

Nonminimum-Phase Zeros

MUCH TO DO ABOUT NOTHING

Classical Control Revisited



In the popular literature there is a certain fascination with the concept of zero [1]–[3]. While today the inconspicuous 0 is taken for granted, the situation was different in the distant past. For example, the Romans had no symbol for 0, a fact memorialized by the jump from 1 B.C. to 1 A.D., a convention instituted in 531 A.D. [4, p. 91]. In contrast, the Mayans had a symbol for zero, and the first day of each Mayan month was day zero [3, p. 18]. The modern zero of mathematics slowly earned its membership in the club of numbers through Indian mathematics, although this acceptance was achieved only through a tortuous process that spanned centuries [3].

A conceptual impediment to the acceptance of zero is the difficulty in understanding the ratio $1/0$. Presumably, this ratio is infinity or ∞ , a much more challenging concept. That 0 and ∞ are close cousins casts suspicion on zero as a valid number. Even in modern times, the zero appears begrudgingly on your telephone keypad *after* the 9. In Europe, the ground floor in a building is routinely labeled 0, and thus the meaning of floor -1 is unambiguous, whereas, in the

United States, there is no floor 0, and negative floor numbers are rarely used. Despite the human reluctance to admit zero as an authentic number, it is as difficult to imagine mathematics today without zero as it is to imagine technology without the wheel and axle.

Although the number zero is well known, the system-theoretic concept of a *system zero* is virtually unknown outside of dynamics and control theory. The purpose of this article is to illuminate the critical role of system zeros in control-system performance for the benefit of a wide audience both inside and outside the control systems community.

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POLES AND ZEROS

Setting aside the notion of zero for the moment, the idea of a *pole* is one of the most fundamental concepts in system theory. Consider the continuous-time single-input, single-output (SISO) system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t),\end{aligned}$$

with scalar-valued input $u(t)$ and scalar-valued output $y(t)$. The $n \times n$ dynamics matrix A represents the dynamics of this system, while the $n \times 1$ column vector B represents the effect of the actuator, and the $1 \times n$ row vector C represents the response of the sensor. Together, (A, B, C) defines the *input-output dynamics* of the system. Taking Laplace transforms yields

$$\hat{y}(s) \triangleq \int_0^\infty y(t)e^{-st} dt = C(sI - A)^{-1}x(0) + G(s)\hat{u}(s),$$

where $C(sI - A)^{-1}x(0)$ is the *initial condition response* of the system and the *transfer function* G is given by

$$G(s) = C(sI - A)^{-1}B.$$

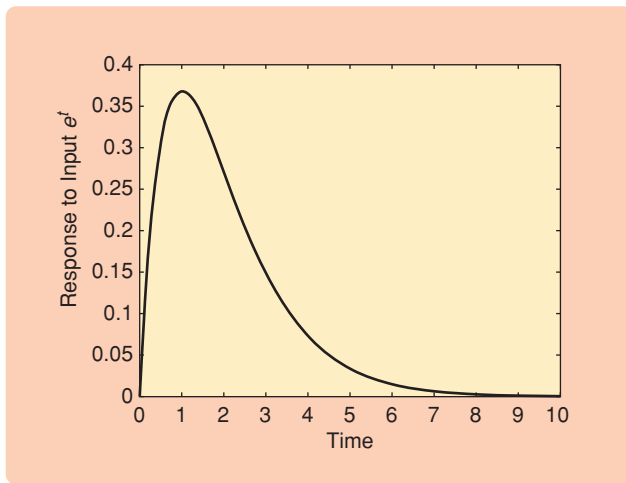


FIGURE 1 Bounded response of a nonminimum-phase system to an unbounded input. The response of the transfer function $G(s) = (s - 1)/(s + 1)^2$ to the unbounded input e^t is bounded and converges to zero due to the open-right-half-plane zero $z = 1$.

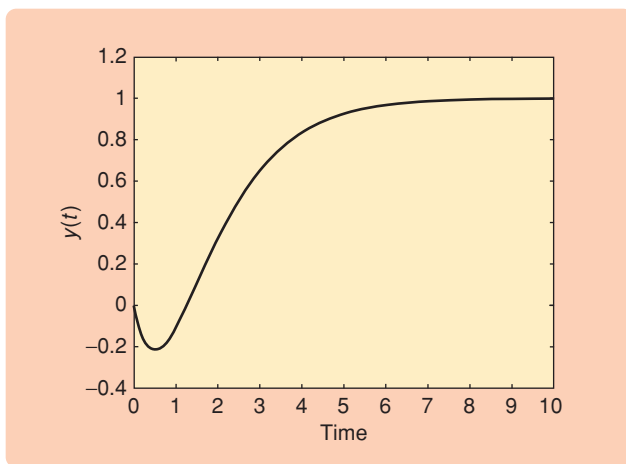


FIGURE 2 “Wrong” initial direction with direction reversal. For the system $G(s) = -(s - 1)/(s + 1)^2$, which has one positive zero, the step response initially moves in the “wrong” direction. Although the operator might be tempted to abandon the control strategy, the system eventually reverses course and reaches the desired steady state. In this case, patience is a virtue.

To examine the poles and zeros of G , write G as

$$G(s) = \frac{N(s)}{D(s)},$$

where the denominator polynomial is $D(s) \triangleq \det(sI - A)$, and the numerator polynomial is

$$N(s) \triangleq C \operatorname{adj}(sI - A)B.$$

Here, $\operatorname{adj}(sI - A)$ is the *adjugate* of $sI - A$ [5, p. 20], [6, p. 42]. The roots of N are the *zeros* of G , while the roots of D are the *poles* of G . This discussion assumes that N and D have no common roots, which is the case if and only if (A, B, C) is a controllable and observable triple. In this case, the n poles of G are precisely the eigenvalues of A .

The poles of G determine whether G is stable or unstable, as well as the decay rate and oscillation frequencies of the initial condition response. The poles of G do not depend on either the input matrix B or the output matrix C . In contrast, the zeros of G are determined by the dynamics matrix A as well as B and C . Hence, the zeros of G depend on the physical placement of the sensors and actuators relative to the underlying dynamics. The concept of a zero distinguishes control theory from dynamical systems theory.

In this article, the term transfer function refers to a proper ratio of polynomials. The transfer function G is strictly proper if and only if $G(\infty) = 0$, whereas G is exactly proper if and only if $G(\infty) \neq 0$. For results that are restricted to either strictly proper or exactly proper transfer functions, the distinction is made explicitly.

THE BLOCKING EFFECT OF A ZERO

What exactly do zeros do, and how do zeros relate to the number zero? A zero of a transfer function is a root of the numerator polynomial of the transfer function, and thus is a real or complex number. When the transfer function is *asymptotically stable*, that is, when all of the roots of the denominator polynomial are in the open left half plane, each zero has a specific effect on the asymptotic response of the transfer function for certain inputs.

To illustrate the effect of a zero on the response of an asymptotically stable transfer function G , consider a step input, so that the response of G approaches a steady state value. If, moreover, the number 0 is a zero of the transfer function G , that is, $G(0) = 0$, then the steady state response of G is zero, that is, the dc gain of the system is zero.

Next, suppose that the input to G is sinusoidal, that is, harmonic, with frequency ω . Then, the asymptotic response is also harmonic with the same frequency of oscillation; this response is the *harmonic steady-state response*. If, moreover, the imaginary number $j\omega$ is a zero of the transfer function G , that is, $G(j\omega) = 0$, then the amplitude of the harmonic steady-state response is zero, and thus the response converges to zero.

Finally, suppose that the input to G is the unbounded signal e^t , in which case one expects the response of the system to be unbounded as well. If, however, the number 1 is a zero of G , that is, $G(1) = 0$, then the response of the system is not only bounded but converges to zero (see Figure 1).

In general, each zero blocks a specific input signal multiplied by an arbitrary constant. In the case of a *non-minimum-phase zero*, that is, an open-right-half-plane zero, the blocked signal is unbounded. The above observations follow from the final value theorem (after all unstable poles of the input are canceled by nonminimum-phase zeros of the system), and, since the system is assumed to be asymptotically stable, are also valid for all nonzero initial conditions of all stabilizable and detectable state-space realizations.

INITIAL UNDERSHOOT DUE TO AN ODD NUMBER OF POSITIVE ZEROS

The effect of *positive* (that is, *real* open-right-half-plane) zeros is evident in the step response of a system. In particular, Figure 2 shows a step response that departs in the nonasymptotic direction; this phenomenon, which is equivalent to initial error growth, is called *initial undershoot*. Note that initial undershoot is defined only for a step response whose initial and asymptotic values are different. A classical result, proved in [7]–[9], states that the step response of an asymptotically stable, strictly proper transfer function exhibits initial undershoot if and only if the system has an odd number of positive zeros.

Undershoot can have significant implications in practice. Suppose, for example, that an economic plan is implemented to boost the economy. In particular, suppose that a central bank implements a step decrease in short-term interest rates, which produces the undesirable effect of initially decreasing gross domestic product, as shown in Figure 2. Supporters of the plan might decide to abandon the plan before its ultimate effect is known, whereas critics of the plan might feel vindicated in having opposed the plan. In reality, however, the input-output dynamics of the economy might have an odd number of positive zeros, in which case the appropriate action is to wait for the system to reverse direction.

On the other hand, as illustrated in Figure 3, it might be the case that the economy initially moves in the “correct” direction, which would suggest that the plan is appropriate. However, the system in Figure 3 eventually reverses course and converges to a negative value, revealing that the plan was inappropriate. To complicate matters even more, the step response of a system with multiple positive zeros can exhibit multiple direction reversals. For example, the step response of a system with two positive zeros, as illustrated in Figure 4, initially moves in the “correct” direction, reverses course to move in the “wrong” direction, and then reverses course yet again to move in the “correct” direction.

An everyday example of positive zeros arises when driving a car backwards, a skill that every young driver must

master. The driver initially moves in one direction, for example, to the right, and later reverses direction, moving to the left. To see this, consider the four-wheeled car model

$$\dot{x} = v \cos \theta, \quad (1)$$

$$\dot{y} = v \sin \theta, \quad (2)$$

$$\dot{\theta} = \frac{v}{\ell} \tan \phi, \quad (3)$$

given in [10] and [11], where (x, y) is the position of the center of the rear wheels, θ is the angle between the car’s longitudinal axis and the x -axis, v is the translational velocity of the rear wheels, ℓ is the distance between the front and rear wheels, and the control input ϕ is the steering

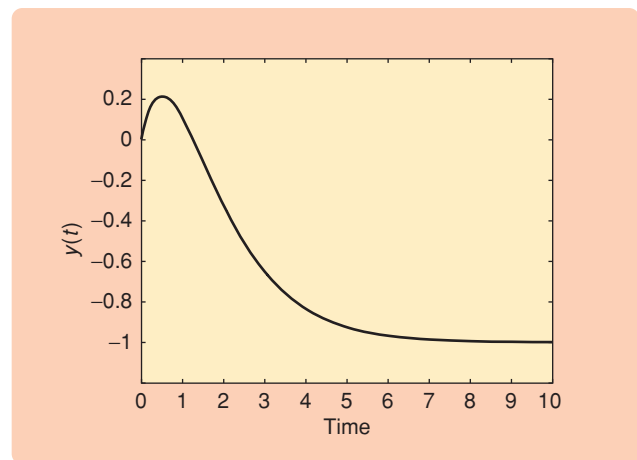


FIGURE 3 “Correct” initial direction with direction reversal. For the system $G(s) = (s - 1)/(s + 1)^2$, which has one positive zero, the step response initially moves in the “correct” direction. However, after more time passes, the system reverses and moves in the “wrong” direction. In this case, initial optimism proves to be unfounded.

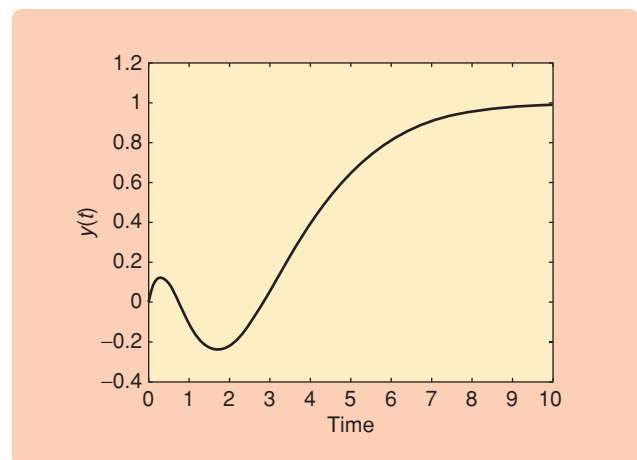


FIGURE 4 “Correct” initial direction with multiple direction reversals. For the system $G(s) = (s - 1)^2/(s + 1)^3$, which has two positive zeros, the step response initially moves in the “correct” direction, reverses course to move in the “wrong” direction, and then reverses course again to move in the “correct” direction. In this case, patience and fortitude are called for.

angle measured from the car's longitudinal axis. The four-wheeled car model contains the nonholonomic constraint

$$\dot{x} \sin \theta - \dot{y} \cos \theta = 0$$

and thus has an uncontrollable linearization about the zero equilibrium. However, if we control the translational acceleration of the rear wheels, that is,

$$\dot{v} = u, \quad (4)$$

where u is a control input, then linearizing (1)–(4) about the equilibrium $(x_0, y_0, \theta_0, v_0)$ yields

$$\begin{aligned} \delta \dot{x} &= \cos(\theta_0) \delta v - v_0 \sin(\theta_0) \delta \theta, \\ \delta \dot{y} &= \sin(\theta_0) \delta v + v_0 \cos(\theta_0) \delta \theta, \\ \delta \dot{\theta} &= \frac{v_0}{\ell} \phi, \\ \delta \dot{v} &= u, \end{aligned}$$

which is controllable for all constant nonzero v_0 .

Now assume that the car has a constant nonzero speed v_0 and $\phi = 0$, and assume that the car's longitudinal axis is initially parallel to the x -axis, that is, $\theta_0 = 0$. Furthermore, the output of the system is the y -axis position of the center of the front wheels, given by $y_{\text{out}} = y + \ell \sin \theta$, which has the linearization $y_{\text{out}} = \delta y + \ell \delta \theta$. Then the transfer function from the steering angle control input ϕ to the lateral position y_{out} is

$$G(s) = v_0 \frac{s + v_0/\ell}{s^2}.$$

If the car is driving in reverse, then G has exactly one positive zero, namely, $z = -v_0/\ell$. Thus, the lateral response to a step input in the steering angle exhibits initial undershoot. A similar result appears in [12, p. 158]. The same effect occurs in some aircraft as can be seen from the step

response of the elevator deflection to pitch angle [13, p. 494]. These examples suggest that one source of nonminimum phase zeros is noncollocation, that is, physical separation, of sensing and actuation [14]–[20].

ZERO CROSSINGS DUE TO POSITIVE ZEROS

The result of [7]–[9] implies that the step response for a strictly proper transfer function having an even number of positive zeros does not exhibit initial undershoot. Nevertheless, the step response shown in Figure 4 for a system with two positive zeros exhibits two direction reversals and two zero crossings, where the term *zero crossing* refers to the situation in which a signal passes through the value of zero. In fact, we now show that if an asymptotically stable transfer function possesses at least one positive zero, then the step response of the system undergoes at least one zero crossing.

Let z denote a positive zero of the asymptotically stable transfer function G . The Laplace transform $\hat{y}(s)$ of the output $y(t)$ for a unit step input is given by

$$\hat{y}(s) = G(s)(1/s).$$

Setting $s = z$ yields $\hat{y}(z) = G(z)(1/z)$. Since $G(z) = 0$, it follows that $\hat{y}(z) = 0$, and thus

$$\int_0^{\infty} e^{-zt} y(t) dt = 0. \quad (5)$$

Since e^{-zt} is positive on $[0, \infty)$, it follows that $y(t)$ must cross zero on $(0, \infty)$. In addition, (5) implies that the weighted “negative area” and “positive area” associated with $y(t)$ are exactly equal. Note that (5) depends on z but does not depend on either the poles or the remaining zeros of G .

Whereas it follows from [7]–[9] that the step response of a strictly proper G with an odd number of positive zeros has initial undershoot and thus at least one zero crossing, it follows from (5) that at least one zero crossing occurs if G is proper and has at least one positive zero. As shown in Figure 4, the step response must possess at least two direction reversals if G has a nonzero even number of positive zeros.

Figures 3 and 4 suggest that the number of zero crossings is equal to the number of positive zeros. In fact, a statement that the number of zero crossings is equal to the number of positive zeros is given in [21, p. 174] and attributed to [22], which in turn attributes the result to [23]. However, a statement of this result does not appear in [23]. Furthermore, in the second edition of [24, p. 184], the result is attributed to [25] rather than [22]. However, the result given in [25] is more restrictive than the statement in [24], since [25] states that the number of zero crossings is equal to the number of positive zeros for strictly proper transfer functions with only real poles and zeros. In fact, the statement in [24] is incorrect for transfer functions with complex zeros. For example, the step response of a system with no positive zeros but two complex nonminimum-phase zeros

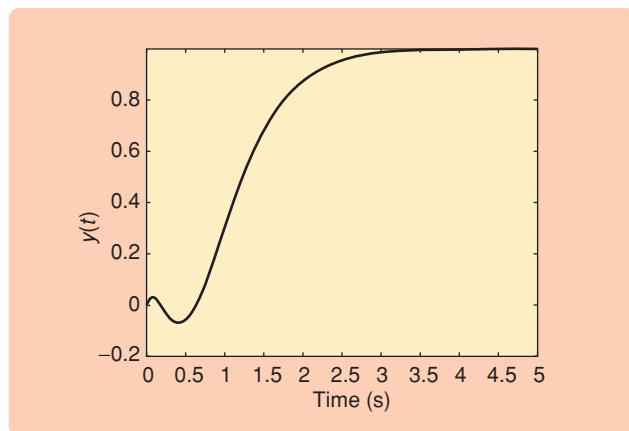


FIGURE 5 Nonmonotonic step response. The step response to the transfer function $G(s) = (s^2 - 10s + 27)/(s + 3)^3$, which has nonreal nonminimum-phase zeros but no positive zeros, has two zero crossings.

can exhibit two zero crossings, as shown in Figure 5. Numerical testing suggests that the number of zero crossings in the step response is greater than or equal to the number of positive zeros. A proof of this conjecture is open.

For a different system with two nonreal nonminimum-phase zeros, Figure 6 shows a nonmonotonic step response with no zero crossings. Furthermore, for yet another system with two nonreal nonminimum-phase zeros, Figure 7 shows a monotonic step response. Hence, the presence of nonminimum-phase zeros does not guarantee the existence of either zero crossings or direction reversals. However, it is shown in [26] that, for asymptotically stable, strictly proper systems with only real poles and real zeros, the number of extrema in the step response (not including $t = 0$) is greater than or equal to the number of zeros to the right of the rightmost pole.

OVERSHOOT DUE TO POSITIVE ZEROS

In addition to initial undershoot and zero crossings, the step response of an asymptotically stable transfer function can exhibit *overshoot*, that is, assume values both greater than and less than the asymptotic value of the step response. In fact, the step response of an asymptotically stable transfer function G exhibits overshoot if $G(s) - G(0)$ has at least one positive zero. To see this, let z be a positive zero of $G(s) - G(0)$. Applying to $G(s) - G(0)$ the same steps used to derive (5), it follows that (see also [27, pp. 213–214])

$$\int_0^{\infty} e^{-zt} [y(t) - y(\infty)] dt = 0. \quad (6)$$

Consequently, $y(t) - y(\infty)$ must change sign on $[0, \infty)$, and thus $y(t)$ overshoots its steady-state value $y(\infty) = \lim_{t \rightarrow \infty} y(t)$. This behavior depends on the positive zero z of $G(s) - G(0)$ but does not depend on any other details of G .

As a special case, consider a system whose step response converges to zero, that is, $y(\infty) = 0$, which arises in control systems with integral action. In this case, G exhibits overshoot if G has at least one positive zero.

Table 1 summarizes the results given above on initial undershoot, zero crossings, and overshoot in the step response of an asymptotically stable transfer function G .

UNDERSHOOT, OVERSHOOT, AND ZERO CROSSINGS IN SERVO SYSTEMS

We now specialize the results given thus far to servo systems. Feedback stabilization of an unstable plant unavoidably gives rise to nonminimum-phase zeros. To see this,

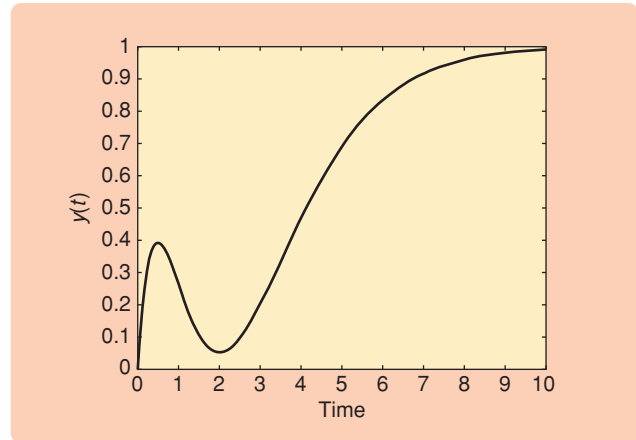


FIGURE 6 Nonmonotonic step response. The step response to the transfer function $G(s) = (2s^2 - s + 1)/(s + 1)^3$, which has two nonreal nonminimum-phase zeros, has two direction reversals, and thus is not monotonic. However, the step response does not exhibit any zero crossings.

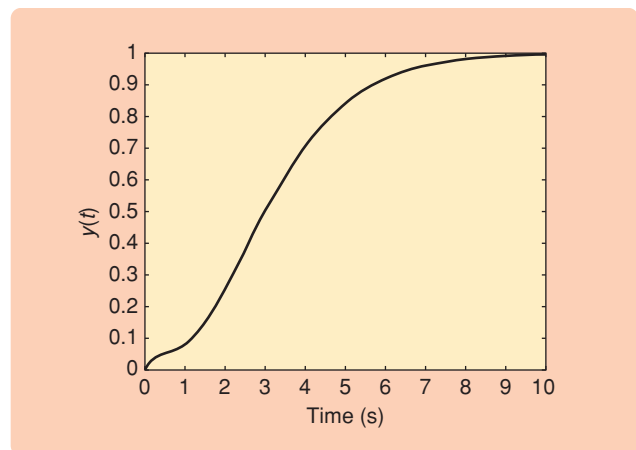


FIGURE 7 Monotonic step response. The step response to the transfer function $G(s) = (s^2 - s + 4)/(s + 1)^3$, which has two nonreal nonminimum-phase zeros, is monotonic.

TABLE 1 Initial undershoot, zero crossing, and overshoot in the step response of an asymptotically stable transfer function G . Results for exactly proper transfer functions are needed to analyze the sensitivity transfer function S , which arises in servo systems and which is not addressed by the classical theory.

	Initial Undershoot	Zero Crossing	Overshoot
Strictly Proper	If and only if G has an odd number of positive zeros	If G has at least one positive zero	If $G(s) - G(0)$ has at least one positive zero
Exactly Proper	If and only if $G(s) - G(\infty)$ has an odd number of positive zeros	If G has at least one positive zero	If $G(s) - G(0)$ has at least one positive zero

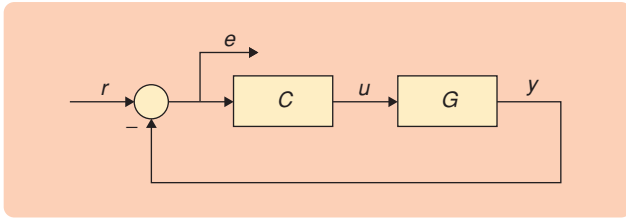


FIGURE 8 Standard servo problem with loop transfer function $L(s) = C(s)G(s)$. The zeros of the sensitivity transfer function $S = 1/(1 + L)$ from command r to error $e = r - y$ are the poles of $L(s)$.

consider the servo problem in Figure 8, where $L(s) = C(s)G(s) = N(s)/D(s)$ represents the *loop transfer function*, that is, the plant G cascaded with a proper feedback controller C under the assumption that $L(s)$ is strictly proper. Then the asymptotically stable closed-loop transfer function from the reference signal r to the error signal e is given by the *sensitivity transfer function* $S(s) = 1/(1 + L(s))$, that is, $\hat{e}(s) = S(s)\hat{r}(s)$. Since

$$S(s) = \frac{D(s)}{N(s) + D(s)}, \quad (7)$$

it follows that the zeros of S are precisely the poles of L . Therefore, if either the plant or the controller is unstable and no unstable pole/zero cancellation occurs (see the discussion below), so that L is also unstable, then the corresponding sensitivity transfer function S has nonminimum-phase zeros. These nonminimum-phase zeros tend to enhance performance by decreasing the magnitude of the sensitivity function (relative to the sensitivity transfer function that would be obtained if these zeros were not present) thereby reducing the transmission of specific signals.

Initial Undershoot

To analyze initial undershoot in the step response of a servo system, one might be tempted to apply the result of [7]–[9] to the sensitivity transfer function S and conclude that the step response of S exhibits initial undershoot if and only if S has an odd number of positive zeros (or equivalently, L has an odd number of positive poles). However, the result of [7]–[9] does not apply because the sensitivity transfer function S is not strictly proper. Nevertheless, “Initial Undershoot Revisited” extends the result of [7]–[9] to include exactly proper transfer functions.

Initial Undershoot Revisited

Initial undershoot describes the qualitative behavior of the step response of a transfer function. A single-input, single-output asymptotically stable transfer function $G(s)$ exhibits initial undershoot if its step response initially moves in the direction that is opposite to the direction of the asymptotic value. Initial undershoot is thus equivalent to initial error growth. In [7]–[9], it is shown that a strictly proper transfer function exhibits initial undershoot if and only if the transfer function has an odd number of positive zeros.

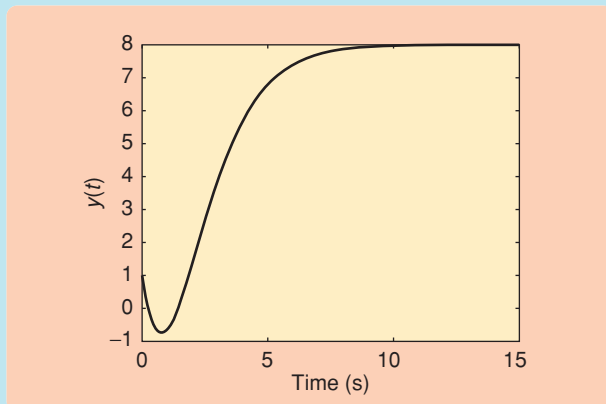


FIGURE S1 The step response of $G_1(s) = (s - 2)^2(s + 2)/(s + 1)^3$, which exhibits initial undershoot due to an odd number of positive zeros in $G_1(s) - G_1(\infty)$. In particular, $G_1(s) - G_1(\infty)$ has one negative zero at -2.0748 and one positive zero at 0.6748 . Note that G_1 exhibits initial undershoot even though G_1 itself has an even number of positive zeros.

To analyze the step response of a servo system, however, it is necessary to extend the definition and classification of initial undershoot to include exactly proper transfer functions, that is, transfer functions whose numerator and denominator polynomials have the same degree.

Let G be an asymptotically stable transfer function with relative degree $d \geq 0$, where $d = 0$ denotes the exactly proper case. Let $y(t)$ be the step response of G . The step response has the initial value $y(0^+) = G(\infty) = \lim_{s \rightarrow \infty} G(s)$ and the asymptotic value $y(\infty) = \lim_{t \rightarrow \infty} y(t) = G(0)$. Then $y(t)$ exhibits *initial undershoot* if

$$y^{(\rho)}(0^+) [y(\infty) - y(0^+)] < 0, \quad (S1)$$

where $\rho \triangleq \min(d, 1)$ and, by the initial value theorem, $y^{(\rho)}(0^+) \triangleq \lim_{s \rightarrow \infty} s^\rho [G(s) - G(\infty)]$. Note that $y(t)$ can exhibit initial undershoot only if the initial value differs from the final value, that is, $y(0^+) \neq y(\infty)$. In the strictly proper case, $d \geq 1$, and thus $\rho = d$ and $y(0^+) = 0$. Hence (S1) becomes

$$y^{(d)}(0^+) y(\infty) < 0,$$

which is the condition given in [7]–[9].

For example, consider the step response of the exactly proper transfer function

$$G_1(s) = \frac{(s - 2)^2(s + 2)}{(s + 1)^3},$$

Specifically, it is shown (with G replaced by S) that the step response of S with L strictly proper exhibits initial undershoot if and only if $S(s) - S(\infty) = S(s) - 1$ has an odd number of positive zeros. In terms of the error signal, initial undershoot means that, after a step servo command is introduced, the error initially grows and thus attains a larger value than the step difference due to the servo command.

Now, define the *complementary sensitivity transfer function* $T(s) = 1 - S(s) = L(s)/(1 + L(s))$ from r to y of the closed-loop system, that is, $\hat{y}(s) = T(s)\hat{r}(s)$, given by

$$T(s) = \frac{N(s)}{N(s) + D(s)}. \quad (8)$$

Thus, the step response of S exhibits initial undershoot if and only if T has an odd number of positive zeros. Furthermore, since L and T have the same zeros, the step response of S exhibits initial undershoot if and only if L has an odd number of positive zeros.

Note that T is strictly proper since C and thus L are strictly proper. Therefore, the step response of T exhibits initial undershoot if and only if T (and thus L) has an odd number of positive zeros.

Zero Crossings

A similar result can be used to show that the error $e(t)$ changes sign and thus has at least one zero crossing when L has at least one positive pole p . To see this, note that p is also a zero of S . Since $S(p) = 0$, applying to S the same steps used to derive (5), which is a valid procedure despite the fact that S is not strictly proper, yields

$$\int_0^{\infty} e^{-pt} e(t) dt = 0.$$

Thus, $e(t)$ must cross zero.

On the other hand, $y(t)$ has at least one zero crossing if L has at least one positive zero z . To see this, note that, since z is a zero of L , it follows that z is also a zero of T , and thus (5) is satisfied, which implies that $y(t)$ must cross zero.

Overshoot

The step response of a servo system can exhibit overshoot, that is, assume values both greater than and less than the asymptotic value. It follows from (6) that the output $y(t)$ of a servo system with a step command $r(t)$ exhibits overshoot if $T(s) - T(0)$ has at least one positive zero. Furthermore,

which exhibits initial undershoot, as shown in Figure S1, even though G_1 has an even number of positive zeros. On the other hand, the step response of the exactly proper transfer function

$$G_2(s) = \frac{(s-2)(s+2)^2}{(s+1)^3},$$

does not exhibit initial undershoot, as shown in Figure S2, even though G_2 has an odd number of positive zeros. Thus, the result of [7]–[9] is not valid for exactly proper transfer functions.

PROPOSITION

The step response $y(t)$ exhibits initial undershoot if and only if $G(s) - G(\infty)$ has an odd number of positive zeros.

PROOF

Let $H(s) = G(s) - G(\infty) = \beta N(s)/D(s)$, where N and D are monic polynomials, β is a real number, and H has relative degree ρ . Thus, $y^{(\rho)}(0^+) = \lim_{s \rightarrow \infty} s^\rho H(s) = \beta$. Next, note that $y(\infty) - y(0^+) = G(0) - G(\infty) = H(0) = \beta N(0)/D(0)$. Thus, (S1) is satisfied, that is, $y(t)$ exhibits initial undershoot, if and only if $\beta^2 N(0)/D(0) < 0$. Since D is Hurwitz, it follows that $D(0)$ is positive, and thus $y(t)$ exhibits initial undershoot if and only if $N(0)$ is negative. Note that $N(0)$ is the product of the negatives of the roots of N , and thus $y(t)$ exhibits initial undershoot if and only if N has an odd number of positive roots. \square

Now, it follows immediately from the proposition that the step response of $G_1(s) = (s-2)^2(s+2)/(s+1)^3$ exhibits initial

undershoot because $G_1(s) - G_1(\infty) = (-5s^2 - 7s + 7)/(s+1)^3$ has exactly one positive zero, whereas the step response of $G_2(s) = (s-2)(s+2)^2/(s+1)^3$ does not exhibit initial undershoot because $G_2(s) - G_2(\infty) = (-s^2 - 7s - 9)/(s+1)^3$ has no positive zeros.

Note that if G is strictly proper, then $G(\infty) = 0$, and the proposition specializes to the result presented in [7]–[9].

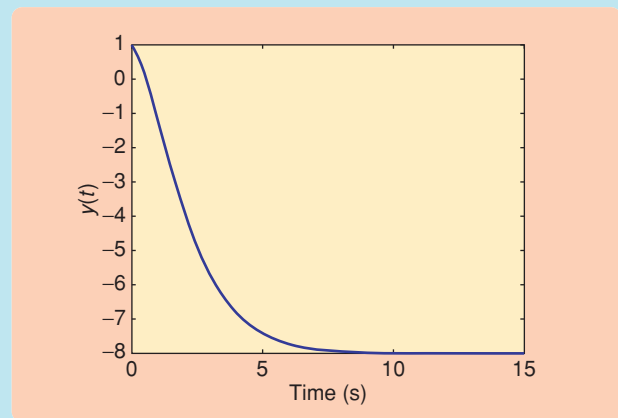


FIGURE S2 The step response of $G_2(s) = (s-2)(s+2)^2/(s+1)^3$, which does not exhibit initial undershoot because $G_2(s) - G_2(\infty)$ has an even number of positive zeros. In particular, $G_2(s) - G_2(\infty)$ has two negative zeros at -5.3028 and -1.6972 and no positive zeros. Note that G_2 does not exhibit initial undershoot even though G_2 itself has an odd number of positive zeros.

A conceptual impediment to the acceptance of zero is the difficulty in understanding the ratio 1/0.

applying to S the same steps used to derive (6), which again is a valid procedure despite the fact that S is not strictly proper, it follows that the error $e(t)$ to a step command $r(t)$ exhibits overshoot if $S(s) - S(0)$ has at least one positive zero. Note that $e(t) = r(t) - y(t)$, and thus $e(t)$ exhibits overshoot if and only if $y(t)$ exhibits overshoot. Therefore, $S(s) - S(0)$ has at least one positive zero if and only if $T(s) - T(0)$ has at least one positive zero.

As a special case, consider a system whose error $e(t)$ converges to zero, that is, $S(0) = 0$. In particular, $e(t)$ converges to zero if the controller C has integral action. In this case, $e(t)$ exhibits overshoot if L has at least one positive pole. To see this, note that it follows from (7) that a positive pole p of L is also a positive zero of the sensitivity $S(s) = S(s) - S(0)$, which implies that $e(t)$ overshoots its steady-state value, namely, zero. Consequently, $y(t)$ also overshoots its steady-state value, namely, the value of the step command.

Table 2 summarizes the results on initial undershoot, zero crossings, and overshoot for the servo system shown in Figure 8, where $u(t)$ is the unit step input, $y(t)$ is the output, and $e(t)$ is the error.

INVERTED PENDULUM ON A CART

To demonstrate the above results, try balancing a long stick in the palm of your hand, so that you (that is, your eye, brain, and arm) are stabilizing an unstable system. You can think of your arm as the force actuator and your arm as the “cart,” which provides a base for the stick. It turns out that the linearized transfer function from the force applied by your hand to the position of your hand has both a positive zero and a positive pole. Specifically, this transfer function is given by [28, p. 86]

$$G(s) = \frac{1}{M} \frac{(s - z)(s + z)}{s^2(s - p)(s + p)},$$

where p and z are the positive pole and positive zero, respectively, given by

$$p = \sqrt{\frac{g}{\ell} + \frac{mg}{M\ell}}, \quad z = \sqrt{\frac{g}{\ell}},$$

where m is the mass of the stick, ℓ is the length of the stick, M is the mass of the cart, and g is the acceleration due to gravity. The servo control system that your eye, brain, and arm implement is shown in Figure 8. Note that $p > z$, that is, the pole is to the right of the zero.

Although the controller implemented by the brain is unknown, we can draw some conclusions about properties that any stabilizing linear time-invariant controller must possess. Since the plant has an odd number of real poles (specifically, one pole at p) to the right of the positive zero z , it follows from the parity interlacing principle [28, p. 80], [29], [30] that the plant cannot be stabilized by a stable controller. In fact, the controller must have an odd number of positive poles to the right of the zero z in order to prevent a closed-loop pole from being attracted to z . Although it is a challenging exercise to use root locus to construct a stabilizing controller, LQG can easily be used.

For a rough test of these conditions, command yourself to move the “cart” several inches to the left. You readily observe that a typical controller implemented by your brain causes your hand to move first to the right before moving to the left. This kind of response is also manifested by bicycle countersteering as discussed in “Bicycle Countersteering Revisited.” The cart position $y(t)$ and the error $e(t)$ both exhibit initial undershoot, in both cases due to the positive zero z in T , which arises from the positive plant zero $z = \sqrt{g/\ell}$, assuming that your brain does not introduce an odd number of additional positive zeros through C . However, with regard to initial undershoot, LQG behaves differently from the typical human brain. In

TABLE 2 Initial undershoot, zero crossing, and overshoot in the servo system shown in Figure 8, where $u(t)$ is the unit step input, $y(t)$ is the output, and $e(t)$ is the error. As shown in the text, the conditions for overshoot of $y(t)$ and $e(t)$ are equivalent.

	Initial Undershoot	Zero Crossing	Overshoot
$y(t)$	If and only if L has an odd number of positive zeros	If L has at least one positive zero	If $T(s) - T(0)$ has at least one positive zero
$e(t)$	If and only if L has an odd number of positive zeros	If L has at least one positive pole	If $S(s) - S(0)$ has at least one positive zero

particular, LQG can be used to design a controller with exactly one positive zero. Thus, L (and T) has two positive zeros, and the cart position $y(t)$ does not exhibit initial undershoot. For example, let $M = m = 0.1$ kg, $l = 1$ m, $g = 9.8$ m/s², and consider the LQG controller

$$C(s) = \frac{-5085s^3 - 21355s^2 + 5820s + 3099}{s^4 + 45.84s^3 + 981.1s^2 - 39678s - 132520}, \quad (9)$$

which has two negative zeros at -4.4273 and -0.2744 and one positive zero at 0.5018 . Consequently, the step response of the cart position $y(t)$ does not exhibit initial undershoot, as shown in Figure 9. In fact, numerical experiments suggest that LQG always yields cart controllers with an odd number of positive zeros.

In addition to initial undershoot, the positive zero z of L implies that the cart position $y(t)$ has at least one zero crossing, whereas the positive pole p of L implies that the step response error $e(t)$ has at least one zero crossing. Furthermore, in bringing the “cart” to its new position, we observe overshoot, which is due to the positive plant pole p . This example is discussed extensively in [31].

ROBUSTNESS AND PERFORMANCE LIMITATIONS OF NONMINIMUM-PHASE ZEROS

In addition to initial undershoot and direction reversals in the step response of a system, nonminimum-phase zeros limit closed-loop performance. This effect can be seen by noting that the poles of the closed-loop system are a “mixture” of the plant poles and zeros; the classical root locus method tells us how this mixture plays itself out as the loop gain increases. In particular, as the loop gain is increased, poles move toward zeros, and thus destabiliza-

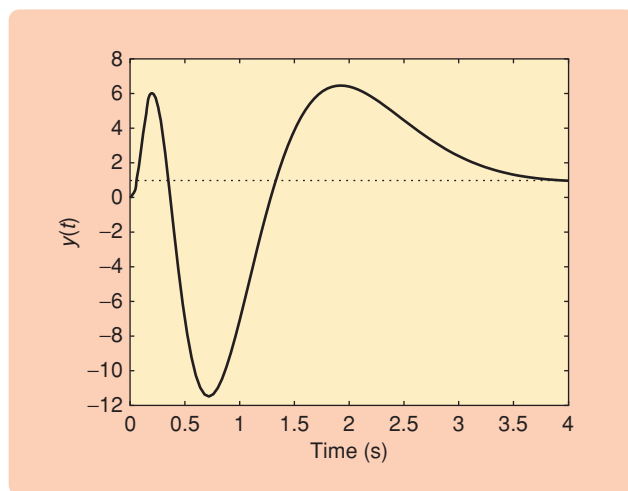


FIGURE 9 The step response of the cart position with an LQG controller. Beginning at 0, the step response initially moves in the correct direction, overshoots the asymptotic value 1, moves in the wrong direction past 0, and finally reverses to reach the desired position.

tion inevitably occurs when the loop transfer function has nonminimum-phase zeros. Hence, feedback control systems have limited gain margin when the loop transfer function has nonminimum-phase zeros, and thus limited gain margin implies a limitation on the robustness of the closed-loop system. A similar limitation on gain margin occurs when proportional feedback is used and the loop transfer function has relative degree greater than 2 [14]. However, controllers can be constructed to have infinite upward gain margin when the loop transfer function is minimum phase [32], [33].

Nonminimum-phase zeros in the loop transfer function also limit bandwidth. To see this, it follows from asymptotic LQG theory [34, p. 369], [35] that nonminimum-phase zeros in the transfer functions from the plant disturbance

Bicycle Countersteering Revisited

The countersteering response in riding a bicycle is an example of initial undershoot. The constant-speed linearization of an open-loop bicycle is unstable with a positive pole, and the positive open-loop pole becomes a positive zero of the sensitivity transfer function. Thus, the step response of the sensitivity transfer function has at least one zero crossing and exhibits overshoot. In addition, the typical rider’s controller results in a loop transfer function with an odd number of positive zeros, and thus the rider-stabilized bicycle exhibits a nonminimum-phase countersteering response, which limits maneuverability. Specifically, in turning the bicycle to the left, the rider commands a left-hand step; however, in response to this step command, the bicycle typically first turns to the right before turning to the left. This initial undershoot behavior is discussed in [S1] and [S2].

A bicycle rider might, however, be able to use an alternative controller that results in a nonzero even number of positive zeros in the loop transfer function. In this case, in response to a left-turn step command, the bicycle turns to the left, quickly turns back to the right, and turns to the left again to complete the left turn without initial undershoot. Figure 9 illustrates this type of response for an LQG-controlled inverted pendulum on a cart. However, as in Figure 9, the rider experiences delayed countersteering since the bike must eventually turn back toward the right and cross zero (see Table 2) before completing the left-hand turn. As long as the relevant transfer function possesses at least one positive zero, a zero crossing cannot be avoided. Numerical demonstration of these properties can be based on models given in [S3].

[S1] K.J. Astrom, R.E. Klein, and A. Lennartsson, “Bicycle dynamics and control,” *IEEE Control Syst. Mag.*, vol. 25, pp. 26–47, Aug. 2005.

[S2] D.G. Wilson, *Bicycling Science*, 3rd ed. Cambridge, MA: MIT Press, 2004.

[S3] D.L.M. Limebeer and R.S. Sharp, “Bicycles, Motorcycles, and Models,” *IEEE Control Syst. Mag.*, vol. 26, pp. 34–61, Oct. 2006.

An everyday example of positive zeros arises when driving a car backwards.

to measurement and from the control to performance variable limit bandwidth in the sense that their mirror images are the asymptotic locations of the closed-loop poles under high gain. A related phenomenon is the *waterbed effect*, which concerns the effect of nonminimum phase zeros on the peak of the sensitivity transfer function [28, p. 98].

Because poles are attracted to zeros, nonminimum-phase zeros limit the use of high-gain feedback. Consequently, open-right-half-plane zeros limit the achievable performance of fixed-gain controllers [36]–[42] as well as adaptive controllers [43].

ZERO CANCELLATION AND HIDDEN UNSTABLE POLES

Mathematically, a zero can cancel a pole when a pair of transfer functions are cascaded. This property corresponds to nothing more than the fact that the stable transfer function $G(s) = (s + 1)/(s + 1)$ is mathematically indistinguishable from the constant transfer function $G(s) = 1$. Likewise, the unstable transfer function $G(s) = (s - 1)/(s - 1)$ is also mathematically indistinguishable from $G(s) = 1$. However, unstable pole-zero cancellation is not an allowable operation in practical plant/controller cascade for the simple reason that an arbitrarily small discrepancy between the zero at 1 and the pole at 1 results in instability.

However, even if there is no discrepancy between an unstable pole and an unstable zero so that mathematical cancellation occurs, the cascaded system generally has an unbounded internal signal. To see why, consider the closed-loop system in Figure 10, whose loop transfer function involves an unstable pole-zero cancellation. For the servo control system shown in Figure 10, the error is given by $\hat{e} = ((s + 1)/(s + 2))\hat{r}$, which seems to indicate stability. However, the transfer function from r to u is given by $\hat{u} = ((s + 1)/((s - 1)(s + 2))\hat{r}$, which is unstable. In fact, this transfer function exposes an otherwise hidden instability in the system. To determine the stability of a system represented in terms of transfer functions, it is thus neces-

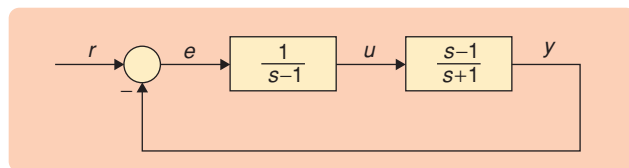


FIGURE 10 Servo feedback system. With a bounded reference signal r , the error $e = [(s + 1)/(s + 2)]r$ is bounded. However, the control input u is generally unbounded due to the instability of the controller. This system possesses a hidden unstable pole-zero cancellation.

sary to examine all transfer functions as discussed in [44, p. 123] and [45].

To demonstrate the effect of a hidden instability, we again consider Figure 10. If we assume that r is the nonzero initial condition response of a linear time-invariant system, then the control signal u is unbounded. Alternatively, if $r(t) \equiv 0$ but the time-domain realization of the transfer function $1/(s - 1)$ has a nonzero initial condition, then the control signal u is still unbounded. Exposing the hidden instability due to unstable pole-zero cancellation is equivalent to recognizing the presence of this unbounded response. Hence, even when the cancellation is perfect, a nonminimum-phase controller zero cannot be used to cancel an unstable plant pole, and an unstable controller pole cannot be used to cancel a nonminimum-phase plant zero.

BLOCKING AND TRANSMISSION ZEROS IN MIMO SYSTEMS

While everything we have said so far applies to SISO systems, the effect of zeros on system behavior and achievable performance is analogous but more complex in MIMO (multiple-input, multiple-output) systems. For treatments of MIMO zeros, see [46]–[57].

For a nonzero $l \times m$ transfer function G , two types of zeros are of interest. A *blocking zero* $z \in \mathbb{C}$ of G has the property that $G(z) = 0$. Hence, $z \in \mathbb{C}$ is a blocking zero of G if and only if $z \in \mathbb{C}$ is a zero of every scalar entry of G . Hence, the blocking zeros of a MIMO transfer function can easily be determined.

The second type of zeros for a MIMO transfer function G are the *transmission zeros*. To characterize transmission zeros, it is useful to consider the *Smith-McMillan form* of a MIMO transfer function [6, p. 140], [42, p. 80]. This result states that every square or rectangular transfer function can be transformed by means of unimodular matrices U_1 and U_2 to a transfer function with nonzero entries appearing only on its main diagonal. (A unimodular matrix has polynomial entries and a constant, nonzero determinant.) The Smith-McMillan form is given by

$$U_1 G U_2 = \begin{bmatrix} \frac{p_1}{q_1} & & & \\ & \ddots & & \\ & & \frac{p_r}{q_r} & \\ & & & 0_{(l-r) \times (m-r)} \end{bmatrix},$$

where p_1, \dots, p_r and q_1, \dots, q_r are monic polynomials (that is, their leading coefficients are unity), p_i and q_i have no common roots, p_i is a factor of p_{i+1} , and q_{i+1} is a factor of q_i . Consequently, the roots of p_r include all of the roots of p_1, \dots, p_{r-1} , while the roots of q_1 include all of the roots of q_2, \dots, q_r . The normal rank of $G(s)$ is r . If $p_1(z) = 0$ then $G(z) = 0$, and thus the roots of p_1 are the blocking zeros of G . Furthermore, note that at least one entry of the Smith-

McMillan form of $G(z)$ is zero for every complex number z that is a root of one of the polynomials p_i . Thus, the roots of p_r are the transmission zeros of G . The analysis of transmission zeros is slightly complicated due to the fact that, as shown by the second example below, a transmission zero can also be a pole.

For example, consider the 2×2 transfer function

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+3} & \frac{1}{s+1} \end{bmatrix},$$

which has no blocking zeros and normal rank 2. To determine the transmission zeros of G , consider the factorization

$$G(s) = \begin{bmatrix} \frac{1}{2}(3s^2 + 13s + 12) & 3 \\ \frac{1}{2}(3s^2 + 13s + 14) & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{(s+1)(s+2)(s+3)} & 0 \\ 0 & \frac{s+5/3}{s+1} \end{bmatrix} \\ \times \begin{bmatrix} -2s-4 & s+3 \\ 1 & -\frac{1}{2} \end{bmatrix}.$$

Note that the first and third matrices are unimodular, while the second matrix is in Smith-McMillan form. Thus, $z = -5/3$ is a transmission zero of G .

As another example, consider

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+2)^2} \end{bmatrix}.$$

Then,

$$G(s) = U_1(s) \begin{bmatrix} \frac{1}{(s+1)^2(s+2)^2} & 0 \\ 0 & s+2 \end{bmatrix} U_2(s),$$

where U_1, U_2 are the unimodular matrices

$$U_1(s) = \begin{bmatrix} (s+2)(s^3 + 4s^2 + 5s + 1) & 1 \\ (s+1)(s^3 + 5s^2 + 8s + 3) & 1 \end{bmatrix}$$

and

$$U_2(s) = \begin{bmatrix} -(s+2) & (s+1)(s^2 + 3s + 1) \\ 1 & -s(s+2) \end{bmatrix}.$$

Hence, the McMillan degree of G is 2, the poles of G are -1 and -2 , the transmission zero of G is -2 , and G has no blocking zeros. Note that -2 is both a pole and a transmission zero of G . Note also that, although G is strictly proper, the Smith-McMillan form of G is improper.

Transmission zeros are usually computed by using a state-space method that involves a minimal realization of $G(s)$. For the SISO transfer function $G(s)$, it is useful to note the identity [6, p. 520]

$$C \operatorname{adj}(sI - A)B = -\det R(s),$$

where $R(s)$ is the Rosenbrock system matrix defined by

**Mathematically, a zero can cancel
a pole when a pair of transfer
functions are cascaded.**

$$R(s) \triangleq \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix}.$$

Consequently, the complex number z is a zero of the SISO transfer function $G(s) = C(sI - A)^{-1}B$ if and only if $\det R(z) = 0$.

Now, suppose that G is an $l \times m$ transfer function, with a minimal realization (A, B, C) , where B is an $n \times m$ matrix and C is a $l \times n$ matrix. Then, the Rosenbrock system matrix R has size $(n+l) \times (n+m)$ and thus is not necessarily square. Now, $z \in \mathbb{C}$ is an *invariant zero* of (A, B, C) if the rank of $R(z)$ is less than the normal rank of R . Furthermore, it is shown in [23, p. 111] that the transmission zeros of G are exactly the invariant zeros of (A, B, C) . Note that in the case of full-state measurement, that is, $C = I$, the rank of $R(s)$ is $n + \operatorname{rank} B$ for all values of s . Hence, in this case, G has no transmission zeros.

By writing $R(s)$ as

$$R(s) = \begin{bmatrix} -A & B \\ C & 0 \end{bmatrix} - s \begin{bmatrix} -I & 0 \\ 0 & 0 \end{bmatrix},$$

it can be seen that the invariant zeros of (A, B, C) are the generalized eigenvalues of the *matrix pencil* [58]

$$\left(\begin{bmatrix} -A & B \\ C & 0 \end{bmatrix}, \begin{bmatrix} -I & 0 \\ 0 & 0 \end{bmatrix} \right).$$

Consequently, while the computation of poles is an eigenvalue problem, the computation of zeros is a generalized eigenvalue problem, which is more difficult [59], [60]. Note that there is no assumption that $R(s)$ is square. We also note that the first n rows of $R(s)$ are $C(s) \triangleq [sI - A \ B]$, which is the *controllability pencil*, while the first n columns of $R(s)$ are $O(s) \triangleq \begin{bmatrix} sI - A \\ C \end{bmatrix}$, which is the *observability pencil*. The PBH tests for controllability and observability are based on $C(s)$ and $O(s)$, respectively.

NONMINIMUM-PHASE ZEROS IN DISCRETE-TIME SYSTEMS

The above discussion is confined to continuous-time systems. For discrete-time systems, nonminimum-phase zeros are zeros that lie outside the unit disk. Such zeros may or may not cause initial undershoot in the step response [61]. In addition, the root-locus rules for discrete-time systems are identical to the rules for continuous-time systems. However, unlike continuous-time systems,

The step response of a system with two positive zeros initially moves in the “correct” direction, reverses course to move in the “wrong” direction, and then reverses course yet again to move in the “correct” direction.

which can have infinite gain margin for a loop transfer function with relative degree less than or equal to two, discrete-time systems with relative degree one or greater have finite gain margins.

Most discrete-time systems arise as sampled continuous-time systems. In this regard it is important to note that sampled minimum-phase transfer functions are often nonminimum phase [62], [63], [64, p. 65]. In particular, sufficiently fast sampling of a continuous-time system with relative degree greater than two gives rise to nonminimum-phase zeros [65]. Techniques for addressing nonminimum-phase sampled-data systems are given in [61] and [66]–[70].

It is clear from root locus that nonminimum-phase zeros in discrete time impose limitations on robustness and performance [71], [72]. Furthermore, as in continuous time, discrete-time nonminimum-phase zeros prevent the use of plant-inversion-based controllers. This limitation is apparent in adaptive control, where many methods are restricted to minimum-phase plants [73]–[76].

CONCLUSIONS

Zeros are a fundamental aspect of systems and control theory; however, the causes and effects of zeros are more subtle than those of poles. In particular, positive zeros can cause initial undershoot (initial error growth), zero crossings, and overshoot in the step response of a system, whereas nonminimum-phase zeros limit bandwidth. Both of these aspects have real-world implications in many applications. Nonminimum-phase zeros exacerbate the tradeoff between the robustness and achievable performance of a feedback control system.

From a control-theoretic point of view, a nonminimum-phase zero in the loop transfer function L is arguably the worst feature a system can possess. Every feedback synthesis methodology must accept limitations due to the presence of open-right-half-plane zeros, and the mark of a good analysis tool is the ability to capture the performance limitations arising from nonminimum-phase zeros.

While the effects of open-right-half-plane poles are evident to every student of control, the lurking dangers and limitations of open-right-half-plane zeros are more subtle and thus more insidious. As control practitioners, we may despise open-right-half-plane zeros because of the difficulties they entail. However, those of us who develop control techniques relish the challenge that open-right-half-plane zeros present in our unique field of endeavor.

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