Initial Conditions in Time- and Frequency-Domain System Identification

IMPLICATIONS OF THE SHIFT OPERATOR VERSUS THE $\mathcal{Z}$ AND DISCRETE FOURIER TRANSFORMS

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This article has two main objectives. The first objective is to demonstrate that, unlike frequency-domain models, time-domain models exactly account for the effect of initial conditions without an explicit expression for the free response. Although this property is obvious in state-space models, it has been the subject of confusion and misconceptions within the context of input–output models. In contrast to time-domain models, frequency-domain models (not to be confused with Laplace-domain and Z-transfer-domain models, which separately include the free response) do not account for initial conditions.

The second objective is to demonstrate the ramifications of this defect for frequency-domain identification in the form of spectral leakage. Spectral leakage is the error in the estimated frequency response function that arises from nonperiodic data, which may be due to the initial conditions, a nonperiodic input, or, if the response is periodic, the use of a noninteger number of periods. Although spectral leakage and leakage-remediation techniques are discussed in [1], the role of initial conditions in contributing to this effect is not considered. This article illustrates various techniques for ameliorating spectral leakage due to initial conditions. Spectral leakage does not arise in time-domain identification.

To demonstrate the effect of initial conditions, we use least-squares techniques for both time-domain and parametric frequency-domain identification as well as spectral analysis for nonparametric frequency-domain identification. All models in this article are single-input, single-output. Although one of the key challenges in system identification is the effect of noise, all data are assumed to be noise-free to focus on the effect of the initial conditions.

As discussed in "Summary," this article is intended as a tutorial for students interested in system identification as well as those engaged in related research, and an overview of the types of models that play a role in discrete-time system identification in both the time and frequency domains is included.

Summary

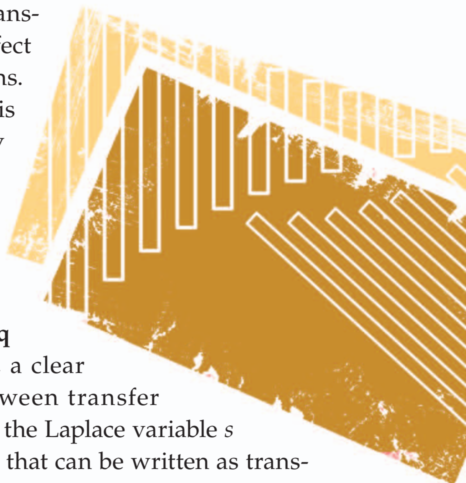
Although initial conditions play a visible role in state-space models, their presence is less obvious in input–output models. In particular, a transfer function written in terms of the Laplace variable represents only the forced response and thus assumes zero initial conditions. By replacing the Laplace variable with the differential operator, a time-domain input–output model has the same appearance as the Laplace transfer function but accounts for both the free and forced responses. In the case of the forward shift operator and the Z transform, the analogous observation clarifies misconceptions in the literature. For time-domain identification, nonzero initial conditions can enhance persistency due to the spectral content of the transient response. However, for frequency-domain identification, the free response and resulting nonperiodic behavior due to nonzero initial conditions cause spectral leakage, which degrades the accuracy of frequency-domain estimates. To overcome this effect, this article proposes an averaging technique for improving the accuracy of nonparametric frequency-domain identification.

Time-Domain, Laplace-Domain, and Frequency-Domain Models

The duality between time-domain and frequency-domain models is an immensely powerful aspect of systems theory. The connections between these types of models are both deep and subtle, and understanding these connections is challenging for students of systems and control theory. Time-domain models in the form of state-space models are relatively easy to comprehend, and their analysis entails eigenvalues, eigenvectors, invariant zeros, and related concepts from matrix theory. At the other end of the "spectrum" lie frequency-domain (Fourier transform) models and their associated Bode plots, which illustrate the harmonic steady-state gain and phase shift of a system with harmonic inputs. Between state-space models and frequency-response plots lie transfer function models, which make poles and transmission zeros evident from their numerator and denominator polynomials. For a transfer function with a minimal state-space realization, the poles and eigenvalues coincide, as do the transmission zeros and invariant zeros.

Within the full range of system models are those that are time domain in character but are not state-space models. Here we are referring to *input–output* models, where higher-order derivatives of the input and output appear without an internal state. Although input–output models do not have a standard name in systems and control literature, they include the autoregressive-moving-average, autoregressive-moving-average with exogenous terms, Box–Jenkins, and related models used extensively in economics and statistics literature [2]–[4].

System identification methods have been extensively developed in both the time domain and frequency domain, and the time-domain methods developed in [5]–[9] naturally account for initial conditions due to the model structure. For time-domain input–output models, however, the manner in which the initial condition is encoded in an input–output model may be puzzling at first glance. For example, the discussion of this encoding in [10]–[12] and [13, p. 522] erroneously states that an input–output model requires an additional input in the form of an impulse to compensate for nonzero initial conditions, which leads to $n+1$ additional parameters in the model. However, that discussion tacitly assumes that input–output models are indistinguishable from transfer functions, which in effect have zero initial conditions. The key to overcoming this confusion is to carefully distinguish between the $\mathcal{Z}$ transform variable z and the forward shift operator $\mathbf{q}$. This distinction is discussed in "Why $\mathbf{p}$ Is Not s and $\mathbf{q}$ Is Not z." In this article, a clear distinction is made between transfer functions that depend on the Laplace variable s and input–output models that can be written as transfer operators that depend on the forward shift operator $\mathbf{q}$. Although these models are superficially similar, they are, in fact, distinct in terms of their treatment of initial conditions.



Why p Is Not s and q Is Not z

The dynamics of a mass-spring-dashpot model are naturally cast in the form

$$m\ddot{q}(t)+c\dot{q}(t)+kq(t)=f(t), \tag{S1}$$

where the position q of the mass is the output. Taking the Laplace transform of (S1) yields

$$\hat{q}(s)=\frac{(ms+c)q(0)+m\dot{q}(0)}{ms^2+cs+k}+\frac{1}{ms^2+cs+k}\hat{f}(s), \tag{S2}$$

which captures both the free response and the forced response. In the special case where $q(0)=0$ and $\dot{q}(0)=0$, (S2) becomes

$$\hat{q}(s)=\frac{1}{ms^2+cs+k}\hat{f}(s). \tag{S3}$$

Since (S3) is the forced response, it is incorrect for nonzero initial conditions.

Although (S2) captures both the free response and the forced response of (S1), an alternative approach that accounts for initial conditions without a separate term involving the initial condition is to avoid the Laplace transform entirely. Instead, by letting $\mathbf{p}$ denote the differential operator $\mathrm{d}/\mathrm{d}t$, (S1) can be written as

$$m\mathbf{p}^2q(t)+c\mathbf{p}q(t)+kq(t)=f(t). \tag{S4}$$

Since (S4) is an exact rewriting of the ordinary differential equation (S1), it accounts for both the input f and the nonzero initial conditions. The next step is to rewrite (S4) *formally* (that is, not rigorously) as

$$q(t)=\frac{1}{m\mathbf{p}^2+c\mathbf{p}+k}f(t). \tag{S5}$$

Since the operator $G(\mathbf{p})=1/(m\mathbf{p}^2+c\mathbf{p}+k)$ in (S5) has exactly the same form as the transfer function $G(s)=1/(ms^2+cs+k)$ in (S3), it is tempting to suggest that $G(\mathbf{p})$ has the same meaning as $G(s)$. However, this is not the case since, as already noted, (S5) accounts for both the free response and the forced response, whereas (S3) accounts for only the forced response.

Resistance to the use of $G(\mathbf{p})$ instead of $G(s)$ tends to arise from the fact that $G(s)$ is a rational function of a complex variable, whereas $G(\mathbf{p})$ may seem unnatural. In fact, it is possible to develop a rigorous mathematical framework for $G(\mathbf{p})$; this is precisely the Mikusinski operational calculus [S1]. As noted in [S2, p. 135], "the effect of the Mikusinski differential operator is that the function f is differentiated but that, in addition, the initial value of the function is taken into account. If the function f takes the value 0 at $t=0$ then s corresponds exactly to Heaviside's differential operator $\mathbf{p}$." This distinction is discussed in [S3] within the context of teaching classical control.

This distinction also arises in the theory of behaviors [S4], where polynomial models in $\mathbf{p}$ are used but without distinguishing between input and output signals. As stated in [S4, p. 46] for $U\in\mathbb{R}^{g\times g}[\xi]$, "For $U^{-1}(\xi)$ may have a proper meaning as a *matrix of rational functions*, but it need not be polynomial, and therefore $U^{-1}(\mathrm{d}/\mathrm{d}t)$ has no meaning in general. (What is the meaning of $(1+(\mathrm{d}/\mathrm{d}t))/(2+(\mathrm{d}/\mathrm{d}t)^2)$?)" The answer to this rhetorical question appears to reside in the operational calculus [S1]–[S3].

The distinction between $\mathbf{q}$ and z is analogous to the distinction between $\mathbf{p}$ and s. A discussion of the relationship between $\mathbf{q}$ and z is given in [S5, Ch. 2].

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DISCRETE-TIME STATE-SPACE AND INPUT–OUTPUT MODELS

This section considers discrete-time state-space and input–output models. Whereas state-space models have an internal state as well as input and output signals, input–output models have no internal state but only input and output signals. It is thus necessary to relate the initial condition of the state-space model to the input and output signals by expressing the initial condition of the state-space model in terms of the initial values of the input and output signals, and vice versa. It should be stressed that this section is confined to time-domain models and does not consider $\mathcal{Z}$-transforms or transfer functions.

Consider the nth-order discrete-time state-space model

$$x(k+1)=Ax(k)+Bu(k), \tag{1}$$
$$x(0)=x_0, \tag{2}$$
$$y(k)=Cx(k)+Du(k), \tag{3}$$

where $x(k)\in\mathbb{R}^n$ is the state, $u(k)\in\mathbb{R}$ is the input, $y(t)\in\mathbb{R}$ is the output, $A\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^n, C\in\mathbb{R}^{1\times n}$, $D\in\mathbb{R}$, and $k\ge 0$. We assume that (A, B, C) is controllable and observable. Solving (1) and (2) for $x(k)$ yields

$$x(k)=A^kx_0+\sum_{i=1}^{k}A^{i-1}Bu(k-i), \tag{4}$$

Whereas state-space models have an internal state as well as input and output signals, input–output models have no internal state but only input and output signals.

and thus y is given by

$$y(k) = y_{\text{free}}(k) + y_{\text{forced}}(k), \tag{5}$$

where the free response y_{free} due to the initial condition x_0 is given by

$$y_{\text{free}}(k) = CA^k x_0 \tag{6}$$

and the forced response y_{forced} due to the input u is given by

$$y_{\text{forced}}(k) = \sum_{i=0}^{k} H_{k-i} u(i) = \sum_{i=0}^{k} H_i u(k-i), \tag{7}$$

where

$$H_i \triangleq \begin{cases} D, & i = 0, \\ CA^{i-1}B, & i \geq 1. \end{cases} \tag{8}$$

Using the forward-shift operator $\mathbf{q}$, (1) can be expressed as

$$\mathbf{q}x(k) = Ax(k) + Bu(k), \tag{9}$$

that is,

$$(\mathbf{q}I - A)x(k) = Bu(k). \tag{10}$$

Multiplying (3) by $\det(\mathbf{q}I - A)$ yields the difference equation

$$\begin{aligned} \det(\mathbf{q}I - A)y(k) &= C\det(\mathbf{q}I - A)I_n x(k) + D\det(\mathbf{q}I - A)u(k) \\ &= [C\,\text{adj}(\mathbf{q}I - A)B + D\det(\mathbf{q}I - A)]u(k). \end{aligned} \tag{11}$$

The difference equation (11) is a discrete-time input–output model whose input is u and output is y. Note that (11) is obtained without dividing by the forward-shift operator $\mathbf{q}$. Moreover, note that the state x does not appear in (11).

Define

$$P(\mathbf{q}) \triangleq C\,\text{adj}(\mathbf{q}I - A)B + D\det(\mathbf{q}I - A) \in \mathbb{R}[\mathbf{q}], \tag{12}$$

$$Q(\mathbf{q}) \triangleq \det(\mathbf{q}I - A) \in \mathbb{R}[\mathbf{q}], \tag{13}$$

where $\mathbb{R}[\mathbf{q}]$ denotes the set of polynomials in $\mathbf{q}$ with real coefficients. Then, (11) can be written as

$$Q(\mathbf{q})y(k) = P(\mathbf{q})u(k). \tag{14}$$

For convenience, define

$$G(\mathbf{q}) \triangleq \frac{P(\mathbf{q})}{Q(\mathbf{q})} \in \mathbb{R}(\mathbf{q}), \tag{15}$$

where $\mathbb{R}(\mathbf{q})$ denotes the set of rational functions in $\mathbf{q}$ with real coefficients, and rewrite (14) as

$$y(k) = G(\mathbf{q})u(k). \tag{16}$$

Since (A, B, C) is controllable and observable, it follows that P and Q are coprime.

Since division by $Q(\mathbf{q})$ in (15) is not meaningful, (16) is only a convenient representation of the difference equation (14). Furthermore, although (15) has the form of a transfer function, (16) is a time-domain relationship. In fact, unlike a transfer function, which captures only the forced response, (16) includes both the free response due to x_0 and the forced response due to u. To illustrate this, let

$$P(\mathbf{q}) = b_0\mathbf{q}^n + b_1\mathbf{q}^{n-1} + \cdots + b_{n-1}\mathbf{q} + b_n, \tag{17}$$

$$Q(\mathbf{q}) = \mathbf{q}^n + a_1\mathbf{q}^{n-1} + \cdots + a_{n-1}\mathbf{q} + a_n. \tag{18}$$

Then, for all $k \geq 0$, (14) can be expressed as

$$\begin{aligned} &\mathbf{q}^n y(k) + a_1\mathbf{q}^{n-1}y(k) + \cdots + a_{n-1}\mathbf{q}y(k) + a_n y(k) \\ &\quad = b_0\mathbf{q}^n u(k) + b_1\mathbf{q}^{n-1}u(k) + \cdots + b_{n-1}\mathbf{q}u(k) + b_n u(k). \end{aligned} \tag{19}$$

That is,

$$\begin{aligned} y(k+n) = &-a_1 y(k+n-1) - \cdots - a_{n-1}y(k+1) - a_n y(k) \\ &+ b_0 u(k+n) + b_1 u(k+n-1) + \cdots \\ &+ b_{n-1}u(k+1) + b_n u(k). \end{aligned} \tag{20}$$

From (6) it is shown that, for all $k \geq 0$, y_{free} satisfies

$$\begin{bmatrix} y_{\text{free}}(k) \\ y_{\text{free}}(k+1) \\ \vdots \\ y_{\text{free}}(k+n-1) \end{bmatrix} = \mathcal{O}(A,C)A^k x_0, \tag{21}$$

where $\mathcal{O}(A,C)$ is the observability matrix. Therefore, if A is nonsingular, then

$$x_0 = A^{-k}\mathcal{O}(A,C)^{-1}\begin{bmatrix} y_{\text{free}}(k) \\ y_{\text{free}}(k+1) \\ \vdots \\ y_{\text{free}}(k+n-1) \end{bmatrix}. \tag{22}$$

Time-Domain Least-Squares Identification of Discrete-Time, Input–Output Models

For the discrete-time, input–output model (20), define

$$\phi(k) \triangleq [-y(k+n-1) \ \cdots \ -y(k) \ \ u(k+n) \ \cdots \ u(k)] \in \mathbb{R}^{1\times(2n+1)}, \tag{S6}$$

$$\Theta \triangleq [a_1 \ \cdots \ a_n \ \ b_0 \ \cdots \ b_n]^{\mathrm{T}} \in \mathbb{R}^{2n+1}. \tag{S7}$$

Then, (20) can be written as

$$y(k+n) = \phi(k)\Theta. \tag{S8}$$

Next, define

$$\Psi_N \triangleq [y(n) \ \cdots \ y(N)]^{\mathrm{T}} \in \mathbb{R}^{N-n+1}, \tag{S9}$$

$$\Phi_N \triangleq [\phi^{\mathrm{T}}(0) \ \cdots \ \phi^{\mathrm{T}}(N-n)]^{\mathrm{T}} \in \mathbb{R}^{(N-n+1)\times(2n+1)}. \tag{S10}$$

Then, (S8) implies that

$$\Psi_N = \Phi_N\Theta, \tag{S11}$$

which has at least one solution Θ. A least-squares solution $\hat{\Theta}_N \in \mathbb{R}^{2n+1}$ of (S11) satisfies

$$\|\Psi_N - \Phi_N\hat{\Theta}_N\|_F \le \min_{\bar{\Theta}\in\mathbb{R}^{2n+1}} \|\Psi_N - \Phi_N\bar{\Theta}\|_F, \tag{S12}$$

where $\|\cdot\|_F$ denotes the Frobenius norm. A minimizer exists because (S11) has a solution. If, however, the data are corrupted by noise and (S11) does not have a solution, then a minimizer still exists because the function

$$f(\Theta_0) \triangleq \|\Psi_N - \Phi_N\Theta_0\|_F \tag{S13}$$

is quadratic in Θ_0 and has a positive-semidefinite Hessian, along with the fact that there exists at least one solution to $f'(\Theta_0) = 0$. If u is persistently exciting of order $2n+1$ [5, p. 412], then Φ_N is left invertible. In this case, (S11), which has at least one solution, has the unique solution

$$\hat{\Theta}_N = (\Phi_N^T\Phi_N)^{-1}\Phi_N^T\Psi_N. \tag{S14}$$

If u is not persistently exciting of order $2n+1$, then Φ_N is not left invertible, and thus (S12) has infinitely many minimizers. In this case, the minimum-norm minimizer is given by

$$\hat{\Theta}_N = \Phi_N^+\Psi_N, \tag{S15}$$

where Φ_N^+ is the pseudoinverse of Φ_N. If Φ_N is left invertible, then (S15) is the unique minimizer and Φ_N^+ is given by

$$\Phi_N^+ = (\Phi_N^{\mathrm{T}}\Phi_N)^{-1}\Phi_N^{\mathrm{T}}. \tag{S16}$$

Setting $k = 0$ in (22) yields

$$x_0 = \mathcal{O}(A,C)^{-1}\begin{bmatrix} y_{\text{free}}(0) \\ y_{\text{free}}(1) \\ \vdots \\ y_{\text{free}}(n-1) \end{bmatrix}. \tag{23}$$

Both (22) and (23) relate the initial condition x_0 of the state-space model (1)–(3) to $y_{\text{free}}(0), y_{\text{free}}(1), \ldots, y_{\text{free}}(n-1)$, and vice versa.

Next, if $u = 0$, then $y = y_{\text{free}}$, and thus it follows from (20) that the free response y_{free} satisfies

$$y_{\text{free}}(k+n) = -a_1 y_{\text{free}}(k+n-1) - \cdots - a_{n-1}y_{\text{free}}(k+1) - a_n y_{\text{free}}(k). \tag{24}$$

The data needed to solve the nth-order difference equation (24) is $y_{\text{free}}(0), y_{\text{free}}(1), \ldots, y_{\text{free}}(n-1)$, which, in the case $u = 0$, also defines the initial condition x_0 of (1) according to (23). Alternatively, if $x_0 = 0$, then $y = y_{\text{forced}}$, and thus it follows from (20) that the forced response y_{forced} satisfies

$$y_{\text{forced}}(k+n) = -a_1 y_{\text{forced}}(k+n-1) - \cdots - a_{n-1}y_{\text{forced}}(k+1) - a_n y_{\text{forced}}(k) + b_0 u(k+n) + b_1 u(k+n-1) + \cdots + b_{n-1}u(k+1) + b_n u(k). \tag{25}$$

For time-domain least-squares identification of discrete-time input–output models, see "Time-Domain Least-Squares Identification of Discrete-Time, Input–Output Models."

$\mathcal{Z}$-TRANSFORM OF DISCRETE-TIME INPUT–OUTPUT MODELS

Consider the state-space models (1)–(3). Using the $\mathcal{Z}$-transform defined by

$$\hat{x}(z) = \mathcal{Z}\{x\} \triangleq \sum_{k=0}^{\infty} x(k)z^{-k}, \tag{26}$$

(1) can be written as

$$z\hat{x}(z) - zx(0) = A\hat{x}(z) + B\hat{u}(z), \tag{27}$$

where $\hat{x}$ and $\hat{u}$ are the $\mathcal{Z}$-transforms of x and u, respectively, and z is the complex $\mathcal{Z}$-transform variable. Therefore,

$$\hat{x}(z) = (zI-A)^{-1}B\hat{u}(z) + z(zI-A)^{-1}x(0). \tag{28}$$

Using the $\mathcal{Z}$-transform, (3) can be expressed as

$$\hat{y}(z) = C\hat{x}(z) + D\hat{u}(z), \tag{29}$$

where $\hat{y}$ is the $\mathcal{Z}$-transform of y. Using (28), (29) can be expressed as

$$\hat{y}(z) = G(z)\hat{u}(z) + zC(zI-A)^{-1}x(0), \tag{30}$$

The discrete Fourier transform arises from the harmonic steady-state response of a linear system driven by a periodic input.

where

$$G(z) \triangleq C(zI - A)^{-1}B + D. \tag{31}$$

If $x(0) = 0$, then (30) becomes $\hat{y}(z) = G(z)\hat{u}(z)$.

Next, consider the input–output model (19). For all $i \geq 1$, the $\mathcal{Z}$-transform of the shifted sequence $(f(k+i))_{k=0}^{\infty}$ is given by

$$\begin{aligned}\mathcal{Z}\{(f(k+i))_{k=0}^{\infty}\} &= \mathcal{Z}\{\mathbf{q}^i(f(k))_{k=0}^{\infty}\} \\ &= z^i\hat{f}(z) - z^i f(0) - z^{i-1}f(1) - \cdots - zf(i-1) \\ &= z^i\hat{f}(z) - \sum_{j=0}^{i-1} z^{i-j}f(j). \end{aligned}\tag{32}$$

Therefore, for all $i \geq 1$, (19) implies

$$\begin{aligned}&(z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n)\hat{y}(z) - \sum_{i=1}^{n}\sum_{j=0}^{i-1} a_{n-i}z^{i-j}y(j) \\ &= (b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1}z + b_n)\hat{u}(z) - \sum_{i=1}^{n}\sum_{j=0}^{i-1} b_{n-i}z^{i-j}u(j),\end{aligned}\tag{33}$$

where $a_0 \triangleq 1$. Solving (33) for $\hat{y}(z)$ yields

$$\hat{y}(z) = G(z)\hat{u}(z) + \frac{\sum_{i=1}^{n}\sum_{j=0}^{i-1} a_{n-i}z^{i-j}y(j) - \sum_{i=1}^{n}\sum_{j=0}^{i-1} b_{n-i}z^{i-j}u(j)}{z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n}, \tag{34}$$

where

$$G(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1}z + b_n}{z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n}. \tag{35}$$

Note that (34) contains the initial conditions of u and y. In particular, $G(z)\hat{u}(z)$ represents the forced response of the system, while the additional term represents the free response. Setting $y(0) = y(1) = \cdots = y(n-1) = 0$ and $u(0) = u(1) = \cdots = u(n-1) = 0$ in (33) yields the forced response

$$\hat{y}(z) = G(z)\hat{u}(z). \tag{36}$$

The second term of the right-hand side of (34) determines the free response.

FREQUENCY-DOMAIN IDENTIFICATION OF DISCRETE-TIME, INPUT–OUTPUT MODELS

This section presents several parametric and nonparametric frequency-domain identification methods. For parametric identification, least squares is considered, where, for nonparametric identification, the $\mathcal{D}$-transforms of the input and output signals and spectral analysis are used.

The discrete Fourier transform arises from the harmonic steady-state response of a linear system driven by a periodic input; see "Harmonic Steady-State Response for Discrete-Time Systems." Moreover, the distinction between the $\mathcal{Z}$-transform and the $\mathcal{D}$-transform is discussed in "Relationship Between the $\mathcal{Z}$-Transform and $\mathcal{D}$-Transform."

Parametric Identification

In this section, N samples of u and y are used to identify G. Since u and y consist of N samples, the discrete Fourier transform (a truncation of the $\mathcal{D}$-transform) of u and y is equivalent to their $\mathcal{D}$-transforms. Consider the parametric frequency-domain model

$$\hat{y}(e^{j\theta_i}) = G(e^{j\theta_i})\hat{u}(e^{j\theta_i}) + T_G(e^{j\theta_i}), \tag{37}$$

where $\theta_i \triangleq \frac{\pi}{N}i \in (-\pi, \pi]$ and $i = -N+1, \ldots, N$. Then, (37) can be expressed as

$$\bar{D}(e^{j\theta_i})\hat{y}(e^{j\theta_i}) = \bar{N}(e^{j\theta_i})\hat{u}(e^{j\theta_i}) + \bar{N}_T(e^{j\theta_i}), \tag{38}$$

where

$$G(e^{j\theta_i}) \triangleq \frac{\bar{N}(e^{j\theta_i})}{\bar{D}(e^{j\theta_i})}, \quad T_G(e^{j\theta_i}) \triangleq \frac{\bar{N}_T(e^{j\theta_i})}{\bar{D}(e^{j\theta_i})}. \tag{39}$$

Note from (S29) and (37) that T_G captures the effect of nonzero initial conditions. Let

$$\bar{N}(e^{j\theta_i}) = b_0 e^{j\theta_i n} + b_1 e^{j\theta_i(n-1)} + \cdots + b_n, \tag{40}$$

$$\bar{D}(e^{j\theta_i}) = e^{j\theta_i n} + a_1 e^{j\theta_i(n-1)} + \cdots + a_n, \tag{41}$$

$$\bar{N}_T(e^{j\theta_i}) = c_0 e^{j\theta_i n} + c_1 e^{j\theta_i(n-1)} + \cdots + c_n. \tag{42}$$

Using (40)–(42), (38) can be expressed as

$$\begin{aligned}e^{j\theta_i n}\hat{y}(e^{j\theta_i}) = &-(a_1 e^{j\theta_i(n-1)} + \cdots + a_n)\hat{y}(e^{j\theta_i}) \\ &+ (b_0 e^{j\theta_i n} + \cdots + b_n)\hat{u}(e^{j\theta_i}) + c_0 e^{j\theta_i n} + \cdots + c_n.\end{aligned}\tag{43}$$

Dividing (43) by $e^{j\theta_i n}$ yields

$$\begin{aligned}\hat{y}(e^{j\theta_i}) = &-(a_1 e^{-j\theta_i} + \cdots + a_n e^{-j\theta_i n})\hat{y}(e^{j\theta_i}) \\ &+ (b_0 + \cdots + b_n e^{-j\theta_i n})\hat{u}(e^{j\theta_i}) + c_0 + \cdots + c_n e^{-j\theta_i n},\end{aligned}\tag{44}$$

which can be written as

$$\hat{y}(e^{j\theta_i}) = \phi(e^{j\theta_i})\Theta, \tag{45}$$

Harmonic Steady-State Response for Discrete-Time Systems

The following result characterizes the response of a discrete-time, asymptotically stable, linear, time-invariant system to a harmonic input with an arbitrary initial condition.

Theorem S1

For $k \geq 0$, consider the discrete-time, linear, time-invariant system (1)–(3), where A is asymptotically stable. Let $u(k) = \mathrm{Re}\, u_0 e^{j\theta_0 k} = A_u \sin(\theta_0 k + \phi)$, where $u_0 \triangleq -A_u j e^{j\phi} \in \mathbb{C}$, A_u and ϕ are real numbers, and $\theta_0 \in (-\pi, \pi]$. Then, $x(k)$ is given by

$$x(k) = A^k (x_0 - \mathrm{Re}[(e^{j\theta_0} I - A)^{-1} B u_0]) + \mathrm{Re}[(e^{j\theta_0} I - A)^{-1} B u_0 e^{j\theta_0 k}]. \tag{S17}$$

Moreover,

$$y(k) = y_{\text{trans}}(k) + y_{\text{hss}}(k), \tag{S18}$$

where

$$y_{\text{trans}}(k) \triangleq C A^k (x(0) - \mathrm{Re}[(e^{j\theta_0} I - A)^{-1} B u_0]), \tag{S19}$$

$$y_{\text{hss}}(k) \triangleq \mathrm{Re}[G(e^{j\theta_0}) u_0 e^{j\theta_0 k}] = M A_u \sin(\theta_0 k + \phi + \gamma), \tag{S20}$$

$M \triangleq |G(e^{j\theta_0})|$, and $\gamma \triangleq \angle G(e^{j\theta_0})$.

The signals y_{trans} and y_{hss} are the transient and harmonic steady-state components of the output y, respectively. If θ_0 is a rational number, then it follows from (S20) that y_{hss} is harmonic with the same frequency as u. Moreover, $|G(e^{j\theta_0})|$ is the amplification of y_{hss} relative to u, and $\angle G(e^{j\theta_0})$ is the phase shift of y_{hss} relative to u. The plots of $|G(e^{j\theta_0})|$ and $\angle G(e^{j\theta_0})$ versus θ_0 are the magnitude and phase Bode plots, respectively. If θ_0 is irrational, then y_{hss} is an almost periodic sequence [S6], [S7].

Next, for all $k \geq 0$, it follows from (S19) that

$$y_{\text{trans}}(k) = y_{\text{free}}(k) + y_{\text{trans,forced}}(k), \tag{S21}$$

where y_{free} is given by (6) and

$$y_{\text{trans,forced}}(k) \triangleq -C A^k \mathrm{Re}[(e^{j\theta_0} I - A)^{-1} B u_0]. \tag{S22}$$

It follows from (5), (S18), and (S21) that, for all $k \geq 0$,

$$\begin{aligned} y_{\text{free}}(k) + y_{\text{forced}}(k) &= y_{\text{trans}}(k) + y_{\text{hss}}(k) \\ &= y_{\text{free}}(k) + y_{\text{trans,forced}}(k) + y_{\text{hss}}(k), \end{aligned}$$

that is,

$$y_{\text{forced}}(k) = y_{\text{trans,forced}}(k) + y_{\text{hss}}(k). \tag{S23}$$

If the initial condition $x(0)$ has the special value

$$x(0) = \mathrm{Re}[(e^{j\theta_0} I - A)^{-1} B u_0], \tag{S24}$$

then, for all $k \geq 0$, $y_{\text{trans}}(k) = 0$ and thus $y(k) = y_{\text{hss}}(k)$.

Finally, if A is asymptotically stable, then it follows from (S19) that

$$\lim_{k\to\infty} y_{\text{trans}}(k) = \lim_{k\to\infty} y_{\text{free}}(k) = \lim_{k\to\infty} y_{\text{trans,forced}}(k) = 0, \tag{S25}$$

and thus

$$\lim_{k\to\infty} [y(k) - y_{\text{hss}}(k)] = 0. \tag{S26}$$

The following result provides the $\mathcal{Z}$-transform of y_{hss}.

Proposition S1

For all $k \geq 0$, let $u(k) = \mathrm{Re}\, u_0 e^{j\theta_0 k} = A_u \sin(\theta_0 k + \phi)$, where $u_0 \triangleq -A_u j e^{j\phi}$, A_u and ϕ are real numbers, and $\theta_0 \in (-\pi, \pi]$, and define $M \triangleq |G(e^{j\theta_0})|$ and $\gamma \triangleq \angle G(e^{j\theta_0})$. Then, for all $z \neq e^{\pm j\theta_0}$,

$$\hat{y}_{\text{hss}}(z) = \frac{A_u}{2j} \frac{e^{j\phi} z(z - e^{-j\theta_0}) G(e^{j\theta_0}) - e^{-j\phi} z(z - e^{j\theta_0}) G(e^{-j\theta_0})}{z^2 - 2\cos(\theta_0) z + 1} \tag{S27}$$

$$= A_u M \left(\frac{\sin(\phi + \gamma) z^2 + \sin(\phi + \gamma - \theta_0) z}{z^2 - 2\cos(\theta_0) z + 1} \right). \tag{S28}$$

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where

$$\phi(e^{j\theta_i}) \triangleq [-\hat{y}(e^{j\theta_i}) e^{-j\theta_i} \ \cdots \ -\hat{y}(e^{j\theta_i}) e^{-j\theta_i n} \ \hat{u}(e^{j\theta_i}) \ \cdots \ \hat{u}(e^{j\theta_i}) e^{-j\theta_i n} \ 1 \ \cdots \ e^{-j\theta_i n}] \in \mathbb{R}^{1\times(3n+2)}, \tag{46}$$

$$\Theta \triangleq [a_1 \ \cdots \ a_n \ b_0 \ \cdots \ b_n \ c_0 \ \cdots \ c_n]^{\mathrm{T}} \in \mathbb{R}^{3n+2}. \tag{47}$$

Next, define

$$\Psi_N \triangleq [\hat{y}(e^{j\theta_0}) \ \cdots \ \hat{y}(e^{j\theta_{N-1}})]^{\mathrm{T}} \in \mathbb{R}^N, \tag{48}$$

$$\Phi_N \triangleq [\phi^{\mathrm{T}}(e^{j\theta_0}) \ \cdots \ \phi^{\mathrm{T}}(e^{j\theta_{N-1}})]^{\mathrm{T}} \in \mathbb{R}^{N\times(3n+2)}. \tag{49}$$

Then, (45) implies that

$$\Psi_N = \Phi_N \Theta. \tag{50}$$

A least-squares solution $\hat{\Theta}_N \in \mathbb{R}^{3n+2}$ of (50) satisfies

$$\|\Psi_N - \Phi_N \hat{\Theta}_N\|_F \leq \min_{\bar{\Theta}\in\mathbb{R}^{3n+2}} \|\Psi_N - \Phi_N \bar{\Theta}\|_F. \tag{51}$$

Nonparametric Identification

This section presents nonparametric frequency-domain identification using the $\mathcal{D}$-transforms of the input and output signals.

Identification Using the Discrete Fourier Transforms of the Input and Output Signals

Let $N \geq 1$. For all $\theta \in (-\pi, \pi]$, consider the discrete Fourier transforms

Relationship Between the $\mathcal{Z}$-Transform and $\mathcal{D}$-Transform

Setting $z=e^{j\theta}$ in (30) yields

$$\hat{y}(e^{j\theta}) = G(e^{j\theta})\hat{u}(e^{j\theta}) + e^{j\theta}C(e^{j\theta}I-A)^{-1}x(0), \tag{S29}$$

where

$$\hat{y}(e^{j\theta}) = \mathcal{D}\{y\} \triangleq \mathcal{Z}_{z=e^{j\theta}}\{y\} \triangleq \sum_{k=0}^{\infty} y(k)e^{-j\theta k}, \tag{S30}$$

$$\hat{u}(e^{j\theta}) = \mathcal{D}\{u\} \triangleq \mathcal{Z}_{z=e^{j\theta}}\{u\} \triangleq \sum_{k=0}^{\infty} u(k)e^{-j\theta k}, \tag{S31}$$

are the $\mathcal{D}$-transforms of y and u, respectively, and $G(e^{j\theta})$ is the discrete-time, frequency-response function. However, note from (S30) and (S31) that if y and u are sinusoidal signals, then neither of the summations in (S30) and (S31) converges. Therefore, to consider harmonic signals, a more general definition of the $\mathcal{D}$-transform based on analytic continuation of the $\mathcal{Z}$-transform is needed. If $x(0)=0$, then (S29) becomes

$$\hat{y}(e^{j\theta}) = G(e^{j\theta})\hat{u}(e^{j\theta}), \tag{S32}$$

which is a specialization of (36) with $z=e^{j\theta}$. In this case, the frequency response function $G(e^{j\theta})$ can be written as

$$G(e^{j\theta}) = \frac{\hat{y}(e^{j\theta})}{\hat{u}(e^{j\theta})}. \tag{S33}$$

Consider a transfer function G with the state-space realization (A,B,C,D), where A is asymptotically stable. For all $k\geq 0$, let $u(k) = \mathrm{Re}\, u_0 e^{j\theta_0 t} = A_u \sin(\theta_0 k + \phi)$. The $\mathcal{Z}$-transform of u is given by

$$\hat{u}(z) = \frac{A_u}{2j}\frac{e^{j\phi}z(z-e^{-j\theta_0}) - e^{-j\phi}z(z-e^{j\theta_0})}{(z-e^{j\theta_0})(z-e^{-j\theta_0})}, \tag{S34}$$

which is valid for all $z\in\mathbb{C}\setminus\{e^{\pm j\theta_0}\}$. Setting $z=e^{j\theta}$ in (S34) yields the $\mathcal{D}$-transform of u given by

$$\hat{u}(e^{j\theta}) = \frac{A_u}{2j}\frac{e^{j\phi}e^{j\theta}(e^{j\theta}-e^{-j\theta_0}) - e^{-j\phi}e^{j\theta}(e^{j\theta}-e^{j\theta_0})}{(e^{j\theta}-e^{j\theta_0})(e^{j\theta}-e^{-j\theta_0})}. \tag{S35}$$

It follows from (S27), which is valid for all $z\in\mathbb{C}\setminus\{e^{\pm j\theta_0}\}$, that the $\mathcal{D}$-transform of y_{hss} is given by

$$\hat{y}_{\mathrm{hss}}(e^{j\theta}) = \frac{A_u}{2j}\left(\frac{e^{j\phi}e^{j\theta}(e^{j\theta}-e^{-j\theta_0})G(e^{j\theta_0}) - e^{-j\phi}e^{j\theta}(e^{j\theta}-e^{j\theta_0})G(e^{-j\theta_0})}{(e^{j\theta}-e^{j\theta_0})(e^{j\theta}-e^{-j\theta_0})}\right). \tag{S36}$$

Note that (S34)–(S36) are defined for all $\theta\in(-\pi,\pi]\setminus\{\pm\theta_0\}$. Moreover, note from (S35) and (S36) that the $\mathcal{D}$-transforms $\hat{u}(e^{j\theta})$ and $\hat{y}_{\mathrm{hss}}(e^{j\theta})$ of u and y_{hss} have frequency content for all $\theta\in(-\pi,\pi]$. Dividing (S36) by (S35) yields

$$\frac{\hat{y}_{\mathrm{hss}}(e^{j\theta})}{\hat{u}(e^{j\theta})} = \frac{e^{j\phi}e^{j\theta}(e^{j\theta}-e^{-j\theta_0})G(e^{j\theta_0}) - e^{-j\phi}e^{j\theta}(e^{j\theta}-e^{j\theta_0})G(e^{-j\theta_0})}{e^{j\phi}e^{j\theta}(e^{j\theta}-e^{-j\theta_0}) - e^{-j\phi}e^{j\theta}(e^{j\theta}-e^{j\theta_0})}. \tag{S37}$$

Note from (S37) that

$$G(e^{j\theta_0}) = \lim_{\theta\to\theta_0^+}\frac{\hat{y}_{\mathrm{hss}}(e^{j\theta})}{\hat{u}(e^{j\theta})} = \lim_{\theta\to\theta_0^-}\frac{\hat{y}_{\mathrm{hss}}(e^{j\theta})}{\hat{u}(e^{j\theta})}, \tag{S38}$$

$$G(e^{-j\theta_0}) = \lim_{\theta\to-\theta_0^+}\frac{\hat{y}_{\mathrm{hss}}(e^{j\theta})}{\hat{u}(e^{j\theta})} = \lim_{\theta\to-\theta_0^-}\frac{\hat{y}_{\mathrm{hss}}(e^{j\theta})}{\hat{u}(e^{j\theta})}, \tag{S39}$$

and thus, by continuity, (S37) holds for all $\theta\in(-\pi,\pi]$.

$$\hat{u}_N(e^{j\theta}) \triangleq \mathcal{D}_N\{u\} = \sum_{k=0}^{N-1} u(k)e^{-j\theta k}, \tag{52}$$

$$\hat{y}_N(e^{j\theta}) \triangleq \mathcal{D}_N\{y\} = \sum_{k=0}^{N-1} y(k)e^{-j\theta k}, \tag{53}$$

where

$$\hat{y}_N(e^{j\theta}) = \hat{y}_{\mathrm{free},N}(e^{j\theta}) + \hat{y}_{\mathrm{forced},N}(e^{j\theta}), \tag{54}$$

$$\hat{y}_{\mathrm{free},N}(e^{j\theta}) \triangleq \mathcal{D}_N\{y_{\mathrm{free}}\} = \sum_{k=0}^{N-1} y_{\mathrm{free}}(k)e^{-j\theta k}, \tag{55}$$

$$\hat{y}_{\mathrm{forced},N}(e^{j\theta}) \triangleq \mathcal{D}_N\{y_{\mathrm{forced}}\} = \sum_{k=0}^{N-1} y_{\mathrm{forced}}(k)e^{-j\theta k}. \tag{56}$$

Note that

$$\lim_{N\to\infty}\hat{u}_N(e^{j\theta}) = \lim_{N\to\infty}\sum_{k=0}^{N-1} u(k)e^{-j\theta k} = \hat{u}(e^{j\theta}), \tag{57}$$

$$\lim_{N\to\infty}\hat{y}_N(e^{j\theta}) = \lim_{N\to\infty}\sum_{k=0}^{N-1} y(k)e^{-j\theta k} = \hat{y}(e^{j\theta}), \tag{58}$$

$$\lim_{N\to\infty}\hat{y}_{\mathrm{forced},N}(e^{j\theta}) = \lim_{N\to\infty}\sum_{k=0}^{N-1} y_{\mathrm{forced}}(k)e^{-j\theta k} = \hat{y}_{\mathrm{forced}}(e^{j\theta}). \tag{59}$$

Define

$$\hat{G}_N(e^{j\theta}) \triangleq \frac{\hat{y}_N(e^{j\theta})}{\hat{u}_N(e^{j\theta})}. \tag{60}$$

Using (54) and (57)–(59), for all $\theta\in(-\pi,\pi]$, (60) implies

$$\lim_{N\to\infty}\hat{G}_N(e^{j\theta}) = \frac{\lim_{N\to\infty}\hat{y}_{\mathrm{free},N}(e^{j\theta})}{\lim_{N\to\infty}\hat{u}_N(e^{j\theta})} + G(e^{j\theta}). \tag{61}$$

Note from (61) that, if $y_{\mathrm{free}}(k)=0$ for all $k\geq 0$ and thus $\hat{y}_{\mathrm{free}}(e^{j\theta})=0$ for all $\theta\in(-\pi,\pi]$, then $\lim_{N\to\infty}\hat{y}_{\mathrm{free},N}(e^{j\theta})=0$ and thus, $\hat{G}_N(e^{j\theta})$ is a consistent estimator of $G(e^{j\theta})$. That is, as the number N of samples used to obtain $\hat{G}_N(e^{j\theta})$ increases, $\hat{G}_N(e^{j\theta})$ converges to $G(e^{j\theta})$. Moreover, if y_{free} is nonzero and G is asymptotically stable, then $\lim_{N\to\infty}\hat{y}_{\mathrm{free},N}(e^{j\theta})$ exists. Therefore, if $\lim_{N\to\infty}|\hat{u}_N(e^{j\theta})|=\infty$, then $\hat{G}_N(e^{j\theta})$ is a consistent estimator of $G(e^{j\theta})$. The examples below show that if u is an impulse or a sinusoidal signal with a frequency $\theta_0\neq\theta$, then $\lim_{N\to\infty}|\hat{u}_N(e^{j\theta})|$ is finite, and thus $\hat{G}_N(e^{j\theta})$ is not a consistent estimator of $G(e^{j\theta})$. However, if u is white noise or a sinusoidal signal with a frequency $\theta_0=\theta$, then $\lim_{N\to\infty}|\hat{u}_N(e^{j\theta})|=\infty$, and thus $\hat{G}_N(e^{j\theta})$ is a consistent estimator of $G(e^{j\theta})$.

Nonparametric Frequency-Domain Identification Using Spectral Analysis

An alternative approach to nonparametric frequency-domain identification is to use (S33) with spectral analysis to estimate the discrete-time frequency response function at a set of frequencies. Consider the system (1)–(3). For all $\theta \in (-\pi, \pi]$, define

$$S_{yu}(e^{j\theta}) \triangleq \sum_{k=0}^{\infty} r_{yu}(k)e^{-j\theta k} = \hat{y}(e^{j\theta})\hat{u}(e^{-j\theta}), \tag{S40}$$

$$S_{uu}(e^{j\theta}) \triangleq \sum_{k=0}^{\infty} r_{uu}(k)e^{-j\theta k} = |\hat{u}(e^{j\theta})|^2, \tag{S41}$$

$$S_{y_{\rm free}u}(e^{j\theta}) \triangleq \sum_{k=0}^{\infty} r_{y_{\rm free}u}(k)e^{-j\theta k} = \hat{y}_{\rm free}(e^{j\theta})\hat{u}(e^{-j\theta}), \tag{S42}$$

$$S_{y_{\rm forced}u}(e^{j\theta}) \triangleq \sum_{k=0}^{\infty} r_{y_{\rm forced}u}(k)e^{-j\theta k} = \hat{y}_{\rm forced}(e^{j\theta})\hat{u}(e^{-j\theta}), \tag{S43}$$

where

$$r_{uu}(k) \triangleq \sum_{i=0}^{\infty} u(i)u(i-k), \quad r_{yu}(k) \triangleq \sum_{i=0}^{\infty} y(i)u(i-k), \tag{S44}$$

$$r_{y_{\rm free}u}(k) \triangleq \sum_{i=0}^{\infty} y_{\rm free}(i)u(i-k), \quad r_{y_{\rm forced}u}(k) \triangleq \sum_{i=0}^{\infty} y_{\rm forced}(i)u(i-k). \tag{S45}$$

Next, let $N \geq 1$ and define

$$S_{uu,N}(e^{j\theta}) \triangleq \sum_{k=0}^{N-1} r_{uu}(k)e^{-j\theta k}, \quad S_{yu,N}(e^{j\theta}) \triangleq \sum_{k=0}^{N-1} r_{yu}(k)e^{-j\theta k}. \tag{S46}$$

Using (5) yields

$$\begin{aligned} S_{yu,N}(e^{j\theta}) &= \sum_{k=0}^{N-1} r_{yu}(k)e^{-j\theta k} \\ &= \sum_{k=0}^{N-1}\sum_{i=0}^{\infty} y(i)u(k-i)e^{-j\theta k} \\ &= \sum_{k=0}^{N-1}\sum_{i=0}^{\infty} y_{\rm free}(i)u(k-i)e^{-j\theta k} \\ &\quad + \sum_{k=0}^{N-1}\sum_{i=0}^{\infty} y_{\rm forced}(i)u(k-i)e^{-j\theta k} \\ &= \sum_{k=0}^{N-1} r_{y_{\rm free}u}(k)e^{-j\theta k} + \sum_{k=0}^{N-1} r_{y_{\rm forced}u}(k)e^{-j\theta k} \\ &= S_{y_{\rm free}u,N}(e^{j\theta}) + S_{y_{\rm forced}u,N}(e^{j\theta}), \end{aligned} \tag{S47}$$

where

$$S_{y_{\rm free}u,N}(e^{j\theta}) \triangleq \sum_{k=0}^{N-1} r_{y_{\rm free}u}(k)e^{-j\theta k}, \quad S_{y_{\rm forced}u,N}(e^{j\theta}) \triangleq \sum_{k=0}^{N-1} r_{y_{\rm forced}u}(k)e^{-j\theta k}, \tag{S48}$$

Moreover,

$$\lim_{N\to\infty} S_{uu,N}(e^{j\theta}) = \lim_{N\to\infty}\sum_{k=0}^{N-1} r_{uu}(k)e^{-j\theta k} = S_{uu}(e^{j\theta}), \tag{S49}$$

$$\lim_{N\to\infty} S_{yu,N}(e^{j\theta}) = \lim_{N\to\infty}\sum_{k=0}^{N-1} r_{yu}(k)e^{-j\theta k} = S_{yu}(e^{j\theta}), \tag{S50}$$

$$\lim_{N\to\infty} S_{y_{\rm free}u,N}(e^{j\theta}) = \lim_{N\to\infty}\sum_{k=0}^{N-1} r_{y_{\rm free}u}(k)e^{-j\theta k} = S_{y_{\rm free}u}(e^{j\theta}), \tag{S51}$$

$$\lim_{N\to\infty} S_{y_{\rm forced}u,N}(e^{j\theta}) = \lim_{N\to\infty}\sum_{k=0}^{N-1} r_{y_{\rm forced}u}(k)e^{-j\theta k} = S_{y_{\rm forced}u}(e^{j\theta}), \tag{S52}$$

$$S_{y_{\rm free}u}(e^{j\theta}) = \hat{y}_{\rm free}(e^{j\theta})\hat{u}(e^{-j\theta}), \tag{S53}$$

$$\begin{aligned} S_{y_{\rm forced}u}(e^{j\theta}) &= \hat{y}_{\rm forced}(e^{j\theta})\hat{u}(e^{-j\theta}) \\ &= G(e^{j\theta})|\hat{u}(e^{j\theta})|^2 \\ &= G(e^{j\theta})S_{uu}(e^{j\theta}). \end{aligned} \tag{S54}$$

Finally, defining

$$\hat{G}_N(e^{j\theta}) \triangleq \frac{S_{yu,N}(e^{j\theta})}{S_{uu,N}(e^{j\theta})}, \tag{S55}$$

and using (S41)–(S43), (S47), (S53), and (S54), it follows from (S55) that

$$\lim_{N\to\infty} \hat{G}_N(e^{j\theta}) = \frac{\lim_{N\to\infty} \hat{y}_{{\rm free},N}(e^{j\theta})}{\lim_{N\to\infty} \hat{u}_N(e^{j\theta})} + G(e^{j\theta}), \tag{S56}$$

which is identical to (61).

For nonparametric frequency-domain identification using spectral analysis, see "Nonparametric Frequency-Domain Identification Using Spectral Analysis."

Using Averaging to Improve Nonparametric Frequency-Domain Estimates of Transfer Functions

In practice, it is often the case that u is a realization of a random process. In this case, numerical experiments show that the estimate of the transfer function obtained using (60) is not smooth [5]. The accuracy of the estimated transfer function can be improved by averaging the estimates of the transfer function obtained from running multiple experiments with different excitation signals [5]. Suppose that $\hat{G}_{N,i}(e^{j\theta})$ is the estimate of the transfer function at frequency $\theta \in (-\pi, \pi]$, obtained from the ith experiment for $i = 1, \ldots, M$. For all $\theta \in (-\pi, \pi]$, the averaged estimate of the transfer function is given by

$$\hat{G}_{{\rm avg},M,N}(e^{j\theta}) \triangleq \frac{1}{M}\sum_{i=1}^{M} \hat{G}_{N,i}(e^{j\theta}). \tag{62}$$

It thus follows from (62) that, for all $\theta \in (-\pi, \pi]$,

$$\lim_{N\to\infty} \hat{G}_{N,i}(e^{j\theta}) = G(e^{j\theta}) + \frac{\hat{y}_{{\rm free},i}(e^{j\theta})}{\hat{u}_i(e^{j\theta})}, \tag{63}$$

where $\hat{y}_{{\rm free},i}(e^{j\theta})$ and $\hat{u}_i(e^{j\theta})$ are the $\mathcal{D}$-transforms of the free response and input from the ith experiment. It then follows from (63) that, for all $\theta \in (-\pi, \pi]$,

Spectral Leakage Effects in Frequency-Domain Identification

Spectral leakage is the error in the estimated frequency-response function that results from using either a periodic input with a noninteger number of periods or a nonperiodic input. As shown in [13, p. 185], leakage errors depend on the initial and final conditions of the system, that is, the state $x(k)$ at $k=0$ and $k=N$, where N is the number of samples used for identification. Example 2.7 in [13, p. 59] shows that leakage errors can be interpreted as a transient effect due to nonzero initial conditions. Theorem 2.6 in [13, p. 59] shows that, if the magnitude of the discrete Fourier transform of the input approaches infinity as the number of samples increases without bound, then the leakage error converges to zero, and thus the frequency response function estimate is asymptotically unbiased. However, the leakage error may be nonzero for each finite data set.

Leakage error can be avoided by using periodic excitation and measurements over an integer number of periods in nonparametric frequency-domain identification. However, in many applications, the excitation signal cannot be specified, and thus leakage errors are unavoidable. Various approaches have been introduced to mitigate the effect of spectral leakage in nonparametric frequency-domain identification. These include the local polynomial approach (that is, seminonparametric identification) [13, Ch.7], [S8], [S9]; Welch's method, which is a weighted average of the estimated frequency response functions [S10]; and the Hanning window [13, p. 41].

On the other hand, parametric frequency-domain identification helps to obtain a more accurate estimate of the frequency-response function than nonparametric frequency-domain identification. Consider [13, p. 59]

$$\hat{y}(e^{j\theta}) = G(e^{j\theta})\hat{u}(e^{j\theta}) + T_G(e^{j\theta})\hat{\delta}(e^{j\theta}), \quad \text{(S57)}$$

where G is the transfer function, $\theta \in (-\pi,\pi]$ is the frequency in rad/sample, and $\hat{u}$ and $\hat{y}$ are the discrete Fourier transforms of the input u and the output y of G, respectively. Furthermore, T_G, which is the transfer function from a fictitious impulse input δ (where $\hat{\delta}(e^{j\theta})=1$ for all $\theta \in (-\pi,\pi]$) to the output y, captures the effect of nonzero initial conditions and spectral leakage in the case of nonzero initial conditions and either a periodic input with a noninteger number of periods or a nonperiodic input [13, p. 185]. Estimation of T_G is thus required [10]–[12]. Since T_G has the same denominator as G, the additional parameters needed to estimate T_G are the coefficients of its numerator. Therefore, if G is of order n, then $3n$ parameters must be estimated to obtain exact estimates of the parameters of G. Examples are shown in [10] and [11].

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$$\begin{aligned}
\lim_{M\to\infty}\frac{1}{M}\sum_{i=1}^{M}\lim_{N\to\infty}\hat{G}_{N,i}(e^{j\theta})
&= \lim_{M\to\infty}\frac{1}{M}\sum_{i=1}^{M}\left(G(e^{j\theta}) + \frac{\hat{y}_{free,i}(e^{j\theta})}{\hat{u}_i(e^{j\theta})}\right)\\
&= G(e^{j\theta}) + \lim_{M\to\infty}\frac{1}{M}\sum_{i=1}^{M}\frac{\hat{y}_{\text{free},i}(e^{j\theta})}{\hat{u}_i(e^{j\theta})}\\
&= G(e^{j\theta}) + \lim_{M\to\infty}\frac{1}{M}\sum_{i=1}^{M}C(e^{j\theta}I-A)^{-1}x_{0,i}\frac{1}{\hat{u}_i(e^{j\theta})}\\
&= G(e^{j\theta}) + C(e^{j\theta}I-A)^{-1}\lim_{M\to\infty}\frac{1}{M}\sum_{i=1}^{M}x_{0,i}\frac{1}{\hat{u}_i(e^{j\theta})}, \qquad (64)
\end{aligned}$$

where $x_{0,i}$ is the initial condition for the ith experiment. Suppose that $(x_{0,i})_{i=0}^{\infty}$ is a realization of the random process $\mathcal{X}_0$. Moreover, suppose that $(1/\hat{u}_i(e^{j\theta}))_{i=0}^{\infty}$ is a realization of a random process $\mathcal{U}$ whose expected value is finite. Since $\mathcal{X}_0$ and $\mathcal{U}$ are uncorrelated, for all $\theta \in (-\pi,\pi]$, (64) becomes

$$\lim_{M\to\infty}\frac{1}{M}\sum_{i=1}^{M}\lim_{N\to\infty}\hat{G}_{N,i}(e^{j\theta}) = G(e^{j\theta}) + C(e^{j\theta}I-A)^{-1}\mathbb{E}[\mathcal{X}_0]\mathbb{E}[\mathcal{U}]. \qquad (65)$$

If $\mathcal{X}_0$ is a zero-mean random process, then (65) implies that, for all $\theta \in (-\pi,\pi]$,

$$\lim_{M\to\infty}\frac{1}{M}\sum_{i=1}^{M}\lim_{N\to\infty}\hat{G}_{N,i}(e^{j\theta}) = G(e^{j\theta}). \qquad (66)$$

That is, averaging the estimates of the transfer function obtained from multiple experiments can help to remove the effect of nonzero initial conditions on the estimate of the transfer function. This is illustrated by a numerical example in the next section.

EFFECT OF INITIAL CONDITIONS ON ESTIMATING THE FREQUENCY RESPONSE FUNCTION

In this section, we investigate the effect of nonzero initial conditions on the estimate of the transfer function obtained using nonparametric frequency-domain methods. This effect is analogous to spectral leakage, which arises due to the use of a noninteger number of periods of a periodic response. For details, see "Spectral Leakage Effects in Frequency-Domain Identification."

The first example considers the case where the input u is an impulse or white noise, and the consistency of the estimate of the transfer function is considered under zero and nonzero initial conditions. The second example shows that the accuracy of the estimated transfer function can be improved by averaging the estimates of the transfer function obtained from running multiple experiments with different excitation signals. The last example compares the estimate of the transfer function obtained using frequency-domain and time-domain methods in this case where the excitation signal is white noise.

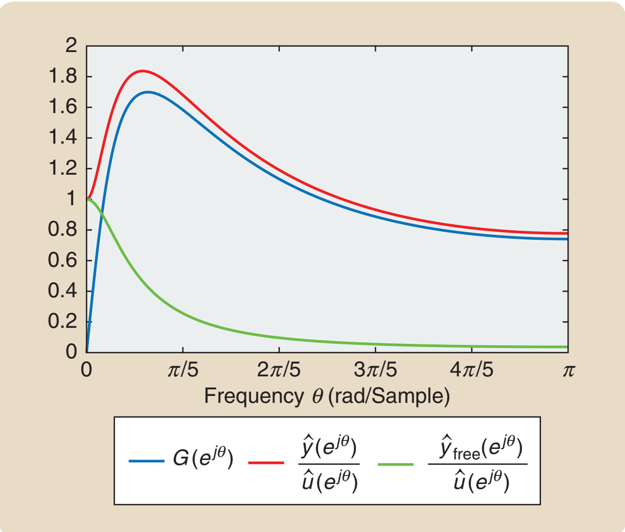

FIGURE 1 Example 1. The magnitude of $G(e^{j\theta})$, $\hat{y}(e^{j\theta})/\hat{u}(e^{j\theta})$, and $\hat{y}_{\rm free}(e^{j\theta})/\hat{u}(e^{j\theta})$. Since the free response is not zero, $G(e^{j\theta})$ and $\hat{y}(e^{j\theta})/\hat{u}(e^{j\theta})$ are not equal.

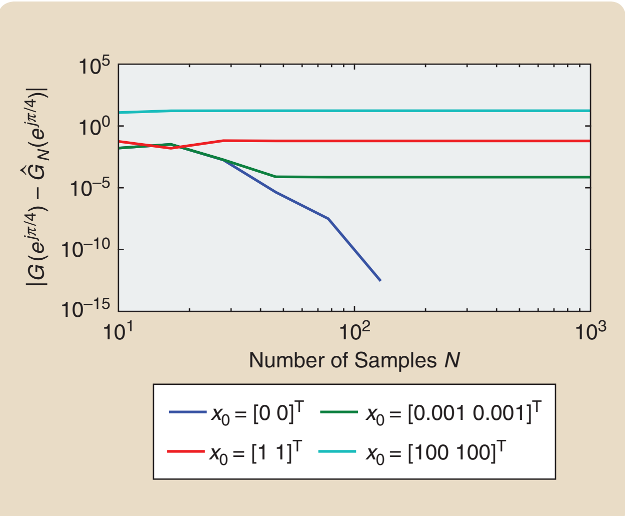

FIGURE 2 Example 1. A plot of $|G(e^{j\pi/4}) - \hat{G}_N(e^{j\pi/4})|$ as N increases, where u is given by (72), and for the initial conditions $x_0 = [0 \;\; 0]^{\rm T}$, $x_0 = [0.001 \;\; 0.001]^{\rm T}$, $x_0 = [1 \;\; 1]^{\rm T}$, and $x_0 = [100 \;\; 100]^{\rm T}$. Note that $\hat{G}_N(e^{j\pi/4})$ is a consistent estimator of $G(e^{j\pi/4})$, only in the case where $x_0 = [0 \;\; 0]^{\rm T}$, that is, the case where $y_{\rm free}(k) = 0$ for all $k \ge 0$.

Example 1

Consider the discrete-time system G with the state-space realization and initial condition given by

$$A = \begin{bmatrix} 1.3 & -0.4 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \;\; -1], \quad D = 0, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{67}$$

Then,

$$G(e^{j\theta}) = C(e^{j\theta}I - A)^{-1}B + D = \frac{e^{j\theta} - 1}{e^{2j\theta} - 1.3e^{j\theta} + 0.4}, \tag{68}$$

and the free response is given by

$$y_{\rm free}(k) = CA^k x_0 = [1 \;\; -1]\begin{bmatrix} 1.3 & -0.4 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3}(0.5^k - 0.8^k). \tag{69}$$

The $\mathcal{Z}$-transform of $y_{\rm free}$ is given by

$$\begin{aligned} \mathcal{Z}\{(y_{\rm free}(k))_{k=0}^{\infty}\} &= \mathcal{Z}\left\{\frac{1}{3}(0.5^k - 0.8^k)\right\} \\ &= \frac{1}{3}\left(\frac{1}{z-0.5} - \frac{1}{z-0.8}\right) = \frac{-0.1}{z^2 - 1.3z + 0.4}. \end{aligned} \tag{70}$$

Replacing z by $e^{j\theta}$ in (70) yields the $\mathcal{D}$-transform

$$\hat{y}_{\rm free}(e^{j\theta}) = \frac{-0.1}{e^{2j\theta} - 1.3e^{j\theta} + 0.4}. \tag{71}$$

Suppose that

$$u(k) = \begin{cases} 1, & k = 0, \\ 0, & k > 0. \end{cases} \tag{72}$$

Then, $\hat{u}(e^{j\theta}) = 1$ for all $\theta \in (-\pi, \pi]$. Therefore,

$$\hat{y}_{\rm forced}(e^{j\theta}) = G(e^{j\theta})\hat{u}(e^{j\theta}) = \frac{e^{j\theta} - 1}{(e^{j\theta} - 0.5)(e^{j\theta} - 0.8)}. \tag{73}$$

Moreover,

$$\begin{aligned} \frac{\hat{y}(e^{j\theta})}{\hat{u}(e^{j\theta})} &= G(e^{j\theta}) + \frac{\hat{y}_{\rm free}(e^{j\theta})}{\hat{u}(e^{j\theta})} \\ &= \frac{e^{j\theta} - 1}{e^{2j\theta} - 1.3e^{j\theta} + 0.4} - \frac{0.1}{e^{2j\theta} - 1.3e^{j\theta} + 0.4} \\ &= \frac{e^{j\theta} - 1.1}{e^{2j\theta} - 1.3e^{j\theta} + 0.4}. \end{aligned} \tag{74}$$

The numerators of (68) and (74) are different. Figure 1 shows the magnitude of $G(e^{j\theta})$, $\hat{y}(e^{j\theta})/\hat{u}(e^{j\theta})$, and $\hat{y}_{\rm free}(e^{j\theta})/\hat{u}(e^{j\theta})$. Since the free response is not zero, it follows that $G(e^{j\theta})$ and $\hat{y}(e^{j\theta})/\hat{u}(e^{j\theta})$ are not equal.

Next, let u be identical to (72), and let $N \ge 1$. For all $\theta \in (-\pi, \pi]$, $\hat{u}(e^{j\theta}) = 1$ and, for all $N \ge 1$, $\hat{u}_N(e^{j\theta}) = 1$. It follows from (60) that $\hat{G}_N(e^{j\theta}) = \hat{y}_N(e^{j\theta})$. Figure 2 shows the difference $|G(e^{j\pi/4}) - \hat{G}_N(e^{j\pi/4})|$ as N increases for the initial conditions $x_0 = [0 \;\; 0]^{\rm T}$, $x_0 = [0.001 \;\; 0.001]^{\rm T}$, $x_0 = [1 \;\; 1]^{\rm T}$, and $x_0 = [100 \;\; 100]^{\rm T}$. Note from Figure 2 that $\hat{G}_N(e^{j\pi/4})$ is a consistent estimator of $G(e^{j\pi/4})$ only in the case where $x_0 = [0 \;\; 0]^{\rm T}$, that is, the case where $y_{\rm free}(k) = 0$ for all $k \ge 0$.

Next, let u be a realization of a stationary white random process with distribution $\mathcal{N}(0,1)$. Using (60), it follows that, for all $N \geq 1$, $\hat{G}_N(e^{j\pi/4}) = \hat{y}_N(e^{j\pi/4})/\hat{u}_N(e^{j\pi/4})$. Moreover, for all $N \geq 1$, (60) implies that $\hat{G}_N(e^{j\pi/6}) = \hat{y}_N(e^{j\pi/6})/\hat{u}_N(e^{j\pi/6})$. Figure 3 shows the difference $|G(e^{j\pi/4}) - \hat{G}_N(e^{j\pi/4})|$ averaged over 1000 experiments as N increases with the initial conditions $x_0 = [0 \;\; 0]^{\mathrm{T}}$, $x_0 = [1 \;\; 1]^{\mathrm{T}}$, $x_0 = [10 \;\; 10]^{\mathrm{T}}$, and $x_0 = [100 \;\; 100]^{\mathrm{T}}$. As shown in Figure 3, a white noise input yields consistent estimates of $G(e^{j\theta})$ for all frequencies $\theta \in (-\pi, \pi]$ and all initial conditions x_0. ■

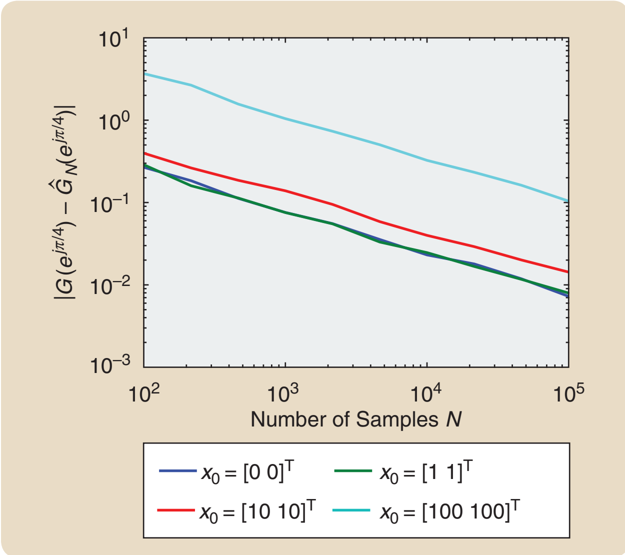

FIGURE 3 Example 1. A plot of $|G(e^{j\pi/4}) - \hat{G}_N(e^{j\pi/4})|$ averaged over 1000 experiments as N increases, where u is a realization of a stationary white random process with distribution $\mathcal{N}(0,1)$ and the initial conditions $x_0 = [0 \;\; 0]^{\mathrm{T}}$, $x_0 = [1 \;\; 1]^{\mathrm{T}}$, $x_0 = [10 \;\; 10]^{\mathrm{T}}$, and $x_0 = [100 \;\; 100]^{\mathrm{T}}$. Note that $\hat{G}_N(e^{j\pi/4})$ is a consistent estimator of $G(e^{j\pi/4})$ for both zero and nonzero initial conditions.

Example 2

Consider the discrete-time system G with the state-space realization (67). Let u be a realization of a stationary white random process with distribution $\mathcal{N}(0,1)$. First consider zero initial conditions and use (60) with 100 different realizations, each of which contains $N = 10^7$ samples of u and y to obtain the estimator $\hat{G}_N(e^{j\pi/4})$ of $G(e^{j\pi/4})$. Figure 4 shows the error $|G(e^{j\pi/4}) - \hat{G}_N(e^{j\pi/4})|$ as a horizontal line. Since the free response is zero, the difference between $G(e^{j\pi/4})$ and $\hat{G}_N(e^{j\pi/4})$ is due to truncating the $\mathcal{D}$-transforms $\hat{u}(e^{j\pi/4})$ and $\hat{y}(e^{j\pi/4})$ of u and y, respectively, to obtain the estimator $\hat{G}_N(e^{j\pi/4})$ of $G(e^{j\pi/4})$.

Next, consider the initial condition $x_0 = [100 \;\; 100]^{\mathrm{T}}$ and use (60) with 100 different realizations, each of which contains $N = 10^7$ samples of u and y to obtain the estimator $\hat{G}_N(e^{j\pi/4})$ of $G(e^{j\pi/4})$. Figure 4 shows the error $|G(e^{j\pi/4}) - \hat{G}_N(e^{j\pi/4})|$ as a horizontal line. The difference between $G(e^{j\pi/4})$ and $\hat{G}_N(e^{j\pi/4})$ is due to the free response of G as well as truncating the $\mathcal{D}$-transforms $\hat{u}(e^{j\pi/4})$ and $\hat{y}(e^{j\pi/4})$ of u and y, respectively, to obtain the estimate $\hat{G}_N(e^{j\pi/4})$.

Consider the initial condition $x_0 = [100 \;\; 100]^{\mathrm{T}}$ and 100 different realizations, each of which contains $N = 10^7$ samples of u and y. For each realization, partition the $N = 10^7$ samples into $M = 1000$ parts, each of which contains $N_p = 10^4$ samples. Then, use (60) with the input and output samples from the kth partition for all $k = 1, \ldots, M$ to obtain the estimator $\hat{G}_{N_p,k}(e^{j\pi/4})$ of $G(e^{j\pi/4})$. The estimates obtained from the first i partitions are then averaged using

$$\hat{G}_{\mathrm{avg},i,N_p}(e^{j\frac{\pi}{4}}) \triangleq \frac{1}{i}\sum_{k=1}^{i} \hat{G}_{N_p,k}(e^{j\frac{\pi}{4}}), \tag{75}$$

where $i = 1, \ldots, M$. Figure 4 shows the error $|G(e^{j\pi/4}) - \hat{G}_{\mathrm{avg},i,N_p}(e^{j\pi/4})|$ for all $i = 1, \ldots, M$. The error in this case decreases as the number of partitions used for averaging increases. ■

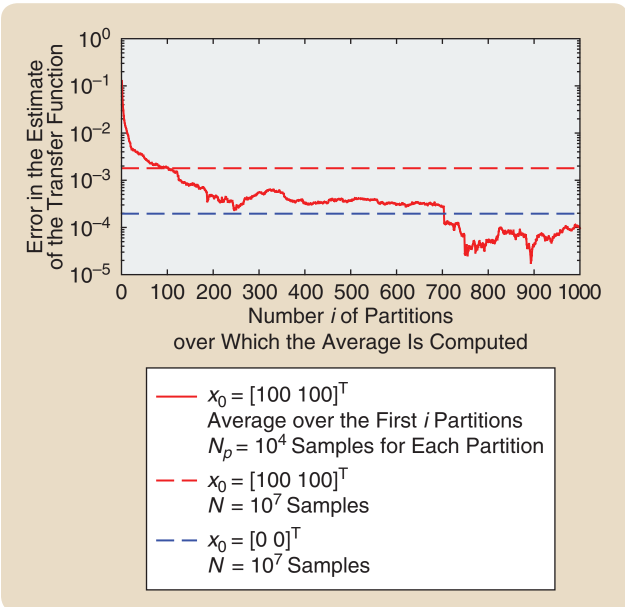

FIGURE 4 A plot of the estimation error $|G(e^{j\pi/4}) - \hat{G}_N(e^{j\pi/4})|$, where the estimate $\hat{G}_N(e^{j\pi/4})$ is obtained using 10^7 samples of the input and output data with zero initial conditions (dashed blue line), 10^7 samples of the input and output data with the initial condition $x_0 = [100 \;\; 100]^{\mathrm{T}}$ (dashed red line), and averaging the estimates obtained from the first i partitions each with 10^4 samples and $i = 1, \ldots, 1000$.

Example 3

Consider the discrete-time asymptotically stable system (1)–(3) with the state-space realization

$$A = \begin{bmatrix} -0.5 & 0.2 \\ 0 & 0.7 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad C = [1.25 \;\; -3], \quad D = 0. \tag{76}$$

Let $x(k) \in \mathbb{R}^2$ be the state vector with the initial state $x(0)$. Let $u_0 \in \mathbb{R}^{1\times N}$ be a realization of a stationary white random process with the Gaussian distribution $\mathcal{N}(0,1)$. Define the input $u \triangleq [u_0 \;\; u_0] \in \mathbb{R}^{1\times 2N}$, that is, u is formed by repeating u_0. Consider zero initial conditions, that is, $x(0) = 0$, and define $y(k) \triangleq Cx(k)$. We partition $y \in \mathbb{R}^{1\times 2N}$ into two

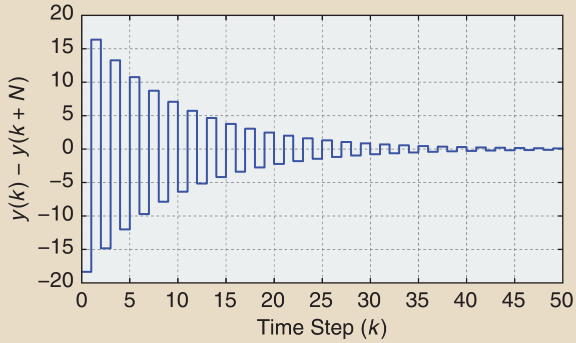

FIGURE 5 A plot of the difference $y(k)-y(k+N)$ for the system (1)–(3) with the realization (76), where $k=0,\ldots,50$, $N=500$, $u=[u_0\ \ u_0]$ is the input, and $x(0)=0$ is the initial state. This plot shows that the difference $y(k)-y(k+N)$ is not zero because $x(k)$ is not zero when data collection begins at time $k=N$.

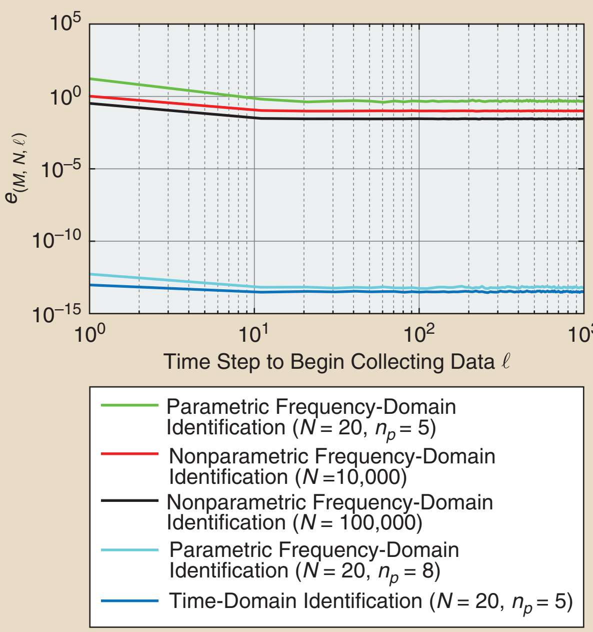

FIGURE 6 A plot of $e_{M,N,\ell}$ obtained using time-domain identification and parametric frequency-domain identification with $N=20$ samples, and nonparametric frequency-domain identification with $N=10{,}000$ and $N=100{,}000$ samples, ℓ varies from 1 to 1000, and $M=100$ experiments. The unknown initial condition is $x(0)=[200\ \ 200]^{\mathrm{T}}$.

halves, where the first half of y is the response of (1)–(3) due to the input u_0 and the zero initial condition $x(0)$, while the second half of y is the response of (1)–(3) due to the input u_0 and the possibly nonzero initial condition $x(N)$. Figure 5 shows the difference $y(k)-y(k+N)$, where $k=0,\ldots,N-1$ and $N=500$ time steps for a given realization u_0. Although $x(0)=0$, the difference $y(k)-y(k+N)$ is not zero, due to the fact that $x(k)$ is not zero when data collection begins at time $k=N$.

Next, define $Y_{N,\ell}\triangleq[y(\ell)\ \cdots\ y(\ell+N-1)]\in\mathbb{R}^{1\times N}$, $U_{N,\ell}\triangleq[u(\ell)\ \cdots\ u(\ell+N-1)]\in\mathbb{R}^{1\times N}$, and $F_N\triangleq 2^p$, where p is the smallest integer such that $2^p\geq N$. For all $k=1,\ldots,F_N$, let $G(e^{j\theta_k})$ be the frequency response of (1)–(3) at θ_k. Moreover, for all $k=1,\ldots,F_N$, let

$$\hat{G}_{\mathrm{avg},M,N,\ell}(e^{j\theta_k})\triangleq\frac{1}{M}\sum_{i=1}^{M}\hat{G}_{N,\ell,i}(e^{j\theta_k}),\tag{77}$$

where M is the number of experiments and $\hat{G}_{N,\ell,i}(e^{j\theta_k})$ is the estimated value of $G(e^{j\theta_k})$ obtained from the ith experiment using either frequency-domain or time-domain identification. For nonparametric frequency-domain identification, $\hat{G}_{N,\ell,i}(e^{j\theta_k})$ is obtained by finding the ratio of the cross-power spectral density of $Y_{N,\ell}$ and $U_{N,\ell}$ to the power spectral density of $U_{N,\ell}$ for the ith experiment. For parametric frequency-domain identification, $\hat{G}_{N,\ell,i}(e^{j\theta_k})$ is obtained from the frequency response of the model constructed using least-squares identification with the frequency-domain data $\hat{U}_{N,\ell}(e^{j\theta})$ and $\hat{Y}_{N,\ell}(e^{j\theta})$, where $\theta\in(-\pi,\pi]$, with T_G in (37) either estimated or set to zero. For $n=2$, the parametric frequency-domain model (37) has $3n+2=8$ parameters. However, setting $T_G=0$ implies that (37) has $2n+1=5$ parameters. For time-domain identification, $\hat{G}_{N,\ell,i}(e^{j\theta_k})$ is obtained from the frequency response of the estimated model constructed using least-squares identification with the time-domain data $U_{N,\ell}$ and $Y_{N,\ell}$. The time-domain model (20) has $2n+1=5$ parameters. Moreover, in the time-domain and parametric frequency-domain models, $b_0=0$, which is justified by the one-step delay in the output data.

Next, define the error

$$e_{M,N,\ell}\triangleq\left(\sum_{k=1}^{F_N}\left|G(e^{j\theta_k})-\hat{G}_{\mathrm{avg},M,N,\ell}(e^{j\theta_k})\right|^2\right)^{1/2},\tag{78}$$

and let n_p be the number of parameters in the model used for identification. Let the unknown initial condition be $x(0)=[200\ \ 200]^{\mathrm{T}}$. Figure 6 shows $e_{M,N,\ell}$ using time-domain identification and parametric frequency-domain identification with T_G is either estimated or set to zero, $N=20$ samples, and ℓ varies from one to 1000. Moreover, Figure 6 shows $e_{M,N,\ell}$ obtained using nonparametric frequency-domain identification with $N=10{,}000$ and $N=100{,}000$ samples as ℓ varies from one to 1000. Note from Figure 6 that the frequency-response function estimates obtained using time-domain identification with $n_p=5$ parameters and parametric frequency-domain identification with $n_p=8$ parameters are much better than those obtained using nonparametric frequency-domain identification and parametric frequency-domain identification with $n_p=5$ parameters. Moreover, although noise-free data is used, Figure 6 shows that waiting for the free response to decay can improve the accuracy of the frequency response function estimates obtained using nonparametric frequency-domain identification, but it does not yield exact estimates. This is due to spectral leakage effects and the effect of nonzero initial condition $x(\ell)$, which occurs

Leakage error can be avoided by using periodic excitation and measurements over an integer number of periods in nonparametric frequency-domain identification.

at the instant that data collection begins and thus corrupts the estimates obtained using finite data sets. On the other hand, Figure 6 shows that the frequency response function estimates obtained using time-domain identification and parametric frequency-domain identification with T_G estimated are not affected by the nonzero initial conditions. Finally, to obtain an exact estimate of G using parametric frequency-domain identification, it is necessary to estimate T_G, which results in $n+1$ more parameters than the number of parameters required by time-domain identification to obtain an exact estimate of G. ■

For the definition of spectral leakage and approaches to avoid it, see "Spectral Leakage Effects in Frequency-Domain Identification."

CONCLUSIONS

The first objective of this article was to emphasize a key distinction between time-domain and frequency-domain models, namely, time-domain models (including both state-space and input–output models) fully account for the initial conditions. Next, the effect of initial conditions on the accuracy of time-domain and frequency-domain models obtained by system identification were investigated. Specifically, it was discussed if, within the context of frequency-domain identification, the effect of nonzero initial conditions can be removed or mitigated by either 1) discarding data collected near the beginning of the experiment or 2) using a sufficiently large data set. Finally, it was shown that, by partitioning the data set and averaging the frequency response estimates, the bias can be removed. Alternatively, semi-nonparametric frequency-domain identification methods [13, Ch. 7] can address leakage errors and the effect of nonzero initial conditions. These methods assume that the frequency response of the system can be locally approximated by a low-order polynomial [13, p. 226].

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