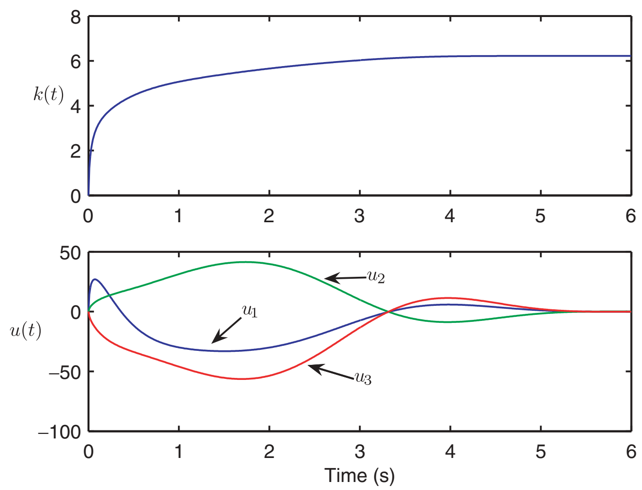

Figure 6. Time histories of the adaptive parameter $k(t)$ and the control signal $u(t)$. The adaptive parameter converges to approximately 6.2.

7. Example: 4th-order non-linear time-varying system

Consider the 4th-order non-linear time-varying system

$$\ddddot{x}(t) + m_3(t)\ddot{x}(t) + m_2(\dot{x})\dot{x}(t) + m_1(t,\ddot{x},\dot{x})\dot{x}(t) + m_0(t)x(t) = bu(t), \qquad (37)$$

where

$$m_0(t) = 10\,\mathrm{sgn}(\sin(\pi t)), \qquad (38)$$

$$m_1(t,\ddot{x},\dot{x}) = 2\cos(16\pi t)\sin(10\ddot{x}\dot{x}), \qquad (39)$$

$$m_2(\ddot{x}) = -13\frac{\sin\ddot{x}}{\ddot{x}}, \qquad (40)$$

$$m_3(t) = N(t) - 10, \qquad (41)$$

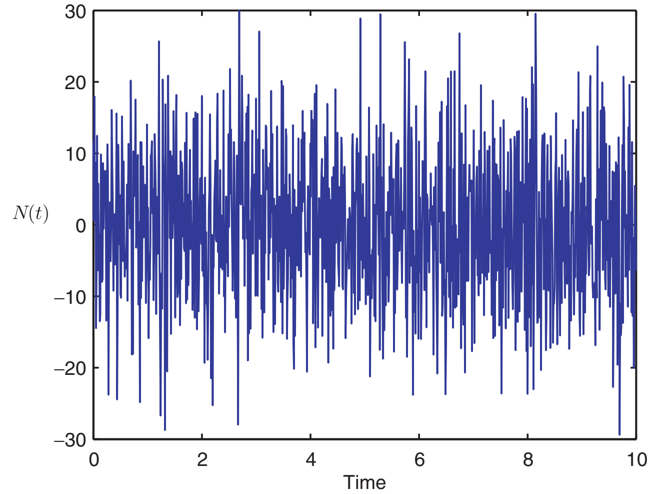

Figure 7. The random signal $N(t)$ that varies between -30 and 30.

$b = -\frac{1}{2}$, and $N(t)$ is a random signal that varies between -30 and 30, as shown in figure 7.

The functions $m_0(t), m_1(t,\ddot{x},\dot{x}), m_2(\ddot{x}), m_3(t)$ are locally Lipschitz in $\ddot{x}, \dot{x}$, piecewise continuous in t, and bounded. Thus, the system (37)–(41) satisfies Assumptions (A1)–(A5). Consider the adaptive controller

$$u(t) = -k(t)\begin{bmatrix} g_3 & g_2 & g_1 & g_0 \end{bmatrix}\begin{bmatrix} \ddot{x}(t) \\ \dot{x}(t) \\ x(t) \\ (t) \end{bmatrix}, \qquad (42)$$

$$k(t) = e^{-\alpha k(t)}\begin{bmatrix} \ddot{x}(t) \\ \dot{x}(t) \\ x(t) \\ (t) \end{bmatrix}^{\mathrm{T}} R \begin{bmatrix} \ddot{x}(t) \\ \dot{x}(t) \\ x(t) \\ (t) \end{bmatrix}, \qquad (43)$$

where $g_0, g_1, g_2, g_3 \in \mathbb{R}$, $\alpha > 0$ and R is positive definite. To satisfy the assumption of Theorem 1, the controller parameters are chosen to be $g_0 = 1$, $g_1 = 3$, $g_2 = 3$, $g_3 = 1$, $\alpha = 0.1$, and $R = I$. The system (37)–(41) with the adaptive controller (42) and (43) connected in feedback is simulated with the initial conditions $k(0) = 0$, $x(0) = 6$, $\dot{x}(0) = 4$, $\ddot{x}(0) = -2$, and $x(0) = 3$. The time history of the state is shown in figure 8. The adaptive controller stabilizes the origin. Time histories of the adaptive parameter $k(t)$ and the control signal $u(t)$ are shown in figure 9. The adaptive parameter converges to approximately 40.0.

8. Conclusions

In this paper, we presented a parameter-monotonic adaptive controller for a class of vector non-linear time-varying systems with full-state feedback.

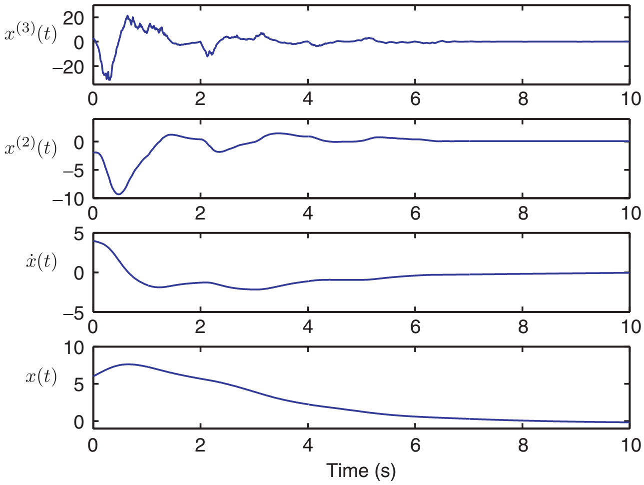

Figure 8. Time histories for the states of the 4th-order non-linear time-varying system with the adaptive controller (42) and (43). The origin of the open-loop system is unstable, and the adaptive controller stabilizes the origin.

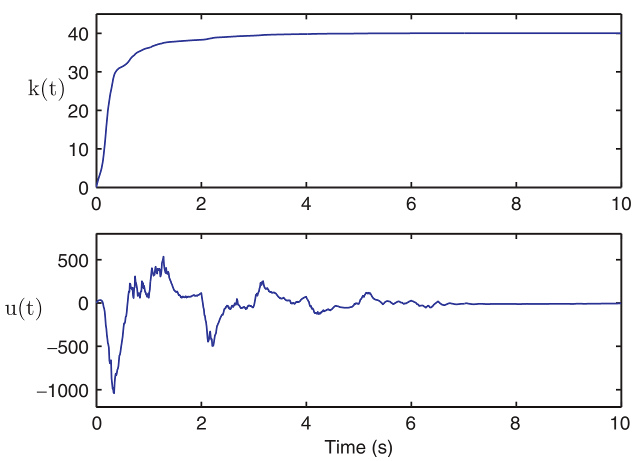

Figure 9. Time histories of the adaptive parameter $k(t)$ and the control signal $u(t)$. The adaptive parameter converges to approximately 40.0.

The adaptive controller was proven to stabilize nth-order vector non-linear time-varying systems with state-and-time-dependent uncertainties. Future research includes extensions to 2nd-order vector non-linear time-varying systems where the non-linear maps are only required to be lower bounded. In addition, adaptive stabilization of non-linear time-varying systems with unmatched uncertainty is an important open problem.

Appendix A: preliminary results

In this appendix, we provide several useful results regarding matrices in controllable canonical form and block controllable canonical form. The following result, given in Betser *et al.* (1995), concerns the solution to the Lyapunov equation for a matrix in controllable canonical form.

Lemma A.1: *Consider the controllable canonical form*

$$\mathcal{A} \triangleq \begin{bmatrix} -d_{n-1} & -d_{n-2} & \cdots & -d_1 & -d_0 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix},$$

and assume that $\mathcal{A}$ is asymptotically stable. Let $P \in \mathbb{R}^{n\times n}$ be the positive-definite solution to the Lyapunov equation $\mathcal{A}^{\mathrm{T}}P + P\mathcal{A} = -Q$, where $Q \in \mathbb{R}^{n\times n}$ is positive definite. Let p_1 denote the first column of P. Then p_1 satisfies

$$2DH\,Dp_1 = q,$$

where

$$H \triangleq \begin{bmatrix} d_{n-1} & 1 & 0 & 0 & \cdots & 0 & 0 \\ d_{n-3} & d_{n-2} & d_{n-1} & 1 & \cdots & 0 & 0 \\ d_{n-5} & d_{n-4} & d_{n-3} & d_{n-2} & \cdots & 0 & 0 \\ d_{n-7} & d_{n-6} & d_{n-5} & d_{n-4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d_1 & d_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & d_0 \end{bmatrix}$$

is the Hurwitz matrix of the characteristic polynomial of $\mathcal{A}$, $D \triangleq \mathrm{diag}(1, -1, 1, -1, \ldots)$, and

$$q \triangleq \begin{bmatrix} \sum_{1\le i,j\le n,\, i+j=2} (-1)^{i-1} Q_{i,j} \\ \sum_{1\le i,j\le n,\, i+j=4} (-1)^{i-2} Q_{i,j} \\ \vdots \\ \sum_{1\le i,j\le n,\, i+j=2n} (-1)^{i-n} Q_{i,j} \end{bmatrix}. \tag{A.1}$$

The following lemma provides asymptotic properties for a matrix in controllable canonical form whose entries depend linearly on a real parameter k.

Lemma A.2: *Let $g(s) \triangleq g_{n-1}s^{n-1} + g_{n-2}s^{n-2} + \cdots + g_0$ be a Hurwitz polynomial, where $g_{n-i} > 0$, and define*

$$A_s(k) \triangleq \begin{bmatrix} -khg_{n-1} & -khg_{n-2} & \cdots & -khg_1 & -khg_0 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

where $k \in \mathbb{R}$ and $h > 0$. Then, the following statements hold.

(i) *There exists* $k_1 > 0$ *such that, for all* $k \geq k_1$, $A_s(k)$ *is asymptotically stable.*

(ii) *For every positive-definite* $Q \in \mathbb{R}^{n\times n}$, *there exists* $P : \mathbb{R} \to \mathbb{R}^{n\times n}$ *such that each entry of* P *is a real rational function, and for all* $k \geq k_1$, $P(k)$ *is positive definite and satisfies*

$$A_s^{\mathrm{T}}(k)P(k) + P(k)A_s(k) = -Q. \tag{A.2}$$

(iii) *Let* $p_1(k)$ *denote the first column of* $P(k)$. *Then* $\lim_{k\to\infty} p_1(k) = 0$.

Proof: Define the Hurwitz matrix associated with the characteristic polynomial of $A_s(k)$ by

$$H(k) \triangleq \begin{bmatrix} khg_{n-1} & 1 & 0 & 0 & \cdots & 0 & 0 \\ khg_{n-3} & khg_{n-2} & khg_{n-1} & 1 & \cdots & 0 & 0 \\ khg_{n-5} & khg_{n-4} & khg_{n-3} & khg_{n-2} & \cdots & 0 & 0 \\ khg_{n-7} & khg_{n-6} & khg_{n-5} & khg_{n-4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & khg_1 & khg_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & khg_0 \end{bmatrix}.$$

The Hurwitz stability conditions for the characteristic polynomial of $A_s(k)$ are polynomials in k given by

$$\Lambda_1(k) \triangleq khg_{n-1} > 0,$$

$$\Lambda_2(k) \triangleq \begin{vmatrix} khg_{n-1} & 1 \\ khg_{n-3} & khg_{n-2} \end{vmatrix} > 0,$$

$$\Lambda_3(k) \triangleq \begin{vmatrix} khg_{n-1} & 1 & 0 \\ khg_{n-3} & khg_{n-2} & khg_{n-1} \\ khg_{n-5} & khg_{n-4} & khg_{n-3} \end{vmatrix} > 0,$$

$$\vdots$$

$$\Lambda_n(k) \triangleq \begin{vmatrix} \Lambda_3 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ & & khg_2 \\ 0 & \cdots\ 0 & khg_0 \end{vmatrix} > 0.$$

For sufficiently large k, the Hurwitz conditions are satisfied since $g(s)$ is Hurwitz with positive leading coefficient. Therefore, there exists $k_1 > 0$ such that, for all $k \geq k_1$, the matrix $A_s(k)$ is asymptotically stable. Then, for all $k \geq k_1$, (A2) has the unique solution

$$P(k) \triangleq -\mathrm{vec}^{-1}\left[(A_s^{\mathrm{T}}(k) \oplus A_s^{\mathrm{T}}(k))^{-1}\mathrm{vec}Q\right] = \int_0^\infty e^{A_s^{\mathrm{T}}(k)\tau} Q e^{A_s(k)\tau}\mathrm{d}\tau.$$

Then the entries of $P(k)$ are real rational functions, and for all $k \geq k_1$, $P(k)$ is positive definite.

Now, we consider the asymptotic properties of $p_1(k)$. For all $k \geq k_1$, the inverse of the Hurwitz matrix exists and can be expressed as

$$H^{-1}(k) = \frac{1}{\det(H(k))} \times \begin{bmatrix} [H(k)]_{1,1} & -[H(k)]_{2,1} & \cdots & (-1)^{n+1}[H(k)]_{n,1} \\ -[H(k)]_{1,2} & [H(k)]_{2,2} & \cdots & (-1)^{n+2}[H(k)]_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}[H(k)]_{1,n} & (-1)^{n+2}[H(k)]_{2,n} & \cdots & [H(k)]_{n,n} \end{bmatrix},$$

where $[H(k)]_{i,j}$ is the (i,j)th minor of $H(k)$. The determinant of $H(k)$ is a degree n polynomial in k, while $[H(k)]_{i,j}$ is a polynomial in k of degree not exceeding $n-1$. Therefore,

$$\lim_{k\to\infty} H^{-1}(k) = 0.$$

Using Lemma A.1 we obtain

$$\lim_{k\to\infty} p_1(k) = \lim_{k\to\infty} \frac{1}{2} D^{-1}H^{-1}(k)D^{-1}q = 0,$$

where q is determined from Q using (A.1) of Lemma A.1. □

The next result, which is an extension of Lemma A.2, provides asymptotic properties for a matrix in block controllable canonical form whose entries depend linearly on a real parameter k.

Lemma A.3: *Let* $g(s) \triangleq g_{n-1}s^{n-1} + g_{n-2}s^{n-2} + \cdots + g_0$ *be a Hurwitz polynomial, where* $g_{n-1} > 0$, *and define*

$$A_s(k) \triangleq \begin{bmatrix} -kg_{n-1}H & -kg_{n-2}H & \cdots & -kg_1H & -kg_0H \\ I_m & 0 & & 0 & 0 \\ 0 & I_m & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & I_m & 0 \end{bmatrix},$$

where $k \in \mathbb{R}$ *and* $H \in \mathbb{R}^{m\times m}$ *is positive definite. Then, the following statements hold.*

(i) *There exists* $k_1 > 0$ *such that, for all* $k \geq k_1$, $A_s(k)$ *is asymptotically stable.*

(ii) *For every positive-definite* $Q \in \mathbb{R}^{n\times n}$, *there exists* $P : \mathbb{R} \to \mathbb{R}^{mn\times mn}$ *such that each entry of* P *is a real rational function, and, for all* $k \geq k_1$, $P(k)$ *is positive definite and satisfies*

$$A_s^{\mathrm{T}}(k)P(k) + P(k)A_s(k) = -Q \otimes I_m. \tag{A.3}$$

(iii) *Let* $P_1(k)$ *denote the first* m *columns of* $P(k)$. *Then* $\lim_{k\to\infty} P_1(k) = 0$.

(iv) *For every* $\alpha > 0$, *there exists* $k_2 \geq k_1$ *such that, for all* $k \geq k_2$, $\alpha P(k) - (\partial P(k)/\partial k)$ *is positive definite.*

Proof: First, we show properties (i) and (ii). Since H is positive definite, there exists $U \in \mathbb{R}^{m\times m}$ such that

$$H = U^{\mathrm{T}} \Lambda U, \tag{A.4}$$

where $U^{\mathrm{T}} = \mathrm{U}^{-1}$, $\Lambda \triangleq \mathrm{diag}(h_1, h_2, \ldots, h_m)$, and $h_1, \ldots, h_m > 0$.

For $i = 1, \ldots, m$, define

$$\tilde{A}_i(k) \triangleq \begin{bmatrix} -kh_i g_{n-1} & -kh_i g_{n-2} & \cdots & -kh_i g_1 & -kh_i g_0 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Lemma A.2 implies that, for $i = 1, \ldots, m$, there exists $\tilde{k}_i > 0$ such that for all $k \geq \tilde{k}_i$, $\tilde{A}_i(k)$ is asymptotically stable. Furthermore, for all positive-definite Q, Lemma A.2 implies that for $i = 1, \ldots, m$, there exists $\tilde{P}_i : \mathbb{R} \to \mathbb{R}^{n\times n}$ such that each entry of $\tilde{P}_i$ is a real rational function, and, for all $k \geq \tilde{k}_i$, $\tilde{P}_i(k)$ is positive definite and satisfies

$$\tilde{A}_i^{\mathrm{T}}(k)\tilde{P}_i(k) + \tilde{P}_i(k)\tilde{A}_i(k) = -Q.$$

Next, define $\tilde{A}(k) \triangleq \mathrm{diag}(\tilde{A}_1(k), \ldots, \tilde{A}_m(k))$, $\tilde{P}(k) \triangleq \mathrm{diag}(\tilde{P}_1(k), \ldots, \tilde{P}_m(k))$, and $k_1 \triangleq \max(\tilde{k}_1, \ldots, \tilde{k}_m)$. Therefore, for all $k \geq k_1$, $\tilde{A}(k)$ is asymptotically stable and satisfies

$$\tilde{A}^{\mathrm{T}}(k)\tilde{P}(k) + \tilde{P}(k)\tilde{A}(k) = -I_m \otimes Q. \tag{A.5}$$

For $i = 1, 2, \ldots, mn$, define $e_i \triangleq [0_{1\times(i-1)}\ 1\ 0_{1\times(mn-i)}]^{\mathrm{T}}$. For $i = 1, \ldots, n$, define

$$V_i \triangleq \begin{bmatrix} e_i & e_{i+n} & e_{i+2n} & \ldots & e_{i+(m-1)n} \end{bmatrix},$$

and define the permutation matrix

$$V \triangleq \begin{bmatrix} V_1 & \ldots & V_n \end{bmatrix}.$$

Note that $V^{\mathrm{T}} = V^{-1}$. Also note that $V^{\mathrm{T}}(I_m \otimes Q)V = Q \otimes I_m$.

Now consider the similarity transformation

$$\bar{A}(k) \triangleq V^{\mathrm{T}}\tilde{A}(k)V = \begin{bmatrix} -kg_{n-1}\Lambda & -kg_{n-2}\Lambda & \cdots & -kg_1\Lambda & -kg_0\Lambda \\ I_m & 0 & & 0 & 0 \\ 0 & I_m & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & I_m & 0 \end{bmatrix}. \tag{A.6}$$

Pre-multiplying (A5) by V^{T} and post-multiplying by V yields

$$\bar{A}^{\mathrm{T}}(k)\bar{P}(k) + \bar{P}(k)\bar{A}(k) = -Q \otimes I_m, \tag{A.7}$$

where $\bar{P}(k) \triangleq V^{\mathrm{T}}\tilde{P}(k)V$.

Next, defining $W \triangleq I_m \otimes U$, it follows from (A.4) and (A6) that $W^{\mathrm{T}}\bar{A}(k)W = A_{\mathrm{s}}(k)$. Therefore, pre-multiplying (A7) by W^{T} and post-multiplying by W yields

$$A_{\mathrm{s}}^{\mathrm{T}}(k)P(k) + P(k)A_{\mathrm{s}}(k) = -Q \otimes I_m, \tag{A.8}$$

where $P(k) \triangleq W^{\mathrm{T}}\bar{P}(k)W$.

Therefore, for all $k \geq k_1$, $A_{\mathrm{s}}(k)$ is asymptotically stable and there exists $P : \mathbb{R} \to \mathbb{R}^{mn\times mn}$, such that each entry of P is a real rational function, and for all $k \geq k_1$, $P(k)$ is positive definite and satisfies (A8).

Next, we show property (iii). For $i = 1, \ldots, m$, write

$$\tilde{P}_i(k) = \begin{bmatrix} \tilde{p}_{i,11}(k) & \tilde{p}_{i,12}(k) & \tilde{p}_{i,13}(k) & \cdots & \tilde{p}_{i,1n}(k) \\ \tilde{p}_{i,12}(k) & \tilde{p}_{i,22}(k) & \tilde{p}_{i,23}(k) & \cdots & \tilde{p}_{i,2n}(k) \\ \tilde{p}_{i,13}(k) & \tilde{p}_{i,23}(k) & \tilde{p}_{i,33}(k) & \cdots & \tilde{p}_{i,3n}(k) \\ \vdots & & & \ddots & \vdots \\ \tilde{p}_{i,1n}(k) & \tilde{p}_{i,2n}(k) & \tilde{p}_{i,3n}(k) & \cdots & \tilde{p}_{i,nn}(k) \end{bmatrix}.$$

Since $\tilde{P}(k) = \mathrm{diag}(\tilde{P}_1(k), \ldots, \tilde{P}_m(k))$ and $\bar{P}(k) = V^{\mathrm{T}}\tilde{P}(k)V$, it follows that,

$$\bar{P}(k) = V^{\mathrm{T}}\tilde{P}(k)V = \begin{bmatrix} \bar{P}_{11}(k) & \bar{P}_{12}(k) & \cdots & \bar{P}_{1n}(k) \\ \bar{P}_{12}(k) & \bar{P}_{22}(k) & \cdots & \bar{P}_{2n}(k) \\ \vdots & & \ddots & \vdots \\ \bar{P}_{1n}(k) & \bar{P}_{2n}(k) & \cdots & \bar{P}_{nn}(k) \end{bmatrix}, \tag{A.9}$$

where, for $i = 1, \ldots, n$ and $j = \mathrm{i}, \ldots, n$, $\bar{P}_{ij}(k) \triangleq \mathrm{diag}(\tilde{p}_{1,ij}(k), \tilde{p}_{2,ij}(k), \ldots, \tilde{p}_{m,ij}(k))$. Lemma A.2 implies that, for $l = 1, \ldots, m$ and $j = 1, \ldots, n$, $\lim_{k\to\infty} \tilde{p}_{l,1j}(k) = 0$. Therefore, for $j = 1, \ldots, n$, $\lim_{k\to\infty} \bar{P}_{1j}(k) = 0$.

Since $P(k) = W^{\mathrm{T}}\bar{P}(k)W$, it follows from (A9) that

$$P(k) = \begin{bmatrix} U^{\mathrm{T}}\bar{P}_{11}(k)U & U^{\mathrm{T}}\bar{P}_{12}(k)U & \cdots & U^{\mathrm{T}}\bar{P}_{1n}(k)U \\ U^{\mathrm{T}}\bar{P}_{12}(k)U & U^{\mathrm{T}}\bar{P}_{22}(k)U & \cdots & U^{\mathrm{T}}\bar{P}_{2n}(k)U \\ \vdots & & \ddots & \vdots \\ U^{\mathrm{T}}\bar{P}_{1n}(k)U & U^{\mathrm{T}}\bar{P}_{2n}(k)U & \cdots & U^{\mathrm{T}}\bar{P}_{nn}(k)U \end{bmatrix}. \tag{A.10}$$

Since, for $j = 1, \ldots, n$, $\lim_{k\to\infty} \bar{P}_{1j}(k) = 0$, it follows from (A10) that $\lim_{k\to\infty} P_1(k) = 0$, where $P_1(k)$ denotes the first m columns of $P(k)$.

Next, we show property (iv). Let $\alpha > 0$. Multiplying (A3) by $-e^{-\alpha k}$ yields

$$-e^{-\alpha k}A_{\mathrm{s}}^{\mathrm{T}}(k)P(k) - e^{-\alpha k}P(k)A_{\mathrm{s}}(k) = e^{-\alpha k}(Q \otimes I_m). \tag{A.11}$$

Differentiating (A11) with respect to k yields

$$A_s^T(k)\hat{P}(k) + \hat{P}(k)A_s(k) = \hat{Q}(k), \tag{A.12}$$

where

$$\begin{aligned} \hat{P}(k) &\triangleq \alpha P(k) - \frac{\partial P(k)}{\partial k}, \\ \hat{Q}(k) &\triangleq -\alpha(Q \otimes I_m) + \Delta^T E_1^T P(k) + P(k)E_1\Delta, \\ \Delta &\triangleq \begin{bmatrix} -g_{n-1}H & \cdots & -g_0 H \end{bmatrix}. \end{aligned} \tag{A.13}$$

Since

$$\begin{aligned} 0 \le &\left[\sqrt{\frac{\alpha}{2}}(Q \otimes I_m)^{1/2} - \sqrt{\frac{2}{\alpha}}(Q \otimes I_m)^{-1/2}P(k)E_1\Delta\right]^T \\ &\times \left[\sqrt{\frac{\alpha}{2}}(Q \otimes I_m)^{1/2} - \sqrt{\frac{2}{\alpha}}(Q \otimes I_m)^{-1/2}P(k)E_1\Delta\right], \end{aligned}$$

it follows that

$$\begin{aligned} \Delta^T E_1^T P(k) + P(k)E_1\Delta &\le \frac{\alpha}{2}(Q \otimes I_m) \\ &+ \frac{2}{\alpha}\Delta^T E_1^T P(k)(Q \otimes I_m)^{-1}P(k)E_1\Delta. \end{aligned} \tag{A.14}$$

Combining (A13) and (A14) yields

$$\begin{aligned} \hat{Q}(k) &\le -\frac{\alpha}{2}(Q \otimes I_m) + \frac{2}{\alpha}\Delta^T E_1^T P(k)(Q \otimes I_m)^{-1}P(k)E_1\Delta \\ &\le -\frac{\alpha}{2}(Q \otimes I_m) + \frac{2}{\alpha}\Delta^T P_1^T(k)(Q \otimes I_m)^{-1}P_1(k)\Delta. \end{aligned}$$

Since $\lim_{k\to\infty} P_1(k) = 0$, let $k_2 \ge k_1$ be such that, for all $k \ge k_2$,

$$\frac{2}{\alpha}\Delta^T P_1^T(k)(Q \otimes I_m)^{-1}P_1(k)\Delta < \frac{\alpha}{2}(Q \otimes I_m),$$

and thus, for all $k \ge k_2$, $\hat{Q}(k)$ is negative definite. Then, it follows from (A12) that, for all $k \ge k_2$,

$$A_s^T(k)\hat{P}(k) + \hat{P}(k)A_s(k) < 0.$$

Since $A_s(k)$ is asymptotically stable, $\hat{P}(k) = \alpha P(k) - (\partial P(k)/\partial k)$ is positive definite. □

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