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Wassim M. Haddad [a]; Jonathan P. How [b]; Steven R. Hall [b]; Dennis S. Bernstein [c]
[a] School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332, U.S.A
[b] Space Engineering Research Center, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.
[c] Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2140, U.S.A.

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Extensions of mixed-μ bounds to monotonic and odd monotonic nonlinearities using absolute stability theory†

WASSIM M. HADDAD‡, JONATHAN P. HOW§, STEVEN R. HALL§ and DENNIS S. BERNSTEIN||

In this paper we make explicit connections between classical absolute stability theory and modern mixed-μ analysis and synthesis. Specifically, using the parameter-dependent Lyapunov function framework of Haddad and Bernstein and the frequency dependent off-axis circle interpretation of How and Hall, we extend previous work on absolute stability theory for monotonic and odd monotonic nonlinearities to provide tight approximations for constant real parameter uncertainty. An immediate application of this framework is the generalization and reformulation of mixed-μ analysis and synthesis in terms of Lyapunov functions and Riccati equations. This observation is exploited to provide robust, reduced-order controller synthesis while avoiding the standard D, $N - K$ iteration and curve-fitting procedures.

Nomenclature

$G(s)$	system transfer function
$\tilde{G}(s)$	transformed system transfer function
A, B, C	state-space realization of $G(s)$
D, N	scaling matrices within the μ-synthesis
$f(\cdot)$	nonlinear function
$\tilde{f}(\cdot)$	transformed nonlinear function
I_m or I	$m \times m$ identity matrix
M_1, M_2	upper and lower slope bounds for $f(\cdot)$
P	Lyapunov function matrix
$r(\cdot,\cdot), R(\cdot,\cdot)$	supply rates
$V_G(\cdot), V_{si}(\cdot)$	system and nonlinearity storage functions
$V(\cdot)$	Lyapunov function
$W(s)$	stability multiplier
u, x, y	system inputs, states and outputs
z_{ij}	filtered outputs of $G(s)$
u_i, u_{ij}	$\tilde{f}_i(\tilde{y}_i)$, $\tilde{f}_i(z_{ij})$
H_j, N_j, S_j	matrices of multiplier coefficients
Δ	uncertainty blocks
$\mathbb{E}$	expected value

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‡ School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332, U.S.A.

§ Space Engineering Research Center, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

|| Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2140, U.S.A.

$\mathbb{N}^r$, $\mathbb{D}^r$, $\mathbb{S}^r$	$r \times r$ non-negative definite, diagonal and symmetric matrices
$\mathcal{K}$	uncertainty block structure
$(\cdot)_i$	ith row of $(\cdot)$
$(\cdot)_{ii}$	(i, i)th element of $(\cdot)$
α, β, γ, η	multiplier coefficients

1. Introduction

Many of the great landmarks of control theory are associated with the theory of absolute stability. The Aizerman conjecture and the Lur'e problem, as well as the circle and Popov criteria, are extensively developed in the classical monographs of Aizerman and Gantmacher (1964), Lefschetz (1965) and Popov (1973). A more modern treatment is given by Safonov (1980), while an excellent textbook treatment is presented in Vidyasagar (1992). The influence of absolute stability on the development of modern robust control is clearly evident from such works as Zames (1966). However, despite continued development of the theory as summarized in the important book by Narendra and Taylor (1973), absolute stability has had limited direct influence on the development of robust control theory. Since absolute stability theory concerns the stability of a system for classes of nonlinearities which, as noted by Siljak (1990) and Haddad and Bernstein (1993), can readily be interpreted as an uncertainty model, it is surprising that modern robust control did not take greater advantage of this wealth of knowledge. There appear to be (at least) three reasons for this state of affairs, namely, $\mathcal{H}_\infty$ theory, state-space Lyapunov function theory, and linear uncertainty.

The development of $\mathcal{H}_\infty$ or bounded real theory as a key component of robust control theory focuses on small-gain arguments for robustness guarantees. Although such conditions can be recast for sector-bounded uncertainty, such connections were rarely made (see Francis 1987). Furthermore, the extensive development of state-space Lyapunov function theory as in Leitmann (1979), Khargonekar *et al.* (1990) and Packard and Doyle (1990), was seemingly remote from absolute stability theory, which involves frequency domain conditions with an emphasis on graphical techniques. Finally, much of modern robust control theory is concerned with linear uncertainty, as distinct from the class of sector-bounded nonlinearities addressed by absolute stability theory.

Several recent developments now allow one to discern the relationship between the classical theory of absolute stability and the modern theory of robust control. First, the state-space formulation of $\mathcal{H}_\infty$ or bounded real theory, as developed by Anderson and Vongpanitlerd (1973), Petersen (1987) and Doyle *et al.* (1989), provides a better understanding of the time domain foundations of absolute stability theory. This was followed by the realization that absolute stability results such as the Popov criterion, when specialized to the linear uncertainty problem, are based on parameter-dependent Lyapunov functions, see Haddad and Bernstein (1993) and Haddad and Bernstein (1991 a). Finally, there was the development of upper bounds for mixed-μ theory in Fan *et al.* (1991), that have recently been interpreted by How and Hall (1992) in terms of frequency dependent off-axis circles, and are thus connected directly to absolute stability theory.

The purpose of the present paper is to make significant progress in understanding the relationship between classical absolute stability theory and modern μ-analysis and synthesis. Our goal is to use the approach of Haddad and Bernstein (1991 a) and How and Hall (1992) to demonstrate that the frequency domain multipliers for the various classes of nonlinearities correspond to specific selections of the D, N-scales that arise in the mixed-μ problem. A key aspect of our development is the construction of parameter-dependent Lyapunov functions that support the mixed-μ results.

The principal limitation of the norm-based $\mathcal{H}_\infty$ theory resides in the fact that uncertainty phase information is discarded, so that constant real parametric plant uncertainty is captured as non-parametric frequency-dependent uncertainty. In the time domain, non-parametric uncertainty is manifested as uncertain real parameters that may be time-varying, and these can destabilize a system, even when the parameter variations are confined to a region in which constant variations are non-destabilizing. Consequently, time-varying models of constant real parametric uncertainties are unnecessarily conservative. Thus, to address the constant real parameter uncertainty problem, it is crucial to restrict the allowable time-variation of the uncertainty. One approach is to construct refined Lyapunov functions that explicitly contain the uncertain parameters, an idea proposed by Haddad and Bernstein (1991 a) and developed in this paper using the same storage function approach employed by Willems (1972). The form of the family of parameter-dependent Lyapunov functions $V(x, \Delta A) = x^{\mathrm{T}} P(\Delta A) x$ is critical since the presence of ΔA restricts the allowable time-variation of the uncertain parameters, and thus exploits phase information. This approach is used by Haddad and Bernstein (1993) to generalize the nonlinearity-dependent Lur'e-Postnikov Lyapunov function of the classical Popov criterion to a parameter-dependent Lyapunov function for constant real parameter uncertainty. Potentially less conservative tests for constant real parameter uncertainty can be obtained from similar generalizations of the Lyapunov functions for the slope restricted monotonic and odd monotonic nonlinearities. In this case, the nonlinear uncertainty set is a much better approximation to the linear uncertainty set, and thus provides a framework for significantly reducing conservatism, as numerically demonstrated by Safonov and Wyetizner (1987).

In this paper, we extend previous work on absolute stability theory for differentiable, slope bounded monotonic and odd monotonic memoryless nonlinearities. This class serves as a much better approximation to the case of constant real parametric uncertainty. We show in § 5 that the choice of certain D, N-scales in mixed-μ theory corresponds to the absolute stability criteria for the monotonic and odd monotonic nonlinearities in Narendra and Neuman (1966), Narendra and Cho (1967), Thathachar *et al.* (1967), Cho and Narendra (1968) and Zames and Falb (1968). A direct benefit of these constructions is the new machinery for mixed-μ analysis and synthesis in terms of parameter-dependent Lyapunov functions and Riccati equations for full- and reduced-order compensator synthesis. Related optimality conditions arising from the chosen class of D, N-scales also play a role in the controller synthesis procedure. The overall framework thus provides an alternative approach to μ-synthesis, while avoiding the standard D, $N - K$ iteration and curve-fitting procedure. Finally, we numerically demonstrate our approach on a lightly damped flexible structure with frequency uncertainty.

2. Mathematical preliminaries

In this section we establish definitions and notation. Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, let $(\cdot)^{\mathrm{T}}$ and $(\cdot)^*$ denote transpose and complex conjugate transpose. Furthermore, we write $\|\cdot\|_2$ for the euclidean norm, $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, $\sigma_{\max}(\cdot)$ for the maximum singular value, $\rho(\cdot)$ for the spectral radius, 'tr' for the trace operator, and $M \geqslant 0$ $(M > 0)$ to denote the fact that the hermitian matrix M is non-negative (positive) definite. In this paper a *real-rational matrix function* is a matrix whose elements are rational functions with real coefficients. Furthermore, a *transfer function* is a real-rational matrix function each of whose elements is proper, i.e. finite at $s = \infty$. A *strictly proper transfer function* is a transfer function that is zero at infinity. Finally, an *asymptotically stable transfer function* is a transfer function with all of its poles in the open left half-plane. The space of asymptotically stable transfer functions is denoted by $\mathcal{RH}_\infty$, i.e. the real-rational subset of $\mathcal{H}_\infty$. Let

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{1}$$

denote a state-space realization of a transfer function $G(s)$, that is $G(s) = C(sI - A)^{-1}B + D$. The notation '$\overset{\min}{\sim}$' is used to denote a minimal realization. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of an asymptotically stable transfer function $G(s)$ are defined as

$$\|G(s)\|_2^2 \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} \|G(\mathrm{j}\omega)\|_{\mathrm{F}}^2 \,\mathrm{d}\omega \tag{2}$$

$$\|G(s)\|_\infty \triangleq \sup_{\omega \in \mathbb{R}} \sigma_{\max}[G(\mathrm{j}\omega)] \tag{3}$$

From Anderson and Vongpanitlerd (1973 p. 216), a square transfer function $G(s)$ is positive real if (1) all poles of $G(s)$ are in the closed left half-plane; and (2) $G(s) + G^*(s)$ is non-negative definite for $\mathrm{Re}[s] > 0$. Furthermore, we use the definition from Lozano-Leal and Joshi (1990) and Wen (1988) that a square transfer function $G(s)$ is strictly positive real if (1) $G(s)$ is asymptotically stable; and (2) $G(\mathrm{j}\omega) + G^*(\mathrm{j}\omega)$ is positive definite for all real ω. Finally, a square transfer function $G(s)$ is strongly positive real if it is strictly positive real and $D + D^{\mathrm{T}} > 0$, where $D \triangleq G(\infty)$. Recall from Anderson (1967) that a minimal realization of a positive real transfer function is stable in the sense of Lyapunov, and from Lozano-Leal and Joshi (1990) that a strictly positive real transfer function is asymptotically stable.

For notational convenience in the paper, G will denote an $l \times m$ transfer function with input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^l$ and internal state $x \in \mathbb{R}^n$. We will omit all matrix dimensions throughout, and assume that all quantities have compatible dimensions.

3. Supply rates, storage functions and system stability

Several definitions are necessary to develop the appropriate tools for the analysis framework. Consider a dynamical system $\mathcal{G}$ of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4}$$

$$y(t) = g(x(t)) + Du(t) \tag{5}$$

where $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^l$ and $x(t) \in \mathbb{R}^n$. In the special case where the output function is linear, $(g(x) = Cx)$, $\mathcal{G} = G(s)$ is an LTI system with realization of the form of (1).

For the dynamical system $\mathcal{G}$ of (4) and (5), a function $r : \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$, is called a *supply rate* if it is locally integrable, so that $\int_{t_1}^{t_2} |r(y(\xi), u(\xi))| \, d\xi < \infty$ for all $t_1, t_2 > 0$, and if $\oint r(y(\xi)), u(\xi)) \, d\xi \geq 0$ for every path that takes the dynamical system from some initial state to the same final state. A more general form for the supply rate presented in Pinzoni and Willems (1992) will be used in this paper. Under the new definition, the supply rate can be a function of the signals (u, y) and, if they exist, their time derivatives.

Definition 1. (Willems 1972): A system $\mathcal{G}$ of the form in (4) and (5), with states $x \in \mathbb{R}^n$ is said to be *dissipative with respect to the supply rate* $r(\cdot, \cdot)$ if there exists a non-negative definite function $V_s : \mathbb{R}^n \to \mathbb{R}$, called a *storage function*, that satisfies the dissipation inequality

$$V_s(x(t_2)) \leq V_s(x(t_1)) + \int_{t_1}^{t_2} r(y(\xi), u(\xi)) \, d\xi \tag{6}$$

for all t_1, t_2 and for all $x(\cdot)$, $y(\cdot)$ and $u(\cdot)$ satisfying (4) and (5). □

If $V_s(x)$ is a differentiable function, then Willems (1972) provides an equivalent statement of dissipativeness of the system $\mathcal{G}$ with respect to the supply rate r, i.e.

$$\dot{V}_s(x(t)) \leq r(y(t), u(t)), \quad t \geq 0 \tag{7}$$

where $\dot{V}$ denotes the total derivative of $V(x)$ along the state trajectory $x(t)$. For a strongly dissipative system, (7) is replaced by the condition $\dot{V}_s(x(t)) < r(y(t), u(t))$ with a similar modification to (6). For the particular example of a mechanical system with force inputs and velocity outputs we can associate the storage function with the stored or available energy in the system, and the supply rate with the net flow of energy into the system. However, the concepts of the supply rates and storage functions also apply to more general systems for which this energy interpretation is no longer valid.

A variety of supply rates have been considered by Willems (1972), Hill and Moylan (1980) and Hill (1988). An appropriate supply rate for testing the passivity of a system $y = G(s)u$ is $r(y, u) = u^{\mathrm{T}} y$. This choice can be motivated by the example of a system with a force input and velocity output, so that the product $u^{\mathrm{T}} y$ is a measure of power. For bounded gain tests, the appropriate supply rate is $r(y, u) = u^{\mathrm{T}} u - \gamma^2 y^{\mathrm{T}} y$. A motivation for this choice follows from the identity

$$\int_0^{\mathrm{T}} r(y, u) \, dt = \int_0^{\mathrm{T}} u^{\mathrm{T}} u \, dt - \gamma^2 \int_0^{\mathrm{T}} y^{\mathrm{T}} y \, dt \tag{8}$$

which consists of two parts, the first associated with the energy at the system input, and the second with the weighted energy at the system output. If the integral on the left-hand side of (8) is positive, indicating that the weighted output energy is less than the input energy at any time T, then the system is gain bounded, since $\|G\|_\infty < \gamma^{-1}$.

As will now be shown, storage functions and supply rates provide a means

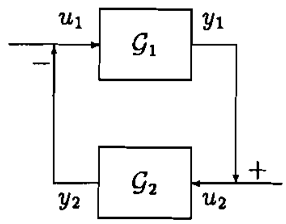

Figure 1. Two interconnected systems.

for developing Lyapunov functions for determining the stability of coupled feedback systems. In particular, if there exists a storage function for each system that is dissipative with respect to an appropriate supply rate, then these functions can be combined to form a Lyapunov function for the interconnected system. A more precise statement of this result for two systems interconnected as in Fig. 1 is provided by the following lemma.

Lemma 1: *Consider two dynamical systems $\mathcal{G}_1$ and $\mathcal{G}_2$ with state-space representation as in* (4) *and* (5), *and input–output pairs (u_1, y_1) and (u_2, y_2) respectively. Assume that the two systems are connected as illustrated in Fig. 1, so that $u_1 = -y_2$ and $u_2 = y_1$. Furthermore, associated with these systems are states x_1, x_2, supply rates $r_1(y_1, u_1)$, $r_2(y_2, u_2)$ and storage functions $V_{s1}(x_1)$, $V_{s2}(x_2)$ respectively. Suppose that both $V_{s1}(x_1)$ and $V_{s2}(x_2)$ are positive definite, and that the supply rates satisfy $r_1(y_1, u_1) + r_2(y_2, u_2) = 0$, for all $u_1 = -y_2$ and $u_2 = y_1$. Then the solution $(x_1, x_2) = 0$ of the feedback interconnection of $\mathcal{G}_1$ and $\mathcal{G}_2$ is Lyapunov stable with Lyapunov function $V = V_{s1} + V_{s2}$.*

Proof: Since $V = V_{s1} + V_{s2}$ is the sum of two positive definite functions, it is positive definite. Furthermore, $\dot{V}(x_1, x_2) = \dot{V}_{s1}(x_1) + \dot{V}_{s2}(x_2) \leq r_1(y_1, u_1) + r_2(y_2, u_2) = 0$. From the positive definiteness of V and the negative semidefiniteness of $\dot{V}$, it follows that V is a Lyapunov function that guarantees the Lyapunov stability of the solution $(x_1, x_2) = 0$. □

In the special case that the states $x_2(t)$ can be written in terms of the states $x_1(t)$ for all $t \geq 0$, then it is possible to relax the assumptions on V_{s2} in Lemma 1. In this case, if the storage function V_{s1} is positive definite and the storage function V_{s2} is non-negative definite, then V is positive definite.

Next, we extend the results of Lemma 1 to the case of interest in this paper, where a single LTI system $G(s)$ is independently interconnected to m systems, as illustrated in Fig. 2. In this case, we consider a system G with m scalar inputs

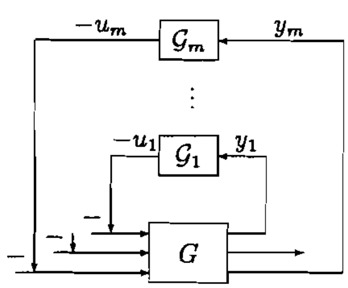

Figure 2. Multiple uncertainties for mixed robustness tests.

and outputs such that each u_i is only influenced by y_i. The systems in this example are special cases of those in Lemma 1 since one is LTI and the other is block diagonal.

Corollary 1: *Consider an LTI system $G(s)$ with inputs u_i, outputs y_i, $i = 1, \ldots, m$, and states x. Introduce the dynamical systems $\mathcal{G}_i$*

$$\dot{x}_i(t) = A_i x_i(t) + B_i y_i(t) \tag{9}$$

$$-u_i(t) = g_i(x_i(t)) + D_i y_i(t) \tag{10}$$

with supply rates $r_i(-u_i(t), y_i(t))$ and positive definite storage functions $V_{si}(x_i(t))$. Define an overall supply rate $R(y_1(t), \ldots, y_m(t), u_1(t), \ldots, u_m(t)) = \sum_{i=1}^{m} \gamma_i r_i(-u_i(t), y_i(t))$, with $\gamma_i > 0$, $i = 1, \ldots, m$. If there exists a positive definite storage function $V_G(x)$ for the system $G(s)$ which is dissipative with respect to the negative of the overall supply rate, then the interconnected system is Lyapunov stable.

Proof: Consider the positive definite function $V(x, x_1, \ldots, x_m) = V_G(x) + \sum_{i=1}^{m} \gamma_i V_{si}(x_i)$. From the definition of storage functions, it follows that $\dot{V}_{si}(x_i(t)) \leqslant r_i(-u_i(t), y_i(t))$, $i = 1, \ldots, m$, and $\dot{V}_G(x) \leqslant -R(y_1(t), \ldots, y_m(t), u_1(t), \ldots, u_m(t))$. Then

$$\begin{aligned} \dot{V}(x(t), x_1(t), \ldots, x_m(t)) &= \dot{V}_G(x(t)) + \sum_{i=1}^{m} \gamma_i \dot{V}_{si}(x_i(t)) \\ &\leqslant -R(\cdot, \cdot) + \sum_{i=1}^{m} \gamma_i r_i(\cdot, \cdot) = 0 \end{aligned}$$

where the arguments of the supply rates are dropped for clarity. □

As before, if the states $x_1, \ldots, x_m$ of the dynamical systems can be written in terms of x, then V is positive definite if V_G is positive definite and V_{si}, $i = 1, \ldots, m$ are non-negative definite. Furthermore, if the positive definite storage function V_G is strongly dissipative with respect to the corresponding supply rate $-R(\cdot, \cdot)$, then the Lyapunov function V is positive definite and $\dot{V}$ is negative definite. It then follows that the interconnected system is asymptotically stable.

The results of Corollary 1 convert the problem of determining the stability of interconnected systems to that of determining appropriate supply rates and storage functions, and then testing for dissipativeness of the independent systems with respect to the supply rates. For the problem addressed here, where insights into determining the supply rates are available from the characteristics of the nonlinearities, this approach greatly facilitates the construction of the Lyapunov functions. Corollary 1 also allows us to incorporate both complex and real uncertainties by mixing the supply rates for the different dynamical systems.

In the next section, we develop a framework in which different classes of nonlinear functions are incorporated into the stability criteria. The absolute stability criteria that are developed are then related to the robust stability and performance problem with real parametric uncertainty.

4. Stability robustness for monotonic and odd monotonic nonlinear functions

While graphical tests have been developed for SISO feedback systems with a single loop nonlinearity in the absolute stability literature, the goal in this paper is to employ a unified framework from which one can develop both state-space and frequency domain robust stability criteria for multivariable systems. The approach will use the concepts of supply rates, storage functions, and the dissipation inequality defined in § 3. This framework was developed by How and Hall (1992) from the original work of Willems (1972), and applied to multivariable nonlinearities with general sector constraints, extending the results of Popov. Haddad and Bernstein (1991 a, 1993) have investigated the multivariable Popov criterion for robust stability and $\mathcal{H}_2$ performance for both linear uncertainties and time-invariant nonlinearities. The following analysis extends these results to encompass both monotonic and odd monotonic restrictions on the nonlinear functions, as illustrated in Fig. 3.

The basis of the stability analysis tests is illustrated in Fig. 4, where $G(s)$ is an LTI system with realization

$$G(s) \sim \left[\begin{array}{c|c} A & B_0 \\ \hline C_0 & 0 \end{array}\right]$$

and $f(\cdot)$ is a nonlinear function. As discussed in the Introduction, this nonlinearity is used to model the uncertainty associated with the system $G(s)$. The transfer function $W(s)$ is an appropriate frequency domain stability multiplier which is selected based on the known properties of the memoryless nonlinearity $f(\cdot)$, such as gain or slope bounds, and its purpose is to modify the region of instability for the system, as discussed by Zames (1966). The system from $\hat{y}$ to u through $W^{-1}(s)$ and $f(\cdot)$ can be written as a dynamical system of the form in (4) and (5). Furthermore, for the independently coupled case discussed previously, $W(s)$ will be a diagonal matrix, and $f(\cdot)$ is a component decoupled nonlinearity.

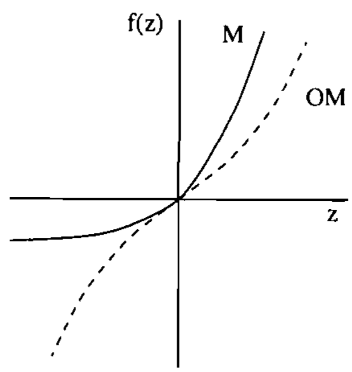

Figure 3. Examples of monotonic (M) and odd monotonic (OM) nonlinear functions.

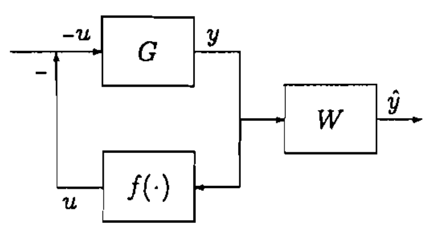

Figure 4. Framework for stability tests with linear systems coupled to nonlinearities.

The process for determining the absolute stability of a system (stability for an entire class of nonlinear functions) can be broken down into several steps. For the particular case of interest, with m nonlinear functions independently interconnected to the LTI system $G(s)$, it follows from Corollary 1 that the storage functions and supply rates can be developed separately. For each input–output pair (u_i, y_i) of the system $G(s)$, a storage function V_{si} based on the states of the multiplier $W_i(s)$ must be shown to be dissipative with respect to the supply rate $r_i(u_i, y_i)$. Then, as in Corollary 1, the condition that the storage function for the linear system $G(s)$ be dissipative, with respect to a supply rate that is the negative sum of the supply rates for the nonlinear systems, leads to a condition for stability of the interconnected system. This criterion for stability can be interpreted as requiring the existence of a positive definite solution of an algebraic Riccati equation. Furthermore, combining the storage functions for the linear and nonlinear systems provides a parameter-dependent Lyapunov function for the interconnected system.

System parametric uncertainty arises from many factors and can exhibit both linear and nonlinear behaviour. For example, phenomena such as backlash, spring hardening, oscillation with stops, integrator wind-up, actuator saturation, and the aerodynamic effect on aircraft dynamics all lead to nonlinear system uncertainty. Some of these examples can be accurately modelled as sector bounded nonlinear functions. However, in this paper, we develop robustness tests with more specific, and thus potentially less conservative, classes of monotonic and odd monotonic nonlinear functions. With these additional classes of functions, the robustness tests developed in this paper cover the continuum of real parametric uncertainties from arbitrary time-invariant sector-bounded nonlinear functions to constant linear functions.

Popov (1961) introduced a multiplier for sector-bounded time-invariant nonlinear functions. Several authors have discussed the appropriate multipliers for monotonic and odd monotonic restrictions on the nonlinear functions. Specifically, Narendra and Taylor (1973), Brockett and Willems (1965), Narendra and Cho (1967), Zames and Falb (1968), Thathachar *et al.* (1967), Thathachar and Srinath (1967) have developed suitable stability multipliers $W(s)$ for monotonic and odd monotonic nonlinear functions. These multipliers are given by the functions in the sets $\mathcal{W}_{\mathrm{RL}}$ and $\mathcal{W}_{\mathrm{RC}}$, which exhibit an interlacing pole–zero pattern on the negative real axis. From Guillemin (1957), the two sets are distinguished by which is closest to the origin, a pole ($\mathcal{W}_{\mathrm{RC}}$) or a zero ($\mathcal{W}_{\mathrm{RL}}$). The standard form of the multiplier for each $i = 1, \ldots, m$ is

$$W_i(s) = \alpha_{i0} + \beta_{i0}s + \sum_{j=1}^{m_{i1}} \alpha_{ij}\left(1 - \frac{\alpha_{ij}}{\beta_{ij}(s + \eta_{ij})}\right) + \sum_{j=m_{i1}+1}^{m_{i2}} \alpha_{ij}\left(1 + \frac{\alpha_{ij}}{\beta_{ij}(s + \eta_{ij})}\right) \tag{11}$$

where the coefficients α_{ij}, β_{ij}, and η_{ij} are non-negative and satisfy $\eta_{ij}\beta_{ij} - \alpha_{ij} \geqslant 0$. To consider just monotonic nonlinearities, take $m_{i2} = m_{i1}$ in (11) (equivalent to eliminating the last summation). For odd monotonic nonlinearities, it is also possible to include multipliers with terms that explicitly contain complex poles and zeros. While the extra freedom associated with this extension will be discussed later, one can develop very general forms of the multiplier $W_i(s)$ with the three main components in (11).

As discussed by How and Hall (1992) and Haddad and Bernstein (1991 a, 1993), the multiplier phase plays a crucial role in determining the conservativeness of the analysis test. The first two terms of (11) correspond to the standard Popov multiplier whose phase angle increases monotonically from 0° and 90°. The first sum in (11) is a partial fraction expansion of the driving point impedance of a resistor–inductor (RL) network. While the phase for this class also lies between 0° to 90°, it is not a monotonically increasing function of frequency. The last summation in (11) is of the form of a driving point impedance of a resistor–capacitor (RC) network, with a pole closest to the origin, and phase between 0° and −90°.

As illustrated in Fig. 4, proving stability of the coupled system requires handling signals of the form $W(s)y$. While obtaining filtered outputs of this form is simple for the Popov multiplier, it is quite complicated for the multipliers in (11). In particular, with these extended multipliers it is necessary to augment the multiplier dynamics to the original system so that the filtered outputs, to be defined later, can be obtained directly from the augmented state vector. The resulting augmented matrix A_a then contains the poles of both the system $G(s)$ and the multipliers $W_i(s)$, $i = 1, \ldots, m$.

While much of absolute stability theory has been developed for infinite sector or slope restrictions on the nonlinearity, the shifting approach discussed by Rekasius and Gibson (1962) and Desoer and Vidyasagar (1975) has been used to handle finite bounds. Define M_1, $M_2 \in \mathbb{R}^{m \times m}$ as diagonal matrices whose non-zero elements represent the upper and lower sector bounds for each input–output loop. The transformations illustrated in Fig. 5 convert the general slope restrictions (M_1, M_2) to a one-sided condition $(0, M_2 - M_1)$, and then finally to an infinite one $(0, \infty)$. For now we consider only the bounds $(0, M_2)$, and a later remark will consider the more general case. The following section outlines the process for shifting these sectors and augmenting the multiplier dynamics. Sections 4.2 and 4.3 present stability tests for the monotonic and odd monotonic nonlinearities.

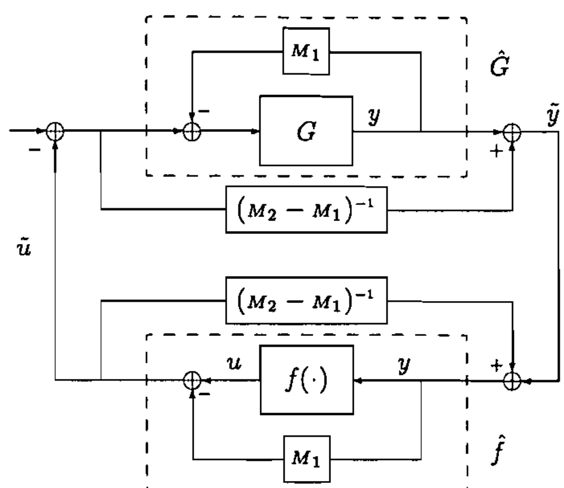

Figure 5. System transformations and the definitions of $\hat{G}$, $\hat{f}(\cdot)$, $\tilde{u}$ and $\tilde{y}$. If the lower slope bound $M_1 = 0$, then $\hat{f}(\cdot) = f(\cdot)$, $\hat{G}(s) = G(s)$ and $\tilde{u} = u$.

4.1. *Multiplier augmentation*

We begin with a discussion of the transformations illustrated in Fig. 5. In the following, take $M_1 = 0$ and $M_2 = M = \mathrm{diag}(M_{11}, \ldots, M_{mm})$, and consider differentiable monotonic and odd monotonic nonlinear functions that satisfy the constraint $(\mathrm{d}f_i(\sigma)/\mathrm{d}\sigma) < M_{ii}$ for all values of σ. Note that this constraint implies that $f_i(\sigma)$ satisfies the sector constraint $0 \leq \sigma f_i(\sigma) < M_{ii}\sigma^2$. From the figure, observe that $f(y) = \tilde{f}(\tilde{y})$, $\tilde{f}(0) = 0$, and

$$\tilde{y} = y - M^{-1}\tilde{f}(\tilde{y}) \tag{12}$$

Desoer and Vidyasagar (1975) presented an excellent discussion of the existence and uniqueness of the solution of this equation. While these issues dominate the discussion for arbitrary nonlinearities, it is known from Thathachar and Srinath (1967) that these properties are automatically satisfied for sector-bounded monotonic nonlinearities. Furthermore, for each nonlinearity $f_i(y_i)$, with $y_i \neq 0$ define

$$\frac{\tilde{f}_i(\tilde{y}_i)}{\tilde{y}_i} = \frac{f_i(y_i)/y_i}{1 - M_{ii}^{-1} f_i(y_i)/y_i} \tag{13}$$

Then, if $f_i(\cdot)$ is sector-bounded by M_{ii}, then the equivalent condition for the shifted nonlinearity is $\tilde{y}_i\tilde{f}_i(\tilde{y}_i) \geq 0$. Also, by the chain rule

$$\frac{\mathrm{d}\tilde{f}_i(\tilde{y}_i)}{\mathrm{d}\tilde{y}_i} = \frac{\mathrm{d}f_i(y_i)/\mathrm{d}y_i}{1 - M_{ii}^{-1}\,\mathrm{d}f_i(y_i)/\mathrm{d}y_i} \tag{14}$$

so that if $f_i(\cdot)$ is differentiable and satisfies the slope restrictions $0 \leq (\mathrm{d}f_i(\sigma)/\mathrm{d}\sigma) < M_{ii}$, then $\tilde{f}_i(\cdot)$ is also differentiable and satisfies $0 \leq (\mathrm{d}\tilde{f}_i(\sigma)/\mathrm{d}\sigma)$, and thus is monotonic. The same transformation can be used for slope-restricted odd monotonic nonlinearities.

The corresponding changes to the LTI system are also illustrated in Fig. 5. In particular, for $M_1 = 0$, the shifted system is given by

$$\tilde{G}(s) = G(s) + M^{-1} \tag{15}$$

with inputs $-u$, and outputs $\tilde{y}$. Each transformed nonlinearity $\tilde{f}_i(\cdot)$ is restricted to lie in the first and third quadrants, so that

$$\sigma\tilde{f}_i(\sigma) \geq 0, \quad \sigma \in \mathbb{R} \tag{16}$$

Furthermore, since the transformed nonlinearities are monotonic, they satisfy

$$0 \leq (\tilde{f}_i(\sigma_1) - \tilde{f}_i(\sigma_2))(\sigma_1 - \sigma_2), \quad \sigma_1, \sigma_2 \in \mathbb{R} \tag{17}$$

Having discussed the transformations, we can now proceed with the multiplier augmentation. As illustrated in Fig. 4 (with $G(s)$, $f(\cdot)$ replaced by $\tilde{G}(s)$, $\tilde{f}(\cdot)$), the stability tests are performed on the systems formed by combining the multiplier with the system $\tilde{G}(s)$ and its inverse with the nonlinearity $\tilde{f}(\cdot)$. The supply rate for these systems will then be a function of the new output (equivalent to $\hat{y}$ in Fig. 4), which is obtained by applying the appropriate multiplier to each element of the output of $\tilde{G}(s)$. Observing the form of the multiplier in (11), it can be seen that in the expression $W_i(s)\tilde{y}_i$, we obtain terms of the form

$$z_{ij} = \frac{\alpha_{ij}\tilde{y}_i}{\beta_{ij}(s + \eta_{ij})} \tag{18}$$

Hence, the corresponding supply rate will involve real signals z_{ij} obtained by passing the system output $\tilde{y}_i$ through a parallel bank of decoupled low-pass filters with time constants $1/\eta_{ij}$ and positive gains $\alpha_{ij}/\beta_{ij}\eta_{ij}$. For the system as in (4) and (5), formed from $W^{-1}(s)$ and $\tilde{f}(\cdot)$, with z_{ij} as the system states, the dynamics of each term in the multiplier can be augmented to the system by rewritinng (18) as

$$\dot{z}_{ij} + \eta_{ij}z_{ij} = \frac{\alpha_{ij}}{\beta_{ij}}\tilde{y}_i = \frac{\alpha_{ij}}{\beta_{ij}}(C_0x - M^{-1}u)_i \tag{19}$$

where $(\cdot)_i$ denotes the ith row of $(\cdot)$. Writing the states in a vector $z_i^{\mathrm{T}} \triangleq [z_{i1}, \ldots, z_{im_{i2}}]$, the dynamics associated with each multiplier $W_i(s)$ can be written as

$$\dot{z}_i = [\hat{C}_i \quad A_i]\begin{bmatrix} x \\ z_i \end{bmatrix} - \hat{M}_i u \tag{20}$$

where $A_i \triangleq \operatorname{diag}(-\eta_{ij})$, $j = 1, \ldots, m_{i2}$, and

$$\hat{C}_i \triangleq \begin{bmatrix} \dfrac{\alpha_{i1}}{\beta_{i1}} \\ \dfrac{\alpha_{i2}}{\beta_{i2}} \\ \vdots \\ \dfrac{\alpha_{im_{i2}}}{\beta_{im_{i2}}} \end{bmatrix}(C_0)_i, \quad \hat{M}_i \triangleq \begin{bmatrix} \dfrac{\alpha_{i1}}{\beta_{i1}} \\ \dfrac{\alpha_{i2}}{\beta_{i2}} \\ \vdots \\ \dfrac{\alpha_{im_{i2}}}{\beta_{im_{i2}}} \end{bmatrix}(M^{-1})_i \tag{21}$$

where $(C_0)_i$ and $(M^{-1})_i$ denote the ith rows of the respective matrices.

With m input–output pairs to the system $G(s)$, we augment the multiplier dynamics to the shifted system $\tilde{G}(s)$ to obtain a state-space representation of $\tilde{G}_{\mathrm{a}}(s)$ given by

$$\dot{x}_{\mathrm{a}} = A_{\mathrm{a}}x_{\mathrm{a}} - B_{\mathrm{a}}u \tag{22}$$

$$\tilde{y} = C_{\mathrm{a}}x_{\mathrm{a}} - M^{-1}u \tag{23}$$

where $x_{\mathrm{a}} \in \mathbb{R}^{n_{\mathrm{a}}}$, $n_{\mathrm{a}} \triangleq n + \sum_{i=1}^{m} m_{i2}$, A_{a}, B_{a} and C_{a} are defined as

$$x_{\mathrm{a}} = \begin{bmatrix} x \\ z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}, \quad A_{\mathrm{a}} = \begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ \hat{C}_1 & A_1 & 0 & & 0 \\ \hat{C}_2 & 0 & A_2 & \ddots & 0 \\ \vdots & & & & \\ \hat{C}_m & 0 & 0 & & A_m \end{bmatrix}, \quad B_{\mathrm{a}} = \begin{bmatrix} B_0 \\ \hat{M}_1 \\ \hat{M}_2 \\ \vdots \\ \hat{M}_m \end{bmatrix} \tag{24}$$

$$C_{\mathrm{a}} = [C_0 \quad 0 \quad 0 \quad \cdots \quad 0]$$

Next, define R_{ij} as an output matrix for this augmented system, designed to access the jth element of z_i so

$$z_{ij} = R_{ij}x_{\mathrm{a}} \tag{25}$$

Then, the only non-zero element of R_{ij} is the $(n + \sum_{l=1}^{i-1} m_{l2} + j)$th term, which is 1.

Note that although extra dynamics associated with the multipliers have been added to the system $\tilde{G}_a(s)$ it can easily be shown that

$$\tilde{G}_a(s) = C_a(sI - A_a)^{-1}B_a + M^{-1} = C_0(sI - A)^{-1}B_0 + M^{-1} = \tilde{G}(s) \quad (26)$$

Hence, by pole–zero cancellation in each input–output loop, the frequency domain representations of $\tilde{G}(s)$ and $\tilde{G}_a(s)$ are equivalent in terms of their input–output properties. The following simple example is used to clarify the preceding notational development.

Example 1: Consider a LTI system with realization

$$G(s) \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & -2 & 1 \\ \hline c_1 & c_2 & 0 \end{array}\right] \quad (27)$$

with $M = 1$, and the multiplier given by

$$W(s) = \alpha_1\left(1 - \frac{\alpha_1}{s + \eta_1}\right) + \alpha_2\left(1 - \frac{\alpha_2}{s + \eta_2}\right) \quad (28)$$

Let z_1, z_2 be the states corresponding to the multiplier dynamics. Then the augmented system is given by

$$x_a = \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix}, \quad A_a = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ \alpha_1 c_1 & \alpha_1 c_2 & -\eta_1 & 0 \\ \alpha_2 c_1 & \alpha_2 c_2 & 0 & -\eta_2 \end{bmatrix}, \quad B_a = \begin{bmatrix} 0 \\ 1 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} \quad (29)$$

$$C_a = [c_1 \ c_2 \ 0 \ 0], \quad R_{11} = [0 \ 0 \ 1 \ 0], \quad R_{12} = [0 \ 0 \ 0 \ 1] \quad (30)$$

Since

$$(sI - A_a)^{-1} = \begin{bmatrix} (sI - A)^{-1} & 0 & 0 \\ c_1'(s) & (s + \eta_1)^{-1} & 0 \\ c_2'(s) & 0 & (s + \eta_2)^{-1} \end{bmatrix} \quad (31)$$

where

$$c_i'(s) = \frac{\alpha_i}{s + \eta_i}[c_1 \ c_2](sI - A)^{-1}, \quad i = 1, 2 \quad (32)$$

it follows that $C_a(sI - A_a)^{-1}B_a = C_0(sI - A)^{-1}B_0$. Furthermore

$$R_{11}(sI - A_a)^{-1}B_a = [0 \ 0 \ 1 \ 0]\begin{bmatrix} (sI - A)^{-1} & 0 & 0 \\ c_1'(s) & (s + \eta_1)^{-1} & 0 \\ c_2'(s) & 0 & (s + \eta_2)^{-1} \end{bmatrix}\begin{bmatrix} B \\ \alpha_1 \\ \alpha_2 \end{bmatrix} \quad (33)$$

$$= [c_1'(s) \ (s + \eta_1)^{-1} \ 0]\begin{bmatrix} B \\ \alpha_1 \\ \alpha_2 \end{bmatrix} \quad (34)$$

$$= \frac{\alpha_1}{s + n_1}\tilde{G}(s) \quad (35)$$

and similarly for R_{12}. □

From Corollary 1, it is known that in the development of the storage functions and supply rates we can consider each input–output pair of $G(s)$ independently. Hence, without loss of generality, we consider the development of supply rates and storage functions for the separate single–input single–output nonlinearities coupled with the appropriate multiplier. Since the goal is to demonstrate that the combination of the nonlinear function and the multiplier, as in Fig. 4, is passive, it follows from § 3 that the appropriate supply rate is the product of the system inputs and outputs. However, as discussed by How and Hall (1992), and How (1993), a modification of this supply rate is required for the multipliers in (11) if $M \neq 0$. In this case, with s denoting the standard Laplace variable and $\hat{y}_i = W_i(s)\tilde{y}_i$, we consider signals of the form

$$\bar{y}_i = W_i(s)\tilde{y}_i + \beta_{i0} M_{ii}^{-1} s u_i \tag{36}$$

It will be seen that this additional term is equivalent to the quadratic term in Narendra and Taylor (1973) which is added to the Lur'e-Postnikov Lyapunov function to account for a direct transmission in the plant dynamics. As will also be seen in the following development, the term su_i is used to cancel an equivalent term from the Popov multiplier in the expression $\hat{y}_i = W_i(s)\tilde{y}_i$. The assumed differentiability of the shifted nonlinearities guarantees that the expression in (36) exists.

4.2. *Monotonic nonlinear functions*

In this section we develop the supply rate and appropriate storage functions for monotonic nonlinearities ($m_{i2} = m_{i1}$), and present robust stability conditions for the full system via algebraic Riccati equations. Using the definitions of $W_i(s)$ in (11), $\tilde{y}$ in (12), and the filtered outputs in (18), the signal in (36) becomes

$$\begin{aligned}\bar{y}_i &= \sum_{j=1}^{m_{i1}} \alpha_{ij}\left(1 - \frac{\alpha_{ij}}{\beta_{ij}(s + \eta_{ij})}\right)\tilde{y}_i + (\alpha_{i0} + \beta_{i0}s)(y - M^{-1}u)_i + \beta_{i0}M_{ii}^{-1}su_i \\ &= \sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + (\alpha_{i0} + \beta_{i0}s)y_i - \alpha_{i0}M_{ii}^{-1}u_i \\ &= \sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}sy_i + \alpha_{i0}\tilde{y}_i \end{aligned} \tag{37}$$

We then construct the supply rate $r_i(\hat{y}_i, u_i)$ in terms of the time domain representation of $\bar{y}_i$ and u_i, which yields

$$r_i(\hat{y}_i, u_i) = \left[\sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i\right]u_i \tag{38}$$

An appropriate storage function for this supply rate is given in the following lemma.

Lemma 2: *Consider a differentiable monotonic nonlinear function* $f_i(\cdot)$ *that satisfies the slope restrictions* $0 \leq \mathrm{d}f_i(\sigma)/\mathrm{d}\sigma < M_{ii}$. *As in* (13), *define the differentiable monotonic nonlinearity* $\tilde{f}_i(\cdot)$ *that satisfies* (16) *and* (17). *Consider the dynamical system* $\mathcal{G}_i$ *which is a combination of* $\tilde{f}_i(\cdot)$ *and* $W_i^{-1}(s)$ *from* (11) ($m_{i2} = m_{i1}$), *with corresponding state-space representation given by* (4) *and* (5).

Then the function V_{si} defined by

$$V_{si}(\tilde{y}_i, z_{i1}, \ldots, z_{im_{i1}}) = \beta_{i0}\left(\int_0^{\tilde{y}_i} \tilde{f}_i(\sigma)\,\mathrm{d}\sigma + \tfrac{1}{2}M_{ii}^{-1}u_i^2\right) + \sum_{j=1}^{m_{i1}} \beta_{ij}\int_0^{z_{ij}} \tilde{f}_i(\sigma)\,\mathrm{d}\sigma \tag{39}$$

is a storage function for the supply rate in (38).

Proof: Since $u_i = \tilde{f}_i(\tilde{y}_i)$, and the dynamics of $\mathcal{G}_i$ can be written in terms of $\tilde{y}_i$ and z_i, $V_{si}(\cdot)$ is a function of the states of the dynamical system. Also, since $\tilde{f}(0) = 0$, it follows that $V_{si}(0) = 0$, and from (16), since β_{ij}, $j = 0, 1, \ldots, m_{i1}$, are non-negative, then V_{si} is non-negative definite. Finally, to show that (39) is a storage function, it must be demonstrated that it is dissipative with respect to the supply rate of (38).

For notational convenience, let $u_{ij} = \tilde{f}_i(z_{ij})$ and $u_i = \tilde{f}_i(\tilde{y}_i)$. Now, dropping the arguments for convenience

$$\dot{V}_{si} = \sum_{j=1}^{m_{i1}} \beta_{ij}u_{ij}\dot{z}_{ij} + \beta_{i0}\left[\dot{\tilde{y}}_i + M_{ii}^{-1}\frac{\mathrm{d}u_i}{\mathrm{d}t}\right]u_i \tag{40}$$

From the definition of $\tilde{y}$ in (12), terms of the form $\mathrm{d}u_i/\mathrm{d}t$ cancel. Next, note that (16) and (17) yield

$$0 \le \alpha_{i0}\tilde{y}_i u_i \tag{41}$$

$$0 \le \alpha_{ij}(\tilde{y}_i - z_{ij})(u_i - u_{ij}) \tag{42}$$

$$0 \le (\eta_{ij}\beta_{ij} - \alpha_{ij})z_{ij}u_{ij} \tag{43}$$

for $j = 1, \ldots, m_{i1}$. Adding (41), (42) and (43) to (40) yields

$$\begin{aligned}\dot{V}_{si} \le {} & \sum_{j=1}^{m_{i1}} u_{ij}(\beta_{ij}\dot{z}_{ij} + (\eta_{ij}\beta_{ij} - \alpha_{ij})z_{ij}) + \sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij})(u_i - u_{ij}) \\ & + (\beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y})u_i\end{aligned} \tag{44}$$

After collecting terms, we obtain

$$\begin{aligned}\dot{V}_{si} \le {} & \sum_{j=1}^{m_{i1}} [\beta_{ij}\dot{z}_{ij} + (\eta_{ij}\beta_{ij} - \alpha_{ij})z_{ij} - \alpha_{ij}(\tilde{y}_i - z_{ij})]u_{ij} \\ & + \left[\sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i\right]u_i\end{aligned} \tag{45}$$

Now, it follows from (18) that

$$\beta_{ij}\dot{z}_{ij} = \alpha_{ij}\tilde{y}_i - \beta_{ij}\eta_{ij}z_{ij} \tag{46}$$

so that the first summation of (45) is zero, and using (38) yields

$$\dot{V}_{si} \le \sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij})u_i + (\beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i)u_i = r_i \tag{47}$$

which demonstrates that the storage function is dissipative with respect to the supply rate. □

One should note that the state-space representation for $\mathcal{G}_i$ is in terms of the states $\tilde{y}_i$ and z_i which can easily be written in terms of x_a because of the augmentation process discussed in the previous section. While it is convenient to consider the supply rates and storage functions for each nonlinear function, these must be combined to form the supply rate for the linear system $\tilde{G}(s)$ and the full multiplier $W(s)$. Vector notation will simplify this development, but we must first define the following matrices. These definitions are complicated by the fact that each $W_i(s)$ can have a different number of expansion terms in (11). This difficulty can be handled by defining extended values of α_{ij}, β_{ij} and η_{ij}. Let $m_1 = \max_i(m_{i1})$. Then, for each $i = 1, \ldots, m$ and $j = 1, \ldots, m_1$, let $\alpha_{ij} = 0$, $\beta_{ij} = 0$, $\eta_{ij} = 0$ and $R_{ij} = 0$ if $j > m_{i1}$. Furthermore, define $H_j = \mathrm{diag}(\alpha_{1j}, \ldots, \alpha_{mj})$, $N_j = \mathrm{diag}(\beta_{j1}, \ldots, \beta_{jm})$, $S_j = \mathrm{diag}(\eta_{j1}, \ldots, \eta_{jm})$, $j = 0, 1, \ldots, m_1$. Finally, let $R_j = [R_{1j}^{\mathrm{T}}, R_{2j}^{\mathrm{T}}, \ldots, R_{mj}^{\mathrm{T}}]^{\mathrm{T}}$. Then, as in Corollary 1, using (38) to form the overall supply rate for the LTI system $G(s)$ yields

$$R(\hat{y}, u) = \sum_{i=1}^{m} r_i(\hat{y}_i, u_i) \tag{48}$$

$$= [u_1 \quad \ldots \quad u_m] \begin{bmatrix} \sum_{j=1}^{m_{11}} \alpha_{1j}(\tilde{y}_1 - z_{1j}) + \beta_{10}\dot{y}_1 + \alpha_{10}\tilde{y}_1 \\ \vdots \\ \sum_{j=1}^{m_{m1}} \alpha_{mj}(\tilde{y}_m - z_{mj}) + \beta_{m0}\dot{y}_m + \alpha_{m0}\tilde{y}_m \end{bmatrix} \tag{49}$$

$$= u^{\mathrm{T}}\left[\sum_{j=1}^{m_1} H_j(\tilde{y} - \hat{z}_j) + N_0\dot{y} + H_0\tilde{y}\right] \tag{50}$$

where $\hat{z}_j = [z_{1j} \cdots z_{mj}]^{\mathrm{T}}$. This representation of $R(\cdot, \cdot)$ can be simplified further by using the definition of R_j to note that for each j, $\hat{z}_j \triangleq R_j x_a$.

Furthermore, using (12) for $\tilde{y}$, and noting that $\dot{y} = C_a\dot{x}_a = C_a(A_a x_a - B_a u)$, the overall supply rate can then be written as

$$R(\hat{y}, u) = u^{\mathrm{T}}\left[H_0 C_a + N_0 C_a A_a + \sum_{j=1}^{m_1} H_j(C_a - R_j)\right]x_a$$
$$- u^{\mathrm{T}}\left(N_0 C_a B_a + \sum_{j=0}^{m_1} H_j M^{-1}\right)u \tag{51}$$

Having developed the overall supply rate, we now present the stability condition for the interconnected system.

Theorem 1: *Consider an LTI system $G(s)$ independently coupled to m differentiable monotonic nonlinearities that satisfy the slope restrictions $0 \le \mathrm{d}f_i(\sigma)/\mathrm{d}\sigma < M_{ii}$. If for each input–output pair (u_i, y_i), there exist multipliers $W_i(s)$ as in (11) and a matrix $R = R^{\mathrm{T}} > 0 \in \mathbb{R}^{n_a \times n_a}$ such that with the preceding definitions of H_j, N_j, and R_j*

(1) $\hat{R}_0 \triangleq [(N_0 C_a B_a + \sum_{j=0}^{m_1} H_j M^{-1}) + (N_0 C_a B_a + \sum_{j=0}^{m_1} H_j M^{-1})^{\mathrm{T}}] > 0$, *and*

(2) *there exists a symmetric matrix $P > 0$, satisfying*

$$0 = A_a^T P + PA_a + R + \left[H_0C_a + N_0C_aA_a + \sum_{j=1}^{m_1} H_j(C_a - R_j) - B_a^T P\right]^T \hat{R}_0^{-1} \left[H_0C_a + N_0C_aA_a + \sum_{j=1}^{m_1} H_j(C_a - R_j) - B_a^T P\right] \quad (52)$$

then the negative feedback interconnection of the system $G(s)$ and the nonlinearities $f_i(\ )$, $i = 1, \ldots, m$, is asymptotically stable. Furthermore, a Lyapunov function for the combined system is given by

$$V(x_a) = x_a^T P x_a + 2\sum_{i=1}^{m}\left[\beta_{i0}\left(\int_0^{\tilde{y}_i} \tilde{f}_i(\sigma)\,d\sigma + \tfrac{1}{2}M_{ii}^{-1}u_i^2\right) + \sum_{j=1}^{m_{i1}} \beta_{ij}\int_0^{z_{ij}} \tilde{f}_i(\sigma)\,d\sigma\right] \quad (53)$$

Proof: For the LTI system $G(s)$, introduce the storage function

$$V_G(x_a) = x_a^T P x_a \quad (54)$$

We proceed using the development of Corollary 1. Note that (39) gives a storage function for each input–output pair (u_i, y_i) which is dissipative with respect to a given supply rate. For asymptotic stability of the interconnected system, it must be demonstrated that the positive definite function V_G is strongly dissipative with respect to the negative sum of these supply rates. Clearly, if P is a positive definite matrix, then V_G is a positive definite function. Next, from (7) and (51), we require that

$$\dot{V}_G(x_a(t)) < -R(\hat{y}, u) \quad (55)$$

Define $\Gamma(x_a, u) \triangleq \dot{V}_G(x_a) + R(\hat{y}, u)$. Then, since $\dot{x}_a = A_a x_a - B_a u$, the condition for dissipativeness is that

$$\Gamma(x_a, u) = 2x_a^T P(A_a x_a - B_a u) + u^T\left[H_0C_a + N_0C_aA_a + \sum_{j=1}^{m_1} H_j(C_a - R_j)\right]x_a - u^T\left(N_0C_aB_a + \sum_{j=0}^{m_1} H_j M^{-1}\right)u \quad (56)$$

Following the procedure in How and Hall (1992), it is now possible to develop the worst case sequence of inputs $u_w(x(t))$ by forming the equation $(\partial\Gamma(x_a, u)/\partial u) = 0$ and solving for u. If condition (1) of the theorem is satisfied, then a solution is known to exist, and the worst case input sequence corresponds to a maximum stationary point since $(\partial^2\Gamma(x_a, u)/\partial^2 u) = -\hat{R}_0 < 0$. To prove that the system is dissipative, it is sufficient to guarantee that the worst case input results in a negative maximum of $\Gamma(x_a, u)$. Thus, select $R = R^T > 0$ and require that $\max_u \Gamma(x_a, u) = -x_a^T R x_a < 0$. Substituting for u_w into $\Gamma(x_a, u)$ yields the condition that P of (54) satisfies

$$A_a^T P + PA_a + \left[H_0C_a + N_0C_aA_a + \sum_{j=1}^{m_1} H_j(C_a - R_j) - B_a^T P\right]^T \hat{R}_0^{-1} \left[H_0C_a + N_0C_aA_a + \sum_{j=1}^{m_1} H_j(C_a - R_j) - B_a^T P\right] < -R \quad (57)$$

which yields (52). Hence, it follows from Corollary 1 that the Lyapunov function for the combined system is

$$V = V_{\mathrm{G}} + \sum_{i=1}^{m} V_{si} \tag{58}$$

Substituting the definitions of these storage functions from (39) and (54) yields (53). □

Remark 1: $V(x_{\mathrm{a}})$ in (53) is an extended Lur'e-Postnikov Lyapunov function since it depends explicitly on the nonlinearity $\tilde{f}_i(\cdot)$. Similarly, in the linear uncertainty case, where $\tilde{f}_i(\tilde{y}_i) = F_i\tilde{y}_i$, $V(x_{\mathrm{a}})$ becomes a parameter-dependent Lyapunov function since the uncertain parameters F_i explicitly appear in the Lyapunov function. In this case, as discussed in the Introduction, the uncertain parameters are not allowed to be arbitrarily time-varying, and the result is a refined framework for constant real parametric uncertainty (Haddad and Bernstein 1991 a). □

4.3. *Odd monotonic nonlinear functions*

We now consider nonlinearities with odd monotonic restrictions as illustrated in Fig. 3. The procedure is identical to the one discussed in the previous section, the main difference now being that the transformed nonlinear function $\tilde{f}(\cdot)$ satisfies (16), (17) and an additional constraint, from Narendra and Neuman (1966), Thathachar *et al.* (1967), Thathachar and Srinath (1967), Narendra and Taylor (1973)

$$0 \leq \sigma_1\tilde{f}(\sigma_1) + \sigma_2\tilde{f}(\sigma_2) + \sigma_1\tilde{f}(\sigma_2) - \sigma_2\tilde{f}(\sigma_1), \quad \sigma_1, \sigma_2 \in \mathbb{R} \tag{59}$$

The definition of the supply rate and multiplier augmentation process are as discussed in § 4.1 and 4.2. In this case, we consider $m_{i2} > m_{i1}$ in $W_i(s)$ of (11).

Using the simplification of $r_i(\hat{y}_i, u_i)$ in (38), the definition of z_{ij} in (18), and noting the form of the multiplier terms in $W_i(s)$ for $j = m_{i1} + 1, \ldots, m_{i2}$, the supply rate can be rewritten as

$$r_i(\hat{y}_i, u_i) = \left[\sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \sum_{j=m_{i1}+1}^{m_{i2}} \alpha_{ij}(\tilde{y}_i + z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i\right]u_i \tag{60}$$

A storage function for this supply rate is given in the following lemma.

Lemma 3: *Consider a differentiable odd monotonic nonlinear function $f_i(\cdot)$ that satisfies the slope restrictions $0 \leq (\mathrm{d}f_i(\sigma)/\mathrm{d}\sigma) < M_{ii}$. As in (13), define the differentiable odd monotonic nonlinearity $\tilde{f}_i(\cdot)$ that satisfies (16) and (59). Consider the dynamical system $\mathcal{G}_i$ which is a combination of $\tilde{f}_i(\cdot)$ and the multiplier $W_i^{-1}(s)$ from (11) ($m_{i2} > m_{i1}$), with corresponding state-space representation given by (4) and (5). Then the function V_{si} defined by*

$$V_{si}(\tilde{y}_i, z_{i1}, \ldots, z_{im_{i2}}) = \beta_{i0}\left(\int_0^{\tilde{y}_i} \tilde{f}_i(\sigma)\,\mathrm{d}\sigma + \tfrac{1}{2}M_{ii}^{-1}u_i^2\right) + \sum_{j=1}^{m_{i2}} \beta_{ij}\int_0^{z_{ij}} \tilde{f}_i(\sigma)\,\mathrm{d}\sigma \tag{61}$$

is a storage function for the supply rate in (60).

Proof: As in the proof of Lemma 2, V_{si} is a non-negative definite function of the states of the system formed by combining $W_i^{-1}(s)$ with the nonlinearity $\tilde{f}_i(\cdot)$ as in Fig. 4. Furthermore, using the results of Lemma 2, note that

$$\dot{V}_{si} \leqslant \left[\sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i\right]u_i + \sum_{j=m_{i1}+1}^{m_{i2}} \beta_{ij}u_{ij}\dot{z}_{ij} \tag{62}$$

Note that (59) yields

$$0 \leqslant \alpha_{ij}((z_{ij} + \tilde{y}_i)u_i + (z_{ij} - \tilde{y}_i)u_{ij}) \tag{63}$$

for $j = m_{i1} + 1, \ldots, m_{i2}$. Next, adding (41), (43) and (63) to (62) yields

$$\dot{V}_{si} \leqslant \left[\sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i\right]u_i + \sum_{j=m_{i1}+1}^{m_{i2}} [\beta_{ij}u_{ij}\dot{z}_{ij} + (\beta_{ij}\eta_{ij} - \alpha_{ij})z_{ij}u_{ij} + \alpha_{ij}((z_{ij} + \tilde{y}_i)u_i + (z_{ij} - \tilde{y}_i)u_{ij})] \tag{64}$$

Next, (46) can be used to replace $\dot{z}_{ij}$, and after cancelling terms, (64) becomes

$$\dot{V}_{si} \leqslant \left[\sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \sum_{j=m_{i1}+1}^{m_{i2}} \alpha_{ij}(\tilde{y}_i + z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i\right]u_i \tag{65}$$

Hence, from the definition of $r_i(\cdot, \cdot)$ in (60), it follows that $\dot{V}_{si} \leqslant r_i(\cdot, \cdot)$, or equivalently, that the storage function is dissipative with respect to the supply rate. □

To prove overall system stability, we again form augmented matrices using m_1 and $m_2 = \max_i(m_{i2})$. Then, from (51) and (60), the overall supply rate can be written as

$$R(\hat{y}, u) = u^{\mathrm{T}}\left[H_0C_{\mathrm{a}} + N_0C_{\mathrm{a}}A_{\mathrm{a}} + \sum_{j=1}^{m_1} H_j(C_{\mathrm{a}} - R_j) + \sum_{j=m_1+1}^{m_2} H_j(C_{\mathrm{a}} + R_j)\right]x_{\mathrm{a}} - u^{\mathrm{T}}\left(N_0C_{\mathrm{a}}B_{\mathrm{a}} + \sum_{j=0}^{m_2} H_jM^{-1}\right)u \tag{66}$$

We can now state the following theorem governing the overall stability of the system.

Theorem 2: *Consider an LTI system $G(s)$ independently coupled to m differentiable odd monotonic nonlinearities that satisfy the slope restrictions $0 \leqslant \mathrm{d}f_i(\sigma)/\mathrm{d}\sigma < M_{ii}$. If for each input–output pair (u_i, y_i), there exist multipliers $W_i(s)$ as in (11) and a matrix $R = R^{\mathrm{T}} > 0 \in \mathbb{R}^{n_{\mathrm{a}} \times n_{\mathrm{a}}}$ such that, with the preceding definitions of H_j, N_j and R_j*

(1) $R_0 \triangleq [(N_0C_{\mathrm{a}}B_{\mathrm{a}} + \sum_{j=0}^{m_2} H_jM^{-1}) + (N_0C_{\mathrm{a}}B_{\mathrm{a}} + \sum_{j=0}^{m_2} H_jM^{-1})^{\mathrm{T}}] > 0$, *and*

(2) *there exists a symmetric matrix $P > 0$, satisfying*

$$0 = A_{\mathrm{a}}^{\mathrm{T}}P + PA_{\mathrm{a}} + R + \left[H_0C_{\mathrm{a}} + N_0C_{\mathrm{a}}A_{\mathrm{a}} + \sum_{j=1}^{m_1} H_j(C_{\mathrm{a}} - R_j) + \sum_{j=m_1+1}^{m_2} H_j(C_{\mathrm{a}} + R_j) - B_{\mathrm{a}}^{\mathrm{T}}P\right]^{\mathrm{T}} R_0^{-1}\left[H_0C_{\mathrm{a}} + N_0C_{\mathrm{a}}A_{\mathrm{a}} + \sum_{j=1}^{m_1} H_j(C_{\mathrm{a}} - R_j) + \sum_{j=m_1+1}^{m_2} H_j(C_{\mathrm{a}} + R_j) - B_{\mathrm{a}}^{\mathrm{T}}P\right] \tag{67}$$

then the negative feedback interconnection of system $G(s)$ and the nonlinearities $f_i(\ )$, $i = 1, \ldots, m$, is asymptotically stable. In this case, a Lyapunov function for the combined system is given by

$$V(x_a) = x_a^T P x_a + 2\sum_{i=1}^{m}\left[\beta_{i0}\left(\int_0^{\tilde{y}_i}\tilde{f}_i(\sigma)\,d\sigma + \tfrac{1}{2}M_{ii}^{-1}u_i^2\right) + \sum_{j=1}^{m_{i2}}\beta_{ij}\int_0^{z_{ij}}\tilde{f}_i(\sigma)\,d\sigma\right] \tag{68}$$

Proof: The proof is similar to the one for Theorem 1 and is omitted. □

Remark 2: To consider nonlinearities with both upper and lower slope constraints, we employ both transformations in Fig. 5. In particular, we define $\tilde{f}(\cdot)$ and $\tilde{G}(s)$ in (13) and (15) in terms of $\hat{f}(\cdot)$ and $\hat{G}(s)$, where

$$\hat{f}(y) = f(y) - M_1 y, \quad y \in \mathbb{R}^m \tag{69}$$

$$\hat{G}(s) = (I + G(s)M_1)^{-1}G(s) \sim \left[\begin{array}{c|c} A - B_0 M_1 C_0 & B_0 \\ \hline C_0 & 0 \end{array}\right] \tag{70}$$

The previous analysis can then be repeated, starting with a system $\hat{G}(s)$ and differentiable (odd) monotonic nonlinearities $\hat{f}(\cdot)$ with upper slope bounds $M_2 - M_1$. The appropriate Riccati equations can then be obtained from Theorems 1 and 2 by redefining A_a in (24), replacing u with $\tilde{u} = u - M_1 y$, and then substituting $M_2 - M_1$ for M. □

In the next section, we make explicit connections between the time and frequency domain stability conditions. As a result, we provide sufficient conditions for the existence of positive definite solutions to (52) and (67). These conditions also enable us to make explicit connections between absolute stability theory and mixed-μ theory.

5. Frequency domain stability conditions

The utility of absolute stability criteria, such as the one in Popov (1961), is the simplicity of its graphical interpretation in the frequency domain. The previous section developed state-space conditions for the stability of an LTI system $G(s)$ coupled with two main classes of nonlinear functions. While recent trends in control theory are toward time domain or state-space Riccati-based tests, one advantage of frequency domain criteria is the insight that they provide to the role of the frequency domain multipliers. The following provides a powerful tool for converting from state-space to frequency domain stability conditions within the supply rate framework.

Lemma 4 (Trentelman and Willems 1991): *Consider an LTI system $y = -Gu$ with supply rate $r(y, u)$. Define a system $z = -Hu$, $z \in \mathbb{R}^{m+l}$ with state-space representation*

$$H(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A_h & B_h \\ \hline C_h & D_h \end{array}\right]$$

and supply rate $\hat{r}(z, u) = z^T L z$, where $L = L^T \in \mathbb{R}^{(m+l)\times(m+l)}$ and $\hat{r}(z, u) \triangleq$

$r(y, u)$. If A_h has no poles on the jω*-axis, then the following statements are equivalent:*

(1) *the system $H(s)$ is dissipative with respect to the quadratic supply rate $\hat{r}(z, u)$;*

(2) $H^*(j\omega)LH(j\omega) \geq 0,\ \omega \in \mathbb{R}$.

Proof: For the proof see Trentelman and Willems (1991). □

The system $H(s)$ represents a modification of the system $G(s)$ to provide functions of both the inputs $-u$ and outputs $y = -Gu$ in the vector z. Consider the simple example of the supply rate $\hat{r}(z, u) = -u^{\mathrm{T}}y$, where

$$z = \begin{bmatrix} y \\ -u \end{bmatrix}, \quad H(s) = \begin{bmatrix} G(s) \\ I_m \end{bmatrix} \quad \text{and } L = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \tag{71}$$

The frequency domain test for dissipativity is then

$$H^*(j\omega)LH(j\omega) = G^*(j\omega) + G(j\omega) \geq 0, \quad \omega \in \mathbb{R} \tag{72}$$

which, along with the condition that $G(s)$ is asymptotically stable, is the standard matrix positive real test for the system $G(s)$. Note that similar statements can also be developed with the strongly dissipative conditions.

Thus, Lemma 4 provides a tool for transforming the time domain stability criteria given in the previous sections into equivalent frequency domain stability criteria. Furthermore, while these tests are interesting in terms of the extra interpretation that they provide for the multipliers, they also provide conditions for the existence of the positive definite matrices P in Theorems 1 and 2.

Next, introduce a system $H(s)$ with outputs that combine to form the negative of the overall supply rate. Let $W(s) = \operatorname{diag}(W_i(s))$, and define the output vector z

$$\begin{aligned} z &= H(s)(-\tilde{u}) + \begin{bmatrix} W(s) & -N_0(M_2 - M_1)^{-1}s \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{y} \\ -\tilde{u} \end{bmatrix} \\ &= \begin{bmatrix} W(s) & -N_0(M_2 - M_1)^{-1}s \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{G}_a \\ I \end{bmatrix} (-\tilde{u}) \end{aligned} \tag{73}$$

With the matrix L as in (71), it follows that the supply rate is $-R(\hat{y}, u) = z^{\mathrm{T}}Lz$. From Lemma 4, the test for dissipativeness is then whether $H^*(j\omega)LH(j\omega) \geq 0\ \forall\omega$. Substituting the definition of $H(s)$ into this condition and noting that N_0, M_1 and M_2 are diagonal, we obtain

$$\begin{aligned} H^*LH &= [\tilde{G}_a^*\ I] \begin{bmatrix} W^*(j\omega) & 0 \\ N_0(M_2 - M_1)^{-1}j\omega & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} W(j\omega) & -N_0(M_2 - M_1)^{-1}j\omega \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{G}_a \\ I \end{bmatrix} \\ &= [\tilde{G}_a^*\ I] \begin{bmatrix} W^*(j\omega) & 0 \\ N_0(M_2 - M_1)^{-1}j\omega & I \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0 & I \\ W(j\omega) & -N_0(M_2 - M_1)^{-1}j\omega \end{bmatrix} \begin{bmatrix} \tilde{G}_a \\ I \end{bmatrix} \\ &= \tilde{G}_a^*W^* + W\tilde{G}_a \end{aligned} \tag{74}$$

Consequently, an equivalent test for stability is that $T_1(s) \triangleq W(s)\tilde{G}_a(s)$ be positive real. Furthermore, since it follows from (26) that $\tilde{G}(s)$ and $\tilde{G}_a(s)$ are equivalent in terms of their input–output properties, we need only consider the positive realness of $T(s) \triangleq W(s)\tilde{G}(s)$. Hence, it follows from the work of Haddad and Bernstein (1991 b) that if A_a is asymptotically stable and $T(s)$ is strongly positive real, then there exists an $n_a \times n_a$ symmetric matrix $P > 0$ which satisfies (52) or (67), depending on the form of the multiplier $W(s)$, as discussed in Theorems 1 and 2. Conversely, for a given selection of $W(s)$, if there exists a $P > 0$ for all $R > 0$, then A_a is asymptotically stable, and $T(s)$ is strongly positive real.

In the following, we consider the case involving finite upper and lower bounds on the slope of the nonlinearity. In particular, assume that the upper bound satisfies $M_2 > 0$ and the lower bound satisfies $M_1 < M_2$. In this case, since the double shift illustrated in Fig. 5 must be utilized, we replace (15) with

$$\tilde{G}(s) = (I + GM_1)^{-1}(I + GM_2)(M_2 - M_1)^{-1} \tag{75}$$

where it is assumed that M_1 is selected so that $I + GM_1$ is invertible at all frequencies. To clarify the physical interpretation of the stability criterion and develop connections with the upper bounds for mixed-μ, the symmetric bound $M_1 = -M_2$ will be used in the following development. Narendra and Taylor (1973) develop frequency domain tests for the case that $M_1 \neq -M_2$.

Theorem 3: *Consider the LTI system $G(s)$ with m independent nonlinearities $\tilde{f}_i(\cdot)$ with appropriate sector bounds given by M_1 and M_2. Assume $M_1 = -M_2 < 0$, and that $I - G(\mathrm{j}\omega)M_2$ is invertible for all $\omega \in \mathbb{R}$. For each $i = 1, \ldots, m$, select the multiplier $W_i(s)$ as in* (11) *based on the characteristics of $\tilde{f}_i(\cdot)$. Furthermore, define $W(s) = W_{\mathrm{Re}}(s) + \mathrm{j}W_{\mathrm{Im}}(s) = \mathrm{diag}(W_i(s))$. If*

$$G^*W_{\mathrm{Re}}M_2G - \mathrm{j}(W_{\mathrm{Im}}G - G^*W_{\mathrm{Im}}) - W_{\mathrm{Re}}M_2^{-1} \leq 0 \tag{76}$$

for all $\omega \in \mathbb{R}$, then the negative feedback interconnection of $G(s)$ and the m nonlinearities as illustrated in Fig. 2 are Lyapunov stable.

Proof: With $M_1 = -M_2$, it can easily be demonstrated that the first two factors of $\tilde{G}(s)$ commute. Then from (74) and (75), the condition for stability is that

$$0 \leq T(\mathrm{j}\omega) + T^*(\mathrm{j}\omega), \quad \omega \in \mathbb{R} \tag{77}$$

where

$$T(s) = W(s)(I + G(s)M_2)(I - G(s)M_2)^{-1}(2M_2)^{-1} \tag{78}$$

Since $I - G(s)M_2$ is assumed to be invertible and M_2 is positive, we can develop an equivalent test by pre- and post-multiplying (77) by $(I - G(\mathrm{j}\omega)M_2)^*M_2$ and $M_2(I - G(\mathrm{j}\omega)M_2)$ respectively. Performing this operation, and substituting for $T(s)$ from (78), the condition of (77) is equivalent to the requirement that, for all $\omega \in \mathbb{R}$

$$\begin{aligned} 0 &\leq (I - GM_2)^*M_2W(I + GM_2) + (I + GM_2)^*W^*M_2(I - GM_2) \\ &= M_2(W_{\mathrm{Re}} + \mathrm{j}W_{\mathrm{Im}}) + M_2(W_{\mathrm{Re}} + \mathrm{j}W_{\mathrm{Im}})GM_2 - M_2G^*M_2(W_{\mathrm{Re}} + \mathrm{j}W_{\mathrm{Im}}) \\ &\quad + M_2(W_{\mathrm{Re}} - \mathrm{j}W_{\mathrm{Im}}) - M_2(W_{\mathrm{Re}} - \mathrm{j}W_{\mathrm{Im}})GM_2 + M_2G^*M_2(W_{\mathrm{Re}} - \mathrm{j}W_{\mathrm{Im}}) \\ &\quad - M_2G^*M_2(W_{\mathrm{Re}} + \mathrm{j}W_{\mathrm{Im}})GM_2 - M_2G^*M_2(W_{\mathrm{Re}} - \mathrm{j}W_{\mathrm{Im}})GM_2 \end{aligned} \tag{79}$$

Collecting terms and dividing through by $2M_2^2$ yields the condition in (76). □

Remark 3: The stability criterion in (76) of Theorem 3 is very general, since it may involve a mixture of the time-invariant sector-bounded nonlinearities as in a Popov test, the differentiable monotonic and odd monotonic nonlinearities discussed in § 4.2 and 4.3, and the time-varying nonlinearities considered in a small gain test. □

Remark 4: As discussed by How and Hall (1992), (76) has a graphical interpretation in the scalar case. Specifically, since $W_{\mathrm{Re}} > 0$, (76) can be rewritten as

$$\left(G + \mathrm{j}\frac{W_{\mathrm{Im}}}{W_{\mathrm{Re}}M_2}\right)^*\left(G + \mathrm{j}\frac{W_{\mathrm{Im}}}{W_{\mathrm{Re}}M_2}\right)W_{\mathrm{Re}}M_2 - W_{\mathrm{Re}}M_2\left(\frac{W_{\mathrm{Im}}}{W_{\mathrm{Re}}M_2}\right)^2 - \frac{W_{\mathrm{Re}}}{M_2} \leq 0 \tag{80}$$

or, equivalently, letting $G = x + \mathrm{j}y$, (80) can be written as

$$x^2 + \left(y + \frac{W_{\mathrm{Im}}}{W_{\mathrm{Re}}M_2}\right)^2 \leq \frac{1}{M_2^2} + \left(\frac{W_{\mathrm{Im}}}{W_{\mathrm{Re}}M_2}\right)^2 \tag{81}$$

This corresponds to a circle with a frequency dependent centre at $-W_{\mathrm{Im}}(\omega^2)/W_{\mathrm{Re}}(\omega^2)M_2$ and constant real axis intercepts at $\pm M_2^{-1}$. This approach is reminiscent of the classical off-axis circle criterion of Cho and Narendra (1968), where a single bounding circle is employed as opposed to a family of frequency dependent circles. Further discussions of the role of the multiplier phase and its relationship to the conservatism of the test are presented in Haddad and Bernstein (1991 a), How and Hall (1992) and How (1993). □

Remark 5: For odd monotonic nonlinearities, Narendra and Taylor (1973), Thathachar *et al.* (1967) and Thathachar and Srinath (1967) discuss multiplier terms that contain complex poles and zeros of the form

$$\widetilde{W}_i(s) = (\text{Equation } 11) + \sum_{j=m_{i2}+1}^{m_{i3}} \alpha_{ij}\frac{s^2 + a_{ij}s + b_{ij}}{s^2 + \lambda_{ij}s + \eta_{ij}} \tag{82}$$

Although this case can be handled exactly as the other terms in the multiplier, the proofs of stability are much more involved and exceed the scope of this paper. However, the benefit of these additional terms in the sense of the rapid phase variations that they allow is readily apparent in the frequency domain test of Theorem 3 and, as will be seen in the next section, they provide even more general parametrizations of the D, N-scales in the mixed-μ theory. □

5.1. *Connections to mixed-μ analysis*

In order to compare the upper bounds for real-μ and the frequency domain stability tests developed in the previous section, we present a brief summary of the notation in Fan *et al.* (1991). For the system matrix $G(s) \in \mathbb{C}^{m\times m}$, let m_r, m_c and $m_C(m_t = m_r + m_c + m_C \leq m)$ define the types and number of uncertainties expected in the system. The positive integers $k_i(\sum_{i=1}^{m_t} k_i = m)$ then define the block structure and repetition of the uncertainties denoted by $\mathcal{K}(m_r, m_c, m_C) = (k_1, \ldots, k_{m_r}, \ldots, k_{m_r+m_c}, \ldots, k_{m_t})$. The set of allowable perturbations for the system G is then defined to be

$$\mathscr{X}_{\mathcal{H}} = \{\Delta = \text{block diag}(\delta_1^r I_{k_1}, \ldots, \delta_{m_r}^r I_{k_{m_r}}, \delta_1^c I_{k_{m_r+1}}, \ldots, \delta_{m_c}^c I_{k_{m_r+m_c}}, \ldots, \Delta_1^C, \ldots, \Delta_{m_C}^C): \delta_i^r \in \mathbb{R}, \delta_i^c \in \mathbb{C}, \Delta_i^C \in \mathbb{C}^{k_l \times k_l}, l = m_r + m_c + i\} \tag{83}$$

Definition 2. (Doyle 1982): For $G \in \mathbb{C}^{m\times m}$, $\mu_{\mathcal{H}}(G)$ is defined as

$$\mu_{\mathcal{H}}(G) = (\min_{\Delta \in \mathscr{X}_{\mathcal{H}}} \{\sigma_{\max}(\Delta): \det(I - \Delta G) = 0\})^{-1} \tag{84}$$

where $\mu_{\mathcal{H}}(G) = 0$ if no $\Delta \in \mathscr{X}_{\mathcal{H}}$ exists such that $\det(I - \Delta G) = 0$. □

The complexity inherent in the definition and computation of $\mu_{\mathcal{H}}(G)$ has led to the use of approximations by both upper and lower bounds. For purely complex uncertainties, the bounds

$$\rho(G) \leq \mu_{\mathcal{H}}(G) \leq \sigma_{\max}(G) \tag{85}$$

involving the spectral radius and the maximum singular value are commonly employed. These bounds are usually refined with frequency dependent scaling matrices D.

As discussed by Doyle (1982) and Young *et al.* (1991), the scaled bounds are exact if $m_r = 0$ and $2m_c + m_C \leq 3$. However, for a larger number of complex uncertainties, results from these papers demonstrate that the bounds are only correct to within approximately 15%, and with real parametric uncertainties, the bounds in (85) can be arbitrarily poor. Recent developments have led to new upper and lower bounds for mixed-μ ($m_r \neq 0$), as discussed by Fan *et al.* (1991) and Young *et al.* (1991). For the upper bound of interest, define the hermitian scaling matrices

$$\mathscr{D}_{\mathcal{H}} = \{\text{block diag}(D_1, \ldots, D_{m_r+m_c}, d_1 I_{k_{m_r+m_c+1}}, \ldots, d_{m_C} I_{k_{m_t}}): 0 < D_i = D_i^* \in \mathbb{C}^{k_i \times k_i}, 0 < d_i \in \mathbb{R}\} \tag{86}$$

$$\mathscr{N}_{\mathcal{H}} = \{\text{block diag}(N_1, \ldots, N_{m_r}, 0_{k_{m_r+1}}, \ldots, 0_{k_{m_t}}): N_i = N_i^* \in \mathbb{C}^{k_i \times k_i}\} \tag{87}$$

which are partitioned to be compatible with the uncertainty structure $\mathscr{X}_{\mathcal{H}}$. The set of matrices $\mathscr{D}_{\mathcal{H}}$ includes elements for all three types of uncertainties, whereas $\mathscr{N}_{\mathcal{H}}$ has non-zero terms only in those parts corresponding to the real uncertainties. Members of both $\mathscr{D}_{\mathcal{H}}$ and $\mathscr{N}_{\mathcal{H}}$ are frequency dependent weighting functions and are constrained to be hermitian. The elements of $\mathscr{D}_{\mathcal{H}}$ are further constrained to be positive. Note that within the block definition of (87), the elements of the scaling matrix N are essentially arbitrary.

An upperbound for mixed-μ was developed by Fan *et al.* (1991) by including constraints on the eigenvectors of the system ΔG where $\Delta \in \mathscr{X}_{\mathcal{H}}$ and $m_r \neq 0$. To compare with the analysis results in the previous section, it is sufficient to note the following definition.

Definition 3 (Young *et al.* 1991): For $G(s) \in \mathbb{C}^{m\times m}$ and compatible uncertainty block structure $\mathcal{H}$, define

$$\alpha_* = \inf_{D\in\mathscr{D}_{\mathcal{H}}, N\in\mathscr{N}_{\mathcal{H}}} [\min_{\alpha\in\mathbb{R}} \{\alpha: (G^*DG + j(NG - G^*N) - \alpha D) \leq 0\}] \tag{88}$$

□

It is shown by Fan *et al.* (1991) and Young *et al.* (1991) that α_* is an upper bound for the mixed-μ problem in the sense that $\mu_{\mathcal{K}}(G) \leq (\max(0, \alpha_*))^{1/2}$.

Corollary 2: *Consider the diagonal case where $k_i = 1$, $i = 1, \ldots, m_t$ in $\mathcal{K}$. Then it follows from Theorem 3 with $W_{\mathrm{Re}} M_2$ replaced by D, W_{Im} replaced by $-N$, and M_2^{-2} bounded by α, that the conditions in* (76) *and* (88) *are identical. In the case of m_r linear time-invariant and $m_c + m_C$ nonlinear time-varying functions, then the bound in* (88) *is recovered. Finally, for sector-bounded nonlinear time-varying functions, take $W_{\mathrm{Re}} > 0$ and $W_{\mathrm{Im}} = 0$ in* (76) *to recover complex-μ.*

Proof: For the particular selection of $\mathcal{K}$, since $W_{\mathrm{Re}} M_2 > 0$ and W_{Im} are real, they are members of the $\mathcal{D}_{\mathcal{K}}$ and $\mathcal{N}_{\mathcal{K}}$ respectively. The equivalence then follows by direct substitution. In the linear case, the only restriction on $W(s)$ is that it be a positive real function, and thus W_{Re} and W_{Im} can be any functions in the sets $\mathcal{D}_{\mathcal{K}}$ and $\mathcal{N}_{\mathcal{K}}$. □

The equivalence of these two stability criteria is even stronger if we recognize that the upper bound for mixed-μ is related to M_2^{-1}. Then, as in (88), minimizing over α for a particular selection of D and N functions is equivalent to determining the largest sector (slope) bound M_2 that will destabilize the system for a given multiplier selection. As discussed by Narendra and Taylor (1973), this process is, of course, the foundation for the absolute stability theory. Extensions of Theorem 3 to the block diagonal case have been addressed for the Popov criterion in Haddad and Bernstein (1991 a).

6. Robust stability and performance analysis

In this section, we specialize the results of § 4 to linear uncertainty, and introduce the robust stability and performance problems. As shown in § 4.1, in order to account for the extra dynamics introduced by the frequency domain multiplier, the resulting state-space model is of increased dimension. Hence, let $\mathcal{U} \subset \mathbb{R}^{n_a \times n_a}$ denote a set of perturbations ΔA_a of a given nominal augmented dynamics matrix $A_a \in \mathbb{R}^{n_a \times n_a}$. Within the context of robustness analysis, it is assumed that A_a is asymptotically stable and $0 \in \mathcal{U}$. We begin by considering the question of whether or not $A_a + \Delta A_a$ is asymptotically stable for all $\Delta A_a \in \mathcal{U}$. First, however, we note that since A_a in (24) is lower block triangular, it follows that if $A_a + \Delta A_a$ is asymptotically stable, then $A + \Delta A$ is asymptotically stable for all perturbations ΔA.

Robust stability problem: Determine whether the linear system

$$\dot{x}_a(t) = (A_a + \Delta A_a) x_a(t), \quad t \in [0, \infty) \tag{89}$$

is asymptotically stable for all $\Delta A_a \in \mathcal{U}$. □

To consider the problem of robust performance, introduce an external disturbance model involving white-noise signals as in standard LQG ($\mathcal{H}_2$) theory. The robust performance problem concerns the worst-case $\mathcal{H}_2$ norm, that is, the worst-case (over $\mathcal{U}$) of the expected value of a quadratic form involving outputs $z(t) = E x_a(t)$, where $E \in \mathbb{R}^{q \times n_a}$, when the system is subjected to a standard white noise disturbance $w(t) \in \mathbb{R}^d$ with weighting $D \in \mathbb{R}^{n_a \times d}$.

Robust performance problem: For the disturbed linear system

$$\dot{x}_a(t) = (A_a + \Delta A_a)x_a(t) + Dw(t), \quad t \in [0, \infty) \tag{90}$$

$$z(t) = Ex_a(t) \tag{91}$$

where $w(\cdot)$ is a zero-mean d-dimensional white-noise signal with intensity I_d, determine a performance bound β satisfying

$$J(\mathcal{U}) \triangleq \sup_{\Delta A_a \in \mathcal{U}} \limsup_{t \to \infty} \mathbb{E}\{\|z(t)\|_2^2\} \leq \beta \tag{92}$$

□

As shown in § 7, (90) and (91) may denote a control system in closed-loop configuration subjected to external white noise disturbances and for which $z(t)$ denotes the state and control regulation error.

Of course, since D and E may be rank deficient, there may be cases in which a finite performance bound β satisfying (92) exists, whereas (89) is not asymptotically stable over $\mathcal{U}$. In practice, however, robust performance is mainly of interest when (89) is robustly stable. In this case, the performance $J(\mathcal{U})$ involves the steady-state second moment of the state. For convenience, define the $n_a \times n_a$ non-negative definite matrices $R \triangleq E^{\mathrm{T}}E$, $V \triangleq DD^{\mathrm{T}}$. The following result is immediate.

Lemma 5: *Suppose $A_a + \Delta A_a$ is asymptotically stable for all $\Delta A_a \in \mathcal{U}$. Then*

$$J(\mathcal{U}) = \sup_{\Delta A_a \in \mathcal{U}} \operatorname{tr} Q_{\Delta A_a} R \tag{93}$$

where the $n_a \times n_a$ matrix $Q_{\Delta A_a} \triangleq \lim_{t\to\infty} \mathbb{E}[x_a(t)x_a^{\mathrm{T}}(t)]$ is given by

$$Q_{\Delta A_a} = \int_0^\infty e^{(A_a + \Delta A_a)t} V e^{(A_a + \Delta A_a)^{\mathrm{T}} t} \, dt \tag{94}$$

which is the unique, non-negative definite solution to

$$0 = (A_a + \Delta A_a)Q_{\Delta A_a} + Q_{\Delta A_a}(A_a + \Delta A_a)^{\mathrm{T}} + V \tag{95}$$

In order to draw connections with traditional Lyapunov function theory, we express the $\mathcal{H}_2$ performance measure in terms of a dual variable $P_{\Delta A_a}$ for which the roles of $A_a + \Delta A_a$ and $(A_a + \Delta A_a)^{\mathrm{T}}$ are interchanged.

Proposition 1: *Suppose $A_a + \Delta A_a$ is asymptotically stable for all $\Delta A_a \in \mathcal{U}$. Then*

$$J(\mathcal{U}) = \sup_{\Delta A_a \in \mathcal{U}} \operatorname{tr} P_{\Delta A_a} V \tag{96}$$

where $P_{\Delta A_a} \in \mathbb{R}^{n_a \times n_a}$ is the unique, non-negative-definite solution to

$$0 = (A_a + \Delta A_a)^{\mathrm{T}} P_{\Delta A_a} + P_{\Delta A_a}(A_a + \Delta A_a) + R \tag{97}$$

Proof: For the proof see Haddad and Bernstein (1991 a). □

Remark 6: In Lemma 5, $Q_{\Delta A_a}$ can also be viewed as the controllability gramian for the pair $(A_a + \Delta A_a, D)$. Similarly, $P_{\Delta A_a}$ in Proposition 1 can be viewed as the observability gramian for the pair $(E, A_a + \Delta A_a)$. □

Remark 7: The stochastic performance measure $J(\mathcal{U})$ given by (92) can also be written as

$$J(\mathcal{U}) = \sup_{\Delta A_a \in \mathcal{U}} \int_0^\infty \|E e^{(A_a + \Delta A_a)t} D\|_F^2 \, dt = \sup_{\Delta A_a \in \mathcal{U}} \|G_{\Delta A_a}(s)\|_2^2 \tag{98}$$

where

$$G_{\Delta A_a}(s) \triangleq E[sI - (A_a + \Delta A_a)]^{-1} D \tag{99}$$

which involves the $\mathcal{H}_2$ norm of the impulse response of (90) and (91). This stochastic performance measure can also be given a deterministic interpretation by letting $w(t)$ denote impulses at time $t = 0$. □

In the present paper our approach is to obtain robust stability as a consequence of sufficient conditions for robust performance. Such conditions are developed in the following sections.

6.1. *Robust stability and performance via parameter-dependent Lyapunov functions*

The key step in obtaining robust stability and performance is to bound the uncertain terms $\Delta A_a^T P_{\Delta A_a} + P_{\Delta A_a} \Delta A_a$ in the Lyapunov equation (97) by means of a parameter-dependent bounding function $\Omega(P, \Delta A_a)$ which guarantees robust stability by means of a family of Lyapunov functions. As shown by Haddad and Bernstein (1991 a), this framework corresponds to the construction of a parameter-dependent Lyapunov function that guarantees robust stability. As discussed by Haddad and Bernstein (1991 a), a key feature of this approach is the fact that it constrains the class of allowable time-varying uncertainties, thus reducing conservatism in the presence of constant real parameter uncertainty. The following result is fundamental and forms the basis for all later developments. For notational convenience, let $\mathbb{S}^r$ and $\mathbb{N}^r$ denote the set of $r \times r$ symmetric and non-negative definite matrices respectively.

Theorem 4: *Let* $\Omega_0: \mathbb{N}^{n_a} \to \mathbb{S}^{n_a}$ *and* $P_0: \mathcal{U} \to \mathbb{S}^{n_a}$ *be such that*

$$\Delta A_a^T P + P \Delta A_a \leq \Omega_0(P) - [(A_a + \Delta A_a)^T P_0(\Delta A_a) + P_0(\Delta A_a)(A_a + \Delta A_a)], \quad \Delta A_a \in \mathcal{U}, \quad P \in \mathbb{N}^{n_a} \tag{100}$$

and suppose there exists $P \in \mathbb{N}^{n_a}$ *satisfying*

$$0 = A^T P + PA + \Omega_0(P) + R \tag{101}$$

and such that $P + P_0(\Delta A_a)$ *is non-negative definite for all* $\Delta A_a \in \mathcal{U}$. *Then*

$$(A_a + \Delta A_a, E) \text{ is detectable}, \quad \Delta A_a \in \mathcal{U} \tag{102}$$

if and only if

$$A_a + \Delta A_a \text{ is asymptotically stable}, \quad \Delta A_a \in \mathcal{U} \tag{103}$$

In this case

$$P_{\Delta A_a} \leq P + P_0(\Delta A_a), \quad \Delta A_a \in \mathcal{U} \tag{104}$$

where $P_{\Delta A_a}$ *is given by* (97). *Therefore*

$$J(\mathcal{U}) \leq \operatorname{tr} PV + \sup_{\Delta A_a \in \mathcal{U}} \operatorname{tr} P_0(\Delta A_a) V \tag{105}$$

If, in addition, there exists $\bar{P}_0 \in \mathbb{S}^n$ *such that*

$$P_0(\Delta A_a) \leq \bar{P}_0, \quad \Delta A_a \in \mathcal{U} \tag{106}$$

then $J(\mathcal{U}) \leq \beta$, *where* $\beta \triangleq \mathrm{tr}\,[(P + \bar{P}_0)V]$.

Proof: For the proof see Haddad and Bernstein (1991 a). □

Note that with $\Omega(P, \Delta A_a)$ denoting the right-hand side of (100), this equation can be rewritten as

$$\Delta A_a^T P + P\Delta A_a \leq \Omega(P, \Delta A_a), \quad \Delta A_a \in \mathcal{U}, \quad P \in \mathbb{N}^{n_a} \tag{107}$$

where $\Omega(P, \Delta A_a)$ is a function of the uncertain parameters ΔA_a. For convenience we shall say that $\Omega(\cdot, \cdot)$ is a parameter-dependent bounding function or, to be consistent with Haddad and Bernstein (1991 a), a parameter-dependent Ω-bound. To apply Theorem 4, we first specify a function $\Omega_0(\cdot)$ and an uncertainty set $\mathcal{U}$ such that (107) holds. If the existence of a non-negative definite solution P to (101) can be determined analytically or numerically and the detectability condition (102) is satisfied, then robust stability is guaranteed and the performance bound (105) can be computed.

Finally, we establish connections between Theorem 4 and the Lyapunov function theory. Specifically, we show that a parameter-dependent Ω-bound establishing robust stability is equivalent to the existence of a parameter-dependent Lyapunov function, which also establishes robust stability. To show this, assume there exists a positive-definite solution to (101), let P_0: $\mathcal{U} \to \mathbb{N}^{n_a}$, and define the parameter-dependent Lyapunov function

$$V(x_a) \triangleq x_a^T(P + P_0(\Delta A_a))x_a \tag{108}$$

Note that since P is positive definite and $P_0(\Delta A_a)$ is non-negative definite, $V(x_a)$ is positive definite. Thus, the corresponding Lyapunov derivative is given by

$$\begin{aligned}\dot{V}(x_a) &= x_a^T[(A_a + \Delta A_a)^T(P + P_0(\Delta A_a)) + (P + P_0(\Delta A_a))(A_a + \Delta A_a)]x_a \\ &= x_a^T[A_a^T P + PA_a + \Delta A_a^T P + P\Delta A_a + A_a^T P_0(\Delta A_a) + P_0(\Delta A_a)A_a \\ &\quad + \Delta A_a^T P_0(\Delta A_a) + P_0(\Delta A_a)\Delta A_a]x_a\end{aligned} \tag{109}$$

or, equivalently, using (101)

$$\begin{aligned}\dot{V}(x_a) = -x_a^T[\Omega_0(P) - \{(A_a + \Delta A_a)^T P_0(\Delta A_a) \\ + P_0(\Delta A_a)(A_a + \Delta A_a)\} + R]x_a\end{aligned} \tag{110}$$

Thus, using (100) it follows that $\dot{V}(x_a) \leq 0$ so that $A_a + \Delta A_a$ is stable in the sense of Lyapunov. To show asymptotic stability using La Salle's Theorem in LaSalle (1960), we need to demonstrate that $\dot{V}(x_a) = 0$ implies $x_a = 0$. Note that $\dot{V}(x_a) = 0$ implies $Rx_a = 0$, or, equivalently, $Ex_a = 0$. Thus, with $\dot{x}_a(t) = (A_a + \Delta A_a)x_a(t)$, $Ex_a = 0$ and the detectability assumption in (102), it follows from the PBH test that $x_a = 0$. Hence asymptotic stability is established.

6.2. *Construction of parameter-dependent Lyapunov functions and connections with stability tests for monotonic and odd monotonic nonlinearities*

Having established the theoretical basis for our approach, we now assign explicit structure to the set $\mathcal{U}$ and the parameter-dependent bounding function

$\Omega(\cdot, \cdot)$. Specifically, the uncertainty set $\mathcal{U}$ is defined by

$$\mathcal{U} \triangleq \{\Delta A_a \in \mathbb{R}^{n_a \times n_a}: \Delta A_a = -B_a F(I + M^{-1}F)^{-1} C_a,\ F \in \mathcal{F}\} \tag{111}$$

where $\mathcal{F}$ satisfies

$$\mathcal{F} \triangleq \{F \in \mathbb{R}^{m \times m}: F \geq 0\} \tag{112}$$

and where $B_a \in \mathbb{R}^{n_a \times m}$ and $C_a \in \mathbb{R}^{m \times n_a}$ are fixed matrices denoting the structure of the uncertainty, $M \in \mathbb{R}^{m \times m}$ is a given diagonal positive-definite matrix, and $F \in \mathbb{R}^{m \times m}$ is a diagonal uncertain matrix.

Next, we digress slightly to provide an alternative characterization of the uncertainty set $\mathcal{U}$. In order to state our next result, define the subset $\hat{\mathcal{F}}$ of $\mathcal{F}$ by

$$\hat{\mathcal{F}} \triangleq \{\hat{F}: \hat{F} = F(I + M^{-1}F)^{-1},\ F \in \mathcal{F}\} \tag{113}$$

where by Lemma 3.2 of Haddad and Bernstein (1991 b), $\det(I + M^{-1}F) \neq 0$.

Proposition 2: *Let $M \in \mathbb{R}^{m \times m}$ be positive definite. Then*

$$\hat{\mathcal{F}} = \{\hat{F} \in \mathbb{R}^{m \times m}: \det(I - \hat{F}M^{-1}) \neq 0 \text{ and } \hat{F}M^{-1}\hat{F} \leq \hat{F}\} \tag{114}$$

Proof: '$\subset$'. Let $\hat{F} \in \hat{\mathcal{F}}$. Then there exists $F \in \mathcal{F}$ such that $\hat{F} = F(I + M^{-1}F)^{-1}$. Hence, $\hat{F}M^{-1} = F(I + M^{-1}F)^{-1}M^{-1}$ so that

$$\operatorname{spec}(\hat{F}M^{-1}) = \operatorname{spec}[F(I + M^{-1}F)^{-1}M^{-1}] \tag{115}$$

$$= \operatorname{spec}[M^{-1}F(I + M^{-1}F)^{-1}] \tag{116}$$

$$= \left\{\frac{\lambda}{1 + \lambda}: \lambda \in \operatorname{spec}(M^{-1}F)\right\} \tag{117}$$

where 'spec' denotes spectrum. Hence, $\operatorname{spec}(\hat{F}M^{-1})$ does not include 1, and $\det(I - \hat{F}M^{-1}) \neq 0$. Next, note that $F = (I - \hat{F}M^{-1})^{-1}\hat{F}$. Hence, it follows that

$$\hat{F} - \hat{F}M^{-1}\hat{F} = \tfrac{1}{2}\hat{F}(I - M^{-1}\hat{F}) + \tfrac{1}{2}(I - \hat{F}M^{-1})\hat{F} \tag{118}$$

$$= \tfrac{1}{2}(I - \hat{F}M^{-1})[(I - \hat{F}M^{-1})^{-1}\hat{F} + \hat{F}(I - M^{-1}\hat{F})](I - M^{-1}\hat{F}) \tag{119}$$

$$= (I - \hat{F}M^{-1})F(I - \hat{F}M^{-1}) \geq 0 \tag{120}$$

which proves '$\subset$'.

'$\supset$'. Let $\hat{F}$ be such that $\det(I - \hat{F}M^{-1}) \neq 0$ and $\hat{F}M^{-1}\hat{F} \leq \hat{F}$. Since $\det(I - \hat{F}M^{-1}) \neq 0$, define $F \triangleq (I - \hat{F}M^{-1})^{-1}\hat{F}$. It then follows that

$$F = \tfrac{1}{2}(I - \hat{F}M^{-1})^{-1}\hat{F} + \tfrac{1}{2}\hat{F}(I - \hat{F}M^{-1})^{-1} \tag{121}$$

$$= \tfrac{1}{2}(I - \hat{F}M^{-1})^{-1}[\hat{F}(I - \hat{F}M^{-1}) + (I - \hat{F}M^{-1})\hat{F}](I - \hat{F}M^{-1})^{-1} \tag{122}$$

$$= \tfrac{1}{2}(I - \hat{F}M^{-1})^{-1}[\hat{F} - \hat{F}M^{-1}\hat{F}](I - \hat{F}M^{-1})^{-1} \geq 0 \tag{123}$$

Hence, $F \in \mathcal{F}$. Furthermore, since $F = (I - \hat{F}M^{-1})^{-1}\hat{F}$ is equivalent to $\hat{F} = F(I + M^{-1}F)^{-1}$, $\hat{F} \in \hat{\mathcal{F}}$, which proves '$\supset$'. □

Finally, we present a key Lemma that shows the equivalence of $0 \leq \hat{F} \leq M$ and the structure presented in Proposition 2.

Lemma 6: *Let $F \in \mathbb{R}^{m \times m}$ be a non-negative definite diagonal matrix and $M \in \mathbb{R}^{m \times m}$ a positive-definite diagonal matrix. Then $\hat{F} M^{-1} \hat{F} \leq \hat{F}$ if and only if $0 \leq \hat{F} \leq M$.*

Proof: The proof is a direct consequence of Lemma 4.4 of Haddad and Bernstein (1991 a). □

Now, it follows from Proposition 2 and Lemma 6 that an equivalent representation for our uncertainty set $\mathcal{U}$ in (111) is

$$\mathcal{U} \triangleq \{\Delta A_a \in \mathbb{R}^{n_a \times n_a} : \Delta A_a = -B_a \hat{F} C_a,\ \hat{F} \in \hat{\mathcal{F}}\} \tag{124}$$

For the structure of $\mathcal{U}$ satisfying (111), the parameter-dependent bound $\Omega(\cdot, \cdot)$ satisfying (100) can now be given a concrete form. Since the elements ΔA_a in $\mathcal{U}$ are parametrized by the elements F in $\mathcal{F}$, for convenience in the following results we shall write $P_0(F)$ in place of $P_0(\Delta A_a)$.

Proposition 3: *Let N_0, H_0, N_j, H_j, $S_j \in \mathbb{R}^{m \times m}$ be non-negative definite diagonal matrices such that, as in Theorem 2, $R_0 > 0$, and*

$$N_j S_j - H_j \geq 0, \quad j = 1, \ldots, m_2 \tag{125}$$

Then the functions

$$\Omega_0(P) = \left[H_0 C_a + N_0 C_a A_a + \sum_{j=1}^{m_1} H_j (C_a - R_j) + \sum_{j=m_1+1}^{m_2} H_j (C_a + R_j) - B_a^{\mathrm{T}} P\right]^{\mathrm{T}} R_0^{-1} \left[H_0 C_a + N_0 C_a A_a + \sum_{j=1}^{m_1} H_j (C_a - R_j) + \sum_{j=m_1+1}^{m_2} H_j (C_a + R_j) - B_a^{\mathrm{T}} P\right] \tag{126}$$

$$P_0(F) = C_a^{\mathrm{T}} (I + M^{-1} F)^{-1} [F N_0 + F M^{-1} N_0 F] (I + M^{-1} F)^{-1} C_a + \sum_{j=1}^{m_2} R_j^{\mathrm{T}} F N_j R_j \tag{127}$$

or, equivalently,

$$P_0(\hat{F}) = C_a^{\mathrm{T}} \hat{F} N_0 C_a + \sum_{j=1}^{m_2} R_j^{\mathrm{T}} (I - \hat{F} M^{-1})^{-1} \hat{F} N_j R_j \tag{128}$$

satisfies (100) *with $\mathcal{U}$ given by* (111).

Proof: The proof is a direct consequence of Theorem 2, with $\tilde{f}(\tilde{y}) = F\tilde{y} = F(I + M^{-1}F)^{-1} C_a x_a$. For further details, on a similar proof, see Haddad and Bernstein (1991 a). □

Next, using Theorem 4 and Proposition 3, we have the following immediate result.

Theorem 5: *Let N_0, H_0, N_j, H_j, $S_j \in \mathbb{R}^{m \times m}$ be non-negative definite diagonal matrices such that $R_0 > 0$ and* (125) *is satisfied. Furthermore, suppose that there exists a non-negative definite matrix P satisfying*

$$0 = A_a^T P + PA_a + R + \left[H_0C_a + N_0C_aA_a + \sum_{j=1}^{m_1} H_j(C_a - R_j) + \sum_{j=m_1+1}^{m_2} H_j(C_a + R_j) - B_a^T P\right]^T R_0^{-1} \left[H_0C_a + N_0C_aA_a + \sum_{j=1}^{m_1} H_j(C_a - R_j) + \sum_{j=m_1+1}^{m_2} H_j(C_a + R_j) - B_a^T P\right] \tag{129}$$

Then

$$(A_a + \Delta A_a, E) \textit{ is detectable}, \quad \Delta A_a \in \mathcal{U} \tag{130}$$

if and only if

$$A_a + \Delta A_a \textit{ is asymptotically stable}, \quad \Delta A_a \in \mathcal{U} \tag{131}$$

In this case

$$J(\mathcal{U}) \leq \operatorname{tr} PV + \sup_{\hat{F}\in\hat{\mathcal{F}}} \operatorname{tr}\left[\left(C_a^T\hat{F}N_0C_a + \sum_{j=1}^{m_2} R_j^T(I - \hat{F}M^{-1})^{-1}\hat{F}N_jR_j\right)V\right] = \operatorname{tr} PV + \sup_{\hat{F}\in\hat{\mathcal{F}}} \operatorname{tr}[(C_a^T\hat{F}N_0C_a)V] \tag{132}$$

Proof: The result is a direct specialization of Theorem 4 using Proposition 3. We only note that $P_0(\Delta A_a)$ now has the form $P_0(\hat{F}) = C_a^T\hat{F}N_0C_a + \sum_{j=1}^{m_2} R_j^T(I - \hat{F}M^{-1})^{-1}\hat{F}N_jR_j$. Since $\hat{F}N_j \geq 0$, $j = 0, \ldots, m_2$, for all $\hat{F} \in \hat{\mathcal{F}}$ it follows that $P + P_0(\hat{F})$ is non-negative definite for all $\hat{F} \in \hat{\mathcal{F}}$ as required by Theorem 4. Finally, (132) follows by noting $R_jV = 0$, $j = 1, \ldots, m_2$. □

Theorem 5 is directly applicable to dynamic systems with m-mixed uncertainties. Specifically, it follows from Theorem 2 that if the nonlinearity m-vector $f(y)$ is composed of n_1 time invariant first and third quadrant functions, $n_2 - n_1$ monotone increasing functions, and $m - n_2$ odd monotone increasing functions, then the nominal system is robustly stable for all such mixed uncertainty. Furthermore, in the linear uncertainty case, $f(y) = \hat{F}y$ it was recently shown by Haddad and Bernstein (1991 a) that under certain compatibility assumptions between N_0 and $\hat{\mathcal{F}}$ (for the Popov case), the set $\mathcal{U}$ allows a richer class of multivariable uncertainties in that F may represent a fully populated uncertainty matrix. Similar extensions for the monotonic and odd monotonic case are possible; however, for simplicity of exposition, we defer these results to a future paper. This of course, allows for non-scalar multiple uncertainty blocks within the analysis and synthesis framework.

7. Robust controller synthesis via static and dynamic output feedback

In this section we introduce the robust stability and performance problem with static output feedback control. As mentioned in the previous section, owing to the extra dynamics introduced by the multiplier, our resulting state-space model is of increased dimension. Hence, this problem involves a set $\mathcal{U} \subset \mathbb{R}^{n_a \times n_a}$ of uncertain perturbations ΔA_a of the nominal augmented system matrix A_a.

Robust stability and performance problem: Given the n_ath-order stabilizable augmented plant with constant real-valued plant parameter variations

$$\dot{x}_a(t) = (A_a + \Delta A_a)x_a(t) + Bu(t) + Dw(t), \quad t \in [0, \infty) \tag{133}$$

$$y(t) = Cx_a(t) \tag{134}$$

where $u(t) \in \mathbb{R}^{m_0}$ and $w(t) \in \mathbb{R}^d$ and $y(t) \in \mathbb{R}^l$, determine an output feedback control law

$$u(t) = Ky(t) \tag{135}$$

that satisfies the following design criteria:

(1) the closed-loop system (133)–(135) is asymptotically stable for all $\Delta A_a \in \mathcal{U}$, that is, $A_a + BKC + \Delta A_a$ is asymptotically stable for all $\Delta A_a \in \mathcal{U}$; and

(2) the performance functional

$$J(K) \triangleq \sup_{\Delta A_a \in \mathcal{U}} \limsup_{t \to \infty} \frac{1}{t} \mathbb{E}\left\{\int_0^t [x_a^{\mathrm{T}}(s) R_{xx} x_a(s) + u^{\mathrm{T}}(s) R_{uu} u(s)]\, \mathrm{d}s\right\} \tag{136}$$

is minimized. □

Since we are only interested in controlling the actual system dynamics, i.e. the non-augmented dynamics, in accordance with the partitioning in (24), our control, measurement, disturbance and state weighting matrices B, C, D and R_{xx}, have the following structure

$$B = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}, \quad C = [\hat{C} \ \ 0], \quad D = \begin{bmatrix} \hat{D} \\ 0 \end{bmatrix}, \quad R_{xx} = \begin{bmatrix} \hat{R}_{xx} & 0 \\ 0 & 0 \end{bmatrix} \tag{137}$$

For each variation $\Delta A_a \in \mathcal{U}$, the closed-loop system (133)–(135) can be written as

$$\dot{x}_a(t) = (\hat{A} + \Delta A_a)x_a(t) + Dw(t), \quad t \in [0, \infty) \tag{138}$$

where

$$\hat{A} \triangleq A_a + BKC \tag{139}$$

and where the white noise disturbance has intensity $V = DD^{\mathrm{T}}$. Finally, note that if $\hat{A}_a + \Delta A_a$ is asymptotically stable for all $\Delta A_a \in \mathcal{U}$ for a given K, then (136) can be written as

$$J(K) = \sup_{\Delta A_a \in \mathcal{U}} \operatorname{tr} P_{\Delta A_a} V \tag{140}$$

where $P_{\Delta} A_a$ satisfies (100) with A_a replaced by $\hat{A}$ and R replaced by

$$\hat{R} \triangleq R_{xx} + C^{\mathrm{T}} K^{\mathrm{T}} R_{uu} KC \tag{141}$$

To apply Theorem 5 to controller synthesis we consider the performance bound (105) in place of the actual worst-case $\mathcal{H}_2$ performance as in Theorem 5 with A_a, R replaced by $\hat{A}$ and $\hat{R}$ to address the closed-loop control problem. This leads to the following optimization problem.

Optimization problem: Determine $K \in \mathbb{R}^{m_0 \times l}$ that minimizes

$$\mathcal{J}(K) \leq \operatorname{tr} PV + \sup_{\hat{F} \in \hat{\mathcal{F}}} \operatorname{tr}[(C_a^{\mathrm{T}} \hat{F} N_0 C_a)V] \tag{142}$$

subject to

$$0 = \hat{A}^{\mathrm{T}}P + P\hat{A} + \hat{R} + \left[H_0C_a + N_0C_a\hat{A} + \sum_{j=1}^{m_1} H_j(C_a - R_j) + \sum_{j=m_1+1}^{m_2} H_j(C_a + R_j) - B_a^{\mathrm{T}}P\right]^{\mathrm{T}} R_0^{-1} \left[H_0C_a + N_0C_a\hat{A} + \sum_{j=1}^{m_1} H_j(C_a - R_j) + \sum_{j=m_1+1}^{m_2} H_j(C_a + R_j) - B_a^{\mathrm{T}}P\right] \tag{143}$$

□

The relationship between the optimization problem and the robust stability and performance problem is straightforward, as shown by the following observation.

Proposition 4: *If $P \in \mathbb{N}^{n_a}$ and $K \in \mathbb{R}^{m_0 \times l}$ satisfy* (143) *and the detectability condition* (102) *holds, then $\hat{A} + \Delta A_a$ is asymptotically stable for all $\Delta A_a \in \mathcal{U}$, and*

$$J(K) \leq \mathcal{J}(K) \tag{144}$$

Proof: Since (143) has a solution $P \in \mathbb{N}^{n_a}$ and the detectability condition (102) holds, the hypotheses of Theorem 4 are satisfied so that robust stability with robust performance bound is guaranteed. The condition in (144) is merely a restatement of (105). □

Note that, since the last term in (142) is not a function of either the controller gain K or the constraint (144), it plays no role in the optimization process. Next, we present sufficient conditions for robust stability and performance for the static output feedback case. For arbitrary $P, Q \in \mathbb{R}^{n_a \times n_a}$ define the notation

$$\tilde{C} \triangleq H_0C_a + N_0C_aA_a + \sum_{j=1}^{m_1} H_j(C_a - R_j) + \sum_{j=m_1+1}^{m_2} H_j(C_a + R_j) \tag{145}$$

$$R_{2a} \triangleq R_{uu} + B^{\mathrm{T}}C_a^{\mathrm{T}}N_0R_0^{-1}N_0C_aB \tag{146}$$

$$P_a \triangleq B^{\mathrm{T}}P + B^{\mathrm{T}}C_a^{\mathrm{T}}N_0R_0^{-1}(\tilde{C} - B_a^{\mathrm{T}}P) \tag{147}$$

$$A_P \triangleq A_a - B_aR_0^{-1}\tilde{C} \tag{148}$$

$$\nu \triangleq QC^{\mathrm{T}}(CQC^{\mathrm{T}})^{-1}C, \quad \nu_\perp \triangleq I_n - \nu \tag{149}$$

when the indicated inverses exist.

Theorem 6: *Assume $R_0 > 0$ and assume* (125) *holds. Furthermore, suppose there exist $n_a \times n_a$ non-negative definite matrices P, Q such that $CQC^{\mathrm{T}} > 0$ and*

$$0 = A_P^{\mathrm{T}}P + PA_P + R_{xx} + \tilde{C}^{\mathrm{T}}R_0^{-1}\tilde{C} + PB_aR_0^{-1}B_a^{\mathrm{T}}P - P_a^{\mathrm{T}}R_{2a}^{-1}P_a + \nu_\perp^{\mathrm{T}}P_a^{\mathrm{T}}R_{2a}^{-1}P_a\nu_\perp \tag{150}$$

$$0 = [A_P + (B_aR_0^{-1}N_0C_a - I)BR_{2a}^{-1}P_a\nu + B_aR_0^{-1}B_a^{\mathrm{T}}P]Q + Q[A_P + (B_aR_0^{-1}N_0C_a - I)BR_{2a}^{-1}P_a\nu + B_aR_0^{-1}B_a^{\mathrm{T}}P]^{\mathrm{T}} + V \tag{151}$$

and let K be given by

$$K = -R_{2a}^{-1} P_a Q C^T (C Q C^T)^{-1} \tag{152}$$

Then $(\hat{A} + \Delta A_a, \hat{R})$ *is detectable for all* $\Delta A_a \in \mathcal{U}$ *if and only if* $\hat{A} + \Delta A_a$ *is asymptotically stable for all* $\Delta A_a \in \mathcal{U}$. *In this case the closed-loop system performance in* (140) *satisfies the parameter-dependent* $\mathcal{H}_2$ *bound*

$$J(K) \leq \operatorname{tr} PV + \sup_{\hat{F} \in \mathcal{F}} \operatorname{tr}[(C_a^T \hat{F} N_0 C_a) V] \tag{153}$$

Proof: The proof follows the one in Haddad and Bernstein (1991 a). □

Remark 8: The definiteness condition $CQC^T > 0$ holds if C has full row rank and Q is positive-definite. Conversely, if $CQC^T > 0$, then C must have full row rank but Q need not necessarily be positive-definite. This condition implies the existence of the static gain projection ν. □

7.1. *Dynamic output feedback controller synthesis*

In this section, we introduce the robust stability and performance dynamic output-feedback control problem. Since the multiplier dynamics increase the plant order from n to n_a, to allow for greater design flexibility, the compensator dimension n_c is fixed to be less than the augmented plant order n_a. Hence, define $\tilde{n} = n_a + n_c$. Note that in this context, an nth-order controller can be regarded as a reduced-order design. As in Hyland and Bernstein (1985), this constraint leads to an oblique projection that introduces additional coupling in the design equations along with additional design equations. This coupling shows that regulator/estimator separation breaks down in the reduced-order controller case.

Dynamic robust stability and performance problem: Given the n_ath-order stabilizable and detectable plant with constant structured real-valued plant parameter variations

$$\dot{x}_a(t) = (A_a + \Delta A_a) x_a(t) + Bu(t) + D_1 w(t), \quad t \geq 0 \tag{154}$$

$$y(t) = Cx_a(t) + D_2 w(t) \tag{155}$$

where $u(t) \in \mathbb{R}^{m_0}$, $w(t) \in \mathbb{R}^d$ and $y(t) \in \mathbb{R}^l$, determine an n_cth-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{156}$$

$$u(t) = C_c x_c(t) \tag{157}$$

that satisfies the following design criteria:

(1) the closed-loop system in (154)–(157) is asymptotically stable for all $\Delta A_a \in \mathcal{U}$; and

(2) the performance functional in (140) with $J(K)$ replaced by $J(A_c, B_c, C_c)$ is minimized.

For each uncertain variation $\Delta A_{\mathrm{a}} \in \mathcal{U}$, the closed-loop system in (154)–(157) can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{D}w(t), \quad t \geq 0 \tag{158}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x_{\mathrm{a}}(t) \\ x_{\mathrm{c}}(t) \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A_{\mathrm{a}} & BC_{\mathrm{c}} \\ B_{\mathrm{c}}C & A_{\mathrm{c}} \end{bmatrix}, \quad \Delta\tilde{A} \triangleq \begin{bmatrix} \Delta A_{\mathrm{a}} & 0_{n_{\mathrm{a}}\times n_{\mathrm{c}}} \\ 0_{n_{\mathrm{c}}\times n_{\mathrm{a}}} & 0_{n_{\mathrm{c}}\times n_{\mathrm{c}}} \end{bmatrix} \tag{159}$$

and where the closed-loop disturbance $\tilde{D}w(t)$ has intensity $\tilde{V} = \tilde{D}\tilde{D}^{\mathrm{T}}$, where

$$\tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_{\mathrm{c}}D_2 \end{bmatrix}$$

$$\tilde{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & B_{\mathrm{c}}V_2B_{\mathrm{c}}^{\mathrm{T}} \end{bmatrix}$$

$V_1 = D_1D_1^{\mathrm{T}}$ and $V_2 = D_2D_2^{\mathrm{T}}$. The closed-loop system uncertainty $\Delta\tilde{A}$ has the form

$$\Delta\tilde{A} = \tilde{B}_{\mathrm{a}}\hat{F}\tilde{C}_{\mathrm{a}} \tag{160}$$

where

$$\tilde{B}_{\mathrm{a}} \triangleq \begin{bmatrix} B_{\mathrm{a}} \\ 0_{n_{\mathrm{c}}\times m} \end{bmatrix}, \quad \tilde{C}_{\mathrm{a}} \triangleq [C_{\mathrm{a}} \;\; 0_{m\times n_{\mathrm{c}}}] \tag{161}$$

Finally, if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A_{\mathrm{a}} \in \mathcal{U}$ for a given compensator $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}})$, then it follows from Proposition 1 that the performance measure in (135) is given by

$$J(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}}) = \sup_{\Delta A_{\mathrm{a}}\in\mathcal{U}} \operatorname{tr} \tilde{P}_{\Delta\tilde{A}}\tilde{V} \tag{162}$$

where $P_{\Delta\tilde{A}}$ satisfies the $\tilde{n} \times \tilde{n}$ Lyapunov equation

$$0 = (\tilde{A} + \Delta\tilde{A})^{\mathrm{T}}\tilde{P}_{\Delta\tilde{A}} + \tilde{P}_{\Delta\tilde{A}}(\tilde{A} + \Delta\tilde{A}) + \tilde{R} \tag{163}$$

where

$$\tilde{E} = [E_1 \;\; E_2C_{\mathrm{c}}], \quad \tilde{R} = \tilde{E}^{\mathrm{T}}\tilde{E} = \begin{bmatrix} R_{xx} & 0 \\ 0 & C_{\mathrm{c}}^{\mathrm{T}}R_{uu}C_{\mathrm{c}} \end{bmatrix} \tag{164}$$

□

Next, we proceed as in the previous section where we replace the Lyapunov equation (163) for the dynamic problem, with a Riccati equation that guarantees that the closed-loop system is robustly stable. Thus, for the dynamic output feedback problem, Theorem 5 holds with A_{a}, R, V replaced by $\tilde{A}$, $\tilde{R}$, $\tilde{V}$. For clarity we state the dynamic optimization problem.

Dynamic optimization problem: Determine $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}})$ that minimizes

$$\mathcal{J}(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}}) \triangleq \operatorname{tr} \tilde{P}\tilde{V} + \sup_{\hat{F}\in\hat{\mathcal{F}}} \operatorname{tr}[(\tilde{C}_{\mathrm{a}}^{\mathrm{T}}\hat{F}N_0\tilde{C}_{\mathrm{a}})\tilde{V}] \tag{165}$$

where $\tilde{P} \in \mathbb{N}^{\tilde{n}}$ satisfies

$$0 = \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \tilde{R}$$
$$+ \left[H_0\tilde{C}_{\mathrm{a}} + N_0\tilde{C}_{\mathrm{a}}\tilde{A} + \sum_{j=1}^{m_1} H_j(\tilde{C}_{\mathrm{a}} - \tilde{R}_j) + \sum_{j=m_1+1}^{m_2} H_j(\tilde{C}_{\mathrm{a}} + \tilde{R}_j) - \tilde{B}_{\mathrm{a}}^{\mathrm{T}}\tilde{P}\right]^{\mathrm{T}} \tilde{R}_0^{-1}$$
$$\left[H_0\tilde{C}_{\mathrm{a}} + N_0\tilde{C}_{\mathrm{a}}\tilde{A} + \sum_{j=1}^{m_1} H_j(\tilde{C}_{\mathrm{a}} - \tilde{R}_j) + \sum_{j=m_1+1}^{m_2} H_j(\tilde{C}_{\mathrm{a}} + \tilde{R}_j) - \tilde{B}_{\mathrm{a}}^{\mathrm{T}}\tilde{P}\right] \tag{166}$$

where $\tilde{R}_j \triangleq [R_j \; 0_{m\times n_c}]$ and

$$\tilde{R}_0 \triangleq \left[\left(N_0\tilde{C}_{\mathrm{a}}\tilde{B}_{\mathrm{a}} + \sum_{j=0}^{m_2} H_j M^{-1}\right) + \left(N_0\tilde{C}_{\mathrm{a}}\tilde{B}_{\mathrm{a}} + \sum_{j=0}^{m_2} H_j M^{-1}\right)^{\mathrm{T}}\right] \tag{167}$$

and such that (A_c, B_c, C_c) is controllable and observable, and (125) holds. □

By deriving necessary conditions for the dynamic optimization problem as in § 6.1 we obtain sufficient conditions for characterizing fixed-order dynamic output feedback controllers that guarantee robust stability and performance. The following lemma is required for the statement of the main theorem.

Lemma 7: *Let $\hat{Q}$, $\hat{P}$ be $n_a \times n_a$ non-negative definite matrices and suppose that rank $\hat{Q}\hat{P} = n_c$. Then there exist $n_c \times n_a$ G, Γ and $n_c \times n_c$ invertible M, unique except for a change of basis in $\mathbb{R}^{n_c}$, such that*

$$\hat{Q}\hat{P} = G^{\mathrm{T}} M \Gamma, \quad \Gamma G^{\mathrm{T}} = I_{n_c} \tag{168}$$

Furthermore, the $n_a \times n_a$ matrices

$$\tau \triangleq G^{\mathrm{T}}\Gamma, \quad \tau_\perp \triangleq I_{n_a} - \tau \tag{169}$$

are idempotent and have rank n_c and $n_a - n_c$ respectively.

Proof: For the proof see Bernstein and Haddad (1990). □

We now state the main results of this section concerning reduced-order controllers. For convenience, recall the definitions of R_0, P_a, A_P, R_{2a}, $\tilde{C}$, and define

$$\bar{\Sigma} \triangleq C^{\mathrm{T}} V_2^{-1} C \tag{170}$$
$$A_{\hat{P}} \triangleq A_a - Q\bar{\Sigma} - B_a R_0^{-1}(\tilde{C} - B_a^{\mathrm{T}} P) \tag{171}$$
$$A_{\hat{Q}} \triangleq A_a - BR_{2a}^{-1}P_a + B_a R_0^{-1} N_0 C_a B R_{2a}^{-1} P_a - B_a R_0^{-1}(\tilde{C} - B_a^{\mathrm{T}} P) \tag{172}$$

for arbitrary Q, $P \in \mathbb{R}^{n_a \times n_a}$.

Theorem 7: *Let $n_c \leq n_a$, and assume $R_0 > 0$ and (125) holds. Furthermore, suppose there exist $n_a \times n_a$ non-negative definite matrices P, Q, $\hat{P}$, $\hat{Q}$ satisfying*

$$0 = A_P^{\mathrm{T}} P + P A_P + R_{xx} + \tilde{C}^{\mathrm{T}} R_0^{-1} \tilde{C} + P B_a R_0^{-1} B_a^{\mathrm{T}} P$$
$$- P_a^{\mathrm{T}} R_{2a}^{-1} P_a + \tau_\perp^{\mathrm{T}} P_a^{\mathrm{T}} R_{2a}^{-1} P_a \tau_\perp \tag{173}$$
$$0 = (A_P + B_a R_0^{-1} B_a^{\mathrm{T}}[P + \hat{P}])Q + Q(A_P + B_a R_0^{-1} B_a^{\mathrm{T}}[P + \hat{P}])^{\mathrm{T}}$$
$$+ V_1 - Q\bar{\Sigma}Q + \tau_\perp^{\mathrm{T}} Q\bar{\Sigma}Q\tau_\perp \tag{174}$$
$$0 = A_{\hat{P}}^{\mathrm{T}}\hat{P} + \hat{P}A_{\hat{P}} + \hat{P}B_a R_0^{-1} B_a^{\mathrm{T}}\hat{P} + P_a^{\mathrm{T}} R_{2a}^{-1} P_a - \tau_\perp^{\mathrm{T}} P_a^{\mathrm{T}} R_{2a}^{-1} P_a \tau_\perp \tag{175}$$

$$0 = A_{\hat{Q}}\hat{Q} + \hat{Q}A_{\hat{Q}}^{T} + Q\bar{\Sigma}Q - \tau_{\perp}Q\bar{\Sigma}Q\tau_{\perp}^{T} \tag{176}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c \tag{177}$$

and let A_c, B_c, C_c be given by

$$A_c = \Gamma[A_{\hat{Q}} - Q\bar{\Sigma}]G^{T} \tag{178}$$

$$B_c = \Gamma QC^{T}V_2^{-1} \tag{179}$$

$$C_c = -R_{2a}^{-1}P_aG^{T} \tag{180}$$

Then $(\tilde{A} + \Delta\tilde{A}, \tilde{E})$ is detectable for all $\Delta A_a \in \mathcal{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A_a \in \mathcal{U}$. In this case the performance of the closed-loop system (158) *satisfies the parameter-dependent $\mathcal{H}_2$ bound*

$$J(A_c, B_c, C_c) \leq \operatorname{tr}[(P + \hat{P})V_1 + \hat{P}Q\bar{\Sigma}Q] \tag{181}$$

$$+ \sup_{\hat{F} \in \hat{\mathcal{F}}} \operatorname{tr}[(C_a^{T}\hat{F}N_0C_a)V_1] \tag{182}$$

Proof: The proof follows as in the proof of Theorem 5.1 of Haddad and Bernstein (1991 a) with additional terms arising due to the reduced-order dynamic compensation structure and odd monotonic constraints. For details of a similar proof, see Bernstein and Haddad (1989). □

Remark 9: Several special cases can immediately be discerned from Theorem 7. For example, in the full-order case, set $n_c = n_a$ so that $\tau = G = \Gamma = I_{n_a}$ and $\tau_{\perp} = 0$. In this case the last term in each of (173)–(176) is zero and (176) is superfluous. Alternatively, letting $B_a = 0$, $C_a = 0$ and retaining the reduced-order constraint $n_c < n_a$, yields the result of Hyland and Bernstein (1985). Finally, setting $m_2 = 0$ yields the results of Haddad and Bernstein (1991 a) for the case in which F is diagonal. □

Theorem 7 provides constructive sufficient conditions that yield reduced-order dynamic feedback gains A_c, B_c, C_c for robust stability and performance. Note that when solving (173)–(176) numerically, the matrices M, N_j, H_j, and S_j, $j = 0, \ldots, m_2$, and the structure of B_a and C_a appearing in the design equations can be adjusted to examine trade-offs between performance and robustness. As discussed by Haddad and Bernstein (1991 a), How *et al.* (1992 b), to reduce conservatism further, one can view the multiplier matrices N_j, H_j and S_j as free parameters and optimize the worst case $\mathcal{H}_2$ performance bound with respect to them. The basic approach is to employ a numerical algorithm to design the optimal compensator and the multipliers simultaneously, thus avoiding the need to iterate between controller design and optimal multiplier evaluation as suggested by Haddad and Bernstein (1991 a) and numerically demonstrated by How *et al.* (1992 b). This approach is demonstrated in the following section with the Popov multiplier.

8. Illustrative numerical example

In this section, we consider a special case of the synthesis procedure in § 7. In particular, we present an algorithm for the design of full-order dynamic compensators for systems with m independent scalar uncertainties and restricted multipliers of the standard Popov form

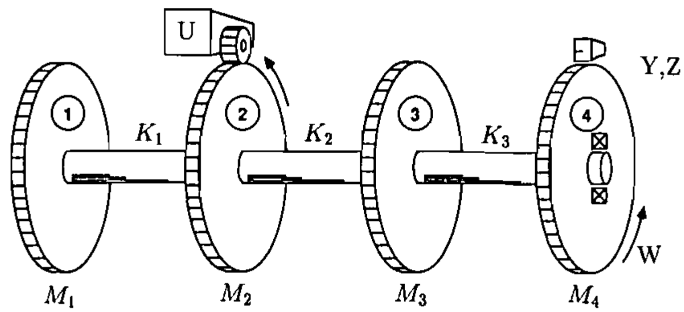

Figure 6. Four-disc oscillator.

$$W_i(s) = \alpha_{i0} + \beta_{i0}s \tag{183}$$

The algorithm is then applied to the Benchmark control problem illustrated in Fig. 6. For this particular form of the multiplier, we have that $m_1 = m_2 = 0$ in (11) and the augmentation process outlined in § 4.1 is not necessary. Recall the definitions of the matrices $H_0 = \mathrm{diag}(\alpha_{10}, \ldots, \alpha_{m0})$ and $N_0 = \mathrm{diag}(\beta_{10}, \ldots, \beta_{m0})$. The general sector uncertainty set discussed in Remark 2 is used in these designs, so the set $\mathcal{U}$ is redefined to be

$$\mathcal{U} \triangleq \{\Delta A \in \mathbb{R}^{n\times n}: \Delta A = -B_0FC_0, \text{ for } F \in \mathcal{F}\} \tag{184}$$

where $\mathcal{F}$ is given by

$$\mathcal{F} \triangleq \{F \in \mathbb{D}^m: M_1 \leq F \leq M_2\} \tag{185}$$

$B_0 \in \mathbb{R}^{n\times m}$ and $C_0 \in \mathbb{R}^{m\times n}$ are fixed matrices denoting the structure of the uncertainty, and $F \in \mathbb{D}^m$ is an uncertain matrix. The augmentation in (159), which accounts for the compensator states in the closed-loop system, is then applied to these B_0 and C_0 matrices. For clarity we restate the dynamic minimization problem.

Dynamic minimization problem: Determine the compensator (A_c, B_c, C_c) that minimizes the overbounding cost

$$\mathcal{J}(A_c, B_c, C_c) \triangleq \mathrm{tr}[\tilde{P} + \tilde{C}_0^{\mathrm{T}}(M_2 - M_1)N_0\tilde{C}_0]\tilde{V} \tag{186}$$

where $\tilde{P} \in \mathbb{N}^{2n}$ satisfies

$$\begin{aligned} 0 = {} & (\tilde{A} - \tilde{B}_0M_1\tilde{C}_0)^{\mathrm{T}}\tilde{P} + \tilde{P}(\tilde{A} - \tilde{B}_0M_1\tilde{C}_0) \\ & + [H_0\tilde{C}_0 + N_0\tilde{C}_0(\tilde{A} - \tilde{B}_0M_1\tilde{C}_0) - \tilde{B}_0^{\mathrm{T}}\tilde{P}]^{\mathrm{T}}R_0^{-1} \\ & \quad [H_0\tilde{C}_0 + N_0\tilde{C}_0(\tilde{A} - \tilde{B}_0M_1\tilde{C}_0) - \tilde{B}_0^{\mathrm{T}}\tilde{P}] + \tilde{R} \end{aligned} \tag{187}$$

and

$$R_0 \triangleq [H_0(M_2 - M_1)^{-1} + N_0C_0B_0] + [H_0(M_2 - M_1)^{-1} + N_0C_0B_0]^{\mathrm{T}} \tag{188}$$

such that (A_c, B_c, C_c) is controllable and observable, H_0, N_0 are diagonal, positive definite and non-negative definite matrices respectively, and $R_0 > 0$. □

By deriving necessary conditions for the dynamic minimization problem we can obtain sufficient conditions for characterizing dynamic output feedback controllers guaranteeing robust stability and performance. To proceed, we use the standard approach of augmenting the cost overbound $\mathcal{J}(A_c, B_c, C_c)$ with the constraint in (187) using the Lagrange multiplier $\tilde{Q} \in \mathbb{N}^{2n}$, to form

$$\mathcal{L}(A_c, B_c, C_c, \tilde{P}, \tilde{Q}) = \mathrm{tr}\,[(\tilde{P} + \tilde{C}_0^{\mathrm{T}}(M_2 - M_1)N_0\tilde{C}_0)\tilde{V} + \tilde{Q}\{(\tilde{A} - \tilde{B}_0M_1\tilde{C}_0)^{\mathrm{T}}\tilde{P} + \tilde{P}(\tilde{A} - \tilde{B}_0M_1\tilde{C}_0) + [H_0\tilde{C}_0 + N_0\tilde{C}_0(\tilde{A} - \tilde{B}_0M_1\tilde{C}_0) - \tilde{B}_0^{\mathrm{T}}\tilde{P}]^{\mathrm{T}}R_0^{-1} [H_0\tilde{C}_0 + N_0\tilde{C}_0(\tilde{A} - \tilde{B}_0M_1\tilde{C}_0) - \tilde{B}_0^{\mathrm{T}}\tilde{P}] + \tilde{R}\}] \tag{189}$$

Since the results are necessary to implement the numerical solution of the optimal controllers, we present the following gradients of the augmented cost in (189) with respect to the free parameters A_c, B_c, C_c, $\tilde{P}$ and $\tilde{Q}$. Clearly, $\partial\mathcal{L}/\partial\tilde{Q}$ recovers the constraint in (187). For convenience, we partition the symmetric matrices $\tilde{P}$ and $\tilde{Q}$ as

$$\tilde{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{bmatrix} \tag{190}$$

and similarly for their product $\tilde{P}\tilde{Q}$, and then present the following gradient expressions

$$0 = \frac{\partial\mathcal{L}}{\partial\tilde{P}} = [\tilde{A} - \tilde{B}_0M_1\tilde{C}_0 - \tilde{B}_0R_0^{-1}(H_0\tilde{C}_0 + N_0\tilde{C}_0(\tilde{A} - \hat{B}_0M_1\tilde{C}_0) - \tilde{B}_0^{\mathrm{T}}\tilde{P})]\tilde{Q} + \tilde{Q}[\tilde{A} - \tilde{B}_0M_1\tilde{C}_0 - \tilde{B}_0R_0^{-1}(H_0\tilde{C}_0 + N_0\tilde{C}_0(\tilde{A} - \tilde{B}_0M_1\tilde{C}_0) - \tilde{B}_0^{\mathrm{T}}\tilde{P})]^{\mathrm{T}} + \tilde{V} \tag{191}$$

$$0 = \tfrac{1}{2}\frac{\partial\mathcal{L}}{\partial A_c} = P_{12}^{\mathrm{T}}Q_{12} + P_{22}Q_{22} = [\tilde{P}\tilde{Q}]_{22} \tag{192}$$

$$0 = \tfrac{1}{2}\frac{\partial\mathcal{L}}{\partial B_c} = P_{22}B_cV_2 + [\tilde{P}\tilde{Q}]_{21}C^{\mathrm{T}} \tag{193}$$

$$0 = \tfrac{1}{2}\frac{\partial\mathcal{L}}{\partial C_c} = B^{\mathrm{T}}(I - B_0R_0^{-1}N_0C_0)^{\mathrm{T}}[\tilde{P}\tilde{Q}]_{12} + B^{\mathrm{T}}C_0^{\mathrm{T}}N_0R_0^{-1}[H_0C_0 + N_0C_0(A - B_0M_1C_0)]Q_{12} + (R_{uu} + B^{\mathrm{T}}C_0^{\mathrm{T}}N_0R_0^{-1}N_0C_0B)C_cQ_{22} \tag{194}$$

As in the previous section, these gradients can be used to derive explicit expressions for the optimal controller in terms of the solutions of three coupled Riccati equations. For convenience in stating the next result, recall the definitions of R_0 in (188) and $\bar{\Sigma} \triangleq C^{\mathrm{T}}V_2^{-1}C$, and define

$$\tilde{C} \triangleq H_0C_0 + N_0C_0(A - B_0M_1C_0) \tag{195}$$

$$R_{2a} \triangleq R_{uu} + B^{\mathrm{T}}C_0^{\mathrm{T}}N_0R_0^{-1}N_0C_0B \tag{196}$$

$$P_a \triangleq B^{\mathrm{T}}P + B^{\mathrm{T}}C_0^{\mathrm{T}}N_0R_0^{-1}(\tilde{C} - B_0^{\mathrm{T}}P) \tag{197}$$

$$A_P \triangleq A - B_0M_1C_0 - B_0R_0^{-1}\tilde{C} \tag{198}$$

$$A_{\hat{P}} \triangleq A_P - Q\bar{\Sigma} + B_0R_0^{-1}B_0^{\mathrm{T}}P \tag{199}$$

$$A_{\hat{Q}} \triangleq A_P + B_0R_0^{-1}P - (I - B_0R_0^{-1}N_0C_0)BR_{2a}^{-1}P_a \tag{200}$$

for arbitrary Q, $P \in \mathbb{R}^{n\times n}$.

Corollary 3: *Let $n_c = n$, $m_1 = m_2 = 0$, $R_0 > 0$, and N_0 and H_0 be diagonal, non-negative and positive definite matrices, respectively. Furthermore, suppose there exist $n \times n$ non-negative definite matrices P, Q, and $\hat{P}$ satisfying*

$$0 = A_P^T P + PA_P + R_{xx} + \tilde{C}^T R_0^{-1} \tilde{C} + PB_0 R_0^{-1} B_0^T P - P_a^T R_{2a}^{-1} P_a \tag{201}$$

$$0 = (A_P + B_0 R_0^{-1} B_0^T [P + \hat{P}])Q + Q(A_P + B_0 R_0^{-1} B_0^T [P + \hat{P}])^T + V_1 - Q\bar{\Sigma} Q \tag{202}$$

$$0 = A_{\hat{P}}^T \hat{P} + \hat{P} A_{\hat{P}} + \hat{P} B_0 R_0^{-1} B_0^T \hat{P} + P_a^T R_{2a}^{-1} P_a \tag{203}$$

and let A_c, B_c and C_c be given by

$$A_c = A_{\hat{Q}} - Q\bar{\Sigma} \tag{204}$$

$$B_c = QC^T V_2^{-1} \tag{205}$$

$$C_c = -R_{2a}^{-1} P_a \tag{206}$$

Then $(\tilde{A} + \Delta\tilde{A}, \tilde{E})$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case the performance of the closed-loop system (158) *satisfies the $\mathcal{H}_2$ bound*

$$J(A_c, B_c, C_c) \leq \operatorname{tr}[(P + \hat{P} + C_0^T(M_2 - M_1)N_0 C_0)V_1 + \hat{P} Q \bar{\Sigma} Q] \tag{207}$$

Proof: The proof is a direct consequence of Theorem 7 with $n_c = n$, $m_1 = m_2 = 0$, and Remark 2 to capture the general sector bounds. □

As discussed earlier, the diagonal matrices M_1, M_2, H_0 and N_0, and the structure in B_0 and C_0 can be used to examine trade-offs between performance and robustness. Also, to reduce conservatism further, one can view the multiplier matrices H_0 and N_0 as free parameters and optimize the worst case $\mathcal{H}_2$ performance bound, see Haddad and Bernstein (1991 a), to obtain

$$\tfrac{1}{2}\frac{\partial \mathcal{L}}{\partial N_0} = \tfrac{1}{2}(M_2 - M_1)\tilde{C}_0 \tilde{V} \tilde{C}_0^T + R_0^{-1}[H_0\tilde{C}_0 + N_0\tilde{C}_0(\tilde{A} - \tilde{B}_0 M_1 \tilde{C}_0) - \tilde{B}_0^T \tilde{P}]\tilde{Q}$$
$$[(\tilde{A} - \tilde{B}_0 M_1 \tilde{C}_0) - \tilde{B}_0 R_0^{-1}(H_0 \tilde{C} + N_0 \tilde{C}_0(\tilde{A} - \tilde{B}_0 M_1 \tilde{C}_0) - \tilde{B}_0^T \tilde{P})]^T \tilde{C}_0^T \tag{208}$$

$$\tfrac{1}{2}\frac{\partial \mathcal{L}}{\partial H_0} = R_0^{-1}[H_0\tilde{C}_0 + N_0\tilde{C}_0(\tilde{A} - \tilde{B}_0 M_1 \tilde{C}_0) - \tilde{B}_0^T \tilde{P}]\tilde{Q}$$
$$[\tilde{C}_0^T - (H_0\tilde{C}_0 + N_0\tilde{C}_0(\tilde{A} - \tilde{B}_0 M_1 \tilde{C}_0) - \tilde{B}_0^T \tilde{P})^T R_0^{-1}(M_2 - M_1)^{-1}] \tag{209}$$

Note that since N_0, $H_0 \in \mathbb{D}^m$, it is only the diagonal elements of $\partial\mathcal{L}/\partial N_0$ and $\partial\mathcal{L}/\partial H_0$ which can be directly influenced through the optimization process, and thus set to zero.

The synthesis approach is from Haddad and Bernstein (1991 a) and How *et al.* (1992 b), and employs a numerical BFGS search algorithm to solve the optimality conditions in (187), (191)–(194), (208) and (209). As discussed in § 7, the optimal compensator and multiplier can be found simultaneously, thus avoiding the need to iterate between controller design and optimal multiplier evaluation. A homotopy algorithm on the robustness bounds M_1 and M_2 is used to achieve the desired values M_{1f} and M_{2f}. With initial values of H_0 and N_0, initial conditions for the optimization are developed from iterative solutions of (201)–(203). Then, for given values of M_1 and M_2, (187), (191)–(194), (208) and (209) are solved to determine the optimal values of A_c, B_c, C_c, H_0 and N_0. If the optimization algorithm converges, M_1 and M_2 are increased and the current design is used as an initial guess for the next iteration. If the optimization fails to converge, the increments in M_1 and M_2 are reduced. The process is continued until M_{1f} and M_{2f} are achieved. The final result is a family of robust control designs which enable an examination of the trade-offs between performance and robustness. Collins, Jr. *et al.* (1994) discuss more sophisticated versions of this homotopy algorithm for the optimal analysis problem, and this can be applied to the synthesis problem.

The following section applies this synthesis algorithm to an illustrative example taken from Cannon and Rosenthal (1984).

8.1. *Benchmark control example*

In this section, we apply the numerical algorithm presented in the previous section to an illustrative numerical example. Specifically, we consider the coupled four-disc system shown in Fig. 6 from Cannon and Rosenthal (1984). In this configuration, the angular sensor and torque actuator are non-collocated. The uncertainty in the system enters the model through errors in the stiffnesses of the springs connecting discs 1–2 and 3–4. The influence of these model errors is illustrated in Fig. 7. It is evident that small modelling errors in the spring stiffnesses, or equivalently, in the modal frequencies, result in large magnitude and phase changes in the plant transfer function. As discussed by Cannon and Rosenthal (1984), the sensitivity of the system phase to these parameter variations means that the four-disc problem represents a very difficult challenge

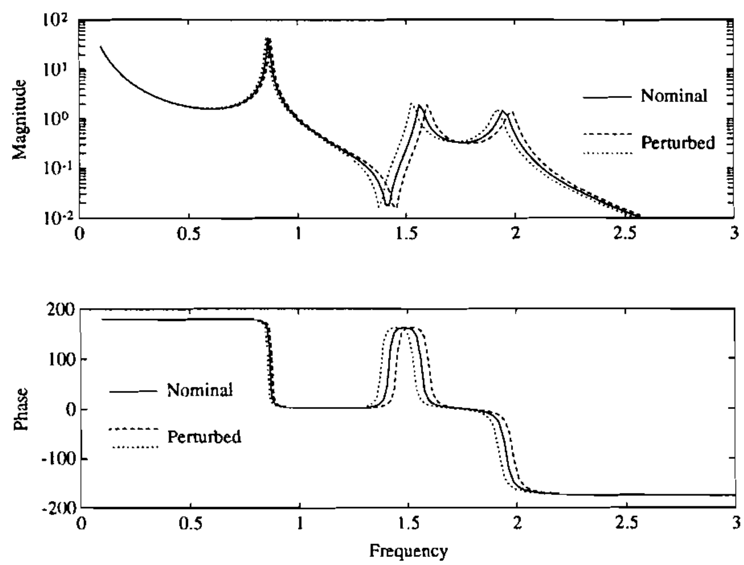

Figure 7. Influence of stiffness uncertainty on transfer function magnitude and phase. The two uncertainties are assumed equal in the analysis. For a lightly damped system, 5% uncertainty in both stiffness values can result in plant phase variations of ± 100°.

to robust control design. This benchmark problem is important because of the similarlity of the uncertainty to the characteristics which are typical of uncertain, lightly damped flexible structures.

For this example, we only consider full-order controllers $n_c = n$, and no consideration is given to the potential spillover to higher frequency unmodelled dynamics. A reduced order design is presented by How *et al.* (1992 a). High frequency, unmodelled dynamics can be addressed by using frequency weighted cost functionals in the specification of the $\mathcal{H}_2$ problem. In the following, we present a state-space model of the system, performance robustness buckets for several values of the stability bounds M_1 and M_2, and an interpretation of how this robustness is achieved. Unlike many control approaches such as $\mathcal{H}_2$ (Doyle 1978) or MEOP designs (Bernstein and Hyland 1988), M_1 and M_2 represent the guaranteed robust stability bounds, which are lower bounds to the actual stability limits achieved.

A state-space model for the four-disc system illustrated in Fig. 6, with states associated with the angular positions of each disc, is given by the matrices

$$A = \begin{bmatrix} 0 & I \\ -J^{-1}K & -J^{-1}D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{1}{m} \\ 0 \\ 0 \end{bmatrix}, \quad C^{\mathrm{T}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \dfrac{1}{m} & 0 \end{bmatrix} \tag{210}$$

and $D_2 = [0 \;\; 1]$, where

$$J = m \begin{bmatrix} 0{\cdot}5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad K = k \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$D = d \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{211}$$

$m = k = 1$, and a low damping value $d = 0{\cdot}01$ is used.

We consider two uncertainties in the spring stiffnesses between discs 1–2 and discs 3–4. Several characteristics make this problem important in the study of controllers for lightly damped flexible structures. First of all, the performance is dominated by the lower frequency rigid body and first flexible modes, which are essentially unaffected by the model uncertainty. Furthermore, the two higher frequency modes and the zero at $1{\cdot}4\ \mathrm{rad\,s^{-1}}$ are highly uncertain, resulting in large phase uncertainties in the system with only 5% variations in the two stiffnesses.

The uncertainty in the dynamics matrix A can then be represented as $\Delta A = -B_0 \Delta K C_0$, where

$$B_0^{\mathrm{T}} = -\begin{bmatrix} 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{212}$$

and $\Delta K = \mathrm{diag}(\Delta k_1, \Delta k_2)$. The general Popov multiplier $H_0 + N_0 s$ (H_0, $N_0 \in \mathbb{D}^2$) was used for this example. To complete the noise and performance specifications for $\mathcal{H}_2$ synthesis, define $R_{xx} = C_1^{\mathrm{T}} C_1$, $V_2 = R_{uu} = \rho$ and $V_1 = D_1 D_1^{\mathrm{T}}$, where $\rho = 0{\cdot}005$ and $C_1 = [0\ 0\ 0\ 1\ 0\ 0\ 0\ 0{\cdot}1]$. Note that the open loop cost is infinite since the rigid body motion is observable in the cost. The cost curves are normalized with respect to the optimal LQG cost for the nominal system.

While the robust control synthesis procedure assumes that the two uncertainties are uncorrelated, for simplicity in the analysis we will consider the case where $\Delta k_1 = \Delta k_2 = \Delta k$. The robust stability and performance results are presented in Fig. 8 and the Table. Note that the compensator in design Gpc3

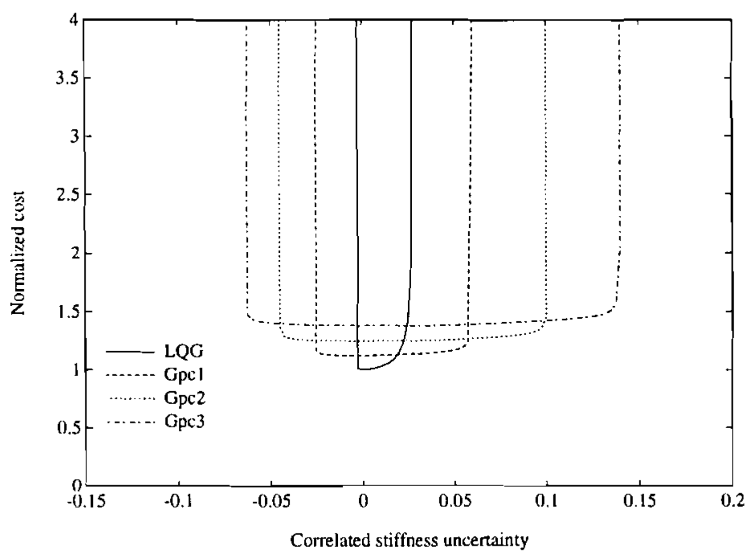

Figure 8. Closed loop robust stability and performance with two stiffness uncertainties. It is assumed for analysis $\Delta k_1 = \Delta k_2$. See the Table for stability bounds.

Fig. 8 label	J_{norm}	Lower bound achieved	Lower bound guaranteed	Upper bound guaranteed	Upper bound achieved
LQG	1·00	−0·003	0	0	0·028
Gpc1	1·12	−0·025	−0·019	0·019	0·060
Gpc2	1·25	−0·045	−0·035	0·035	0·100
Gpc3	1·38	−0·063	−0·051	0·051	0·140

Closed loop robust stability and performance with two stiffness uncertainties. It was assumed that $\Delta k_1 = \Delta k_2$ for the analysis.

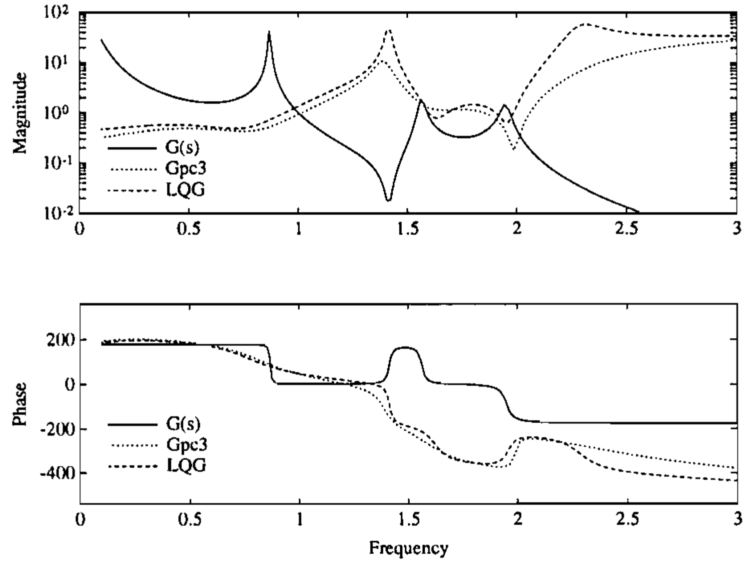

Figure 9. Comparison of optimal LQG (Glc) and Popov (Gpc3) compensators for the system with two stiffness uncertainties. See the Table for stability bounds.

guarantees stability for 5% independent variations in the stiffness values, and this value corresponds to approximately 100° phase variations in the system and represents a significant improvement over values that the LQG design actually achieves. The optimal multiplier for Gpc3 is $W_{\rm opt}(s) = {\rm diag}(1 + 0{\cdot}16s, 2{\cdot}9(1 + 0{\cdot}27s))$.

In many robust control problems, there is a 'stiff' uncertainty direction which is more difficult than other directions. In this example, it is the negative uncertainty values which are more difficult, as can be seen by the closeness of the guaranteed and achieved lower bounds in the Table. While the discrepancy in the guaranteed and achieved upper bounds is, to some extent, a measure of the conservatism in the technique, it is also a reflection of the relative ease of 'robustifying' the system to this type of uncertainty.

The optimal LQG and Gpc3 compensators are compared in Fig. 9. The uncertainty in the zero–pole combination at approximately $1{\cdot}4\,{\rm rad\,s^{-1}}$ is reflected in the Popov compensator by lower compensator gains, and a much smoother phase than the LQG design. Note that the LQG compensator pole at approximately $1.4\,{\rm rad\,s^{-1}}$ is shifted away from the plant zero in the Popov design. This avoids the pole–zero cancellation of plant inversion, which is extremely sensitive to plant uncertainties. Further comparisons of the two design approaches have been presented by How *et al.* (1992 a).

9. Conclusions

The goal of this paper has been to make explicit connections between classical absolute stability theory and modern mixed-μ analysis and synthesis. To this end, we extended previous results on absolute stability theory for monotonic and odd monotonic nonlinearities to provide a tight approximation for constant real parametric uncertainty. Specifically, by using a parameter-dependent Lyapunov function framework in which the uncertain parameters appear explicitly in the Lyapunov function, the allowable time-variation of the parameters is restricted, thereby reducing conservatism with respect to constant real parametric uncertainty.

Connections to μ-analysis are made through frequency domain tests, demonstrating that the stability multipliers are parametrizations of the D, N-scales in mixed-μ. Combining the parameter-dependent Lyapunov functions with fixed-order optimization techniques leads to a Riccati equation characterization for robust $\mathcal{H}_2$ controllers. An advantage of this approach is that reduced-order controllers with optimal frequency domain multipliers can be designed directly, thereby avoiding the standard D, $N - K$ iteration and curve-fitting procedure.

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