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Generalized mixed-μ bounds for real and complex multiple-block uncertainty with internal matrix structure

WASSIM M. HADDAD†‡, DENNIS S. BERNSTEIN§¶ and VIJAYA-SEKHAR CHELLABOINA†||

New absolute stability results that unify and extend existing structured singular value bounds for mixed uncertainty are developed using frequency-domain arguments. These bounds generalize prior upper bounds for mixed-μ by permitting the treatment of non-diagonal real uncertain blocks, as well as accounting for internal matrix structure in the uncertainty. Several specializations to well known absolute stability criteria from the classical literature are given, which allow for a systematic comparison of these results in addressing the problem of robust stability for real parameter uncertainty.

1. Introduction

The ability of the structured singular value to account for complex, real and mixed uncertainty provides a powerful framework for robust stability and performance problems in both analysis and synthesis (see Doyle 1982, Fan *et al.* 1991, Young *et al.* 1991, Packard and Doyle 1993, Safonov and Chiang 1993, and the references therein). Since exact computation of the structured singular value is, in general, an intractable problem, the development of practically implementable bounds remains a high priority in robust control research. Recent work in this area includes upper and lower bounds for mixed uncertainty (Fan *et al.* 1991, Lee and Tits 1991 and Young *et al.* 1991) as well as LMI-based computational techniques (Gahinet and Nemirovskii 1993, Boyd *et al.* 1994).

An alternative approach to developing bounds for the structured singular value is to specialize absolute stability criteria for sector-bounded nonlinearities to the case of linear uncertainty. This approach, which has been explored by Chiang and Safonov (1992), Haddad and Bernstein (1991a, 1993, 1994, 1995a, b), How and Hall (1993) and Haddad *et al.* (1994), demonstrates the direct applicability of the classical theory of absolute stability to the modern structured singular value framework. In particular, the rich theory of multiplier-based absolute stability criteria due to Luré and Postnikov (Aizerman and Gantmacher 1964, Lefschetz 1965, Narendra and Taylor 1978, Popov 1973), Popov (1962), Yakubovitch (1965a, b), Zames and Falb (1968), and numerous others can be seen to have a close and fundamental relationship with recently developed structured singular value bounds.

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The objective of this paper is to develop new absolute stability results that unify and extend existing structured singular value bounds for mixed uncertainty. Since these results are developed for the case of linear uncertainty, the development of these results is based entirely on frequency-domain arguments. In contrast, many of the proofs of classical absolute stability criteria that address specific classes of nonlinearities require the construction of specialized Lyapunov functions (Narendra and Neuman 1966, Narendra and Taylor 1978, Popov 1962, Thathachar and Srinath 1967, Thathahar *et al.* 1967, and Yakubovitch 1965a, b). The resulting simplification has already been demonstrated by Bernstein *et al.* (1995) with a new proof of the classical Popov criterion. The generalized absolute stability criterion given herein by Theorem 1 encompasses virtually all of the classical absolute stability criteria.

Theorem 1 provides the foundation for obtaining generalizations of existing bounds for the structured singular value. These bounds generalize prior upper bounds for mixed-μ by permitting the treatment of fully populated real uncertain blocks which may, in addition, possess internal structure. Such problems arise in a variety of applications, such as the study of modal dynamics, in which transformation to 'standard' diagonal form may introduce additional conservatism, computational complexity, as well as destroying the parameter space of the original uncertainty characterization (see Example 2). The ability to address real uncertain blocks is based on the use of an appropriate class of multipliers whose structure is compatible with the real block uncertainty. Hence, choosing multipliers tailored to the structure of the uncertainty not only leads to the ability to address more general uncertainty characterizations but can also lead to less conservative stability conditions than the standard mixed-μ upper bound. This more general class of multipliers, which appears in the context of the multivariable Popov criterion of Haddad and Bernstein (1991a, 1995a), has no counterpart in Fan *et al.* (1991), Young *et al.* (1991) or Safonov and Chiang (1993).

Although the results of the present paper address sector-bounded linear matrix uncertainty for finite-dimensional linear time-invariant systems, the present framework can be extended to robustness analysis using quadratic separating functions establishing a topological separation between the graph of a linear nominal system and the set of inverse graphs of all possible plant uncertainty (Megretski 1993). In this case it could be shown that the integral quadratic constraint approach for robustness analysis (Megretski 1993) can be tailored to capture internal structure in plant uncertainty.

In this paper we use the following standard notation. Let $\mathbb{R}$ and $\mathbb{C}$ denote real and complex numbers, and let $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ denote real and complex $n\times m$ matrices. Let $\mathbf{A}^{\mathrm{T}}$ and $\mathbf{A}^*$ denote the transpose and complex conjugate transpose of $\mathbf{A}$. $\mathbf{M}\geqslant 0$ ($\mathbf{M}>0$) denotes the fact that the hermitian matrix $\mathbf{M}$ is non-negative (positive) definite. The maximum and minimum singular values of an arbitrary matrix $\mathbf{M}$ are denoted by $\sigma_{\max}(\mathbf{M})$ and $\sigma_{\min}(\mathbf{M})$, respectively. The hermitian and skew-hermitian parts of an arbitrary complex matrix $\mathbf{G}$ are defined by $\mathrm{He}\,\mathbf{G}\triangleq\frac{1}{2}(\mathbf{G}+\mathbf{G}^*)$ and $\mathrm{Sh}\,\mathbf{G}\triangleq\frac{1}{2}(\mathbf{G}-\mathbf{G}^*)$, respectively. Finally, $\mathbf{A}\otimes\mathbf{B}$ denotes the Kroneker product of matrices $\mathbf{A}$ and $\mathbf{B}$.

2. Absolute stability criterion with generalized positive real stability multipliers

In this section we state and prove an absolute stability criterion for multivariable systems with generalized positive real frequency-dependent stability multipliers.

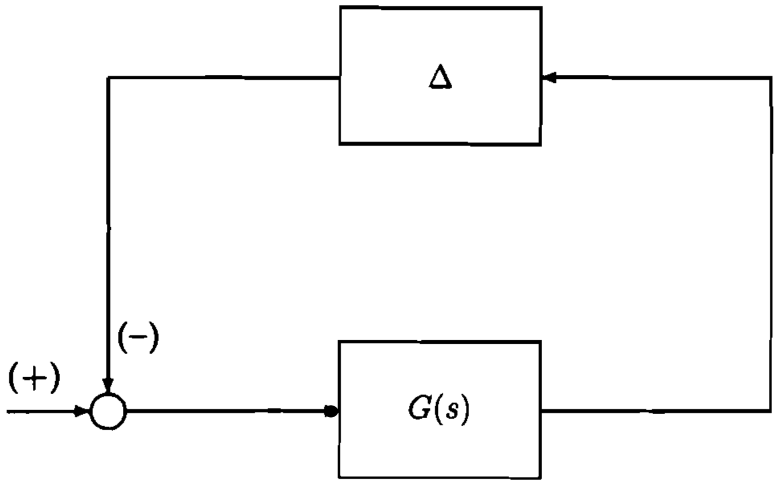

Figure 1. Interconnection of transfer function $\mathbf{G}(s)$ with uncertain matrix Δ.

This criterion involves a square nominal transfer function $\mathbf{G}(s)$ in a negative feedback interconnection with a complex, square, uncertain matrix Δ, as shown in Fig. 1. Now we consider the set of block-diagonal matrices with possibly repeated blocks defined by

$$\Delta_{\mathrm{bs}} \triangleq \{\Delta \in \mathbb{C}^{m\times m} : \Delta = \text{block-diag}(\mathbf{I}_{l_1} \otimes \Delta_1^r, \ldots, \mathbf{I}_{l_r} \otimes \Delta_r^r, \mathbf{I}_{l_{r+1}} \otimes \Delta_{r+1}^c, \ldots, \mathbf{I}_{l_{r+c}} \otimes \Delta_{r+c}^c),$$
$$\Delta_i^r \in \mathbb{R}^{m_i\times m_i}, i = 1, \ldots, r, \Delta_i^c \in \mathbb{C}^{m_i\times m_i}, i = r+1, \ldots, r+c\}$$

where the dimension m_i of each block and the number of repetitions l_i of each block are given such that $\sum_{i=1}^{r+c} l_i m_i = m$. Furthermore, define the subset $\boldsymbol{\Delta} \subseteq \Delta_{\mathrm{bs}}$ consisting of sector-bounded matrices

$$\boldsymbol{\Delta} \triangleq \{\Delta \in \Delta_{\mathrm{bs}} : 2(\Delta - \mathbf{M}_1)^*(\mathbf{M}_2 - \mathbf{M}_1)^{-1}(\Delta - \mathbf{M}_1) \leqslant (\Delta - \mathbf{M}_1) + (\Delta - \mathbf{M}_1)^*\}$$

where $\mathbf{M}_1, \mathbf{M}_2 \in \Delta_{\mathrm{bs}}$ are Hermitian matrices such that $\mathbf{M} \triangleq \mathbf{M}_2 - \mathbf{M}_1$ is positive definite. Note that $\mathbf{M}_1$ and $\mathbf{M}_2$ are elements of $\boldsymbol{\Delta}$. The following lemma provides alternative characterizations of $\boldsymbol{\Delta}$.

Lemma 1: *Let $\Delta \in \Delta_{\mathrm{bs}}$. The following statements are equivalent.*

(*a*) $\Delta \in \boldsymbol{\Delta}$

(*b*) $\begin{bmatrix} \mathrm{He}\,\Delta - \mathbf{M}_1 & \Delta^* - \mathbf{M}_1 \\ \Delta - \mathbf{M}_1 & \mathbf{M} \end{bmatrix} \geqslant 0$

(*c*) $\begin{bmatrix} \mathrm{He}\,\Delta - \mathbf{M}_1 & \Delta - \mathbf{M}_1 \\ \Delta^* - \mathbf{M}_1 & \mathbf{M} \end{bmatrix} \geqslant 0$

(*d*) $\begin{bmatrix} \mathrm{He}\,\Delta - \mathbf{M}_1 & \mathrm{Sh}\,\Delta \\ -\mathrm{Sh}\,\Delta & \mathbf{M}_2 - \mathrm{He}\,\Delta \end{bmatrix} \geqslant 0$

If, in addition, $\det(\mathbf{M}_2 - \mathrm{He}\,\Delta) \neq 0$, *then the following condition is equivalent to the above:*

(*e*) $\mathbf{M}_1 - \mathrm{Sh}\,\Delta(\mathbf{M}_2 - \mathrm{He}\,\Delta)^{-1}\mathrm{Sh}\,\Delta \leqslant \mathrm{He}\,\Delta \leqslant \mathbf{M}_2$

Furthermore, the following statements hold:

(f) $\Delta \in \boldsymbol{\Delta}$ *if and only if* $\Delta^* \in \boldsymbol{\Delta}$

(g) $\Delta = \Delta^* \in \boldsymbol{\Delta}$ *if and only if* $\mathbf{M}_1 \leqslant \Delta \leqslant \mathbf{M}_2$

Proof: The proof of (a)–(d) follows from elementary congruence transformations. Statement (f) follows from (b) and (c), and (g) is a direct consequence of (d). □

2.1. *Absolute stability criterion*

In this subsection we prove the multivariable absolute stability criterion for sector-bounded uncertain matrices. To state this criterion, we define the sets $\mathscr{D}$ and $\mathscr{N}$ of hermitian frequency-dependent scaling matrix functions by

$$\mathscr{D} \triangleq \{D\colon \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{C}^{m\times m}\colon D(\mathrm{j}\omega) \geqslant 0,\ D(\mathrm{j}\omega)\Delta = \Delta D(\mathrm{j}\omega), \omega \in \mathbb{R}, \Delta \in \boldsymbol{\Delta}_{\mathrm{bs}}\}$$

$$\mathscr{N} \triangleq \{N\colon \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{C}^{m\times m}\colon N(\mathrm{j}\omega) = N^*(\mathrm{j}\omega),\ N(\mathrm{j}\omega)\Delta = \Delta^* N(\mathrm{j}\omega), \omega \in \mathbb{R}, \Delta \in \boldsymbol{\Delta}_{\mathrm{bs}}\}$$

Furthermore, define the set $\mathscr{Z}$ of complex multiplier matrix functions by

$$\mathscr{Z} \triangleq \{Z\colon \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{C}^{m\times m}\colon Z(\mathrm{j}\omega) = D(\mathrm{j}\omega) - \mathrm{j}N(\mathrm{j}\omega), \quad D(\cdot) \in \mathscr{D}, N(\cdot) \in \mathscr{N}\}$$

Note that if $Z(\cdot) \in \mathscr{Z}$, $D(\cdot) \in \mathscr{D}$ and $N(\cdot) \in \mathscr{N}$, then $Z(\mathrm{j}\omega) = D(\mathrm{j}\omega) - \mathrm{j}N(\mathrm{j}\omega)$ if and only if $D(\mathrm{j}\omega) = \mathrm{He}\, Z(\mathrm{j}\omega)$ and $N(\mathrm{j}\omega) = \mathrm{j}\,\mathrm{Sh}\, Z(\mathrm{j}\omega)$. Hence, since $D(\mathrm{j}\omega) \geqslant 0$, $\omega \in \mathbb{R} \cup \infty$, $Z(\cdot) \in \mathscr{Z}$ consists of arbitrary generalized positive real functions (Anderson and Moore 1968). For $\Delta \in \boldsymbol{\Delta}_{\mathrm{bs}}$, $\mathscr{D}$ and $\mathscr{N}$ are given by

$$\mathscr{D} = \{D\colon \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{C}^{m\times m}\colon D = \text{block-diag}(D_1 \otimes \mathbf{I}_{m_1}, D_2 \otimes \mathbf{I}_{m_2}, \ldots, D_{r+c} \otimes \mathbf{I}_{m_{r+c}}),\quad 0 \leqslant D_i \in \mathbb{C}^{l_i \times l_i}, i = 1, \ldots, r+c\} \tag{1}$$

$$\mathscr{N} = \{N\colon \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{C}^{m\times m}\colon N = \text{block-diag}(N_1 \otimes \mathbf{I}_{m_1}, N_2 \otimes \mathbf{I}_{m_2}, \ldots, N_{r+c} \otimes \mathbf{I}_{m_{r+c}}),\quad N_i = N_i^* \in \mathbb{C}^{l_i \times l_i}, N_i \otimes \Delta_i = N_i \otimes \Delta_i^*, i = 1, \ldots, r+c\} \tag{2}$$

Remark 1: Although the condition $D(\mathrm{j}\omega)\Delta = \Delta D(\mathrm{j}\omega)$ in $\mathscr{D}$ arises in complex and mixed-μ analysis (Fan *et al.* 1991), the condition $N(\mathrm{j}\omega)\Delta = \Delta^* N(\mathrm{j}\omega)$ in $\mathscr{N}$ has no counterpart in Fan *et al.* (1991). As will be seen in §3 this condition generalizes mixed-μ analysis to address non-diagonal real matrices which are not considered in standard mixed-μ theory. The condition $N(\mathrm{j}\omega)\Delta = \Delta^* N(\mathrm{j}\omega)$ is an extension of the condition used by Haddad and Bernstein (1991 a) for Popov controller synthesis with constant real matrix uncertainty.

Next, we introduce the following key lemma.

Lemma 2: *Let* $Z(\cdot) \in \mathscr{Z}$, *let* $\omega \in \mathbb{R} \cup \infty$, *and suppose that* $\det(\mathbf{I} + \mathbf{G}(\mathrm{j}\omega)\mathbf{M}_1) \neq 0$. *If*

$$\mathrm{He}[Z(\mathrm{j}\omega)(\mathbf{M}^{-1} + (\mathbf{I} + \mathbf{G}(\mathrm{j}\omega)\mathbf{M}_1)^{-1}\mathbf{G}(\mathrm{j}\omega))] > 0 \tag{3}$$

then $\det(\mathbf{I} + \mathbf{G}(\mathrm{j}\omega)\Delta) \neq 0$, *for all* $\Delta \in \boldsymbol{\Delta}$.

Proof: For notational convenience we write D for $D(\mathrm{j}\omega)$ and N for $N(\mathrm{j}\omega)$. Suppose that there exists $\Delta \in \boldsymbol{\Delta}$ such that $\det(\mathbf{I} + \mathbf{G}(\mathrm{j}\omega)\Delta) = 0$. Then there exists $\boldsymbol{x} \in \mathbb{C}^m$, $\boldsymbol{x} \neq 0$, such that $(\mathbf{I} + \Delta\mathbf{G}(\mathrm{j}\omega))\boldsymbol{x} = 0$. Hence, $-\boldsymbol{x} = \Delta\mathbf{G}(\mathrm{j}\omega)\boldsymbol{x}$ and $-\boldsymbol{x}^* = \boldsymbol{x}^*\mathbf{G}^*(\mathrm{j}\omega)\Delta^*$.

Since $2(\Delta-\mathbf{M}_1)^*\mathbf{M}^{-1}(\Delta-\mathbf{M}_1)\leqslant(\Delta-\mathbf{M}_1)^*+(\Delta-\mathbf{M}_1)$, it follows that

$$\mathrm{He}\,[(\Delta-\mathbf{M}_1)^*\mathbf{M}^{-1}(\Delta-\mathbf{M}_1)-(\Delta-\mathbf{M}_1)^*]\leqslant 0$$

or, equivalently,

$$\mathrm{He}\,[\Delta^*\mathbf{M}^{-1}\Delta-2\Delta^*\mathbf{M}^{-1}\mathbf{M}_1+\mathbf{M}_1\mathbf{M}^{-1}\mathbf{M}_1-\Delta^*+\mathbf{M}_1]\leqslant 0 \tag{4}$$

Now, since $D(\cdot)\in\mathscr{D}$ and $\mathbf{M}_1,\mathbf{M}_2\in\mathbf{\Delta}$ it follows that $D\mathbf{M}_1=\mathbf{M}_1 D$ and $D\mathbf{M}^{-1}=\mathbf{M}^{-1}D$. Next note that

$$\begin{aligned}D\,\mathrm{He}\,[\Delta^*\mathbf{M}^{-1}\Delta-2\Delta^*\mathbf{M}^{-1}\mathbf{M}_1+\mathbf{M}_1\mathbf{M}^{-1}\mathbf{M}_1-\Delta^*+\mathbf{M}_1]&\\ =\mathrm{He}\,[\Delta^*\mathbf{M}^{-1}\Delta-2\Delta^*\mathbf{M}^{-1}\mathbf{M}_1+\mathbf{M}_1\mathbf{M}^{-1}\mathbf{M}_1-\Delta^*+\mathbf{M}_1]\,D&\end{aligned}$$

Hence forming $D(4)$ yields

$$\mathrm{He}\,[\Delta^*D\mathbf{M}^{-1}\Delta-2\Delta^*D\mathbf{M}^{-1}\mathbf{M}_1+\mathbf{M}_1D\mathbf{M}^{-1}\mathbf{M}_1-\Delta^*D+D\mathbf{M}_1]\leqslant 0 \tag{5}$$

Furthermore, forming $x^*\mathbf{G}^*(\mathrm{j}\omega)(5)\,\mathbf{G}(\mathrm{j}\omega)x$ yields

$$\begin{aligned}x^*\,\mathrm{He}\,[D\mathbf{M}^{-1}+2D\mathbf{M}^{-1}\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)+\mathbf{G}^*(\mathrm{j}\omega)\mathbf{M}_1D\mathbf{M}^{-1}\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)+D\mathbf{G}(\mathrm{j}\omega)&\\ +\mathbf{G}^*(\mathrm{j}\omega)D\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)]\,x\leqslant 0&\end{aligned} \tag{6}$$

Next, note that $\mathrm{He}\,[Z(\mathrm{j}\omega)(\mathbf{M}^{-1}+(\mathbf{I}+\mathbf{G}(\mathrm{j}\omega)\mathbf{M}_1)^{-1}\mathbf{G}(\mathrm{j}\omega))]>0$ is equivalent to

$$\mathrm{He}\,[Z(\mathrm{j}\omega)(\mathbf{M}^{-1}+\mathbf{G}(\mathrm{j}\omega)(\mathbf{I}+\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega))^{-1}]>0$$

Now, pre- and post-multiplying the above inequality by $\mathbf{I}+\mathbf{G}^*(\mathrm{j}\omega)\mathbf{M}_1$ and $\mathbf{I}+\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)$ we obtain

$$\mathrm{He}\,[(\mathbf{I}+\mathbf{G}^*(\mathrm{j}\omega)\mathbf{M}_1)Z(\mathrm{j}\omega)(\mathbf{M}^{-1}+\mathbf{M}^{-1}\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)+\mathbf{G}(\mathrm{j}\omega))]>0 \tag{7}$$

Since $N(\cdot)\in\mathscr{N}$ and $\mathbf{M}_1,\mathbf{M}_2\in\mathbf{\Delta}$ it follows that $N\mathbf{M}_1=\mathbf{M}_1N$ and $N\mathbf{M}_2=\mathbf{M}_2N$. Thus $N\mathbf{M}=\mathbf{M}N$ and hence $N\mathbf{M}^{-1}=\mathbf{M}^{-1}N$. By using these relations, (7) is simplified to

$$\begin{aligned}\mathrm{He}\,[D\mathbf{M}^{-1}+2D\mathbf{M}^{-1}\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)+\mathbf{G}^*(\mathrm{j}\omega)\mathbf{M}_1D\mathbf{M}^{-1}\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)+D\mathbf{G}(\mathrm{j}\omega)&\\ +\mathbf{G}^*(\mathrm{j}\omega)D\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)]>\mathrm{He}\,[\mathrm{j}N\mathbf{G}(\mathrm{j}\omega)]&\end{aligned} \tag{8}$$

Now forming $x^*(8)\,x$ yields

$$\begin{aligned}x^*\,\mathrm{He}\,[D\mathbf{M}^{-1}+2D\mathbf{M}^{-1}\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)+\mathbf{G}^*(\mathrm{j}\omega)\mathbf{M}_1D\mathbf{M}^{-1}\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)+D\mathbf{G}(\mathrm{j}\omega)&\\ +\mathbf{G}^*(\mathrm{j}\omega)D\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)]\,x>\mathrm{He}\,[\mathrm{j}x^*N\mathbf{G}(\mathrm{j}\omega)\mathrm{x}]=-x^*\mathbf{G}^*(\mathrm{j}\omega)\,\mathrm{He}\,[\mathrm{j}\Delta^*N]\,\mathbf{G}(\mathrm{j}\omega)\,x&\end{aligned}$$

Since $N(\cdot)\in\mathscr{N}$, it follows that $\mathrm{He}\,[\mathrm{j}\Delta^*N]=\frac{j}{2}(\Delta^*N-N\Delta)=0$ and hence

$$\begin{aligned}x^*\,\mathrm{He}\,[D\mathbf{M}^{-1}+2D\mathbf{M}^{-1}\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)+\mathbf{G}^*(\mathrm{j}\omega)\mathbf{M}_1D\mathbf{M}^{-1}\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)&\\ +D\mathbf{G}(\mathrm{j}\omega)+\mathbf{G}^*(\mathrm{j}\omega)D\mathbf{M}_1\mathbf{G}(\mathrm{j}\omega)]\,x>0&\end{aligned}$$

which contradicts (6). Consequently, $\det(\mathbf{I}+\mathbf{G}(\mathrm{j}\omega)\Delta)\neq 0$ for all $\Delta\in\mathbf{\Delta}$. □

We now state our main result.

Theorem 1: *Suppose that* $\mathbf{G}_s(s) \triangleq (\mathbf{I} + \mathbf{G}(s)\mathbf{M}_1)^{-1}\mathbf{G}(s)$ *is asymptotically stable. If there exists* $Z(\cdot) \in \mathscr{Z}$ *such that*

$$\mathrm{He}\,[Z(s)(\mathbf{M}^{-1} + \mathbf{G}_s(s))] > 0 \tag{9}$$

for all $s = \mathrm{j}\omega$, $\omega \in \mathbb{R} \cup \infty$, *then the negative feedback interconnection of* $\mathbf{G}(s)$ *and* Δ *is asymptotically stable for all* $\Delta \in \mathbf{\Delta}$.

Proof: The proof follows from Lemma 2 using similar arguments as in the proof of Theorem 2.1 of Bernstein *et al.* (1995). □

3. Specialization to norm-bounded, internally block-structured uncertainty

In this section we specialize Theorem 1 to the case of norm-bounded uncertainty in order to draw connections with the structured singular value for real and complex block-structured uncertainty. Letting $\mathbf{M}_1 = -\gamma^{-1}\mathbf{I}$ and $\mathbf{M}_2 = \gamma^{-1}\mathbf{I}$, where $\gamma > 0$, it follows that $\mathbf{M} = 2\gamma^{-1}\mathbf{I}$ so that $\mathbf{M}^{-1} = \frac{1}{2}\gamma\mathbf{I}$. The set $\mathbf{\Delta}$ thus becomes

$$\mathbf{\Delta}_\gamma = \{\Delta \in \mathbf{\Delta}_{\mathrm{bs}}: \gamma(\Delta + \gamma^{-1}\mathbf{I})^*(\Delta + \gamma^{-1}\mathbf{I}) \leqslant (\Delta + \gamma^{-1}\mathbf{I}) + (\Delta + \gamma^{-1}\mathbf{I})^*\}$$

Now, $\Delta \in \mathbf{\Delta}_\gamma$ if and only if $\sigma_{\max}(\Delta) \leqslant \gamma^{-1}$. Therefore, $\mathbf{\Delta}_\gamma$ is given by

$$\mathbf{\Delta}_\gamma = \{\Delta \in \mathbf{\Delta}_{\mathrm{bs}}: \sigma_{\max}(\Delta) \leqslant \gamma^{-1}\}$$

Now, we consider a special case of the sets $\mathbf{\Delta}_{\mathrm{bs}}$, $\mathscr{D}$ and $\mathscr{N}$ where $\Delta \in \mathbf{\Delta}_{\mathrm{bs}}$. In particular, let $\mathbf{\Delta}_{\mathrm{bs}}$ be the set of block-structured matrices with possibly repeated real scalar elements, complex scalar elements and complex blocks given by

$$\begin{aligned}\mathbf{\Delta}_{\mathrm{bs}} = \{\Delta \in \mathbb{C}^{m\times m}: \Delta = \text{block-diag}\,(&\delta_1^{\mathrm{r}}\mathbf{I}_{l_1}, \ldots, \delta_r^{\mathrm{r}}\mathbf{I}_{l_r}; \delta_{r+1}^{\mathrm{c}}\mathbf{I}_{l_{r+1}}, \ldots, \delta_{r+q}^{\mathrm{c}}\mathbf{I}_{l_{r+q}}; \Delta_{r+q+1}^{C}, \ldots, \Delta_{r+c}^{C})\\ &\delta_i^{\mathrm{r}} \in \mathbb{R}, i = 1, \ldots, r, \delta_i^{\mathrm{c}} \in \mathbb{C}, i = r+1, \ldots, r+q, \Delta_i^{C} \in \mathbb{C}^{\hat{m}_i \times \hat{m}_i}, i = r+q+1, \ldots, r+c\}\end{aligned} \tag{10}$$

Then $\mathscr{D}$ and $\mathscr{N}$ are the sets of frequency-dependent positive definite and hermitian matrices, respectively, given by

$$\begin{aligned}\mathscr{D} = \{D: \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{C}^{m\times m}: D = \text{block-diag}\,(&D_1, \ldots, D_{r+q}, d_{r+q+1}\mathbf{I}_{\hat{m}_{r+q+1}}, \ldots, d_{r+c}\mathbf{I}_{\hat{m}_{r+c}})\\ &0 < D_i \in \mathbb{C}^{l_i \times l_i}, i = 1, \ldots, r+q, 0 < d_i \in \mathbb{R}, i = r+q+1, \ldots, r+c\}\end{aligned} \tag{11}$$

$$\begin{aligned}\mathscr{N} = \{N: \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{C}^{m\times m}: N = \text{block-diag}\,(&N_1, \ldots, N_r, 0_{r+1}, \ldots, 0_{r+c})\\ &N_i = N_i^* \in \mathbb{C}^{l_i \times l_i}, i = 1, \ldots, r\}\end{aligned} \tag{12}$$

Note that this special case is equivalent to the mixed-μ set considered by Fan *et al.* (1991). Furthermore, with $D(\cdot) \in \mathscr{D}$ and $N(\cdot) \in \mathscr{N}$ given by (11) and (12), respectively, the compatibility conditions required in $\mathscr{D}$ and $\mathscr{N}$ are automatically satisfied for $\mathbf{\Delta}_{\mathrm{bs}}$ given by (10).

Alternatively, let $\Delta_{\rm bs}$ be given by (10) with the additional constraint that the complex blocks possess internal matrix structure. Then $\mathscr{D}$ and $\mathscr{N}$ are given by

$$\mathscr{D} = \{D\colon \mathrm{j}\mathbb{R}\cup\infty \to \mathbb{C}^{m\times m}\colon D = \text{block-diag}\,(D_1,\ldots,D_{r+c})$$
$$0 < D_i \in \mathbb{C}^{l_i\times l_i}, i = 1,\ldots,r+c, D_i\Delta_i^C = \Delta_i^C D_i, i = r+q+1,\ldots,r+c\}$$
$$\mathscr{N} = \{N\colon \mathrm{j}\mathbb{R}\cup\infty \to \mathbb{C}^{m\times m}\colon N = \text{block-diag}\,(N_1,\ldots,N_r,0_{r+1},\ldots,0_{r+q},N_{r+q+1},\ldots,N_{r+c})$$
$$N_i = N_i^* \in \mathbb{C}^{l_i\times l_i}, i = 1,\ldots,r+c; N_i\Delta_i^C = \Delta_i^{C*}N_i, i = r+q+1,\ldots,r+c\}$$

For example, if $\Delta_i^C = \Delta_i^{C*} \in \Delta_{\rm bs}$ then we can choose $D_i = d_i\mathbf{I}$, $d_i \in \mathbb{R}$ and $N_i = n_i\mathbf{I}$, $n_i \in \mathbb{R}$, $i = r+q+1,\ldots,r+c$.

Next, we present a key lemma which is important in connecting the absolute stability criterion given in Theorem 1 to the mixed-μ upper bounds.

Lemma 3: *Let $Z(\cdot)\in\mathscr{Z}$, $D(\cdot)\in\mathscr{D}$, $N(\cdot)\in\mathscr{N}$, and $\omega\in\mathbb{R}\cup\infty$. Then the following statements are equivalent.*

(*a*) $\mathrm{He}[Z(\mathrm{j}\omega)(\tfrac{1}{2}\gamma\mathbf{I}+\mathbf{G}_s(\mathrm{j}\omega))] > 0$

(*b*) $\mathbf{G}^*(\mathrm{j}\omega)D\mathbf{G}(\mathrm{j}\omega)+\mathrm{j}\gamma(N\mathbf{G}^*(\mathrm{j}\omega)-\mathbf{G}^*(\mathrm{j}\omega)N)-\gamma^2 D < 0$

Proof: The result follows by simple algebraic manipulations. □

4. Real structured singular value upper bound

We now obtain upper bounds for the structured singular value for real and complex multiple-block structured uncertainty. These bounds are based upon the absolute stability criterion of Theorem 1 for norm-bounded, block-structured, uncertain matrices. The structured singular value of a complex matrix $\mathbf{G}(\mathrm{j}\omega)$ for mixed real and complex uncertainty is defined by (Fan *et al.* 1991)

$$\mu(\mathbf{G}(\mathrm{j}\omega)) \triangleq (\min_{\Delta\in\Delta_{\rm bs}}\{\sigma_{\max}(\Delta)\colon \det(\mathbf{I}+\mathbf{G}(\mathrm{j}\omega)\Delta) = 0\})^{-1}$$

and if $\det(\mathbf{I}+\mathbf{G}(\mathrm{j}\omega)\Delta) \neq 0$ for all $\Delta\in\Delta_{\rm bs}$ then $\mu(\mathbf{G}(\mathrm{j}\omega)) \triangleq 0$. Hence, a necessary and sufficient condition for robust stability of the feedback interconnection of $\mathbf{G}(s)$ and Δ is given by the following theorem.

Theorem 2: *Let $\gamma > 0$ and suppose that $\mathbf{G}(s)$ is asymptotically stable. Then the negative feedback interconnection of $\mathbf{G}(s)$ and Δ is asymptotically stable for all $\Delta\in\Delta_\gamma$ if and only if*

$$\mu(\mathbf{G}(\mathrm{j}\omega)) < \gamma, \quad \omega\in\mathbb{R}\cup\infty \tag{13}$$

Proof: Let

$$\mathbf{G}(s) \sim \left[\begin{array}{c|c}\mathbf{A} & \mathbf{B}\\ \hline \mathbf{B} & \mathbf{D}\end{array}\right]$$

where $\mathbf{A}$ is Hurwitz, and suppose that the negative feedback interconnection of $\mathbf{G}(s)$ and Δ given by

$$(\mathbf{I}+\mathbf{G}(s)\Delta)^{-1}\mathbf{G}(s) \sim \left[\begin{array}{c|c} \mathbf{A}-\mathbf{B}\Delta(\mathbf{I}+\mathbf{D}\Delta)^{-1}\mathbf{C} & \mathbf{B}-\mathbf{B}\Delta(\mathbf{I}+\mathbf{D}\Delta)^{-1}\mathbf{D} \\ \hline (\mathbf{I}+\mathbf{D}\Delta)^{-1}\mathbf{C} & (\mathbf{I}+\mathbf{D}\Delta)^{-1}\mathbf{D} \end{array}\right]$$

is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\gamma$. Next, note that, for all $\Delta \in \mathbf{\Delta}_\gamma$ and $\omega \in \mathbb{R} \cup \infty$

$$\begin{aligned} \det[\mathbf{I}+\mathbf{G}(\mathrm{j}\omega)\Delta] &= \det[\mathbf{I}+(\mathbf{C}(\mathrm{j}\omega\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D})\Delta] \\ &= \det(\mathbf{I}+\mathbf{D}\Delta)\det[\mathbf{I}+(\mathrm{j}\omega\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\Delta(\mathbf{I}+\mathbf{D}\Delta)^{-1}\mathbf{C}] \\ &= \det(\mathbf{I}+\mathbf{D}\Delta)\det(\mathrm{j}\omega\mathbf{I}-\mathbf{A})^{-1}\det[\mathrm{j}\omega\mathbf{I}-(\mathbf{A}-\mathbf{B}\Delta(\mathbf{I}+\mathbf{D}\Delta)^{-1}\mathbf{C})] \\ &\neq 0 \end{aligned}$$

Hence by definition $\mu(\mathbf{G}(\mathrm{j}\omega)) < \gamma$ for all $\omega \in \mathbb{R} \cup \infty$.

Conversely, suppose that $\mu(\mathbf{G}(\mathrm{j}\omega)) < \gamma$ for all $\omega \in \mathbb{R} \cup \infty$ and assume that

$$\mathbf{G}(s) \sim \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$$

is minimal. Then, by definition, $\det(\mathbf{I}+\mathbf{G}(\infty)\Delta) = \det(\mathbf{I}+\mathbf{D}\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}_\gamma$. Now, suppose that there exists $\Delta \in \mathbf{\Delta}_\gamma$ such that $(\mathbf{I}+\mathbf{G}(s)\Delta)^{-1}\mathbf{G}(s)$ is not asymptotically stable and hence $\mathbf{A}-\mathbf{B}\Delta(\mathbf{I}+\mathbf{D}\Delta)^{-1}\mathbf{C}$ is not Hurwitz. Since $\mathbf{G}(s)$ is assumed to be asymptotically stable, it follows that $\mathbf{A}$ is Hurwitz and thus there exists $\varepsilon \in (0, 1)$ such that $\mathbf{A}-\varepsilon\mathbf{B}\Delta(\mathbf{I}+\varepsilon\mathbf{D}\Delta)^{-1}\mathbf{C}$ has an imaginary eigenvalue $\mathrm{j}\hat{\omega}$. Hence

$$\det[\mathbf{I}+\varepsilon\mathbf{G}(\mathrm{j}\hat{\omega})\Delta] = \det(\mathbf{I}+\varepsilon\mathbf{D}\Delta)\det(\mathrm{j}\hat{\omega}\mathbf{I}-\mathbf{A})^{-1}\det[\mathrm{j}\hat{\omega}\mathbf{I}-(\mathbf{A}-\varepsilon\mathbf{B}\Delta(\mathbf{I}+\varepsilon\mathbf{D}\Delta)^{-1}\mathbf{C})] = 0$$

However, since $\varepsilon\Delta \in \mathbf{\Delta}_\gamma$ it follows from the definition of $\mu(\mathbf{G}(\mathrm{j}\omega))$ that $\det[\mathbf{I}+\varepsilon\mathbf{G}(\mathrm{j}\hat{\omega})\Delta] \neq 0$, which is a contradiction. □

Next, define $\mu_{\mathrm{abs}}(\mathbf{G}(\mathrm{j}\omega))$ by

$$\mu_{\mathrm{abs}}(\mathbf{G}(\mathrm{j}\omega)) \triangleq \inf\{\gamma > 0\text{: there exists } Z(\cdot) \in \mathscr{Z} \text{ such that } \mathrm{He}[Z(\mathrm{j}\omega)(\tfrac{1}{2}\gamma\mathbf{I}+\mathbf{G}_s(\mathrm{j}\omega))] > 0\}$$

or, equivalently, using Lemma 3

$$\mu_{\mathrm{abs}}(\mathbf{G}(\mathrm{j}\omega)) = \inf\{\gamma > 0\text{: there exist } D(\cdot) \in \mathscr{D} \text{ and } N(\cdot) \in \mathscr{N} \text{ such that } \mathbf{G}^*(\mathrm{j}\omega)D\mathbf{G}(\mathrm{j}\omega)+\mathrm{j}\gamma(N\mathbf{G}(\mathrm{j}\omega)-\mathbf{G}^*(\mathrm{j}\omega)N)-\gamma^2 D < 0\} \quad (14)$$

To show that $\mu_{\mathrm{abs}}(\mathbf{G}(\mathrm{j}\omega))$ is an upper bound to $\mu(\mathbf{G}(\mathrm{j}\omega))$, we require the following immediate result.

Lemma 4: *Let $\omega \in \mathbb{R} \cup \infty$. If there exists $Z(\cdot) \in \mathscr{Z}$ such that*

$$\mathrm{He}[Z(\mathrm{j}\omega)(\tfrac{1}{2}\gamma\mathbf{I}+\mathbf{G}_s(\mathrm{j}\omega))] > 0 \quad (15)$$

then $\gamma \geqslant \mu_{\mathrm{abs}}(\mathbf{G}(\mathrm{j}\omega))$ and $\det(\mathbf{I}+\mathbf{G}(\mathrm{j}\omega)\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}_\gamma$. Conversely, if $\gamma > \mu_{\mathrm{abs}}(\mathbf{G}(\mathrm{j}\omega))$ then there exists $Z(\cdot) \in \mathscr{Z}$ such that (15) *holds and* $\det(\mathbf{I}+\mathbf{G}(\mathrm{j}\omega)\Delta) \neq 0$ *for all $\Delta \in \mathbf{\Delta}_\gamma$.*

Proof: Suppose that there exists $Z(\cdot)\in\mathscr{Z}$ such that (15) holds. Since $\mu_{\text{abs}}(\mathbf{G}(j\omega))$ is the infimum over all γ such that there exists $Z(\cdot)\in\mathscr{Z}$ and (15) holds, it follows that $\gamma \geqslant \mu_{\text{abs}}(\mathbf{G}(j\omega))$. Conversely, suppose that $\gamma > \mu_{\text{abs}}(\mathbf{G}(j\omega))$. Then there exists $\hat{\gamma}$ satisfying $\mu_{\text{abs}}(\mathbf{G}(j\omega)) < \hat{\gamma} \leqslant \gamma$ and $Z(\cdot)\in\mathscr{Z}$ such that $\operatorname{He}[Z(j\omega)(\frac{1}{2}\hat{\gamma}\mathbf{I}+\mathbf{G}_s(j\omega))] > 0$. Now, using the fact that $D(j\omega) = \operatorname{He} Z(j\omega) \geqslant 0$, it follows that

$$\operatorname{He}[Z(j\omega)(\tfrac{1}{2}\gamma\mathbf{I}+\mathbf{G}_s(j\omega))] = \tfrac{1}{2}(\gamma-\hat{\gamma})\operatorname{He} Z(j\omega) + \operatorname{He}[Z(j\omega)(\tfrac{1}{2}\hat{\gamma}\mathbf{I}+\mathbf{G}_s(j\omega))] > 0$$

Finally, applying Lemma 2 with $\mathbf{M}^{-1} = \frac{1}{2}\gamma\mathbf{I}$ and $\mathbf{\Delta} = \mathbf{\Delta}_\gamma$, it follows that $\det(\mathbf{I}+\mathbf{G}(j\omega)\Delta) \neq 0$ for all $\Delta\in\mathbf{\Delta}_\gamma$. □

Theorem 3: *Let $\omega\in\mathbb{R}\cup\infty$ and let $\mathbf{G}(j\omega)$ be a complex matrix. Then*

$$\mu(\mathbf{G}(j\omega)) \leqslant \mu_{\text{abs}}(\mathbf{G}(j\omega)) \tag{16}$$

Proof: Suppose $\mu_{\text{abs}}(\mathbf{G}(j\omega)) < \mu(\mathbf{G}(j\omega))$ and let $\gamma > 0$ satisfy $\mu_{\text{abs}}(\mathbf{G}(j\omega)) < \gamma \leqslant \mu(\mathbf{G}(j\omega))$. Then, from the definition of $\mu(\mathbf{G}(j\omega))$ it follows that $\min_{\Delta\in\mathbf{\Delta}_{\text{bs}}}\{\sigma_{\max}(\Delta)\colon \det(\mathbf{I}+\mathbf{G}(j\omega)\Delta)=0\} \leqslant \gamma^{-1}$. It thus follows that there exists $\Delta\in\mathbf{\Delta}_\gamma$ such that $\det(\mathbf{I}+\mathbf{G}(j\omega)\Delta)=0$. However, using Lemma 4 we know that $\mu_{\text{abs}}(\mathbf{G}(j\omega)) < \gamma$, then $\det(\mathbf{I}+\mathbf{G}(j\omega)\Delta) \neq 0$ for all $\Delta\in\mathbf{\Delta}_\gamma$, which is a contradiction. Hence $\mu(\mathbf{G}(j\omega)) \leqslant \mu_{\text{abs}}(\mathbf{G}(j\omega))$. □

Next, in order to provide a systematic comparison of mixed-μ bounds for a fixed, internally block-structured uncertainty set $\mathbf{\Delta}_{\text{bs}}$ define $\mu_i(\mathbf{G}(j\omega))$ by

$$\mu_i(\mathbf{G}(j\omega)) \triangleq \inf\{\gamma > 0\colon \text{there exist } D(\cdot)\in\mathscr{D}_i \text{ and } N(\cdot)\in\mathscr{N}_i \text{ such that}$$
$$\mathbf{G}^*(j\omega)\,D\mathbf{G}(j\omega) + j\gamma(N\mathbf{G}(j\omega)-\mathbf{G}^*(j\omega)\,N) - \gamma^2 D < 0\} \tag{17}$$

where $\mathscr{D}_i$ and $\mathscr{N}_i$ correspond to the pairs of frequency-dependent scaling matrix functions tailored to a fixed uncertainty structure $\mathbf{\Delta}_{\text{bs}}$. The following result is immediate.

Lemma 5: *Let $\mathbf{G}(j\omega)$ be a complex matrix and let $\mathscr{D}_i, \mathscr{N}_i$ and $\mathscr{D}_j, \mathscr{N}_j$ be frequency-dependent scaling matrix sets associated with a fixed uncertainty structure $\mathbf{\Delta}_{\text{bs}}$. Suppose $\mathscr{D}_j \subseteq \mathscr{D}_i$ and $\mathscr{N}_j \subseteq \mathscr{N}_i$. Then*

$$\mu_i(\mathbf{G}(j\omega)) \leqslant \mu_j(\mathbf{G}(j\omega)) \tag{18}$$

Proof: Suppose that $\mu_j(\mathbf{G}(j\omega)) < \mu_i(\mathbf{G}(j\omega))$ and let some $\gamma > 0$ satisfy $\mu_j(\mathbf{G}(j\omega)) < \gamma \leqslant \mu_i(\mathbf{G}(j\omega))$. Then by definition of $\mu_j(\mathbf{G}(j\omega))$ there exist $D(\cdot)\in\mathscr{D}_j$ and $N(\cdot)\in\mathscr{N}_j$ such that $\mathbf{G}^*(j\omega)\,D\mathbf{G}(j\omega)+j\gamma(N\mathbf{G}(j\omega)-\mathbf{G}^*(j\omega)\,N)-\gamma^2 D < 0$. Now, since $\mathscr{D}_j \subseteq \mathscr{D}_i$ and $\mathscr{N}_j \subseteq \mathscr{N}_i$, it follows that $D(\cdot)\in\mathscr{D}_i$ and $N(\cdot)\in\mathscr{N}_i$ and hence $\mu_i(\mathbf{G}(j\omega)) < \gamma$ which is a contradiction. Hence, $\mu_i(\mathbf{G}(j\omega)) \leqslant \mu_j(\mathbf{G}(j\omega))$. □

Remark 2: Since $\mu_{\text{abs}}(\mathbf{G}(j\omega))$ given by (15) has the same form as the standard mixed-μ bound (Fan *et al.* 1991) it is quasiconvex and hence can be set up as a generalized eigenvalue problem which can be solved using standard LMI techniques (Boyd *et al.* 1994, Gahinet and Nemirovskii 1993). Furthermore, in the complex uncertainty case, i.e. $r = 0$, we take $D > 0$ and $N = 0$ so that the complex-μ upper bound (Doyle 1982) given by

$$\mu_{\rm abs}(\mathbf{G}(\mathrm{j}\omega)) = \inf\{\gamma > 0\colon \text{there exist } D(\cdot)\in\mathscr{D} \text{ such that } \mathbf{G}^*(\mathrm{j}\omega)D\mathbf{G}(\mathrm{j}\omega)-\gamma^2 D < 0\}$$

or, equivalently,

$$\mu_{\rm abs}(\mathbf{G}(\mathrm{j}\omega)) = \inf_{D(\cdot)\in\mathscr{D}} \sigma_{\max}(D^{1/2}\mathbf{G}(\mathrm{j}\omega)\,D^{-1/2})$$

is recovered. In this case the upper bound can be computed via a convex optimization problem (Sezginer and Overton 1990).

5. Connections to classical absolute stability criteria

In this section we specialize the absolute stability criterion given in Theorem 1 to several well known versions of absolute stability criteria from the classical literature. The stability criterion of Theorem 1 with particular choices of the stability multiplier $Z(s)$ forms the basis of classical absolute stability theory of feedback systems involving a linear time-invariant system and a memoryless (possibly time-varying) nonlinearity. Within the context of nonlinear absolute stability theory the multiplier $Z(s)$ in (9) is crucial in distinguishing the class of allowable feedback nonlinearities.

As is well known, constructing general positive real stability multiplier parametrizations effectively place less restrictive conditions on the linear (nominal) part of the system, and as a result the absolute stability criterion guarantees stability for a refined class of feedback nonlinearities. For example, the case in which $Z(s) = \mathbf{I}$ in Theorem 1 corresponds to the classical circle criterion which applies to arbitrarily time-varying nonlinearities, while for $Z(s) = \mathbf{I} + Ns$, where N is a non-negative-definite diagonal matrix, Theorem 1 yields the multivariable Popov criterion which applies to time-invariant sector-bounded nonlinearities. Classical absolute stability results extend the positive real Popov stability multiplier for sector-bounded time-invariant feedback nonlinearities to more refined classes of nonlinearities including monotonic, odd monotonic, and locally slope-restricted nonlinear functions. Specifically, suitable positive real stability multipliers are constructed as driving-point impedances of passive electrical networks involving resistor–inductor (RL), resistor–capacitor (RC), and inductor–capacitor (LC) combinations which exhibit interlacing pole-zero patterns on the negative real axis and imaginary axis (Brockett and Willems 1965, Haddad *et al.* 1994, Narendra and Neuman 1966, Narendra and Taylor 1978, Thathachar and Srinath 1967, Thathachar *et al.* 1967, Zames and Falb 1968). For the case of linear uncertainty, Theorem 1 implies that the multiplier can be taken as a generalized positive real (but otherwise arbitrary) transfer function.

Next, we provide explicit comparisons and multivariable generalizations of some of the key stability multipliers addressed in the literature along with their ramifications for the problem of robust stability with real parameter uncertainty. In order to address the above problems we constrain $\mathbf{\Delta}_{\rm bs}$ to the set of real diagonal uncertainties so that

$$\mathbf{\Delta}_{\rm bs} = \{\Delta\in\mathbb{R}^{m\times m}\colon \Delta = \mathrm{diag}(\delta_1,\delta_2,\ldots,\delta_m), \delta_i\in\mathbb{R}, i = 1,\ldots,m\}$$

Next, we define several special cases for the frequency-dependent scaling matrix functions in $\mathscr{D}$ and $\mathscr{N}$ corresponding to small gain ($\mathscr{D}_{\rm sg}$, $\mathscr{N}_{\rm sg}$), Popov ($\mathscr{D}_{\rm P}$, $\mathscr{N}_{\rm P}$) (Haddad *et al.* 1994, Haddad and Bernstein 1995 a), monotonic ($\mathscr{D}_{\rm RC}$, $\mathscr{N}_{\rm RC}$) (Hadded *et al.* 1994, Narendra and Neuman 1966, Narendra and Taylor 1978), odd-monotonic ($\mathscr{D}_{\rm RLC}$,

$\mathcal{N}_{\mathrm{RLC}}$) (Haddad *et al.* 1994, Narendra and Neuman 1966, Narendra and Taylor 1978), generalized odd-monotonic ($\mathcal{D}_{\mathrm{GRLC}}, \mathcal{N}_{\mathrm{GRLC}}$) (Thathachar and Srinath 1967, Thathachar *et al.* 1967), and LC multipliers ($\mathcal{D}_{\mathrm{LC}}, \mathcal{N}_{\mathrm{LC}}$) (Brockett and Willems 1965). Specifically

$$\mathcal{D}_{\mathrm{sg}} \triangleq \{D \in \mathbb{R}^{m\times m}: D = \mathbf{I}\}$$

$$\mathcal{N}_{\mathrm{sg}} \triangleq \{N \in \mathbb{R}^{m\times m}: N = 0\}$$

$$\mathcal{D}_{\mathrm{P}} \triangleq \{D \in \mathbb{R}^{m\times m}: D = \mathrm{diag}(\alpha_{i0}), 0 < \alpha_{i0} \in \mathbb{R}, i = 1, \ldots, m\}$$

$$\mathcal{N}_{\mathrm{P}} \triangleq \{N: \mathbb{R} \to \mathbb{R}^{m\times m}: N(\omega) = \mathrm{diag}(-\omega\beta_{i0}), \beta_{i0} \in \mathbb{R}, i = 1, \ldots, m\}$$

$$\mathcal{D}_{\mathrm{RC}} \triangleq \Bigg\{D: \mathbb{R} \to \mathbb{R}^{m\times m}: D(\omega) = D_{\mathrm{P}} + \mathrm{diag}\left(\sum_{j=1}^{m_{i_1}} \alpha_{ij}\left(1 - \frac{\alpha_{ij}\eta_{ij}}{\beta_{ij}(\omega^2+\eta_{ij}^2)}\right)\right)$$
$$0 \leqslant \alpha_{ij}, \beta_{ij}, \eta_{ij} \in \mathbb{R}, \eta_{ij}\beta_{ij} - \alpha_{ij} \leqslant 0, i = 1, \ldots, m, j = 1, \ldots, m_{i_1}, D_{\mathrm{P}} \in \mathcal{D}_{\mathrm{P}}\Bigg\}$$

$$\mathcal{N}_{\mathrm{RC}} \triangleq \Bigg\{N: \mathbb{R} \to \mathbb{R}^{m\times m}: N(\omega) = N_{\mathrm{P}}(\omega) + \mathrm{diag}\left(\sum_{j=1}^{m_{i_1}} - \frac{\alpha_{ij}^2\omega}{\beta_{ij}(\omega^2+\eta_{ij}^2)}\right)$$
$$0 \leqslant \alpha_{ij}, \beta_{ij}, \eta_{ij} \in \mathbb{R}, \eta_{ij}\beta_{ij} - \alpha_{ij} \leqslant 0, i = 1, \ldots, m; j = 1, \ldots, m_{i_1}, N_{\mathrm{P}}(\cdot) \in \mathcal{N}_{\mathrm{P}}\Bigg\}$$

$$\mathcal{D}_{\mathrm{RLC}} \triangleq \Bigg\{D: \mathbb{R} \to \mathbb{R}^{m\times m}: D(\omega) = D_{\mathrm{RC}}(\omega) + \mathrm{diag}\left(\sum_{j=m_{i_1}+1}^{m_{i_2}} \alpha_{ij}\left(1 + \frac{\alpha_{ij}\eta_{ij}}{\beta_{ij}(\omega^2+\eta_{ij}^2)}\right)\right)$$
$$0 \leqslant \alpha_{ij}, \beta_{ij}, \eta_{ij} \in \mathbb{R}, i = 1, \ldots, m; j = m_{i_1}+1, \ldots, m_{i_2}, D_{\mathrm{RC}}(\cdot) \in \mathcal{D}_{\mathrm{RC}}\Bigg\}$$

$$\mathcal{N}_{\mathrm{RLC}} \triangleq \Bigg\{N: \mathbb{R} \to \mathbb{R}^{m\times m}: N(\omega) = N_{\mathrm{RC}}(\omega) + \mathrm{diag}\left(\sum_{j=m_{i_1}+1}^{m_{i_2}} \frac{\alpha_{ij}^2\omega}{\beta_{ij}(\omega^2+\eta_{ij}^2)}\right)$$
$$0 \leqslant \alpha_{ij}, \beta_{ij}, \eta_{ij} \in \mathbb{R}, i = 1, \ldots, m; j = m_{i_1}+1, \ldots, m_{i_2}, N_{\mathrm{RC}}(\cdot) \in \mathcal{N}_{\mathrm{RC}}\Bigg\}$$

$$\mathcal{D}_{\mathrm{GRLC}} \triangleq \Bigg\{D: \mathbb{R} \to \mathbb{R}^{m\times m}: D(\omega) = \mathrm{diag}\left(\sum_{j=m_{i_2}+1}^{m_{i_3}} \alpha_{ij}\frac{\omega^4 + \omega^2(a_{ij}\lambda_{ij} - b_{ij} - \eta_{ij}) + b_{ij}\eta_{ij}}{(\eta_{ij}-\omega^2)^2 + \lambda_{ij}^2\omega^2}\right)$$
$$+ D_{\mathrm{RLC}}(\omega), a_{ij} \in \mathbb{R}, 0 \leqslant \alpha_{ij}, \beta_{ij}, \eta_{ij} \in \mathbb{R}, a_{ij}\lambda_{ij} - b_{ij} - \eta_{ij} \geqslant 0$$
$$i = 1, \ldots, m, j = m_{i_2}+1, \ldots, m_{i_3}, D_{\mathrm{RLC}}(\cdot) \in \mathcal{D}_{\mathrm{RLC}}\Bigg\}$$

$$\mathcal{N}_{\mathrm{GRLC}} \triangleq \Bigg\{N: \mathbb{R} \to \mathbb{R}^{m\times m}: N(\omega) = \mathrm{diag}\left(\sum_{j=m_{i_2}+1}^{m_{i_3}} -\omega\alpha_{ij}\frac{\omega^2(a_{ij}-\lambda_{ij}) + (a_{ij}\eta_{ij} - \lambda_{ij}b_{ij})}{(\eta_{ij}-\omega^2)^2 + \lambda_{ij}^2\omega^2}\right)$$
$$+ N_{\mathrm{RLC}}(\omega), a_{ij} \in \mathbb{R}, 0 \leqslant \alpha_{ij}, \beta_{ij}, \eta_{ij} \in \mathbb{R}, a_{ij}\lambda_{ij} - b_{ij} - \eta_{ij} \geqslant 0$$
$$i = 1, \ldots, m, j = m_{i_2}+1, \ldots, m_{i_3}, N_{\mathrm{RLC}}(\cdot) \in \mathcal{N}_{\mathrm{RLC}}\Bigg\}$$

$$\mathcal{D}_{\mathrm{LC}} \triangleq \{D \in \mathbb{R}^{m\times m}: D = 0\}$$

$$\mathcal{N}_{\mathrm{LC}} \triangleq \left\{N: \mathbb{R} \to \mathbb{R}^{m\times m}: N(\omega) = \mathrm{diag}\left(\mp\,\omega^{\pm 1}\frac{\Pi(\alpha_i^2-\omega^2)}{\Pi(\beta_i^2-\omega^2)}\right)\alpha_i, \beta_i \in \mathbb{R}, \beta_i \neq 0, i = 1, \ldots, m\right\}$$

Now, applying Theorem 1 with pairs of frequency-dependent scaling functions D and N from the above pairs of $\mathcal{D}$ and $\mathcal{N}$ yields multivariable generalizations of the corresponding absolute stability criteria for linear matrix uncertainty. Since, in this case, the frequency-dependent scaling functions D and N and the uncertainty Δ are real and diagonal the compatibility assumptions discussed in §2 are automatically satisfied. It is important to note, however, that since Theorem 1 does not require that the frequency-dependent scaling matrices in $\mathscr{D}$ and $\mathscr{N}$ be diagonal, the absolute stability criteria corresponding to the above D and N frequency-dependent scaling functions can easily be generalized to address non-diagonal linear matrix real uncertainties with appropriate compatibility assumptions. For example, in the non-diagonal multivariable Popov case, $Z(s) = H + Ns$, where H is a positive-definite matrix and N is a symmetric matrix. In this case $\mathscr{D}_{\mathrm{P}}$ and $\mathscr{N}_{\mathrm{P}}$ become

$$\mathscr{D}_{\mathrm{P}} = \{H \in \mathbb{R}^{m\times m}: H > 0, H\Delta = \Delta H, \Delta \in \mathbf{\Delta}_{\mathrm{bs}}\}$$
$$\mathscr{N}_{\mathrm{P}} = \{-\omega N: N \in \mathbb{R}^{m\times m}, N = N^{\mathrm{T}}, N\Delta = \Delta^{\mathrm{T}}N, \omega \in \mathbb{R}, \Delta \in \mathbf{\Delta}_{\mathrm{bs}}\}$$

which, with $H = \mathbf{I}$, are precisely the conditions used by Haddad and Bernstein (1991 a, 1995 a) to derive a multivariable generalization of the Popov criterion for linear matrix uncertainty using a parameter-dependent Lyapunov function approach.

Next, in order to provide a systematic comparison of the real-μ bounds resulting from the above absolute stability criteria, recall the definition of $\mu_i(\mathbf{G}(\mathrm{j}\omega))$ given by (17) along with Lemma 5. Here, i corresponds to μ bounds predicated on fixed frequency-dependent scaling functions $\mathscr{D}_i$ and $\mathscr{N}_i$. For example, $i = \mathrm{P}$ corresponds to the Popov frequency-dependent scaling functions so that

$$\mu_{\mathrm{P}}(\mathbf{G}(\mathrm{j}\omega)) \triangleq \inf\{\gamma > 0: \text{there exist } D(\cdot) \in \mathscr{D}_{\mathrm{P}} \text{ and } N(\cdot) \in \mathscr{N}_{\mathrm{P}} \text{ such that}$$
$$\mathbf{G}^*(\mathrm{j}\omega) D\mathbf{G}(\mathrm{j}\omega) + \mathrm{j}\gamma(N\mathbf{G}(\mathrm{j}\omega) - \mathbf{G}^*(\mathrm{j}\omega) N) - \gamma^2 D < 0\}$$

With the above pairs of frequency-dependent scaling functions, the following theorem is immediate.

Theorem 4: *Let* $\mathbf{G}(\mathrm{j}\omega)$ *be a complex matrix. Then*

$$\mu(\mathbf{G}(\mathrm{j}\omega)) \leqslant \mu_{\mathrm{abs}}(\mathbf{G}(\mathrm{j}\omega)) \leqslant \mu_{\mathrm{GRLC}}(\mathbf{G}(\mathrm{j}\omega)) \leqslant \mu_{\mathrm{RLC}}(\mathbf{G}(\mathrm{j}\omega))$$
$$\leqslant \mu_{\mathrm{RC}}(\mathbf{G}(\mathrm{j}\omega)) \leqslant \mu_{\mathrm{P}}(\mathbf{G}(\mathrm{j}\omega)) \leqslant \mu_{\mathrm{sg}}(\mathbf{G}(\mathrm{j}\omega))$$

Proof: It need only be noted that $\mathscr{D}_{\mathrm{sg}} \subset \mathscr{D}_{\mathrm{P}} \subset \mathscr{D}_{\mathrm{RC}} \subset \mathscr{D}_{\mathrm{RLC}} \subset \mathscr{D}_{\mathrm{GRLC}} \subset \mathscr{D}$ and $\mathscr{N}_{\mathrm{sg}} \subset \mathscr{N}_{\mathrm{P}} \subset \mathscr{N}_{\mathrm{RC}} \subset \mathscr{N}_{\mathrm{RLC}} \subset \mathscr{N}_{\mathrm{GRLC}} \subset \mathscr{N}$. The result now is an immediate consequence of Lemma 5. □

Similarly, for $\mathbf{G}(\mathrm{j}\omega) \in \mathbb{C}^{m\times m}, \mu(\mathbf{G}(\mathrm{j}\omega)) \leqslant \mu_{\mathrm{abs}}(\mathbf{G}(\mathrm{j}\omega)) \leqslant \mu_{\mathrm{LC}}(\mathbf{G}(\mathrm{j}\omega))$ follows from Lemma 5 by noting that $\mathscr{D}_{\mathrm{LC}} \subset \mathscr{D}$ and $\mathscr{N}_{\mathrm{LC}} \subset \mathscr{N}$.

Remark 3: The benefit of considering successively more general stability multipliers is that they provide additional degrees of freedom in constructing more general parametrizations of D, N-scales in mixed-μ theory. In this case, as surmised from Theorem 4, the extra degrees of freedom in the D, N-scales allows for more rapid phase variations in the frequency domain test which effectively places fewer restrictive conditions on the nominal part of the system and more restrictive conditions on the uncertainty resulting in successively tighter bounds for real parameter uncertainty. However, as noted by Haddad and Bernstein (1995a), Haddad *et al.* (1994) and Safonov and Chiang (1993), by *a priori* fixing the structure of the dynamic stability multipliers or, equivalently, the D, N-scales, within the controller synthesis framework, eliminates the need of both iterating between controller design and optimal multiplier evaluation and curve fitting procedures.

6. Illustrative numerical examples

In this section we consider two examples to demonstrate the utility of the proposed generalized μ-bound accounting for uncertainty with internal matrix structure. As shown in §4 this bound can be written as a generalized eigenvalue problem involving a linear matrix inequality. For the following computations we utilize the LMI Lab (Gahinet and Nemirovskii 1993) and for notational convenience we denote the μ-bound given by Fan *et al.* (1991) by $\mu_{\mathrm{FTD}}(\mathbf{G}(\mathrm{j}\omega))$ and the μ-bound enforcing internal matrix structure in the uncertainty by $\mu_{\mathrm{abs}}(\mathbf{G}(\mathrm{j}\omega))$.

Example 1: Our first example considers real and complex uncertainty with internal matrix structure. Specifically, let

$$\mathbf{G}(s) \sim \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & 0 \end{array}\right]$$

where

$$\mathbf{A} = \begin{bmatrix} 0\cdot0 & 0\cdot0 & 1\cdot0 & 0\cdot0 \\ 0\cdot0 & 0\cdot0 & 0\cdot0 & 1\cdot0 \\ -13\cdot0 & 10\cdot0 & -0\cdot8 & 0\cdot6 \\ 10\cdot0 & -10\cdot0 & 0\cdot6 & -0\cdot6 \end{bmatrix}, \quad \mathbf{B} = \mathbf{C} = \mathbf{I}$$

$$\boldsymbol{\Delta}_{\mathrm{bs}} = \{\Delta : \Delta = \text{block-diag}\,(\delta_1, \delta_2, \Delta_1), \delta_1, \delta_2 \in \mathbb{R}, \Delta_1 = \Delta_1^* \in \mathbb{C}^{2\times2}\}$$

In order to compute $\mu_{\mathrm{FTD}}(\mathbf{G}(\mathrm{j}\omega))$ we require

$$\mathscr{D} = \{D : \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{R}^{4\times4} : D = \text{block-diag}\,(d_1, d_2, d_3\mathbf{I}_2), d_1, d_2, d_3 : \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{R}\} \quad (19)$$

$$\mathscr{N} = \{N : \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{R}^{4\times4} : N = \text{block-diag}\,(n_1, n_2, 0), n_1, n_2 : \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{R}\} \quad (20)$$

while in order to enforce the hermitian structure in the complex block we choose $\mathscr{D}$ given by (19) and

$$\mathscr{N} = \{N : \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{R}^{4\times4} : N = \text{block-diag}\,(n_1, n_2, n_3\mathbf{I}_2), n_1, n_2, n_3 : \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{R}\}$$

in the computation of $\mu_{\mathrm{abs}}(\mathbf{G}(\mathrm{j}\omega))$. Using the above frequency-dependent scaling sets $\mu_{\mathrm{FTD}}(\mathbf{G}(\mathrm{j}\omega))$ and $\mu_{\mathrm{abs}}(\mathbf{G}(\mathrm{j}\omega))$ are compared in Fig. 2.

Example 2: In this example we demonstrate the effectiveness of the proposed generalized mixed-μ bound along with the condition $N\Delta = \Delta^* N$ necessary in capturing real, internally structured full-block uncertainties. Specifically, we consider a special class of real full-block uncertainties arising in structural systems in real normal form with frequency and damping uncertainty (Haddad and Bernstein 1991 b). In this case

$$\boldsymbol{\Delta}_{\mathrm{bs}} = \left\{\Delta : \Delta = \begin{bmatrix} \delta_1 & \delta_2 \\ -\delta_2 & \delta_1 \end{bmatrix}, \quad \delta_1, \delta_2 \in \mathbb{R}\right\} \quad (21)$$

Hence, the frequency-dependent scaling sets $\mathscr{D}$ and $\mathscr{N}$ are given by

$$\mathscr{D} = \left\{D : \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{C}^{2\times2} : D(\mathrm{j}\omega) = \begin{bmatrix} d_1(\mathrm{j}\omega) & \mathrm{j}d_2(\mathrm{j}\omega) \\ -\mathrm{j}d_2(\mathrm{j}\omega) & d_1(\mathrm{j}\omega) \end{bmatrix}, \quad d_1, d_2 : \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{R}\right\} \quad (22)$$

$$\mathscr{N} = \left\{N : \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{R}^{2\times2} : N(\mathrm{j}\omega) = \begin{bmatrix} n_1(\mathrm{j}\omega) & n_2(\mathrm{j}\omega) \\ n_2(\mathrm{j}\omega) & -n_1(\mathrm{j}\omega) \end{bmatrix}, \quad n_1, n_2 : \mathrm{j}\mathbb{R} \cup \infty \to \mathbb{R}\right\} \quad (23)$$

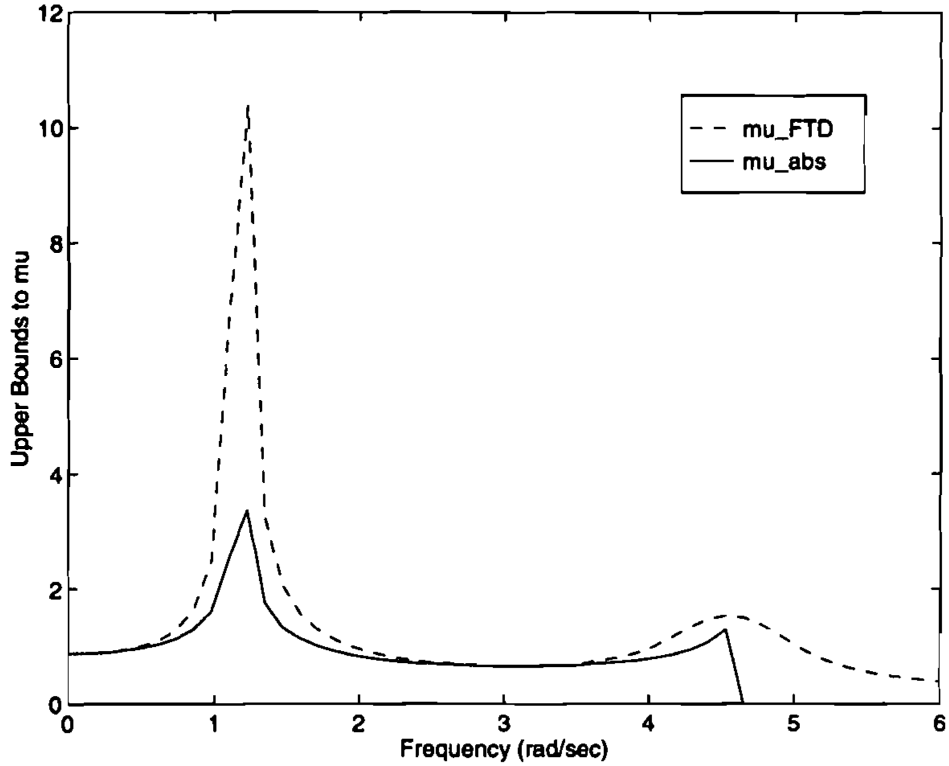

Figure 2. Upper bounds to μ: Example 1.

Now, with

$$N(\mathrm{j}\omega) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad N(\mathrm{j}\omega) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

the condition $N(\mathrm{j}\omega)\Delta = \Delta^* N(\mathrm{j}\omega)$ for all $\Delta \in \mathbb{R}^{2\times 2}$ implies that $\Delta \in \mathbf{\Delta}_{\mathrm{bs}}$ given by (21). Hence, if $\mathcal{N}$ is given by (23) then $\mathbf{\Delta}_{\mathrm{bs}}$ given by (21) is the largest uncertainty set such that $N(\mathrm{j}\omega)\Delta = \Delta^* N(\mathrm{j}\omega)$ holds.

Next, consider the control of a damped harmonic oscillator given by

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t), \quad u(t) = -\mathbf{K}y(t), \quad y(t) = \mathbf{C}x(t)$$

where

$$\mathbf{A} = \begin{bmatrix} -\nu & \omega \\ -\omega & -\nu \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \mathbf{C}^{\mathrm{T}} = [0 \quad 1]^{\mathrm{T}}$$

so that the closed-loop system is given by $\dot{x}(t) = \tilde{\mathbf{A}}x(t)$ where

$$\tilde{\mathbf{A}} = \begin{bmatrix} -\nu & \omega \\ -\omega & -\nu - \mathbf{K} \end{bmatrix}$$

Now, we assume uncertainties δ_ν and δ_ω in ν and ω, respectively, so that the actual

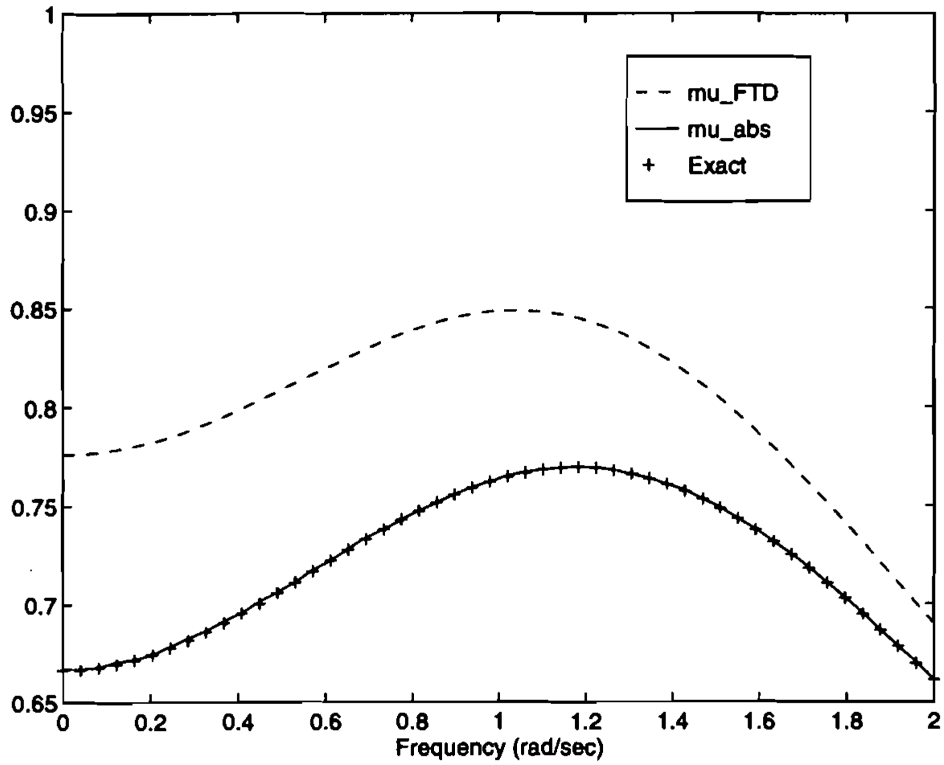

Figure 3. Exact and upper bounds to μ: Example 2.

closed-loop system is written as $\dot{x}(t) = (\tilde{\mathbf{A}}_0 + \mathbf{B}_0 \Delta \mathbf{C}_0) x(t)$ where $\tilde{\mathbf{A}}_0$ is the nominal system-matrix given by

$$\tilde{\mathbf{A}}_0 = \begin{bmatrix} -v_0 & \omega_0 \\ -\omega_0 & -v_0 - \mathbf{K} \end{bmatrix}$$

$\mathbf{B}_0 = \mathbf{C}_0 = \mathbf{I}$ and Δ is an internally structured real block perturbation given by

$$\Delta = \begin{bmatrix} -\delta_v & \delta_\omega \\ -\delta_\omega & -\delta_v \end{bmatrix}$$

In order to analyse the robust stability of this system we compute $\mu_{\text{abs}}(\mathbf{G}(j\omega))$ using $\mathscr{D}$ and $\mathscr{N}$ given above with

$$\tilde{\mathbf{G}}(s) \sim \left[\begin{array}{c|c} \tilde{\mathbf{A}}_0 & \mathbf{B}_0 \\ \hline \mathbf{C}_0 & 0 \end{array}\right]$$

$v_0 = 0{\cdot}75$, $\omega_0 = 1{\cdot}3$, and $\mathbf{K} = 1{\cdot}1$. The result is shown in Fig. 3. It is interesting to note that in this case the exact value of $\mu(\mathbf{G}(j\omega))$ can be computed, and it is equal to $\mu_{\text{abs}}(\mathbf{G}(j\omega))$ (see Fig. 3). Since Δ is a full matrix block, in order to compute $\mu_{\text{FTD}}(\mathbf{G}(j\omega))$ we require

$$\mathscr{D} = \{D : j\mathbb{R} \cup \infty \to \mathbb{R}^{2\times 2} : D = d\mathbf{I}_2, d : j\mathbb{R} \cup \infty \to \mathbb{R}\}, \quad \mathscr{N} = \{N \in \mathbb{C}^{2\times 2} : N = 0\}$$

which corresponds to the standard complex μ-bound (Doyle 1982). This result is also shown in Fig. 3.

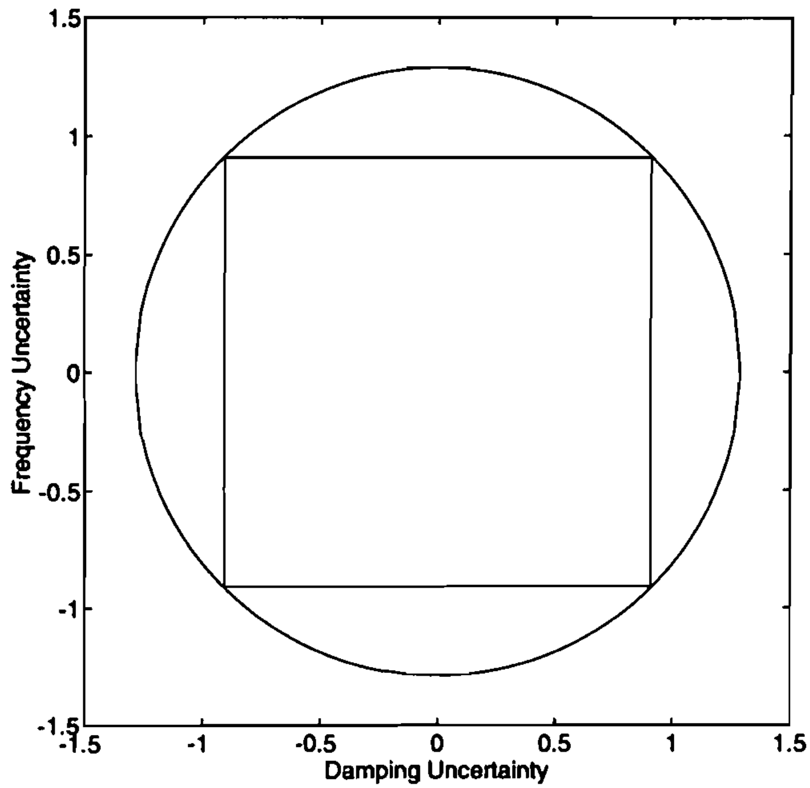

Figure 4. Allowable uncertainty regions in Example 2: circle μ_{abs}, square μ_{FTD}.

Of course, one can always represent the uncertainty as $\Delta = \text{diag}(\delta_\nu, \delta_\nu, \delta_\omega, \delta_\omega)$, by rewriting the above uncertain closed-loop system as $\dot{x}(t) = (\tilde{\mathbf{A}}_0 + \hat{\mathbf{B}}_0 \Delta \hat{\mathbf{C}}_0)\, x(t)$ where

$$\hat{\mathbf{B}}_0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \hat{\mathbf{C}}_0 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$$

and use the mixed-μ bound given by Fan *et al.* (1991). In this case, to analyse the robust stability of this system we compute $\mu_{\text{abs}}(\hat{\mathbf{G}}(j\omega)) = \mu_{\text{FTD}}(\hat{\mathbf{G}}(j\omega))$, where

$$\hat{\mathbf{G}}(s) \sim \left[\begin{array}{c|c} \tilde{\mathbf{A}}_0 & \hat{\mathbf{B}}_0 \\ \hline \hat{\mathbf{C}}_0 & 0 \end{array}\right]$$

by using the frequency-dependent scaling sets $\mathscr{D}$ and $\mathscr{N}$ given by

$$\mathscr{D} = \{D\colon D = \text{block-diag}(D_1, D_2), D_i\colon j\mathbb{R} \cup \infty \to \mathbb{R}^{2\times 2}, D_i \geqslant 0, i = 1, 2\}$$

$$\mathscr{N} = \{N\colon N = \text{block-diag}(N_1, N_2), N_i\colon j\mathbb{R} \cup \infty \to \mathbb{R}^{2\times 2}, N_i = N_i^*, i = 1, 2\}$$

However, in this case the original uncertainty characterization that defines a circular parameter space is destroyed in favour of an uncertainty region defined by a square. This is shown in Fig. 4. Furthermore, the numerical complexity is increased since the new uncertainty characterization increases the overall dimensionality of the optimization problem. For completeness we note that in this case $\mu_{\text{abs}}(\mathbf{G}(j\omega)) = 0{\cdot}7761 < \mu_{\text{abs}}(\hat{\mathbf{G}}(j\omega)) = \mu_{\text{FTD}}(\hat{\mathbf{G}}(j\omega)) = 1{\cdot}1016$.

7. Conclusion

A new absolute stability criterion has been developed that unifies and extends existing structured singular value bounds for mixed uncertainty. These bounds generalize prior upper bounds for mixed-μ by permitting the treatment of non-diagonal real uncertain blocks which may, in addition, possess internal structure. These bounds are constructed by choosing multipliers that are tailored to the structure of the uncertainty and lead to less conservative stability conditions than the standard mixed-μ upper bound. The criterion presented in this paper virtually encompasses all of the classical absolute stability criteria for linear uncertainty and allows for a systematic comparison of these results in addressing the problem of robust stability with real parameter uncertainty. The utility of the proposed generalized μ-bound was demonstrated through several examples.

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