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l-Delay Input and Initial-State Reconstruction for Discrete-Time Linear Systems

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Abstract Prior results on input reconstruction for multi-input, multi-output discrete-time linear systems are extended by defining l-delay input and initial-state observability. This property provides the foundation for reconstructing both unknown inputs and unknown initial conditions, and thus is a stronger notion than l-delay left invertibility, which allows input reconstruction only when the initial state is known. These properties are linked by the main result (Theorem 4), which states that a MIMO discrete-time linear system with at least as many outputs as inputs is l-delay input and initial-state observable if and only if it is l-delay left invertible and has no invariant zeros. In addition, we prove that the minimal delay for input and state reconstruction is identical to the minimal delay for left invertibility. When transmission zeros are present, we numerically demonstrate l-delay input and state reconstruction to show how the input-reconstruction error depends on the locations of the zeros. Specifically, minimum-phase zeros give rise to decaying input reconstruction error, nonminimum-phase zeros give rise to growing reconstruction error, and zeros on the unit circle give rise to persistent reconstruction error.

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1 Introduction

Input reconstruction, that is, the problem of inferring the inputs to a system from available measurements, impacts numerous applications, ranging from machine tool control to cryptography, filtering, and coding. In particular, when unknown inputs represent disturbances or the effects of system uncertainties, estimates can be used to improve the control system performance. In addition, when unknown inputs represent the effect of actuator failures, estimates can be used to implement fault-tolerant control, and thereby enhance system reliability [6]

The reconstruction of unknown inputs under the assumption that the initial state of the system is either known or zero is addressed by left inversion, a long-studied problem in the systems literature [14, 17, 18, 22, 23]. In particular, necessary and sufficient conditions were obtained in [17, 18, 22, 23] for the existence of a linear time-invariant dynamical system that, when cascaded with the original system, produces as its output the input to the original system. These conditions are given in terms of a rank condition on matrices made up of either the system matrices or the system Markov parameters. The issue of internal stability of the resulting cascade system was addressed in [14], which considers both left inversion and the dual problem of right inversion. The issue of internal stability and the effect of nonminimum-phase zeros on input reconstruction were addressed within the context of right inversion along with the related notions of noncausal inversion, preview, preaction, and steering along zeros [4, 7, 8, 12, 16, 20, 25]. A unified approach to noncausal right and left inversion is given in [13].

Although the problem of left inversion has been extensively studied, the more realistic problem of determining the unknown inputs from measurements when the initial state is nonzero has received less attention. In the continuous-time case, [1] defines the notion of an unknown-state, unknown-input-reconstructible system, which is characterized through a necessary and sufficient geometric condition, which implies, in particular, that the system has no invariant zeros. Later, [9] and [3] present input and state asymptotic reconstructors. In particular, the reconstructor of [3] does not require differentiators and ensures convergence up to an arbitrary degree of accuracy. More recently, [24] introduces a modified version of the reconstructor of [3], which provides an asymptotic estimate of both a function of the state and the unknown input; in fact, in [3] it is observed that, in general, the entire state is not needed to reconstruct the unknown inputs.

For discrete-time, minimum-phase systems, [5] introduces a constructive algorithm for establishing whether the system is finite-time observable and left invertible with or without sampling delays, and provides a procedure for obtaining an estimator that allows both the unknown input and state to be retrieved after a finite number of sampling intervals.

The input-reconstruction problem is distinct from, but closely related to, the problem of state observation with unknown inputs. In both problems, the inputs are unknown. However, unlike the input-reconstruction problem, the problem of state observation with unknown inputs does not seek estimates of the unknown inputs. The literature on unknown-input observers is extensive; see, for example, [19, 21].

In this paper we extend the results of [15] on one-step-delay exact input reconstruction in discrete-time systems by deriving a reconstructor for l-step-delay input and initial-state reconstruction, where the delay l accounts for the relative degree of the system. This reconstructor uses an input-output model that depends on the observability matrix and a block-Toeplitz matrix of Markov parameters. The main result, given by Theorem 4, states that a MIMO discrete-time linear system with at least as many outputs as inputs is l-delay input and initial-state observable if and only if it is l-delay left invertible and has no invariant zeros. A computational procedure is given by Proposition 7. In addition, we prove that the minimal delay for input and state reconstruction is identical to the minimal delay for left inversion. When invariant zeros are present, we demonstrate l-delay input and initial-state reconstruction on several numerical examples to show how the input-reconstruction error depends on the locations of the zeros. Specifically, minimum-phase zeros give rise to decaying input-reconstruction error, nonminimum-phase zeros give rise to growing input-reconstruction error, and zeros on the unit circle give rise to persistent input-reconstruction error.

We apply the l-step-delay reconstructor to several numerical examples to show how the input-reconstruction error depends on the locations of the zeros. Specifically, we show that minimum-phase zeros give rise to decaying input-reconstruction error, nonminimum-phase zeros give rise to growing reconstruction error, and zeros on the unit circle give rise to persistent reconstruction error. The effect of nonminimum-phase zeros shows that this feature increases the delay needed to obtain accurate estimates of the unknown inputs.

In Sect. 2, we state the objective of the paper and introduce notation. In Sect. 3, we review necessary and sufficient conditions for l-delay left invertibility. These results unify and extend standard conditions for left invertibility, and thus provide the foundation for subsequent results on input reconstruction. In Sect. 4, we focus on the problem of input reconstruction with l-step delay when the initial conditions of the system are nonzero and unknown. The basic notion of l-delay input and initial-state observability is defined and characterized through necessary and sufficient algebraic conditions. Sections 5 and 6 illustrate the effect of invariant zeros on the solvability of the input and initial-state reconstruction problem. Several numerical examples help to clarify the new notions and illustrate the effectiveness of the proposed methods.

A preliminary version of some of the results in this paper appeared in the conference paper [10].

2 Problem Statement

Consider the linear discrete-time system

$$x_{k+1} = Ax_k + Bu_k, \tag{1}$$

$$y_k = Cx_k + Du_k, \tag{2}$$

where k is a nonnegative integer, $u_k \in \mathbb{R}^m$, $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$. We assume that (A, B, C, D) is minimal. The $p \times m$ transfer function of (1), (2) is $G(z) = C(zI_n - A)^{-1}B + D$. For each nonnegative integer r, we define the output sequence $Y_{[k:k+r]} \in \mathbb{R}^{p(r+1)}$ and the input sequence $U_{[k:k+r]} \in \mathbb{R}^{m(r+1)}$ by

$$Y_{[k:k+r]} \triangleq \begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+r} \end{bmatrix}, \qquad U_{[k:k+r]} \triangleq \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+r} \end{bmatrix}. \tag{3}$$

The objective is to use measurements of the output vector $Y_{[0:r_1]}$ to determine the initial state x_0 and the input vector $U_{[0:r_2]}$, where $r_1 \geq r_2 \geq 0$. The goal is to determine values of r_1 and r_2 for which input reconstruction is possible.

Note that $Y_{[0:r]}$, $U_{[0:r]}$, and x_0 are related by

$$Y_{[0:r]} = \Gamma_r x_0 + M_{r,r} U_{[0:r]} = \Psi_{r,r} \begin{bmatrix} x_0 \\ U_{[0:r]} \end{bmatrix}, \tag{4}$$

where, for $r_1 \geq r_2 \geq 0$, $\Gamma_{r_1} \in \mathbb{R}^{p(r_1+1)\times n}$, $M_{r_1,r_2} \in \mathbb{R}^{p(r_1+1)\times m(r_2+1)}$, and $\Psi_{r_1,r_2} \in \mathbb{R}^{p(r_1+1)\times[n+m(r_2+1)]}$ are defined by

$$\Gamma_{r_1} \triangleq \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{r_1} \end{bmatrix}, \qquad M_{r_1,r_2} \triangleq \begin{bmatrix} H_0 & 0 & \dots & 0 \\ H_1 & H_0 & \ddots & \vdots \\ H_2 & H_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & H_0 \\ \vdots & \vdots & \vdots & \vdots \\ H_{r_1} & H_{r_1-1} & \dots & H_{r_1-r_2} \end{bmatrix}, \tag{5}$$

and

$$\Psi_{r_1,r_2} \triangleq [\, \Gamma_{r_1} \;\; M_{r_1,r_2} \,], \tag{6}$$

and where

$$H_k \triangleq \begin{cases} D, & k = 0, \\ CA^{k-1}B, & k \geq 1, \end{cases} \tag{7}$$

denote the Markov parameters of the system. Note that $M_{r,0} \in \mathbb{R}^{p(r+1)\times m}$. Furthermore, for all $z \in \mathbb{C}$ such that $|z| > \rho(A)$, where $\rho(A)$ denotes the spectral radius of A, it follows that

$$G(z) = \sum_{i=0}^{\infty} z^{-i} H_i. \tag{8}$$

Throughout this paper, d denotes the relative degree of $G(z)$, that is, the smallest nonnegative integer i such that $H_i \neq 0$.

3 *l*-Delay Left Invertibility of Transfer Functions

In this section we introduce and characterize the notion of l-delay left invertibility, where l refers to the number of time steps that must pass before the inverse can be causally realized. These results set the stage for results on l-delay input reconstruction. A transfer function $F(z)$ is *proper* if it is either exactly proper ($F(\infty) \neq 0$) or strictly proper ($F(\infty) = 0$).

Definition 1 Let l be a nonnegative integer. Then $G(z)$ is *l-delay left invertible* if there exists an $m \times p$ proper transfer function $G_l(z)$ (called an *l-delay left inverse* of $G(z)$) such that $G_l(z)G(z) = z^{-l} I_m$ for almost all $z \in \mathbb{C}$. In this case, the smallest nonnegative integer l_0 for which $G(z)$ is l_0-delay left invertible is the *minimal delay* of G. Finally, G is *left invertible* if there exists a nonnegative integer q such that G is q-delay left invertible.

Note that if $G(z)$ is l-delay left invertible, then $G(z)$ is r-delay left invertible for all $r \geq l$. Recall that the normal rank of a polynomial matrix or transfer function in z is the maximal rank over all values of z.

Theorem 1 *The following statements are equivalent*:

(i) *$G(z)$ is left invertible.*
(ii) *The normal rank of $G(z)$ is m.*
(iii) *The normal rank of $\begin{bmatrix} zI-A & -B \\ C & D \end{bmatrix}$ is $n+m$.*

Proof To prove (i) implies (ii), suppose $G(z)$ is l-delay left invertible for some nonnegative integer l. Then there exists an $m \times p$ proper transfer function $G_l(z)$ such that $z^l G_l(z)G(z) = I_m$ for almost all $z \in \mathbb{C}$. Therefore, the normal rank of $G(z)$ is m.

To prove (ii) implies (i), note that, since the normal rank of $G(z)$ is m, it follows that the Smith–McMillan form of $G(z)$ has the form

$$G(z) = S_1(z) \left[\begin{array}{ccc} \frac{p_1(z)}{q_1(z)} & & \\ & \ddots & \\ & & \frac{p_m(z)}{q_m(z)} \\ \hline & 0_{(p-m)\times m} & \end{array} \right] S_2(z), \tag{9}$$

where off-diagonal zero terms are omitted, and S_1 and S_2 are $p \times p$ and $m \times m$ unimodular matrices, respectively. Now define

$$G_l(z) \triangleq z^{-l} S_2^{-1}(z) \left[\begin{array}{c|c} \begin{array}{ccc} \frac{q_1(z)}{p_1(z)} & & \\ & \ddots & \\ & & \frac{q_m(z)}{p_m(z)} \end{array} & 0_{m\times(p-m)} \end{array} \right] S_1^{-1}(z),$$

where l is a positive integer chosen to be sufficiently large such that $G_l(z)$ is proper. Then, $G_l(z)G(z) = z^{-l} I$ for almost all $z \in \mathbb{C}$.

Finally, the equivalence of (ii) and (iii) follows from Proposition 12.10.3 of [2]. □

Theorem 1 shows that, if $G(z)$ is l-delay left invertible, then $G_l(z)G(z) = z^{-l}I_m$ holds for all $z \in \mathbb{C}$ except the poles of $G(z)$ and $G_l(z)$ and, if $l \geq 1$, then $z = 0$.

Now, for $r \geq l \geq 0$, partition $M_{r,r}$ as

$$M_{r,r} = \begin{bmatrix} M_{r,r-l} & \tilde{M}_{r,l} \end{bmatrix}, \tag{10}$$

where $\tilde{M}_{r,l} \in \mathbb{R}^{p(r+1)\times ml}$. Noting that

$$\Psi_{r,r} = \begin{bmatrix} \Gamma_r & M_{r,r} \end{bmatrix} = \begin{bmatrix} \Gamma_r & M_{r,r-l} & \tilde{M}_{r,l} \end{bmatrix} = \begin{bmatrix} \Psi_{r,r-l} & \tilde{M}_{r,l} \end{bmatrix}, \tag{11}$$

it follows that (4) can be written as

$$Y_{[0:r]} = \Gamma_r x_0 + M_{r,r-l}U_{[0:r-l]} + \tilde{M}_{r,l}U_{[r-l+1:r]} \tag{12}$$

$$= \Psi_{r,r-l}\begin{bmatrix} x_0 \\ U_{[0:r-l]} \end{bmatrix} + \tilde{M}_{r,l}\,U_{[r-l+1:r]}. \tag{13}$$

Furthermore,

$$\tilde{M}_{r,l} = \begin{cases} 0_{p(r+1)\times 0}, & l = 0, \\ \begin{bmatrix} 0_{p(r-l+1)\times ml} \\ M_{l-1,l-1} \end{bmatrix}, & l \geq 1, \end{cases} \tag{14}$$

and thus, for all $l \geq 1$,

$$\operatorname{rank}(\tilde{M}_{r,l}) = \operatorname{rank}(M_{l-1,l-1}). \tag{15}$$

Note that $\tilde{M}_{r,0} \in \mathbb{R}^{p(r+1)\times 0}$ is an empty matrix whose range is $\{0\}$ and rank is 0. Let $\mathcal{R}$ denote range and $\mathcal{N}$ denote null space.

Theorem 2 *Let l be a nonnegative integer. Then the following statements are equivalent*:

(i) *$G(z)$ is l-delay left invertible.*

(ii) *There exists a matrix $K \in \mathbb{R}^{m\times p(l+1)}$ such that*

$$KM_{l,l} = \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix}. \tag{16}$$

(iii) *u_0 is uniquely determined by $Y_{[0:l]}$ and x_0.*

(iv) *$\mathcal{N}(M_{l,l}) \subseteq \mathcal{R}\left(\begin{bmatrix} 0_{m\times ml} \\ I_{ml} \end{bmatrix}\right)$.*

(v) *$M_{l,0}$ has full column rank, and*

$$\mathcal{R}(M_{l,0}) \cap \mathcal{R}(\tilde{M}_{l,l}) = \{0\}. \tag{17}$$

(vi) *$p \geq m$ and*

$$m = \begin{cases} \operatorname{rank}(M_{0,0}), & l = 0, \\ \operatorname{rank}(M_{l,l}) - \operatorname{rank}(M_{l-1,l-1}), & l \geq 1. \end{cases} \tag{18}$$

Proof To prove (i) implies (ii), let $G_l(z)$ be an $m \times p$ proper transfer function such that $G_l(z)G(z) = z^{-l}I$ for almost all $z \in \mathbb{C}$. Then there exists $\rho_0 > 0$ such that, for all $z \in \mathbb{C}$ satisfying $|z| > \rho_0$, it follows that $G_l(z) = \sum_{i=0}^{\infty} z^{-i}\tilde{H}_i$, where $\tilde{H}_0, \tilde{H}_1, \ldots$ denote the $m \times p$ Markov parameters of $G_l(z)$. Now it follows from (8) that

$$\begin{aligned} z^{-l}I = G_l(z)G(z) &= \left(\sum_{i=0}^{\infty} z^{-i}\tilde{H}_i\right)\sum_{j=0}^{\infty} z^{-j}H_j \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} z^{-(i+j)}\tilde{H}_i H_j = \sum_{k=0}^{\infty}\sum_{i=0}^{k} z^{-k}\tilde{H}_i H_{k-i}. \end{aligned}$$

Therefore, $\sum_{i=0}^{l}\tilde{H}_i H_{l-i} = I_m$ and, for all $k \ge 0$ such that $k \neq l$, $\sum_{i=0}^{k}\tilde{H}_i H_{k-i} = 0$. Let $K \triangleq [\,\tilde{H}_l\ \tilde{H}_{l-1}\ \cdots\ \tilde{H}_0\,]$. Then $KM_{l,l} = [\,I_m\ 0 \cdots 0\,]$.

To prove (ii) implies (iii), consider input sequences $U_{[0:l]}$ and $\bar{U}_{[0:l]}$ such that

$$M_{l,l}U_{[0:l]} = Y_{[0:l]} - \Gamma_l x_0$$

and

$$M_{l,l}\bar{U}_{[0:l]} = Y_{[0:l]} - \Gamma_l x_0.$$

Therefore,

$$M_{l,l}(\bar{U}_{[0:l]} - U_{[0:l]}) = 0.$$

Hence

$$\begin{aligned} \bar{u}_0 - u_0 &= \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix}(\bar{U}_{[0:l]} - U_{[0:l]}) \\ &= KM_{l,l}(\bar{U}_{[0:l]} - U_{[0:l]}) = 0. \end{aligned}$$

Therefore, $U_{[0:0]} = u_0$ is uniquely determined by $Y_{[0:l]}$ and x_0.

To prove (iii) implies (iv), let $\xi \in \mathcal{N}(M_{l,l})$, and consider the input sequences $U_{[0:l]}$ and $\bar{U}_{[0:l]} \triangleq U_{[0:l]} + \xi$, which satisfy $M_{l,l}U_{[0:l]} = M_{l,l}\bar{U}_{[0:l]} = Y_{[0:l]} - \Gamma_l x_0$. Now, it follows from (iii) that $u_0 = \bar{u}_0$, which implies that $\xi = \bar{U}_{[0:l]} - U_{[0:l]} = \left[\begin{smallmatrix} 0_{m\times 1} \\ \bar{U}_{[1:l]} - U_{[1:l]} \end{smallmatrix}\right]$. Therefore, $\xi \in \mathcal{R}\left(\left[\begin{smallmatrix} 0_{m\times ml} \\ I_{ml} \end{smallmatrix}\right]\right)$.

To prove (iv) implies (v), let $\nu \in \mathcal{R}(M_{l,0}) \cap \mathcal{R}(\tilde{M}_{l,l})$. Then there exists an input sequence $U_{[0:l]}$ such that $M_{l,0}u_0 = \tilde{M}_{l,l}(-U_{[1:l]}) = \nu$, that is, $U_{[0:l]} \in \mathcal{N}(M_{l,l})$. Since by (iv), $U_{[0:l]} \in \mathcal{R}\left(\left[\begin{smallmatrix} 0_{m\times ml} \\ I_{ml} \end{smallmatrix}\right]\right)$, it follows that $u_0 = 0$, and thus $\nu = 0$. Therefore, $\mathcal{R}(M_{l,0}) \cap \mathcal{R}(\tilde{M}_{l,l}) = \{0\}$. Next, let $u_0 \in \mathcal{N}(M_{l,0})$. Then, with $U_{[0:l]} \triangleq \left[\begin{smallmatrix} u_0 \\ 0_{ml\times 1} \end{smallmatrix}\right]$, it follows that $U_{[0:l]} \in \mathcal{N}(M_{l,l})$. Now (iv) implies that $U_{[0:l]} \in \mathcal{R}\left(\left[\begin{smallmatrix} 0_{m\times ml} \\ I_{ml} \end{smallmatrix}\right]\right)$, that is, $u_0 = 0$. Therefore, $\mathcal{N}(M_{l,0}) = \{0\}$, that is, $M_{l,0}$ has full column rank.

To prove (v) implies (i), note that $\mathcal{R}(\tilde{M}_{l,l})$ is a k-dimensional subspace of $\mathbb{R}^{p(l+1)}$, where $k \triangleq \operatorname{rank}(\tilde{M}_{l,l})$. Let $Q \in \mathbb{R}^{p(l+1)\times[p(l+1)-k]}$ be such that the columns of Q

form an orthonormal basis of $\mathcal{R}(\tilde{M}_{l,l})^{\perp}$. Then

$$Q^{\mathrm{T}}\tilde{M}_{l,l}=0. \tag{19}$$

Since $M_{l,0}$ has full column rank and $\mathcal{R}(M_{l,0})\cap\mathcal{R}(\tilde{M}_{l,l})=\{0\}$, it follows that $Q^{\mathrm{T}}M_{l,0}$ has full column rank. Then the generalized inverse $(Q^{\mathrm{T}}M_{l,0})^{\dagger}=(M_{l,0}^{\mathrm{T}}QQ^{\mathrm{T}}M_{l,0})^{-1}\times M_{l,0}^{\mathrm{T}}Q$ satisfies

$$\left(Q^{\mathrm{T}}M_{l,0}\right)^{\dagger}Q^{\mathrm{T}}M_{l,0}=I_m. \tag{20}$$

Now, it follows from (12), (19), and (20) that

$$\begin{aligned}\left(Q^{\mathrm{T}}M_{l,0}\right)^{\dagger}Q^{\mathrm{T}}[Y_{[0:l]}-\Gamma_l x_0]&=\left(Q^{\mathrm{T}}M_{l,0}\right)^{\dagger}Q^{\mathrm{T}}[M_{l,0}u_0+\tilde{M}_{l,l}U_{[1:l]}]\\&=\left(Q^{\mathrm{T}}M_{l,0}\right)^{\dagger}Q^{\mathrm{T}}M_{l,0}u_0=u_0.\end{aligned}$$

Now, partition $(Q^{\mathrm{T}}M_{l,0})^{\dagger}Q^{\mathrm{T}}$ as

$$\left(Q^{\mathrm{T}}M_{l,0}\right)^{\dagger}Q^{\mathrm{T}}=[\,\tilde{K}_l\quad \tilde{K}_{l-1}\quad\cdots\quad\tilde{K}_0\,],$$

where $\tilde{K}_i\in\mathbb{R}^{m\times p}$, $i=0,1,\dots,l$. Then it follows from (20) that $\sum_{i=0}^{l}\tilde{K}_iH_{l-i}=I_m$. Next it follows from (19) that $(Q^{\mathrm{T}}M_{l,0})^{\dagger}Q^{\mathrm{T}}\tilde{M}_{l,l}=0$, that is, for all k such that $1\le k<l$, $\sum_{i=0}^{k}\tilde{K}_iH_{k-i}=0$. Now, let $\tilde{K}_i=0$ for all $i>l$, and define $G_l(z)\triangleq\sum_{i=0}^{\infty}z^{-i}\tilde{K}_i$. Then

$$\begin{aligned}G_l(z)G(z)&=\left(\sum_{i=0}^{\infty}z^{-i}\tilde{K}_i\right)\sum_{j=0}^{\infty}z^{-j}H_j=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}z^{-(i+j)}\tilde{K}_iH_j\\&=\sum_{k=0}^{\infty}\sum_{i=0}^{k}z^{-k}\tilde{K}_iH_{k-i}=z^{-l}\sum_{i=0}^{l}\tilde{K}_iH_{l-i}=z^{-l}I_m.\end{aligned}$$

Hence, $G_l(z)$ is an l-delay left inverse of $G(z)$.

To prove that (v) implies (vi), note that rank $(M_{l,0})=m$. For $l=0$, it follows that rank $(M_{0,0})=m$. Hence, (vi) holds for $l=0$. Next, for $l>0$, it follows from (15) and (17) that

$$\begin{aligned}\operatorname{rank}(M_{l,l})&=\operatorname{rank}\left[\,M_{l,0}\ \tilde{M}_{l,l}\,\right]=\operatorname{rank}(\tilde{M}_{l,l})+m\\&=\operatorname{rank}(M_{l-1,l-1})+m.\end{aligned}$$

To prove (vi) implies (v), suppose $l=0$. Then it follows from (18) that $M_{0,0}$ has full column rank. Since $\tilde{M}_{0,0}$ is an empty matrix, $\mathcal{R}(M_{0,0})\cap\mathcal{R}(\tilde{M}_{0,0})=\{0\}$. Hence, (17) holds for $l=0$. Next, for $l\ge 1$ note that, since rank $(M_{l,l})\le$ rank $(\tilde{M}_{l,l})+$ rank $(M_{l,0})$, it follows from (15) that

$$\operatorname{rank}(M_{l,l})\le\operatorname{rank}(M_{l-1,l-1})+\operatorname{rank}(M_{l,0}). \tag{21}$$

Using (vi) it follows that

$$\operatorname{rank}(M_{l,l}) = \operatorname{rank}(M_{l-1,l-1}) + m. \tag{22}$$

Now, it follows from (21) and (22) that $m \le \operatorname{rank}(M_{l,0})$. Therefore,

$$\operatorname{rank}(M_{l,0}) = m, \tag{23}$$

that is, $M_{l,0}$ has full column rank. Next, it follows from (15) with $r = l$, (22), and (23) that

$$\operatorname{rank}(M_{l,l}) = \operatorname{rank}(\tilde{M}_{l,l}) + \operatorname{rank}(M_{l,0}). \tag{24}$$

Furthermore, since

$$\begin{aligned}\operatorname{rank}(M_{l,l}) &= \operatorname{rank}\begin{bmatrix} M_{l,0} & \tilde{M}_{l,l} \end{bmatrix} \\ &= \operatorname{rank}(M_{l,0}) + \operatorname{rank}(\tilde{M}_{l,l}) - \dim\big(\mathcal{R}(M_{l,0}) \cap \mathcal{R}(\tilde{M}_{l,l})\big),\end{aligned}$$

it follows from (24) that $\dim(\mathcal{R}(M_{l,0}) \cap \mathcal{R}(\tilde{M}_{l,l})) = 0$, and thus (17) holds. □

The equivalence of (i) and (v) of Theorem 2 implies that $G(z)$ is l-delay left invertible if and only if each of the first m columns of $M_{l,l}$ is linearly independent of the remaining columns of $M_{l,l}$.

Note that if $G(z)$ is SISO, then $G_d(z) \stackrel{\triangle}{=} z^{-d}G^{-1}(z)$ is exactly proper and satisfies $G_d(z)G(z) = z^{-d}$, and thus $l_0 = d$. However, the following example shows that l_0 and d may be different in the MIMO case.

Example 1 Consider the transfer function

$$G(z) = \begin{bmatrix} z^{-1} & z^{-2} \\ z^{-2} & z^{-2} \end{bmatrix}$$

with the minimal realization

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Note that $H_0 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $H_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ (and thus the relative degree $d = 1$), $H_2 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, $H_i = 0$ for all $i > 2$, and

$$G^{-1}(z) = \frac{1}{z-1}\begin{bmatrix} z^2 & -z^2 \\ -z^2 & z^3 \end{bmatrix}.$$

Next, define $G_l(z) = z^{-l}G^{-1}(z)$. Then $G_l(z)$ is proper if and only if $l \ge 2$. Hence $l_0 = 2$. Since $d = 1$, it follows that $l_0 > d$.

The following result gives a necessary and sufficient condition for $l_0 = d$.

Proposition 1 *Let l_0 denote the minimal delay of $G(z)$, and let $l \geq l_0$. Then,*

$$\operatorname{rank}\begin{bmatrix} H_0 & \cdots & H_l \end{bmatrix} \geq m, \tag{25}$$

$$\operatorname{rank}\begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_l \end{bmatrix} = m, \tag{26}$$

and $l_0 \leq n$. Furthermore, $l_0 \geq d$ with equality if and only if $\operatorname{rank}(H_d) = m$.

Proof Suppose $l = l_0 = 0$ so that $l_0 < n$. Then it follows from condition (vi) of Theorem 2 that $\operatorname{rank}(M_{0,0}) = \operatorname{rank}(H_0) = m$. Therefore, (25) and (26) hold. Now suppose $l_0 \geq 0$ and $l \geq 1$. Then it follows from condition (vi) of Theorem 2 that $\operatorname{rank}(M_{l,l}) - \operatorname{rank}(M_{l-1,l-1}) = m$. Since

$$M_{l,l} = \left[\begin{array}{ccc|c} & M_{l-1,l-1} & & 0 \\ \hline H_l & \cdots & H_1 & H_0 \end{array}\right], \tag{27}$$

it follows that $\operatorname{rank}(M_{l-1,l-1}) + \operatorname{rank}[\, H_0 \cdots H_l \,] \geq \operatorname{rank}(M_{l,l})$, which implies (25).

Furthermore, since $G(z)$ is l-delay left invertible, it follows from condition (v) of Theorem 2 that $M_{l,0}$ has full column rank. Since

$$M_{l,0} = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_l \end{bmatrix},$$

(26) holds.

Next let $k \geq 0$ and note that

$$\begin{aligned} \operatorname{rank}(M_{k,k}) &= \operatorname{rank}\begin{bmatrix} M_{k,0} & \tilde{M}_{k,k} \end{bmatrix} \\ &\leq \operatorname{rank}(M_{k,0}) + \operatorname{rank}(\tilde{M}_{k,k}) \\ &\leq m + \operatorname{rank}(\tilde{M}_{k,k}). \end{aligned} \tag{28}$$

It then follows from (14) that

$$\operatorname{rank}(M_{k,k}) - \operatorname{rank}(M_{k-1,k-1}) \leq m. \tag{29}$$

Since $G(z)$ is l-delay left invertible, it follows from condition (vi) of Theorem 2 that (29) holds with equality if and only if $k \geq l$. Next, define $q \stackrel{\triangle}{=} \dim(\mathcal{N}(D))$, and note that

$$\operatorname{rank}(M_{0,0}) = \operatorname{rank}(D) = m - q. \tag{30}$$

Suppose $q = 0$. Then rank $(M_{0,0}) = m$, and it follows from (vi) of Theorem 2 that $l_0 = 0$ and thus $l_0 \leq n$. Now suppose $q \geq 1$. Then it follows from (28) with $k = l_0$ that

$$\operatorname{rank}(M_{l_0,l_0}) \leq l_0(m-1) + (m-q) + 1. \tag{31}$$

Next, define $Q \stackrel{\triangle}{=} [\, A^{l_0}B \cdots B\,]$. Since $G(z)$ is left invertible, it follows from (31) that

$$0 = \dim\left(\mathcal{N}\left(\begin{bmatrix} Q \\ M_{l_0,l_0} \end{bmatrix}\right)\right) \geq (l_0 + q - 1) - n. \tag{32}$$

Since $q \geq 1$ it follows that $l_0 \leq n - q + 1 \leq n$.

Since $H_i = 0$ for all $i < d$, it follows from (25) that $l_0 \geq d$. Now suppose $l_0 = d$. Then (25) implies that $m \leq \operatorname{rank}[\, H_0 \cdots H_d\,] = \operatorname{rank}(H_d) \leq m$, and thus rank $(H_d) = m$. Conversely, suppose rank $(H_d) = m$. Then $M_{d,0}$ has full column rank and $\tilde{M}_{d,d} = 0$. Therefore, condition (v) of Theorem 2 holds with $l = d$, that is, $l_0 \leq d$. Since $l_0 \geq d$, it follows that $l_0 = d$. □

Proposition 1 shows that, if $G(z)$ is l-delay left invertible, then both (25) and (26) are satisfied. However, in the following example, both (25) and (26) hold for $l = 1$, whereas $l_0 = 2$.

Example 2 Consider the transfer function

$$G(z) = \begin{bmatrix} z^{-1} & 1 & z^{-2} \\ z^{-2} & 1 & z^{-1} \\ z^{-2} & z^{-1} & z^{-2} \end{bmatrix},$$

for which $n = 5$ and $d = 0$. Note that

$$H_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

and $H_i = 0$ for all $i > 2$. Then rank $(M_{0,0}) = 1$, rank $(M_{1,1}) = 3$, and rank $(M_{2,2}) = 6$. Since rank $(M_{1,1}) -$ rank $(M_{0,0}) < 3$, and rank $(M_{2,2}) -$ rank $(M_{1,1}) = 3$, it follows from condition (vi) of Theorem 2 that $l_0 = 2$. However, (25) holds for $l = 1$, that is, rank $[\, H_0\ H_1\,] = 3$, and (26) holds for $l = 1$, that is, rank $\begin{bmatrix} H_0 \\ H_1 \end{bmatrix} = 3$. Note that $d \leq l_0 \leq n$, as implied by Proposition 1. In fact, $d < l_0 < n$.

Theorems 1 and 2 provide necessary and sufficient conditions under which the identity transfer function can be attained by a left inverse after l steps. However, this noncausal left inverse has limited value for input reconstruction for two reasons. First, the transfer function formulation of a dynamical system does not account for the initial condition of the state of a corresponding state space realization. Therefore, we must consider a state space formulation in order to have a more complete picture of the free response of the system, which is present in practice. Furthermore, within a state space formulation, if $G(z)$ has a transmission zero, then $G(z)$ can have a nonzero input such that, for some nonzero initial condition of a corresponding state space model, the output is identically zero. We explore these issues in the following sections.

4 l-Delay Input Reconstruction with Known Initial State

In this section we show that if $G(z)$ is l-delay left invertible then we can achieve input reconstruction with an l-step delay from a known initial condition x_0 and output measurements $Y_{[0:r]}$.

Lemma 1 *Assume that $G(z)$ is l-delay left invertible. Then, for all $r \geq l$,*

$$\mathcal{N}(M_{r,r}) \subseteq \mathcal{R}\left(\begin{bmatrix} 0_{m(r-l+1)\times ml} \\ I_{ml} \end{bmatrix}\right). \tag{33}$$

Proof Note that

$$M_{l+1,l+1} = \left[\begin{array}{c|c} M_{l+1,0} & \begin{matrix} 0_{p\times m(l+1)} \\ M_{l,l} \end{matrix} \end{array}\right] = \left[\begin{array}{c|ccc|c} M_{l,0} & & \tilde{M}_{l,l} & & 0_{p(l+1)\times m} \\ \hline H_{l+1} & H_l & \cdots & H_1 & H_0 \end{array}\right]. \tag{34}$$

Since $G(z)$ is l-delay left invertible it follows from condition (v) of Theorem 2 that $M_{l,0}$ has full column rank. Then it follows from (34) that $M_{l+1,0}$ has full column rank. Since $\mathcal{R}(M_{l,0}) \cap \mathcal{R}(\tilde{M}_{l,l}) = \{0\}$ and $M_{l,0}$ has full column rank, it follows from (34) that $\mathcal{R}(M_{l+1,0}) \cap \mathcal{R}(\left[\begin{smallmatrix} 0_{p\times m(l+1)} \\ M_{l,l} \end{smallmatrix}\right]) = \{0\}$. Then $M_{l+1,1} = \left[M_{l+1,0} \middle| \begin{smallmatrix} 0_{p\times m} \\ M_{l,0} \end{smallmatrix}\right]$ has full column rank and $\mathcal{R}(M_{l+1,1}) \cap \mathcal{R}(\tilde{M}_{l+1,l}) = \{0\}$. Let $\xi_1 \in \mathbb{C}^{2m}$ and $\xi_2 \in \mathbb{C}^{ml}$ be such that $M_{l+1,1}\xi_1 + \tilde{M}_{l+1,l}\xi_2 = 0$, that is $\left[\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right] \in \mathcal{N}(M_{l+1,l+1})$. Since $\mathcal{R}(M_{l+1,1}) \cap \mathcal{R}(\tilde{M}_{l+1,l}) = \{0\}$, it follows that $M_{l+1,1}\xi_1 = 0$. Since $M_{l+1,1}$ has full column rank, it follows that $\xi_1 = 0$. Therefore, $\mathcal{N}(M_{l+1,l+1}) \subseteq \mathcal{R}(\left[\begin{smallmatrix} 0_{2m\times ml} \\ I_{ml} \end{smallmatrix}\right])$. Hence, (33) holds with $r = l+1$. Proceeding similarly, we attain $\mathcal{N}(M_{r,r}) \subseteq \mathcal{R}(\left[\begin{smallmatrix} 0_{m(r-l+1)\times ml} \\ I_{ml} \end{smallmatrix}\right])$ for all $r \geq l$. □

Theorem 3 *Assume that $G(z)$ is l-delay left invertible. Then, for all $r \geq l$, $U_{[0:r-l]}$ is uniquely determined by x_0 and $Y_{[0:r]}$. Furthermore, the unique solution of (4) with reconstruction delay l is*

$$U_{[0:r-l]} = \mathrm{row}_{[1:(r-l+1)m]}\left[M_{r,r}^{\dagger}(Y_{[0:r]} - \Gamma_r x_0)\right]. \tag{35}$$

Proof Note that (4) implies

$$M_{r,r}U_{[0:r]} = Y_{[0:r]} - \Gamma_r x_0. \tag{36}$$

Then it follows from Proposition 6.1.7(v) of [2] that

$$U_{[0:r]} = M_{r,r}^{\dagger}(Y_{[0:r]} - \Gamma_r x_0) + (I - M_{r,r}^{\dagger}M_{r,r})U_{[0:r]}. \tag{37}$$

Using Proposition 6.1.6(viii) of [2] it follows that $\mathcal{R}(I - M^\dagger_{r,r} M_{r,r}) = \mathcal{N}(M_{r,r})$. Then,

$$\begin{aligned} U_{[0:r-l]} &= \begin{bmatrix} I_{m(r-l+1)} & 0 \end{bmatrix} \big(M^\dagger_{r,r}(Y_{[0:r]} - \Gamma_r x_0) + (I - M^\dagger_{r,r} M_{r,r}) U_{[0:r]} \big) \\ &\in \Big\{ \begin{bmatrix} I_{m(r-l+1)} & 0 \end{bmatrix} M^\dagger_{r,r}(Y_{[0:r]} - \Gamma_r x_0) \\ &\quad + \begin{bmatrix} I_{m(r-l+1)} & 0 \end{bmatrix} \mathcal{R}\left(\begin{bmatrix} 0_{m(r-l+1)\times ml} \\ I_{ml} \end{bmatrix} \right) \Big\} \\ &= \big\{ \begin{bmatrix} I_{m(r-l+1)} & 0 \end{bmatrix} M^\dagger_{r,r}(Y_{[0:r]} - \Gamma_r x_0) \big\}. \end{aligned}$$

Therefore, $U_{[0:r-l]} = \mathrm{row}_{[1:(r-l+1)m]}[M^\dagger_{r,r}(Y_{[0:r]} - \Gamma_r x_0)]$. □

5 l-Delay Input and Initial-State Reconstruction

In this section we take into account the unknown and possibly nonzero initial condition of (1), (2) and consider the problem of input and initial-state reconstruction. The following definition provides the foundation for input and initial-state reconstruction. For the following definition, recall the definition of Ψ_{r_1,r_2} given by (6).

Definition 2 Let l be a nonnegative integer. The system (1), (2) is *l-delay input and initial-state observable* (IISO) if there exists $r \geq l$ such that

$$\mathcal{N}(\Psi_{r,r}) \subseteq \mathcal{R}\left(\begin{bmatrix} 0_{[n+m(r-l+1)]\times lm} \\ I_{lm} \end{bmatrix} \right). \tag{38}$$

In this case, the smallest nonnegative integer l_0' for which (1), (2) is l_0'-delay input and initial-state observable is the *minimal IISO delay*. Furthermore, (1), (2) is *input and initial-state observable* if there exists a nonnegative integer q such that (1), (2) is q-delay input and initial-state observable.

In Definition 2, $r+1$ is the length of the output data window used to reconstruct the input and initial state. Specifically, if $Y_{[0:r]} = 0$ in (4), that is,

$$\begin{bmatrix} x_0 \\ U_{[0:r]} \end{bmatrix} = \begin{bmatrix} x_0 \\ U_{[0:r-l]} \\ U_{[r-l+1:r]} \end{bmatrix} \in \mathcal{N}(\Psi_{r,r}),$$

then (38) implies that $\begin{bmatrix} x_0 \\ U_{[0:r-l]} \end{bmatrix} = 0$, and thus x_0 and $U_{[0:r-l]}$ can be uniquely reconstructed from $Y_{[0:r]}$.

The following result provides necessary and sufficient conditions for l-delay input and initial-state observability.

Proposition 2 *Let l be a nonnegative integer. The system* (1), (2) *is l-delay input and initial-state observable if and only if there exists $r \geq l$ such that both of the following statements hold*:

(i) $\operatorname{rank}(\Psi_{r,r}) = \operatorname{rank}(\Psi_{r,r-l}) + \operatorname{rank}(\tilde{M}_{r,l})$.
(ii) $\Psi_{r,r-l}$ *has full column rank.*

Proof To prove (i), let $\nu \in \mathcal{R}(\Psi_{r,r-l}) \cap \mathcal{R}(\tilde{M}_{r,l})$. Then there exist $\xi_1 \in \mathbb{R}^{n+m(r-l+1)}$ and $\xi_2 \in \mathbb{R}^{lm}$ such that $\Psi_{r,r-l}\xi_1 = \tilde{M}_{r,l}(-\xi_2) = \nu$, that is, $\left[\begin{smallmatrix}\xi_1\\ \xi_2\end{smallmatrix}\right] \in \mathcal{N}(\Psi_{r,r})$. From (38) it follows that $\xi_1 = 0$, and thus $\nu = 0$. Therefore, $\mathcal{R}(\Psi_{r,r-l}) \cap \mathcal{R}(\tilde{M}_{r,l}) = \{0\}$. Consequently, it follows from Fact 2.11.9 in [2] that $\operatorname{rank}(\Psi_{r,r}) = \operatorname{rank}([\,\Psi_{r,r-l}\ \tilde{M}_{r,l}\,]) = \operatorname{rank}(\Psi_{r,r-l}) + \operatorname{rank}(\tilde{M}_{r,l})$.

Next, to prove (ii), it follows from (11) and (38) that

$$\left\{\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} : \xi_1 \in \mathcal{N}(\Psi_{r,r-l}),\ \xi_2 \in \mathcal{N}(\tilde{M}_{r,l})\right\} \subseteq \mathcal{N}(\Psi_{r,r}) \subseteq \mathcal{R}\left(\begin{bmatrix}0_{[n+m(r-l+1)]\times lm}\\ I_{lm}\end{bmatrix}\right).$$

Therefore, $\mathcal{N}(\Psi_{r,r-l}) = \{0\}$, that is, $\Psi_{r,r-l}$ has full column rank.

Conversely, assume that both (i) and (ii) hold. Let $\left[\begin{smallmatrix}\xi_1\\ \xi_2\end{smallmatrix}\right] \in \mathcal{N}(\Psi_{r,r})$, that is, $\Psi_{r,r-l}\xi_1 + \tilde{M}_{r,l}\xi_2 = 0$. It follows from (i) that $\mathcal{R}(\Psi_{r,r-l}) \cap \mathcal{R}(\tilde{M}_{r,l}) = \{0\}$, and thus $\Psi_{r,r-l}\xi_1 = 0$ and $\tilde{M}_{r,l}\xi_2 = 0$. Since, by (ii), $\Psi_{r,r-l}$ has full column rank, it follows that $\xi_1 = 0$, and thus $\mathcal{N}(\Psi_{r,r}) \subseteq \mathcal{R}\left(\left[\begin{smallmatrix}0_{[n+m(r-l+1)]\times lm}\\ I_{lm}\end{smallmatrix}\right]\right)$. Hence (1), (2) is l-delay input and initial-state observable. □

Lemma 2 *Let* $x_0 \in \mathbb{C}^n$, $\zeta \in \mathbb{C}$, $u_0 \in \mathbb{C}^m$, $r \geq 0$ *and*

$$U_{[0:r]} \triangleq \left[\,u_0^{\mathrm{T}}\ \zeta u_0^{\mathrm{T}}\ \cdots\ \zeta^r u_0^{\mathrm{T}}\,\right]^{\mathrm{T}} \in \mathbb{C}^{(r+1)m}. \tag{39}$$

Let $Y_{[0:r]} = \Psi_{r,r}\left[\begin{smallmatrix}x_0\\ U_{[0:r]}\end{smallmatrix}\right]$. *Then*

$$\begin{bmatrix}\zeta I - A & -B\\ C & D\end{bmatrix}\begin{bmatrix}x_0\\ u_0\end{bmatrix} = 0 \tag{40}$$

if and only if, for all $r \geq 0$,

$$Y_{[0:r]} = 0. \tag{41}$$

Proof For $r \geq 1$, define $x_r \triangleq Ax_{r-1} + B\zeta^{r-1}u_0$. Noting that (40) implies $Cx_0 + Du_0 = 0$ and $(\zeta I - A)x_0 - Bu_0 = 0$, it follows that $x_1 = \zeta x_0$. Hence, (40) holds with $\left[\begin{smallmatrix}x_0\\ u_0\end{smallmatrix}\right]$ replaced by $\left[\begin{smallmatrix}x_1\\ \zeta u_0\end{smallmatrix}\right]$. Proceeding similarly, it follows that $x_r = \zeta^r x_0$ and $Cx_r + D\zeta^r u_0 = 0$ for all $r \geq 0$. Next, note that

$$Y_{[0:r]} = \begin{bmatrix}Cx_0 + Du_0\\ \vdots\\ Cx_r + D\zeta^r u_0\end{bmatrix} = 0,$$

which proves that (41) holds for all $r \geq 0$. The converse follows by reversing these steps. □

Recall that $\zeta \in \mathbb{C}$ is an *invariant zero* of (1), (2) if rank $\left[\begin{smallmatrix} \zeta I-A & -B \\ C & D \end{smallmatrix}\right]$ is less than the normal rank of $\left[\begin{smallmatrix} zI-A & -B \\ C & D \end{smallmatrix}\right]$.

Lemma 3 *Assume that* (1), (2) *is input and initial-state observable. Then* (1), (2) *has no invariant zeros.*

Proof Suppose that (1), (2) has an invariant zero $\zeta \in \mathbb{C}$. Then rank $\left[\begin{smallmatrix} \zeta I-A & -B \\ C & D \end{smallmatrix}\right] < n+m$, that is, there exists nonzero $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathbb{C}^{n+m}$ that satisfies (40). Then Lemma 2 implies that $\left[\begin{smallmatrix} x_0 \\ U_{[0:r]} \end{smallmatrix}\right]$ satisfies (41) for all $r \geq 0$. Therefore, (1), (2) is not input and initial-state observable. □

Theorem 4 *The system* (1), (2) *is l-delay input and state observable if and only if $G(z)$ is l-delay left invertible and* (1), (2) *has no invariant zeros. Furthermore,*

$$l_0 = l_0' \leq n. \tag{42}$$

Proof To prove sufficiency, note that, since $G(z)$ is l-delay left invertible, it follows from (iii) of Theorem 1 that the normal rank of $\left[\begin{smallmatrix} zI-A & -B \\ C & D \end{smallmatrix}\right]$ is $n+m$. Since (1), (2) has no invariant zeros, it follows that rank $\left[\begin{smallmatrix} \zeta I-A & -B \\ C & D \end{smallmatrix}\right] = n+m$ for all $\zeta \in \mathbb{C}$. Then it follows from Theorem A.1 that there exists $r \geq n$ such that $\mathcal{R}(\Gamma_r) \cap \mathcal{R}(M_{r,r}) = \{0\}$ and Γ_r has full column rank. Since $G(z)$ is l-delay left invertible, it follows from condition (v) of Theorem 2 that, for all $r \geq l_0$, $\mathcal{R}(M_{r,r-l_0}) \cap \mathcal{R}(\tilde{M}_{r,l_0}) = \{0\}$ and $M_{r,r-l_0}$ has full column rank. Furthermore, it follows from Proposition 1 that $l_0 \leq n$. Therefore, for some $r \geq n$, $\mathcal{R}(\Psi_{r,r-l_0}) \cap \mathcal{R}(\tilde{M}_{r,l_0}) = \{0\}$ and $\Psi_{r,r-l_0}$ has full column rank. Hence, it follows from Proposition 2 with $l = l_0 \leq r$, that (1), (2) is l-delay input and initial-state observable.

To prove necessity, suppose that (1), (2) is l-delay input and initial-state observable for some $l \geq 0$. Then it follows from Lemma 3 that (1), (2) has no invariant zeros. Next, it follows from condition (i) of Proposition 2 that there exists $r \geq l$ such that $\mathcal{R}(\Psi_{r,r-l}) \cap \mathcal{R}(\tilde{M}_{r,l}) = \{0\}$, and thus (since $\mathcal{R}(M_{r,r-l}) \subseteq \mathcal{R}(\Psi_{r,r-l})$) $\mathcal{R}(M_{r,r-l}) \cap \mathcal{R}(\tilde{M}_{r,l}) = \{0\}$. Note that

$$M_{r,r-l} = \left[\begin{array}{c|c} M_{r,r-l-1} & \begin{matrix} 0_{p(r-l)\times m} \\ M_{l,0} \end{matrix} \end{array}\right], \qquad \tilde{M}_{r,l} = \begin{bmatrix} 0_{p(r-l)\times ml} \\ \tilde{M}_{l,l} \end{bmatrix}. \tag{43}$$

It then follows that

$$\begin{aligned} \{0\} &\subseteq \{0_{p(r-l)}\} \times \big(\mathcal{R}(M_{l,0}) \cap \mathcal{R}(\tilde{M}_{l,l})\big) \\ &= \mathcal{R}\left(\begin{bmatrix} 0_{p(r-l)\times m} \\ M_{l,0} \end{bmatrix}\right) \cap \mathcal{R}\left(\begin{bmatrix} 0_{p(r-l)\times ml} \\ \tilde{M}_{l,l} \end{bmatrix}\right) \\ &\subseteq \big(\mathcal{R}(M_{r,r-l}) \cap \mathcal{R}(\tilde{M}_{r,l})\big) = \{0\}. \end{aligned}$$

Hence (17) is satisfied. Next, it follows from condition (ii) of Proposition 2 that $M_{r,r-l}$ has full column rank. Thus, it follows from (43) that $M_{l,0}$ has full column rank. It then follows from condition (v) of Theorem 2 that $G(z)$ is l-delay left invertible.

Finally, it follows from Proposition 1 and the arguments used in the previous paragraphs that $l_0 = l_0' \le n$. □

Let $\operatorname{def}(M)$ denote the defect of the matrix M, that is, the dimension of the null space of M.

Proposition 3 *Assume that $G(z)$ is SISO and (1), (2) is input and initial-state observable. Then*

$$l_0 = l_0' = d = n. \tag{44}$$

Furthermore, for all $r \ge n-1$, $\operatorname{def}(\Psi_{r,r}) = n$.

Proof Let $l \ge 0$ be such that (1), (2) is l-delay input and state observable. Then it follows from Theorem 4 that $G(z)$ is l-delay left invertible and has no invariant zeros. Since $G(z)$ is SISO it follows that $z^{-d}G^{-1}(z)$ is exactly proper, and thus $l \ge l_0 = d$. Since $G(z)$ has no zeros it follows that $n = d$. It then follows from Theorem 4 that $l_0 = l_0' = d = n$.

Finally, let $r \ge n$. Then $\Psi_{r,r} = \left[\Gamma_r \middle| M_{r,r-n} \middle| {0_{(r+1-n)\times n} \atop \tilde{M}_{n,n}}\right]$. Since $H_n \ne 0$ it follows that $M_{r,r-n}$ has full column rank and thus $\Psi_{r,r-n} = [\,\Gamma_r \; M_{r,r-n}\,]$ has full column rank. Since $\tilde{M}_{n,n} = 0$ it follows that

$$\operatorname{def}(\Psi_{r,r}) = \dim\bigl(\mathcal{N}(\Psi_{r,r})\bigr) = \operatorname{rank}\begin{bmatrix} 0_{(r+1-n)\times n} \\ I_n \end{bmatrix} = n.$$ □

Proposition 4 *Let $m = 1$ and assume that (1), (2) is input and initial-state observable. Then*

$$l_0 = l_0' = d \le n. \tag{45}$$

Proof Let $l \ge 0$ be such that (1), (2) is l-delay input and state observable. Then it follows from Theorem 4 that $G(z)$ is l-delay left invertible and (1), (2) has no invariant zeros. Since $G(z)$ is SIMO it follows that H_d has full column rank. Then it follows from Proposition 1 that $l_0 = d$. Therefore, it follows from Theorem 4 that $l_0 = l_0' = d \le n$. □

The following SIMO example is input and initial-state observable and satisfies $d < n$.

Example 3 Consider the transfer function

$$G(z) = \begin{bmatrix} \frac{1}{z} \\ \frac{1}{z^2} \end{bmatrix},$$

with the minimal realization

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then $H_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ has full column rank and $d = 1$. It then follows from Proposition 1 that $G(z)$ is 1-delay left invertible and $l_0 = 1$. Next it follows from Proposition 4 that $l_0 = l_0' = d \le n = 2$. In fact, $d < n$.

Proposition 5 *Assume that* (1), (2) *is input and initial-state observable. Then*

$$d \le l_0 = l_0' \le n. \tag{46}$$

Proof Since (1), (2) is input and initial-state observable, it follows from Theorem 4 that $l_0 = l_0' \le n$ and $G(z)$ is l_0-delay left invertible. It then follows from Proposition 1 that $d \le l_0$. Hence (46) is satisfied. □

The following result provides necessary conditions for l-delay input and initial-state observability.

Proposition 6 *Let l be a nonnegative integer, and assume that* (1), (2) *is l-delay input and initial-state observable. Then the following statements hold*:

(i) (1), (2) *is k-delay input and initial-state observable for all $k \ge l$.*
(ii) $\operatorname{rank}(\Psi_{r,r-l}) = \operatorname{rank}(\Psi_{r,r-l-1}) + m$ *for all* $r \ge n$.
(iii) $m \le p$.
(iv) *If* $m = p$, *then* $n \le ml$.
(v) $\operatorname{rank}(\Gamma_{n-1}) = n$.

Proof To prove (i), since (1), (2) is l-delay input and initial-state observable, there exists $r \ge l$ such that (38) holds. Now, note that

$$\Psi_{r+1,r-l} = \left[\begin{array}{cccc} \multicolumn{4}{c}{\Psi_{r,r-l}} \\ \hline CA^{r+1} & H_{r+1} & \cdots & H_{l+1} \end{array}\right] \tag{47}$$

and

$$\tilde{M}_{r+1,l+1} = \left[\begin{array}{c|c} 0 & \\ \vdots & 0_{p\times ml} \\ H_0 & \\ \vdots & \tilde{M}_{r,l} \\ H_l & \end{array}\right] = \left[\begin{array}{ccc|c} & \tilde{M}_{r,l} & & 0_{p(r+1)\times m} \\ H_l & \cdots & H_1 & H_0 \end{array}\right]. \tag{48}$$

Then it follows from condition (ii) of Proposition 2 that $\Psi_{r,r-l}$ has full column rank. It then follows from (47) that $\Psi_{r+1,r-l}$ has full column rank. Next, it follows from condition (i) of Proposition 2 that $\mathcal{R}(\Psi_{r,r-l}) \cap \mathcal{R}(\tilde{M}_{r,l}) = \{0\}$. Then it follows from (47)–(48) that $\mathcal{R}(\Psi_{r+1,r-l}) \cap \mathcal{R}(\tilde{M}_{r+1,l+1}) = \{0\}$. Since

$$\begin{bmatrix} 0 \\ \vdots \\ H_0 \\ \vdots \\ H_l \end{bmatrix}$$

has full column rank, it follows from (48) that $\Psi_{r+1,r+1-l}$ has full column rank and $\operatorname{rank}(\Psi_{r+1,r+1}) = \operatorname{rank}(\Psi_{r+1,r+1-l}) + \operatorname{rank}(\tilde{M}_{r+1,l})$. Hence, $\mathcal{N}(\Psi_{r+1,r+1}) \subseteq \mathcal{R}\left(\begin{bmatrix} 0_{[n+(r-l+2)m)]\times lm} \\ I_{lm} \end{bmatrix}\right)$. Proceeding similarly, for all $s \geq r$,

$$\mathcal{N}(\Psi_{s,s}) \subseteq \mathcal{R}\left(\begin{bmatrix} 0_{[n+(s-l+1)m)]\times lm} \\ I_{lm} \end{bmatrix}\right). \tag{49}$$

Let $s \geq k \geq l$. Then

$$\mathcal{R}\left(\begin{bmatrix} 0_{[n+(s-l+1)m)]\times lm} \\ I_{lm} \end{bmatrix}\right) \subseteq \mathcal{R}\left(\begin{bmatrix} 0_{[n+(s-k+1)m)]\times km} \\ I_{km} \end{bmatrix}\right),$$

and it follows from (38) that

$$\mathcal{N}(\Psi_{s,s}) \subseteq \mathcal{R}\left(\begin{bmatrix} 0_{[n+(s-k+1)m)]\times km} \\ I_{km} \end{bmatrix}\right).$$

Hence, (1), (2) is k-delay input and initial-state observable.

Next, to prove (ii), it follows from (ii) of Proposition 2 that

$$\Psi_{r,r-l} = \begin{bmatrix} \Psi_{r,r-l-1} & \operatorname{col}_{[n+(r-l)m+1:n+(r-l+1)m]}(\Psi_{r,r-l}) \end{bmatrix}$$

has full column rank and thus $\Psi_{r,r-l-1}$ and the last m columns of $\Psi_{r,r-l}$ have full column rank. Therefore, $\operatorname{rank}(\Psi_{r,r-l}) = \operatorname{rank}(\Psi_{r,r-l-1}) + m$.

Next, to prove (iii), since Proposition 2 holds for all $r \geq n$, it follows that $\Psi_{r,r-l}$ has full column rank, that is, $n + m(r-l+1) \leq p(r+1)$ for all $r > l$. Hence, $m \leq p$.

Next, to prove (iv), it follows from (ii) of Proposition 2 that $n + m(r-l+1) \leq p(r+1)$. Now with $m = p$, it follows that $n \leq ml$.

Next, to prove (v), it follows from (ii) of Proposition 2 that $\Psi_{r,r-l}$ has full column rank, and thus Γ_r has full column rank. Therefore, $\operatorname{rank}(\Gamma_{n-1}) = n$. □

The following result provides an explicit reconstructor for the l-delay input and initial-state reconstruction problem.

Proposition 7 *Assume that* (1), (2) *is l-delay input and initial-state observable and* (38) *holds for some $r \geq l$. Then x_0 and $U_{[0:r-l]}$ are uniquely determined by $Y_{[0:r]}$. Furthermore, the unique solution of* (12) *with reconstruction delay l is given by*

$$\begin{bmatrix} x_0 \\ U_{[0:r-l]} \end{bmatrix} = \left(Q^{\mathrm{T}}\Psi_{r,r-l}\right)^{\dagger} Q^{\mathrm{T}} Y_{[0:r]}, \tag{50}$$

where † represents the Moore–Penrose generalized inverse, $k \triangleq \operatorname{rank}(\tilde{M}_{r,l})$, *and the columns of* $Q \in \mathbb{R}^{p(r+1)\times[p(r+1)-k]}$ *are an orthonormal basis of* $\mathcal{N}(\tilde{M}_{r,l}^{\mathrm{T}})$.

Proof Let $r \geq l \geq d$, and $\xi = \begin{bmatrix} x_0 \\ U_{[0:r-l]} \end{bmatrix}$. By assumption, each row of Q^{T} is an element of $\mathcal{R}(\tilde{M}_{r,l})^{\perp}$. Therefore,

$$Q^{\mathrm{T}}\tilde{M}_{r,l} = 0. \tag{51}$$

Since by (ii) of Proposition 2, $\Psi_{r,r-l}$ has full column rank, it follows that $Q^{\mathrm{T}}\Psi_{r,r-l}$ has full column rank. Then the generalized inverse $(Q^{\mathrm{T}}\Psi_{r,r-l})^{\dagger} = (\Psi_{r,r-l}^{\mathrm{T}}Q \times Q^{\mathrm{T}}\Psi_{r,r-l})^{-1}\Psi_{r,r-l}^{\mathrm{T}}Q$ satisfies

$$\left(Q^{\mathrm{T}}\Psi_{r,r-l}\right)^{\dagger}Q^{\mathrm{T}}\Psi_{r,r-l} = I_{n+(r-l+1)m}. \tag{52}$$

Now, it follows from (12), (51), and (52) that

$$\begin{aligned}\left(Q^{\mathrm{T}}\Psi_{r,r-l}\right)^{\dagger}Q^{\mathrm{T}}Y_{[0:r]} &= \left(Q^{\mathrm{T}}\Psi_{l,0}\right)^{\dagger}Q^{\mathrm{T}}\left(\Psi_{r,r-l}\xi + \tilde{M}_{r,l}U_{[r-l+1:r]}\right) \\ &= \left(Q^{\mathrm{T}}\Psi_{r,r-l}\right)^{\dagger}Q^{\mathrm{T}}\Psi_{r,r-l}\xi = \xi.\end{aligned}$$

□

The special case $l = 1$, $d = 1$ is given in [15]. Proposition 2 shows that, for this special case (1), (2) is 1-delay input and initial-state observable if and only if there exists $r \geq 1$ such that $\Psi_{r,r-1}$ has full column rank. Furthermore, since $\tilde{M}_{r,1} = 0_{(r+1)p\times m}$, it follows from Proposition 5 that $Q = I_{(r+1)p}$ and it follows from (50) that the unique solution of (12) for each $r \geq 1$ is given by

$$\begin{bmatrix} x_0 \\ U_{[0:r-1]} \end{bmatrix} = \Psi_{r,r-1}^{\dagger}Y_{[0:r]}. \tag{53}$$

Example 4 Consider the transfer function

$$G(z) = \frac{1}{(z-0.5)(z-0.6)},$$

with the minimal realization

$$A = \begin{bmatrix} 1.1 & -0.6 \\ 0.5 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \end{bmatrix}.$$

Since $\Psi_{2,0}$ has full column rank, $\Psi_{2,1}$ does not have full column rank, and $\operatorname{rank}(\Psi_{2,2}) = \operatorname{rank}(\Psi_{2,0}) + \operatorname{rank}(\tilde{M}_{2,2})$, it follows from Proposition 2 that $G(z)$ is 2-delay input and initial-state observable but not 1-delay input and initial-state observable. Therefore, $l_0' = 2$. Since $G(z)$ is SISO with $d = 2$, it follows that $l_0 = d$ and thus $l_0 = l_0' = 2$. For both cases we compute the least-squares expression in (50), and Fig. 1 compares 1-delay and 2-delay input and state reconstruction. As expected, 2-delay input and state reconstruction described by (50) correctly estimates the input and initial state, whereas, for $l = 1$, the solution given by (50) fails.

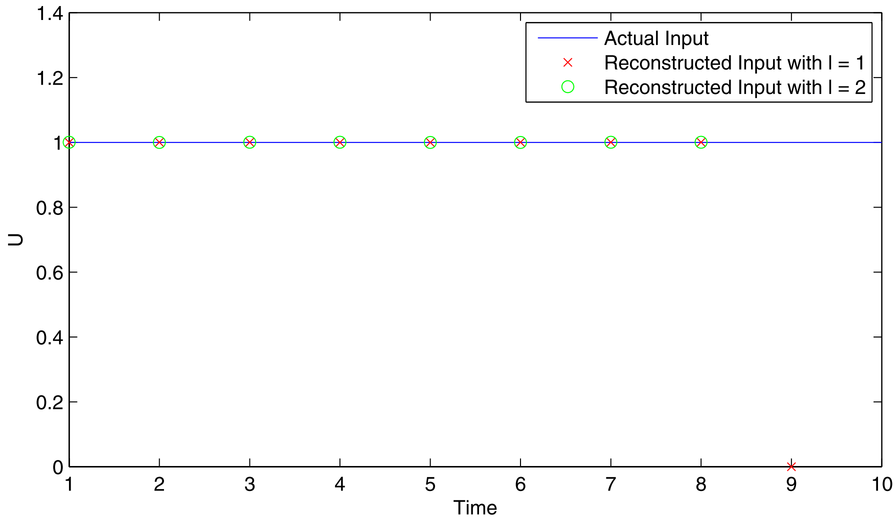

Fig. 1 Comparison between the actual input sequence $U_{[0:10]}$ and the reconstructed input sequence $\hat{U}_{[0:9]}$ for $l=1$ and $U_{[0:8]}$ for $l=2$ delay input and state reconstruction in Example 4

Example 4 demonstrates the case $l_0 = l_0'$, and Fig. 1 shows 1-delay and 2-delay input and state reconstruction for a 2-delay input and initial-state observable system with $d=1$. In this case, 1-delay input and state reconstruction estimate is given by

$$\begin{bmatrix} \hat{x}_0 \\ \hat{U}_{[0:r-1]} \end{bmatrix} = \Psi_{r,r-1}^{\dagger} Y_{[0:r]}. \tag{54}$$

6 Invariant Zeros and Unobservable Inputs

If the system (1), (2) has invariant zeros, then it follows from Lemma 3 that (1), (2) is not input and initial-state observable. In this case, the input and initial state cannot be exactly reconstructed from output measurements. In this section we relate the error in input reconstruction to the locations of the invariant zeros relative to the unit circle.

Definition 3 Let $r \geq 0$. Then the input $U_{[0:r]}$ is *unobservable* if there exists a nonzero initial condition x_0 such that $\left[\begin{smallmatrix} x_0 \\ U_{[0:r]} \end{smallmatrix}\right] \in \mathcal{N}(\Psi_{r,r})$.

Theorem A.1 implies that, if $G(z)$ is left invertible and (1), (2) has at least one invariant zero, then there exists an initial condition and an input sequence, not both zero, such that the output is identically zero. In fact, the following immediate result shows that such initial conditions and input sequences are nonzero elements in the null space of $\Psi_{r,r}$. These initial conditions and input sequences are thus unobservable for the purpose of input and initial-state reconstruction.

Proposition 8 *Let* $\xi \in \mathbb{R}^n$ *and* $\nu \in \mathbb{R}^{(r+1)m}$ *be such that*

$$\Psi_{r,r}\begin{bmatrix}\xi\\ \nu\end{bmatrix} = \Gamma_r\xi + M_{r,r}\nu = 0. \tag{55}$$

Then, with $x_0 = \xi$ *and* $U_{[0:r]} = \nu$, *it follows that* $Y_{[0:r]} = 0$.

For SIMO systems that have a zero outside of the spectral radius of A, the following result provides an explicit expression for an unobservable input and initial state.

Proposition 9 *Let* $m = 1$ *and let* $\zeta \in \mathbb{C}$ *be an invariant zero of* (1), (2) *such that* $|\zeta| > \rho(A)$. *Furthermore, let* $c \in \mathbb{C}$, *define the input sequence*

$$u_k = \mathrm{Re}\big(c\zeta^k\big), \quad k = 0, 1, \ldots, \tag{56}$$

and let

$$x_0 = \mathrm{Re}\left[c\sum_{i=1}^{\infty}\zeta^{-i}A^{i-1}B\right]. \tag{57}$$

Then, for all $r \geq 0$, $Y_{[0:r]} = 0$, *that is*,

$$\begin{bmatrix}x_0\\ U_{[0:r]}\end{bmatrix} \in \mathcal{N}(\Psi_{r,r}). \tag{58}$$

Proof Since $G(\zeta) = 0$, it follows from (8) that

$$c\zeta^r\left(\sum_{i=0}^{r}\zeta^{-i}H_i + \sum_{i=r+1}^{\infty}\zeta^{-i}H_i\right) = 0, \tag{59}$$

that is,

$$[\,H_r \quad H_{r-1} \quad \cdots \quad H_0\,]\nu_{[0:r]} + CA^r x_0 = 0, \tag{60}$$

where

$$\nu_{[0:r]} \triangleq \mathrm{Re}\left(c\begin{bmatrix}1\\ \zeta^1\\ \vdots\\ \zeta^r\end{bmatrix}\right).$$

Note that

$$\sum_{i=1}^{\infty}\zeta^{-i}A^{i-1}B = \sum_{i=0}^{\infty}\zeta^{-(i+1)}A^iB.$$

Now dividing (59) by ζ yields

$$c\zeta^r\left(\sum_{i=0}^{r-1}\zeta^{-(i+1)}H_i + \sum_{i=r}^{\infty}\zeta^{-(i+1)}H_i\right) = 0,$$

that is,

$$[\,H_{r-1} \quad H_{r-2} \quad \cdots \quad H_0\,]\nu_{[0:r-1]} + CA^{r-1}x_0 = 0. \tag{61}$$

Proceeding similarly, it follows from (60), (61) and (8) that

$$\Psi_{r,r}\begin{bmatrix} x_0 \\ \nu_{[0:r]} \end{bmatrix} = 0.$$

Now, with $U_{[0:r]} = \nu_{[0:r]}$ it follows that there exists x_0 such that $\left[\begin{smallmatrix} x_0 \\ U_{[0:r]} \end{smallmatrix}\right] \in \mathcal{N}(\Psi_{r,r})$. □

Note that, if ζ is a zero of (1), (2), then rank $\left[\begin{smallmatrix} \zeta I - A & -B \\ C & D \end{smallmatrix}\right] < n + m$. Furthermore, the values of x_0 given by (57) satisfy

$$\begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x_0 \\ c \end{bmatrix} = 0. \tag{62}$$

Note that if ζ is an element of the open unit disk, then (56) decays, whereas (56) grows if ζ is outside the unit disk. If however, ζ is on the unit disk, then (56) neither grows nor decays. The following examples illustrate these properties.

Consider a realization of the type of (1), (2), where A, B are given by

$$A = \begin{bmatrix} 0.1 & 1.7029 & 0 \\ 0 & 0.2 & 1 \\ 0 & 0 & 0.3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \tag{63}$$

and where C is chosen to set the locations of the invariant zeros relative to the unit circle. Note that $x_0 = [\,10 \quad 5 \quad 15\,]$ is used only for simulation, but knowledge of x_0 is not assumed to be available for input reconstruction.

Example 5 Consider the transfer function

$$G(z) = \frac{z^2 - z + 0.5}{(z-0.1)(z-0.2)(z-0.3)}, \tag{64}$$

which has minimum-phase zeros $0.5 + 0.5\jmath$, $0.5 - 0.5\jmath$ and $C = [\,0.25 \quad -0.5 \quad 2\,]$. Figure 2 compares the 1-delay reconstructed input sequence with the actual input sequence. Note that the input-reconstruction error is largest for small values of k, which corresponds to the fact that the system has minimum-phase zeros.

Example 6 Consider the transfer function

$$G(z) = \frac{z - 1}{(z-0.1)(z-0.2)(z-0.3)}, \tag{65}$$

for which $C = [\,0 \quad 0.5 \quad -4\,]$. Figure 3 compares the 2-delay reconstructed input sequence with the actual input sequence. Note that the input-reconstruction error is

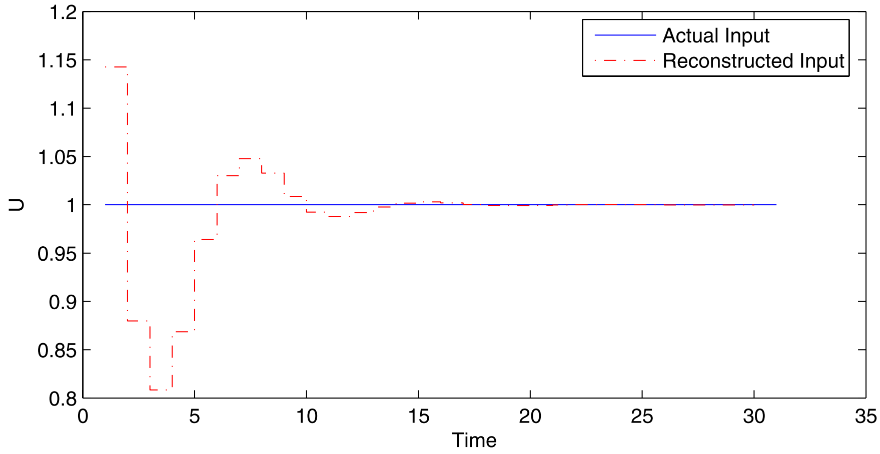

Fig. 2 Comparison between the actual input sequence and the 1-delay reconstructed input of the system given by Example 5, with minimum-phase zeros $0.5 + 0.5\jmath$ and $0.5 - 0.5\jmath$. In this case, the input-reconstruction error is decaying and oscillatory

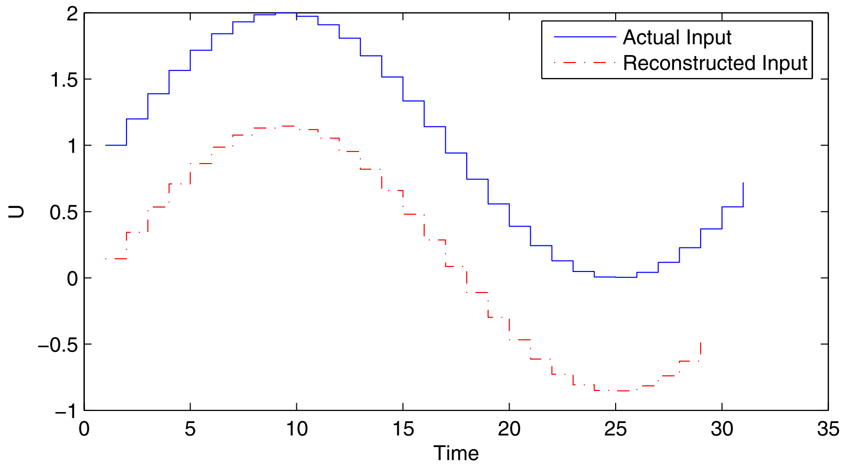

Fig. 3 Comparison between the actual input sequence and 2-delay reconstructed input of the system given by Example 6, with zero 1. In this case the input-reconstruction error is persistent; in fact, it is constant over the interval

persistent over the interval, which is a consequence of the fact that the zero is located on the unit circle. In addition, the sign of the error is constant due to the fact that the zero is located at 1.

Example 7 Consider the nonminimum-phase transfer function

$$G(z) = \frac{z-3}{(z-0.1)(z-0.2)(z-0.3)}, \tag{66}$$

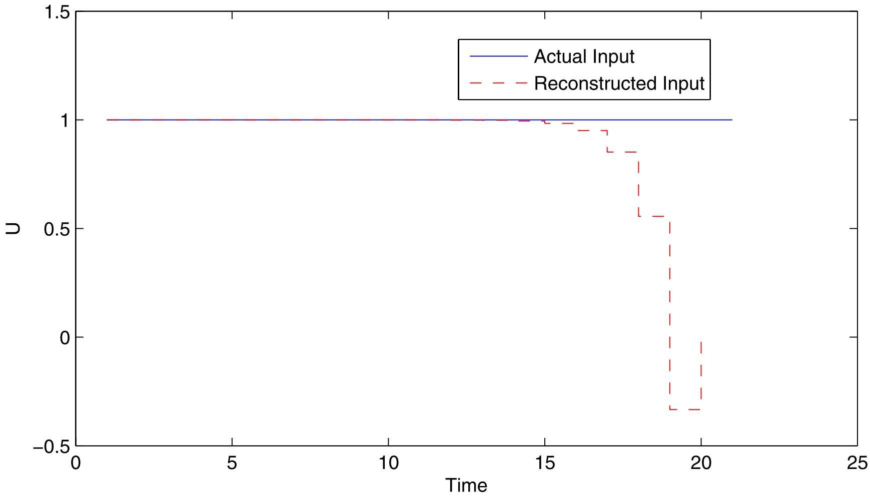

Fig. 4 Comparison between the actual input sequence and 1-delay reconstructed input of the system given by Example 7, with nonminimum-phase zero 3. The input-reconstruction error is monotonically increasing due to the real nonminimum-phase zero

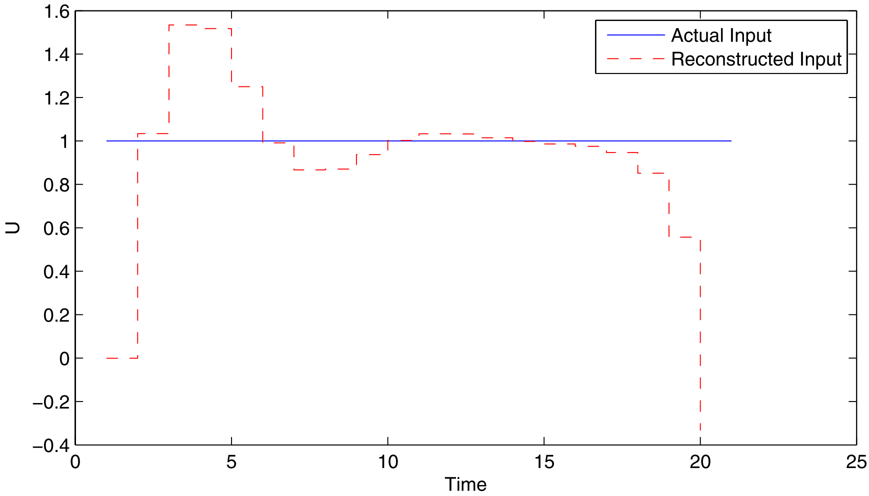

Fig. 5 Comparison between the actual input and the 0-delay reconstructed input for the system given by Example 8, with nonminimum-phase zero 3 and minimum-phase zeros $0.5 + 0.5\jmath$ and $0.5 - 0.5\jmath$. The input-reconstruction error is smallest in the interior of the reconstruction interval due to the presence of both nonminimum-phase and nonminimum-phase zeros

for which $C = [\,-0.8515 \quad 0.5000 \quad 0\,]$. Figure 4 compares the 1-delay reconstructed input sequence with the actual input sequence. Note that the input-reconstruction error is monotonically increasing due to the real nonminimum-phase zero.

Example 8 Consider the transfer function

$$G(z) = \frac{(z-3)(z^2-z+0.5)}{(z-0.1)(z-0.2)(z-0.3)}, \tag{67}$$

which has nonminimum-phase zero 3, minimum-phase zero 0.5, and $C = [0.4552 \quad -0.7772 \quad -1.6432]$. Figure 5 compares the 0-delay reconstructed input sequence with the actual input sequence. Note that the input-reconstruction error is smallest in the interior of the reconstruction interval due to the presence of both nonminimum-phase and nonminimum-phase zeros.

7 Conclusions

In this paper we defined l-delay input and initial-state observability, and we characterized this property through necessary and sufficient conditions. We proved that l-delay input and initial-state observability is equivalent to left invertibility and the absence of invariant zeros, and that the minimal delay for input and initial-state reconstruction is equal to the minimal delay for left invertibility. An explicit reconstructor was given for reconstructing the input and initial state. For systems with zeros, numerical examples showed how the input-reconstruction error depends on the locations of the zeros, specifically, the speed of convergence depends on the distance of the zeros from the unit circle.

Future research and open questions include: refining the bounds (45) for the minimal reconstruction delay for MIMO systems; the effects of modeling errors and noisy measurements on the reconstructed input and initial state; developing reconstructors that cancel minimum-phase zeros; and developing strategies for minimizing the effect of nonminimum-phase zeros. A technique for addressing the effects of nonminimum-phase zeros is discussed in [11].

Appendix

Lemma A.1 *Let $r \geq n$, $x_0 \in \mathbb{C}^n$, $u_0, \ldots, u_r \in \mathbb{C}^m$ and*

$$Y_{[0:r]} = \Psi_{r,r} \begin{bmatrix} x_0 \\ u_0 \\ \vdots \\ u_r \end{bmatrix}.$$

Assume that $\operatorname{rank}(H_d) = m$. *Then for all $r \geq n$,*

$$Y_{[0:r]} = 0 \tag{68}$$

if and only if

$$x_0 \in \bigcap_{l=0}^{n-1} \mathcal{N}\big[(I_p - DD^\dagger)CK_d^l\big], \tag{69}$$

and, for all $k \geq 0$,

$$u_k = -H_d^{\dagger} C A^d K_d^k x_0, \tag{70}$$

where

$$K_d \triangleq A - B H_d^{\dagger} C A^d. \tag{71}$$

In this case, let $x_{k+1} \triangleq Ax_k + Bu_k$ *for all* $k \geq 0$. *Then, for all* $k \geq 0$, $x_k = K_d^k x_0$.

Proof First, assume that $Y_{[0:r]} = 0$ for all $r \geq 0$. Then it follows that, for all $k \geq 0$,

$$0 = Cx_k + Du_k. \tag{72}$$

Now, suppose $d = 0$. Since $H_d = H_0 = D$ has full column rank, it follows that, for all $k \geq 0$,

$$u_k = -D^{\dagger} C x_k. \tag{73}$$

Then $x_{k+1} = (A - BD^{\dagger}C)x_k$ and it follows from (72) that $DD^{\dagger}Cx_k = Cx_k$, that is, $x_k \in \mathcal{N}[(I_p - DD^{\dagger})C]$ for all $k \geq 0$. Note that, for all $k \geq 0$,

$$y_k = (I_p - DD^{\dagger})C(A - BD^{\dagger}C)^k x_0 = 0. \tag{74}$$

Using the Cayley-Hamilton theorem it follows that, for all $k \geq n$, there exist $\alpha_{0,k}, \ldots, \alpha_{n-1,k}$ such that $(A - BD^{\dagger}C)^k = \sum_{l=0}^{n-1} \alpha_{l,k}(A - BD^{\dagger}C)^l$. It then follows from (74) that $x_0 \in \bigcap_{l=0}^{n-1} \mathcal{N}[(I_p - DD^{\dagger})CK_d^l]$.

Now suppose $d \geq 1$. Then note that, for all $r \geq 0$,

$$Y_{[0:r]} = C \begin{bmatrix} x_0 \\ \vdots \\ x_r \end{bmatrix} = 0. \tag{75}$$

Since $CB = \cdots = CA^{d-2}B = 0$, it follows that, for all $k \in \mathbb{N}$ and $i \in \{0, \ldots, d-2\}$,

$$CA^i x_{k+1} = CA^{i+1} x_k.$$

Since $Cx_{k+d} = 0$ for all $k \geq 0$, $Cx_{k+d} = CA^{d-1}x_{k+1} = CA^d x_k + CA^{d-1}Bu_k$ and, $H_d = CA^{d-1}B$ has full column rank, it follows that

$$u_k = -H_d^{\dagger} C A^d x_k.$$

Hence, $x_{k+1} = K_d x_k$ and, for all $k \geq 0$,

$$y_k = CK_d^k x_0 = 0. \tag{76}$$

Using the Cayley–Hamilton theorem it follows that, for all $k \geq n$, there exist $\alpha_{0,k}, \ldots, \alpha_{n-1,k}$ such that $K_d^k = \sum_{l=0}^{n-1} \alpha_{l,k} K_d^l$. It then follows from (76) that $x_0 \in \bigcap_{l=0}^{n-1} \mathcal{N}[CK_d^l]$.

Conversely, suppose $d = 0$. Then using Fact 2.12.20 of [2] it follows from (70) that, for all $k \geq 0$,

$$y_k = \left[CA^k - \sum_{l=0}^{k-1} CA^l BD^\dagger CK_d^{k-1-l} - DD^\dagger CK_d^k \right] x_0$$

$$= C\left[A^k - \sum_{l=0}^{k-d} A^l BD^\dagger C(A - BD^\dagger C)^{k-l-1} \right] x_0 - DD^\dagger CK_d^k x_0$$

$$= (I_p - DD^\dagger)CK_d^k x_0.$$

Next it follows from (69) that (68) is satisfied.

Now, suppose $d \geq 1$. Then, using Fact 2.12.20 of [2], it follows from (70) that, for all $k \geq 0$,

$$y_k = \left[CA^k - \sum_{l=0}^{k-d} CA^l BH_d^\dagger CA^d K_d^{k-1-l} \right] x_0$$

$$= C\left[A^k - \sum_{l=0}^{k-d} A^l BH_d^\dagger CA^d (A - BH_d^\dagger CA^d)^{k-l-1} \right] x_0$$

$$= CK_d^k x_0.$$

Next it follows from (69) that (68) is satisfied. □

Define the discrete-time system

$$x_{k+1} = K_d x_k, \tag{77}$$

$$y_k = (I_p - DD^\dagger)Cx_k, \tag{78}$$

where K_d is given by (71).

Lemma A.2 *Let $r \geq n$. Then there exist $x_0 \in \mathbb{R}^n$ and $u_0, \ldots, u_r \in \mathbb{R}^m$ such that $x_0 \neq 0$ and, for all $r \geq n$,*

$$Y_{[0:r]} = \Psi_{r,r} \begin{bmatrix} x_0 \\ u_0 \\ \vdots \\ u_r \end{bmatrix} = 0 \tag{79}$$

if and only if (77), (78) is unobservable. Furthermore, let $x_{k+1} \triangleq Ax_k + Bu_k$ for all $k \geq 0$. Then, for all $k \geq 0$, $x_k = K_d^k x_0$.

Proof First, assume that (77), (78) is unobservable. Now, let $s \triangleq \operatorname{rank}(H_d)$. Then there exist $\tilde{H}_1 \in \mathbb{R}^{p\times s}$, $\tilde{H}_2 \in \mathbb{R}^{s\times m}$ such that $H_d = \tilde{H}_1 \tilde{H}_2$. Note that $\operatorname{rank}(\tilde{H}_1) =$

rank $(\tilde{H}_2) = s$. Now, let u_k be given by

$$u_k = \tilde{H}_2^{\mathrm{T}} v_k, \tag{80}$$

where $v_k \in \mathbb{R}^s$. Suppose $d = 0$. Then $D \neq 0$, and thus

$$x_{k+1} = Ax_k + B' v_k, \tag{81}$$

$$y_k = Cx_k + D' v_k, \tag{82}$$

where $B' \triangleq B\tilde{H}_2^{\mathrm{T}}$ and $D' \triangleq D\tilde{H}_2^{\mathrm{T}} = \tilde{H}_1\tilde{H}_2\tilde{H}_2^{\mathrm{T}}$. Note that $\tilde{H}_2\tilde{H}_2^{\mathrm{T}}$ is positive definite, and thus D' has full column rank. Alternatively, suppose $d > 0$. Then $D = 0$, and thus

$$x_{k+1} = Ax_k + B' v_k, \tag{83}$$

$$y_k = Cx_k, \tag{84}$$

where $B' \triangleq B\tilde{H}_2^{\mathrm{T}}$. Note that $CA^{d-1}B' = H_d\tilde{H}_2^{\mathrm{T}}$, which has full column rank.

Now, let

$$H'_d \triangleq \begin{cases} D', & d = 0, \\ CA^{d-1}B', & d \geq 1, \end{cases} \tag{85}$$

and

$$K'_d \triangleq A - B'(H'_d)^{\dagger} CA^{d-1}. \tag{86}$$

Then note that $K'_d = K_d$, where K_d is given by (71). Since $\bigcap_{l=0}^{n-1} \mathcal{N}[(I_p - DD^{\dagger})CK_d^l] \neq \{0\}$, it follows from Lemma A.1 that there exists $x_0 \neq 0$ such that, for all $k \geq 0$, u_k given by

$$u_k = -\tilde{H}_2^{\mathrm{T}}(H'_d)^{\dagger} C(K'_d)^k x_0$$

satisfies (79). Now, with $x_1 = Ax_0 + Bu_0$ it follows that $x_1 = K_d x_0$. Proceeding similarly, it follows that $x_k = K_d^k x_0$.

Conversely, suppose that there exists $x_0 \neq 0$ such that, for all $r \geq n$, $Y_{[0:r]} = 0$. Then it follows from Lemma A.1 that $\bigcap_{l=0}^{n-1} \mathcal{N}[(I_p - D'(D')^{\dagger})C(K'_d)^l] \neq \{0\}$. Since $K'_d = K_d$ and $D'(D')^{\dagger} = DD^{\dagger}$, it follows that (77), (78) is unobservable. □

Theorem A.1 *Assume that, for all* $\zeta \in \mathbb{C}$, rank $\left[\begin{smallmatrix} \zeta I - A & -B \\ C & D \end{smallmatrix}\right] = n + m$. *Then* $G(z)$ *is left invertible, and there exists* $r \geq n$ *such that* $\mathcal{R}(\Gamma_r) \cap \mathcal{R}(M_{r,r}) = \{0\}$.

Proof Suppose $G(z)$ is not left invertible. Then it follows from (iii) of Theorem 1 that the normal rank of $\left[\begin{smallmatrix} zI - A & -B \\ C & D \end{smallmatrix}\right]$ is less than $n + m$. Therefore, there exists $\zeta \in \mathbb{C}$ such that rank $\left[\begin{smallmatrix} \zeta I - A & -B \\ C & D \end{smallmatrix}\right] < n + m$. Next, suppose that, for all $r \geq n$, $\mathcal{R}(\Gamma_r) \cap \mathcal{R}(M_{r,r}) \neq \{0\}$. Then it follows that there exists $x_0 \in \mathbb{C}^n$ and $U_{[0:r]} \in \mathbb{C}^{m(r+1)}$ such that $x_0 \neq 0$ and $\Gamma_r x_0 + M_{r,r} U_{[0:r]} = 0$ for all $r \geq n$. It then follows from Lemma A.2 that (77),

(78) is unobservable. Now suppose ζ is an eigenvalue of K_d such that $K_d x_0 = \zeta x_0$. It then follows from Lemma A.2 that

$$x_1 = Ax_0 + Bu_0 = K_d x_0 = \zeta x_0.$$

Therefore, there exists nonzero $\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \mathbb{R}^{n+m}$ satisfying $\begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$. Hence, rank $\begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} < n + m$. □

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