LQG Cost Bounds in Discrete-Time $\mathcal{H}_2/\mathcal{H}_\infty$ Control

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Abstract

It is shown that a full-state feedback gain that was originally derived to satisfy a discrete-time, closed-loop $\mathcal{H}_\infty$-norm bound, also minimizes three distinct LQG cost bounds.

1 Introduction

Combined $\mathcal{H}_2/\mathcal{H}_\infty$ feedback controllers can provide both robust stability (via a closed-loop $\mathcal{H}_\infty$-norm bound) and nominal performance (via a closed-loop LQG cost bound or $\mathcal{H}_2$-norm bound). Consequently they are of considerable interest, and—at least for the continuous-time case—there has recently been much work on the subject. See, for example, Mustafa and Glover (1988, 1990), Bernstein and Haddad (1989), Grimble (1989), Rotea and Khargonekar (1990), Zhou *et al.*, (1990), and MacMartin *et al.*, (1990).

It was shown by Mustafa (1989a) that if the continuous-time $\mathcal{H}_\infty$-norm and LQG cost bound are applied to the same closed-loop system—the 'equalized LQG/$\mathcal{H}_\infty$ weights' case—then the LQG cost bound ('auxiliary cost') considered by Bernstein and Haddad (1989) is equal to the entropy considered by Mustafa and Glover (1990). The present paper was motivated by the fact that this equality does *not* hold in the discrete-time case. Indeed, as pointed out by Iglesias *et al.*, (1990), the discrete-time minimum entropy $\mathcal{H}_\infty$ control problem is not simply related to the continuous-time case via the usual bilinear transform. This motivates the need for derivations wholly in discrete-time.

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Although the discrete-time auxiliary cost (defined by Haddad *et al.,* (1991)) and the discrete-time entropy (see e.g., Mustafa (1989a), Iglesias *et al.,* (1990)) are generally different, we will see in Section 2 that they are nevertheless closely related and indeed both can be evaluated in terms of the same discrete-time algebraic Riccati equation. Furthermore, the LQG cost bound in the independent work of Bambang *et al.,* (1990) can also be defined in terms of the same algebraic Riccati equation. Thus one apparently has *three* distinct auxiliary costs and so three different mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems. By restricting the discussion to static full-state feedback, we will show in Section 3 that the *same* gain solves all of these problems. This gain turns out to be the gain derived by Başar (1989) to satisfy an $\mathcal{H}_\infty$-norm bound (see also Yaesh and Shaked (1990)). Thus the present paper gives a combined LQG/$\mathcal{H}_\infty$ interpretation of that control law which was not apparent in the work of Başar (1989).

2 Problem Statement

2.1 The plant and control law

Consider a discrete-time plant

$$\begin{aligned} x(k+1) &= A_0x(k) + B_1w(k) + B_2u(k) \\ z(k) &= C_1x(k) + D_{12}u(k) \end{aligned}$$

where $k = 0, 1, 2, \ldots$ and the disturbances $w \in \mathbb{R}^{m_1}$, the input $u \in \mathbb{R}^{m_2}$, the state $x \in \mathbb{R}^n$ and the outputs $z \in \mathbb{R}^{p_1}$. With static full-state feedback

$$u(k) = Kx(k)$$

where $K \in \mathbb{R}^{m_2 \times n}$ is to be determined, the closed-loop system from disturbances w to outputs z is

$$\begin{aligned} x(k+1) &= \overbrace{(A_0 + B_2K)}^{A}x(k) + \overbrace{B_1}^{B}w(k) \\ z(k) &= \underbrace{(C_1 + D_{12}K)}_{C}x(k) \end{aligned}$$

with z-transfer function $H(z) := C(zI - A)^{-1}B$, which we will occasionally denote by $H = (A, B, C)$.

2.2 Assumptions

A necessary and sufficient assumption for the existence of a gain K that makes the closed-loop system asymptotically stable (i.e., makes $|\lambda(A_0 + B_2K)| < 1$) is that

- (A_0, B_2) is stabilizable.

In common with Başar (1989) we assume that

- (A_0, B_1) is stabilizable and (C_1, A_0) is detectable.

It is also assumed that

- $D_{12}^T\begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.

This is a now fairly standard assumption which clarifies the exposition substantially by normalizing the control weighting and eliminating cross-weightings between control signal and state.

2.3 Basic problem statement

Before delving into the details of defining LQG cost bounds, it is pertinent to give a statement of the basic problem of interest, to have a goal in sight.

The aim is to find a gain K that satisfies certain criteria. The first is of course that K makes the closed-loop stable:

(i) $A = A_0 + B_2K$ is asymptotically stable.

Now the Small Gain Theorem tells us that if also

(ii) $\|H(z)\|_\infty \leq \gamma$,

for some given $\gamma > 0$, then the system is *robustly stable* in that any stable transfer matrix $\Delta(z)$ satisfying $\|\Delta(z)\|_\infty < \gamma^{-1}$ may be connected from z back to w without destabilizing the closed-loop so formed. Here $\|H(z)\|_\infty = \sup_\theta \lambda_{max}^{1/2}(H^T(e^{-j\theta})H(e^{j\theta}))$ is the usual discrete-time $\mathcal{H}_\infty$-norm. For background on $\mathcal{H}_\infty$ control, the reader is referred to Francis (1987) for the continuous-time case and Iglesias and Glover (1990) and Stoorvogel (1991) for the discrete-time case.

One gain that satisfies (i) and (ii) is given in the following result, taken from Lemma 5.2 of Başar (1989).

Proposition 2.1 (Başar, 1989) *Consider the system of Section 2.1 under the assumptions of Section 2.2. Suppose there exists a matrix $P \in \mathbb{R}^{n \times n}$ satisfying*

$$\begin{aligned} P &\geq 0, \\ \gamma^2 I - B_1^T P B_1 &> 0, \\ P &= A_0^T P(I + B_2 B_2^T P - \gamma^{-2} B_1 B_1^T P)^{-1} A_0 + C_1^T C_1. \end{aligned}$$

Then the full-state feedback gain

$$K = -B_2^T P(I + B_2 B_2^T P - \gamma^{-2} B_1 B_1^T P)^{-1} A_0$$

satisfies

(i) $A = A_0 + B_2K$ *is asymptotically stable,*

(ii) $\|H\|_\infty \leq \gamma$.

As may be seen in Iglesias and Glover (1990), usually a class of controllers achieve closed-loop stability and satisfy the $\mathcal{H}_\infty$-norm bound. So we may impose an additional criterion. One possibility would be to minimize *nominal performance* as defined by the Linear Quadratic Gaussian cost

$$C(H) := \lim_{k \to \infty} \mathrm{E}[z^T(k)z(k)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{trace}[H^T(e^{-j\theta})H(e^{j\theta})]d\theta$$

in which case w is interpreted as zero-mean Gaussian white noise with unit untensity. However, this interesting problem is, as far as the present authors are aware, still unsolved. Instead, subject to (i) and (ii), we seek to

(iii) Minimize $J(H, \gamma)$, a suitably defined *upper bound* on the LQG cost.

The precise definition of three different LQG cost bounds of interest to us will be deferred until Section 2.5: firstly we address the $\mathcal{H}_\infty$-norm bound (ii).

2.4 The $\mathcal{H}_\infty$-norm bound

Satisfaction of the $\mathcal{H}_\infty$-norm bound is known to be intimately connected to the existence of solutions to a certain discrete-time algebraic Riccati equation. Consider the set of strictly-proper discrete-time systems of McMillan degree n,

$$\mathcal{D}_n^{p_1,m_1} := \{(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m_1} \times \mathbb{R}^{p_1\times n}\},$$

and the set of asymptotically stable, strictly-proper, discrete-time systems of McMillan degree n,

$$\mathcal{D}_{n;stable}^{p_1,m_1} := \{(A,B,C) \in \mathcal{D}_n^{p_1,m_1} \quad : \quad |\lambda(A)| < 1\}.$$

Given $(A,B,C) \in \mathcal{D}_n^{p_1,m_1}$ and $\gamma > 0$, consider the following conditions on a matrix $P \in \mathbb{R}^{n\times n}$:

$$P \geq 0, \tag{1}$$
$$\gamma^2 I - B^T PB > 0, \tag{2}$$
$$P = A^T PA + A^T PB(\gamma^2 I - B^T PB)^{-1} B^T PA + C^T C. \tag{3}$$

Then we have the following definitions.

- If P satisfies (1), (2) and (3) then we will say that P is a *solution* of the algebraic Riccati equation (3).
- If P is a solution of (3) and also $|\lambda(A + B(\gamma^2 I - B^T PB)^{-1} B^T PA)| \leq 1$ then P is said to be a *strong solution* of the algebraic Riccati equation (3).
- If P is a solution of (3) and also $|\lambda(A + B(\gamma^2 I - B^T PB)^{-1} B^T PA)| < 1$ then P is said to be a *stabilizing solution* of the algebraic Riccati equation (3).
- If P is a solution of (3) and also $P \leq \tilde{P}$ for all other solutions $\tilde{P}$ of (3), then P is said to be a *minimal solution* of the algebraic Riccati equation (3).

The following lemmas state some useful results and connections to $\mathcal{H}_\infty$-norm bounds, which are analogous to well known continuous-time results.

Lemma 2.2 *Let $(A,B,C) \in \mathcal{D}_n^{p_1,m_1}$ and $\gamma > 0$ be given. Suppose there exists a solution P to (3). Then we have the following.*

(a) *There exists a minimal solution to (3), which is:*

  (i) *Unique.*

  (ii) *Strong if (A,B) is stabilizable.*

(b) *The following are equivalent:*

  (i) *(C,A) is detectable.*

  (ii) *A is asymptotically stable.*

(c) *If A is asymptotically stable then:*

  (i) $\|C(zI - A)^{-1}B\|_\infty \leq \gamma$.

  (ii) *$P \geq S$ where $S = A^T SA + C^T C$.*

Proof Part (a) is from Theorem 3.1 of Ran and Vreugdenhil (1987), whereas parts (b) and (c) are from Lemma 2.1 of Haddad *et al.*, (1991). □

Lemma 2.3 *Let* $(A, B, C) \in \mathcal{D}^{p_1,m_1}_{n;stable}$ *and* $\gamma > 0$ *be given.*

(a) *The following are equivalent:*

(i) $\|C(zI - A)^{-1}B\|_\infty < \gamma$.

(ii) *There exists a stabilizing solution* P *to (3).*

(iii) *There exists exactly one stabilizing solution* P *to (3).*

(b) *If a(i)–a(iii) hold, the following are equivalent:*

(i) P *is the strong solution of (3).*

(ii) P *is the stabilizing solution of (3).*

(iii) P *is the minimal solution of (3).*

Proof These follow from Theorem 2 of Molinari (1975), Lemma 4 of Gu *et al.*, (1989), and Lemma 2.2a. (The continuous-time version of Part (a) may be found in Theorem 2.2 of Petersen *al.*, (1990).) □

The next step is to construct upper bounds on the LQG cost using P solving (3).

2.5 LQG cost bounds

It is well known that the LQG cost of $H = (A, B, C) \in \mathcal{D}^{p_1,m_1}_{n;stable}$ is given by

$$C(H) = \text{trace}[B^T S B] \tag{4}$$

where

$$S = A^T S A + C^T C. \tag{5}$$

Suppose that there exists a solution P to (3). Then Lemma 2.2 gives that $\|H\|_\infty \le \gamma$, so the desired $\mathcal{H}_\infty$-norm bound is satisfied, and $P \ge S$. Thus trace$[B^T P B] \ge$ trace$[B^T S B]$ which gives an LQG cost overbound. Other cost bounds are possible, as we now see.

Proposition 2.4 *Let* $H = (A, B, C) \in \mathcal{D}^{p_1,m_1}_{n;stable}$ *and* $\gamma > 0$ *be given. Suppose there exists a solution* P *to (3). Then* $\det(I - \gamma^{-2} B^T P B) > 0$ *so we can define the following LQG cost bounds:*

$$J_1(H,\gamma) := \text{trace}[B^T P B], \tag{6}$$
$$J_2(H,\gamma) := -\gamma^2 \ln\det(I - \gamma^{-2} B^T P B), \tag{7}$$
$$J_3(H,\gamma) := \text{trace}[B^T P (I - \gamma^{-2} B B^T P)^{-1} B]. \tag{8}$$

For $i = 1, 2, 3$ *we have*

(i) $J_i(H,\gamma) \ge C(H)$.

(ii) $J_i(H,\gamma) = C(H) + O(\gamma^{-2})$.

(iii) $J_i(H,\infty) = C(H)$.

Moreover, $J_3(H,\gamma) \ge J_2(H,\gamma) \ge J_1(H,\gamma)$.

Proof See Appendix A. □

Remark 2.5 The cost $J_1(H,\gamma)$ was defined by Haddad *et al.,* (1990). □

Remark 2.6 The cost $J_2(H,\gamma)$ for a system $H = (A,B,C) \in \mathcal{D}^{p_1,m_1}_{n;stable}$ which satisfies $\|H\|_\infty < \gamma$ is in fact the *entropy* of H. The entropy of such an H is defined by

$$I(H,\gamma) := -\frac{\gamma^2}{2\pi}\int_{-\pi}^{\pi} \ln|\det(I - \gamma^{-2}H^T(e^{-j\theta})H(e^{j\theta})|d\theta.$$

For background details, see Mustafa and Glover (1990) for continuous-time entropy, and Iglesias *et al.,* (1990) for discrete-time entropy. Using Lemma 6.2.6 of Mustafa (1989a) to evaluate the entropy in terms of the stabilizing solution P to (3), one obtains

$$I(H,\gamma) = -\gamma^2 \ln\det(I - \gamma^{-2}B^TPB) = J_2(H,\gamma)$$

as claimed. □

Remark 2.7 The cost $J_3(H,\gamma)$ is equal to the LQG cost bound defined in the independent work of Bambang *et al.,* (1990) as we now show. Let P be a solution of (3), then we can define

$$X := P(I - \gamma^{-2}BB^TP)^{-1}. \tag{9}$$

It is easy to show that

$$X \geq 0, \tag{10}$$

$$\gamma^2 I + B^TXB > 0, \tag{11}$$

$$X = A^TXA + XB(\gamma^2 I + B^TXB)^{-1}B^TX + C^TC. \tag{12}$$

Such an X is said to be a *solution* of (12). Conversely, if X is a solution of (12) then

$$P := X(I + \gamma^{-2}BB^TX)^{-1}$$

defines a solution of (3). For X solving (12), Bambang *et al.,* (1990) defined an LQG cost bound

$$J_4(H,\gamma) := \mathrm{trace}[B^TXB]. \tag{13}$$

In view of (9) and (13), we see that

$$J_4(H,\gamma) = \mathrm{trace}[B^TP(I - \gamma^{-2}BB^TP)^{-1}B] = J_3(H,\gamma)$$

as claimed. □

Remark 2.8 Part (iii) of Proposition 2.4 shows that the true LQG cost is recovered from each of the $J_i(H,\gamma)$ when the $\mathcal{H}_\infty$-norm bound γ is completely relaxed ($\gamma \to \infty$), as might be hoped. □

The following lemma shows that, over the class of solutions of (3), the smallest $J_i(H,\gamma)$ is obtained using the minimal solution of (3) in (6)–(8). Since the eventual aim is to minimize the $J_i(H,\gamma)$, it is therefore implicit that the minimal solution of (3) is used in (6)–(8).

Lemma 2.9 *Let $J_i(H,\gamma)$ be as defined in (6)–(8), but with the requirement that the minimal solution of (3) is used. If any other solution of (3) is used in (6)–(8), denote the $J_i(H,\gamma)$ so obtained by $\tilde{J}_i(H,\gamma)$. Then, for $i = 1, 2, 3$,*

$$J_i(H,\gamma) \leq \tilde{J}_i(H,\gamma).$$

Proof See Appendix B. □

3 Minimization of the LQG cost bounds

We will establish the following strengthening of Proposition 2.1.

Proposition 3.1 *Consider the system of Section 2.1 under the assumptions of Section 2.2. Suppose there exists a matrix $P \in \mathbb{R}^{n\times n}$ satisfying*

$$P \geq 0, \tag{14}$$
$$\gamma^2 I - B_1^T P B_1 > 0, \tag{15}$$
$$P = A_0^T P(I + B_2 B_2^T P - \gamma^{-2} B_1 B_1^T P)^{-1} A_0 + C_1^T C_1, \tag{16}$$

such that

$$\bar{A} := (I + B_2 B_2^T P - \gamma^{-2} B_1 B_1^T P)^{-1} A_0 \text{ is asymptotically stable,} \tag{17}$$

and suppose that $(\bar{A}, B_1)$ is controllable. Then the full-state feedback gain

$$K = -B_2^T P(I + B_2 B_2^T P - \gamma^{-2} B_1 B_1^T P)^{-1} A_0 \tag{18}$$

minimizes the LQG cost bounds $J_i(H,\gamma)$, $i = 1, 2, 3$ subject to

(i) *$A = A_0 + B_2 K$ is asymptotically stable,*

(ii) *$\|H\|_\infty \leq \gamma$,*

and in fact $\|H\|_\infty < \gamma$.

To evaluate the minimized costs, simply use P from the above proposition in the definitions of the $J_i(H,\gamma)$ given in (6)–(8).

3.1 Minimization of $J_1(H,\gamma)$

To minimize $J_1(H,\gamma)$ subject to the constraint equation (3), form the Lagrangian

$$\begin{aligned}\mathcal{L}_1(K,P,Q) := {} & \alpha J_1(H,\gamma) \\ & +\text{trace}[\{-P + A^T P A + A^T P B(\gamma^2 I - B^T P B)^{-1} B^T P A + C^T C\} Q]\end{aligned}$$

where the Lagrange multipliers $\alpha \geq 0$ and $Q \in \mathbb{R}^{n\times n}$ are not both zero, and $A = A_0 + B_2 K$, $B = B_1$ and $C = C_1 + D_{12} K$. Note carefully that

$$\frac{\partial J_1}{\partial K} = 0, \quad \frac{\partial J_1}{\partial P} = B_1 B_1^T, \text{ and } \frac{\partial J_1}{\partial Q} = 0.$$

(Here and in the sequel we use the method of Geering (1976) to do the differentiation.) Now the necessary conditions for optimality are

$$\frac{\partial \mathcal{L}_1}{\partial K} = 0, \quad \frac{\partial \mathcal{L}_1}{\partial P} = 0, \text{ and } \frac{\partial \mathcal{L}_1}{\partial Q} = 0.$$

These are, respectively,

$$0 = \{(I + B_2^T P(I - \gamma^{-2} B_1 B_1^T P)^{-1} B_2) K + B_2^T P(I - \gamma^{-2} B_1 B_1^T)^{-1} A_0\} Q \tag{19}$$
$$Q = \bar{A} Q \bar{A}^T + \alpha B_1 B_1^T, \tag{20}$$
$$P = A^T P A + A^T P B(\gamma^2 I - B^T P B)^{-1} B^T P A + C^T C, \tag{21}$$

where $\bar{A} := A + B(\gamma^2 I - B^T P B)^{-1} B^T P A$. It is easy to show that (19) is satisfied by (18), and then (21) becomes (16). By Proposition 2.1 we therefore have that $A = A_0 + B_2 K$ is asymptotically stable and $\|C(zI - A)^{-1}B\|_\infty \leq \gamma$.

We need the minimal solution of (16), otherwise $J_1(H, \gamma)$ is not minimized. Lemma 2.3a tells us that this is also the stabilizing solution (i.e., $|\lambda(\bar{A})| < 1$), and furthermore $\|C(zI - A)^{-1}B\|_\infty < \gamma$. Since $|\lambda(\bar{A})| < 1$ there exists a unique $Q \geq 0$ satisfying (20), using the discrete-time version of Lemma 12.1 of Wonham (1979). Furthermore, if $\alpha = 0$ then $Q = 0$, hence without loss of generality we may set $\alpha = 1$.

To show that K is the minimizing solution, follow the method used in Section 3 of Yaesh and Shaked (1989). Define

$$M := I + B_2^T P (I - \gamma^{-2} B_1 B_1^T P)^{-1} B_2.$$

Then

$$\frac{\partial^2 \mathcal{L}_1}{(\partial K)_{ij}^2} = M_{ii} Q_{jj}.$$

Assume that $(\bar{A}, B_1)$ is in fact controllable. Then the discrete-time version of Lemma 12.2 of Wonham (1979) applied to (20) gives that $Q > 0$. Also, M is clearly positive-definite. It follows that $M_{ii} > 0$ and $Q_{jj} > 0$ (see page 398 of Horn and Johnson (1985), for example). Therefore

$$\frac{\partial^2 \mathcal{L}_1}{(\partial K)_{ij}^2} > 0,$$

as needed.

3.2 Minimization of $J_2(H, \gamma)$

The steps taken are identical to those taken in Section 3.1 to minimize $J_1(H, \gamma)$. The Lagrangian becomes

$$\begin{aligned}\mathcal{L}_2(K, P, Q) &:= \alpha J_2(H, \gamma) \\ &\quad + \text{trace}[\{-P + A^T P A + A^T P B(\gamma^2 I - B^T P B)^{-1} B^T P A + C^T C\} Q]\end{aligned}$$

where the Lagrange multipliers $\alpha \geq 0$ and $Q \in \mathbb{R}^{n \times n}$ are not both zero. Note carefully that

$$\frac{\partial J_2}{\partial K} = 0, \quad \frac{\partial J_2}{\partial P} = B_1 (I - \gamma^{-2} B_1 P B_1^T)^{-1} B_1^T, \quad \text{and} \quad \frac{\partial J_2}{\partial Q} = 0.$$

Thus the necessary conditions for optimality are identical to (19), (20) and (21), except the second stationarity condition gives

$$Q = \bar{A} Q \bar{A}^T + \alpha B_1 (I - \gamma^{-2} B_1 P B_1^T)^{-1} B_1^T$$

in place of (20). The remaining details are the same as Section 3.1; one only needs to note that $(\bar{A}, B_1 (I - \gamma^{-2} B_1^T P B_1)^{-1/2})$ is controllable whenever $(\bar{A}, B_1)$ is controllable.

3.3 Minimization of $J_3(H,\gamma)$

Again, the steps taken are identical to those taken in Section 3.1 to minimize $J_1(H,\gamma)$. The Lagrangian becomes

$$\begin{aligned}\mathcal{L}_3(K,P,Q) := {} & \alpha J_3(H,\gamma) \\ & +\text{trace}[\{-P + A^TPA + A^TPB(\gamma^2 I - B^TPB)^{-1}B^TPA + C^TC\}Q]\end{aligned}$$

where the Lagrange multipliers $\alpha \geq 0$ and $Q \in \mathbb{R}^{n\times n}$ are not both zero. Note carefully that

$$\frac{\partial J_3}{\partial K} = 0, \quad \frac{\partial J_3}{\partial P} = (I - \gamma^{-2}B_1B_1^TP)^{-1}B_1B_1^T(I - \gamma^{-2}B_1B_1^TP)^{-T}, \text{ and } \frac{\partial J_3}{\partial Q} = 0.$$

The necessary conditions for optimality are identical to (19), (20) and (21), except the second stationarity condition gives

$$Q = \bar{A}Q\bar{A}^T + \alpha(I - \gamma^{-2}B_1B_1^TP)^{-1}B_1B_1^T(I - \gamma^{-2}B_1B_1^TP)^{-T}$$

in place of (20). The remaining details are the same as Section 3.1; one only needs to note that $(\bar{A}, (I - \gamma^{-2}B_1B_1^TP)^{-1}B_1)$ is controllable whenever $(\bar{A}, B_1)$ is controllable, because $(I - \gamma^{-2}B_1B_1^TP)^{-1}B_1 = B_1(I - \gamma^{-2}B_1^TPB_1)^{-1}$.

4 Concluding Remark

We have shown that a certain discrete-time full-state feedback gain, initially designed to satisfy a closed-loop $\mathcal{H}_\infty$ norm bound, also minimizes three distinct LQG cost bounds. An obvious open problem is

- What other LQG cost bounds, if any, are also minimized?

Furthermore, it would be interesting to see if the results of this paper generalize to include the case of

- Dynamic output feedback, and/or
- LQG cost bounds and $\mathcal{H}_\infty$-norm bounds applied to different closed-loops.

Finally, in the spirit of Zhou *et al.,* (1990), one wonders:

- What is the induced-norm interpretation of the LQG cost bounds?

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A Appendix: Proof of Proposition 2.4

If P is a solution of (3) then $I - \gamma^{-2}B^TPB > 0$ so $\det(I - \gamma^{-2}B^TPB) > 0$ and also $\det(I - \gamma^{-2}BB^TP) > 0$. Therefore the $J_i(H,\gamma)$ are certainly well-defined.

Part (i) Subtracting (5) from (3) gives

$$P - S = A^T(P - S)A + A^T PB(\gamma^2 I - B^T PB)^{-1} B^T PA$$

which gives

$$P - S = \sum_{k=1}^{\infty} (A^{k+1})^T PB(\gamma^2 I - B^T PB)^{-1} B^T PA^{k+1}. \tag{22}$$

Hence $P - S \geq 0$ so $J_1(H, \gamma) \geq C(H)$. Using the matrix inversion lemma to write

$$P(I - \gamma^{-2} BB^T P)^{-1} = P + PB(\gamma^2 I - B^T PB)^{-1} B^T P$$

leads to

$$J_3(H, \gamma) - J_1(H, \gamma) = \text{trace}[B^T PB(\gamma^2 I - B^T PB)^{-1} B^T PB], \tag{23}$$

which is non-negative. Hence $J_3(H, \gamma) \geq J_1(H, \gamma) \geq C(H)$. An application of Lemma A.2.1(ii) of Mustafa and Glover (1990) gives $J_2(H, \gamma) \geq J_1(H, \gamma)$ immediately, and recalling that $J_1(H, \gamma) \geq C(H)$ completes the proof of part (i).

Part (ii) We know that

$$(\gamma^2 I - B^T PB)^{-1} = \gamma^{-2} \left\{ I + \sum_{j=1}^{\infty} (\gamma^{-2} B^T PB)^k \right\}. \tag{24}$$

Putting this in (22) implies that trace$[P - S] = O(\gamma^{-2})$ and therefore that $J_1(H, \gamma) - C(H) = O(\gamma^{-2})$. Similarly, using (24) in (23) gives $J_3(H, \gamma) - J_1(H, \gamma) = O(\gamma^{-2})$, hence $J_3(H, \gamma) = C(H) + O(\gamma^{-2})$. An application of Lemma A.2.1(i) of Mustafa and Glover (1990) to $J_2(H, \gamma)$ gives $J_2(H, \gamma) = J_1(H, \gamma) + O(\gamma^{-2})$ so also $J_2(H, \gamma) = C(H) + O(\gamma^{-2})$.

Part (iii) This follows from part (ii) on taking the limit as $\gamma \to \infty$.

It remains to prove that $J_3(H, \gamma) \geq J_2(H, \gamma) \geq J_1(H, \gamma)$. That $J_2(H, \gamma) \geq J_1(H, \gamma)$ was shown in the proof of part (i). Now let $\kappa_i := \lambda_i\{\gamma^{-2} B^T PB\}$. It is straightforward to show that

$$J_3(H, \gamma) - J_2(H, \gamma) = \gamma^2 \sum_{i=1}^{n} \left((1 - \kappa_i)^{-1} \kappa_i + \ln(1 - \kappa_i) \right).$$

Since $I \geq I - \gamma^{-2} B^T PB > 0$ we have that $0 \leq \kappa_i < 1$. Hence, as shown on page 11 of Mustafa and Glover (1990),

$$(1 - \kappa_i)^{-1} \kappa_i + \ln(1 - \kappa_i) \geq 0.$$

It follows that $J_3(H, \gamma) \geq J_2(H, \gamma)$. □

B Appendix: Proof of Lemma 2.9

Let $\tilde{P}$ be a solution of (3) which is not the minimal solution P. By definition we have that

$$P \leq \tilde{P}, \tag{25}$$
$$P \leq 0 \text{ and } I - \gamma^{-2} B^T PB > 0, \tag{26}$$
$$\tilde{P} \leq 0 \text{ and } I - \gamma^{-2} B^T \tilde{P} B > 0, \tag{27}$$

which we will need in the following proof. There are three cases, corresponding to $J_1(H, \gamma)$, $J_2(H, \gamma)$ and $J_3(H, \gamma)$, respectively.

Case 1 Simply note that

$$\tilde{J}_1(H,\gamma) - J_1(H,\gamma) = \text{trace}[B^T\tilde{P}B] - \text{trace}[B^TPB] = \text{trace}[B^T(\tilde{P}-P)B] \geq 0,$$

using (25).

Case 2 Using (25)–(27) and Corollary 7.7.4a,c of Horn and Johnson (1985) it is easily shown that

$$(I - \gamma^{-2}B^T\tilde{P}B)^{-1} \geq (I - \gamma^{-2}B^TPB)^{-1} \geq I \tag{28}$$

and

$$\lambda_i\{(I - \gamma^{-2}B^T\tilde{P}B)^{-1}\} \geq \lambda_i\{(I - \gamma^{-2}B^TPB)^{-1}\} \geq 1. \tag{29}$$

Therefore

$$\begin{aligned}
\tilde{J}_2(H,\gamma) &= -\gamma^2 \ln\det(I - \gamma^{-2}B^T\tilde{P}B) \\
&= \gamma^2 \sum_{i=1}^{n} \ln \lambda_i\{(I - \gamma^{-2}B^T\tilde{P}B)^{-1}\} \\
&\geq \gamma^2 \sum_{i=1}^{n} \ln \lambda_i\{(I - \gamma^{-2}B^TPB)^{-1}\} \quad \text{using (29)} \\
&= -\gamma^2 \ln\det(I - \gamma^{-2}B^TPB) \\
&= J_2(H,\gamma).
\end{aligned}$$

Case 3 Using standard properties of trace[·] and Corollary 7.7.4b of Horn and Johnson (1985), we have

$$\begin{aligned}
\tilde{J}_3(H,\gamma) &= \text{trace}[B^T\tilde{P}(I - \gamma^{-2}BB^T\tilde{P})^{-1}B] \\
&= \text{trace}[B^T\tilde{P}B(I - \gamma^{-2}B^T\tilde{P}B)^{-1}] \\
&= \text{trace}[(B^T\tilde{P}B)^{1/2}(I - \gamma^{-2}B^T\tilde{P}B)^{-1}(B^T\tilde{P}B)^{1/2}] \\
&\geq \text{trace}[(B^T\tilde{P}B)^{1/2}(I - \gamma^{-2}B^TPB)^{-1}(B^T\tilde{P}B)^{1/2}] \quad \text{using (28)} \\
&= \text{trace}[(I - \gamma^{-2}B^TPB)^{-1/2}B^T\tilde{P}B(I - \gamma^{-2}B^TPB)^{-1/2}] \\
&\geq \text{trace}[(I - \gamma^{-2}B^TPB)^{-1/2}B^TPB(I - \gamma^{-2}B^TPB)^{-1/2}] \quad \text{using (25)} \\
&= \text{trace}[B^TP(I - \gamma^{-2}BB^TP)^{-1}B] \\
&= J_3(H,\gamma),
\end{aligned}$$

as claimed. □

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