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Parameter consistency and quadratically constrained errors-in-variables least-squares identification

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In this article, we investigate the consistency of parameter estimates obtained from least-squares identification with a quadratic parameter constraint. For generality, we consider infinite impulse-response systems with coloured input and output noise. In the case of finite data, we show that there always exists a possibly indefinite quadratic constraint depending on the noise realisation that results in a constrained optimisation problem that yields the true parameters of the system when a persistency condition is satisfied. When the noise covariance matrix is known to within a scalar multiple, we prove that solutions of the quadratically constrained least-squares (QCLs) estimator with a semidefinite constraint matrix are both unbiased and consistent in the sense that the averaged problem and limiting problem produce, respectively, unbiased and true (with probability 1) estimators. In addition, we provide numerical results that illustrate these properties of the QCLS estimator.

Keywords: errors-in-variables identification; least-squares identification; time-series model identification; consistency; unbiasedness; Koopmans-Levin method

1. Introduction

Least squares is undoubtedly one of the most widely developed, extensively applied and generally useful techniques in science and engineering (Draper and Smith 1981; Lawson and Hanson 1995; Björck 1996). In its simplest form, least squares seeks 'approximate solutions' to $Ax = b$ when this equation has no solution per se due to inherent inconsistency or due to errors in the *regressor matrix* A or the *output vector* b. More precisely, least squares seeks a vector x that minimises the Euclidean norm $\|Ax - b\|_2$, which can be viewed as seeking the smallest *residual* w in the equation $Ax = b + w$.

In a stochastic setting, where A, b or w are random variables, the solution $\hat{x}$ to the least-squares problem is also a random variable. In this case, questions of interest concern the *unbiasedness* and *consistency* of the least-squares estimator $\hat{x}$, where unbiasedness refers to the situation in which the expected value of $\hat{x}$ is the true value x_{true}, while consistency refers to the convergence, with probability 1, of $\hat{x} = \hat{x}_N – x_{\text{true}}$ as the number N of data points used to construct A and b increases without bound. Although unbiasedness and consistency are independent conditions, consistency is usually obtained by constructing an unbiased estimator whose variance converges to zero as the amount of available data increases without bound. The best linear unbiased estimator (BLUE) has been extensively studied; see, for example, Campbell and Meyer (1991, chap. 6).

In system identification, least squares is the key tool for estimating the coefficients of time-series and state-space models (Hsia 1977; Söderström and Stoica 1989; Juang 1993; van Overschee and de Moor 1996; Ljung 1999). For time-series models, a difficulty arises from the fact that the components of the residual w are past values of the process noise, while the entries of the regressor matrix A include past values of the output. Consequently, the residual is correlated with the regressor, resulting in a biased estimator $\hat{x}$. This case is not addressed by the classical theory of BLUE. Hence, time-series estimation poses difficulties that transcend the central results of classical linear estimation theory.

In time-series identification, correlation between the regressor matrix and residual is absent in two very special cases, namely, when the time-series model has an equation error with known spectrum (Söderström and Stoica 1989, pp. 66, 187; Ljung 1999, p. 205) and when the dynamics are finite impulse response. As shown in Ljung (1999, p. 205), when the time-series model has an equation error with known spectrum, the time-series model can be written using the notation of Ljung (1999) as

$$A(\mathbf{q})\, y(t) = B(\mathbf{q})u(t) + \kappa(\mathbf{q})e(t),$$

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where $\mathbf{q}$ is the backward shift operator, $A(\mathbf{q})$ and $B(\mathbf{q})$ are unknown polynomials in $\mathbf{q}$, $\kappa(\mathbf{q})$ is a known filter and $e(t)$ is white filter. Then the time-series model can be rewritten as

$$A(\mathbf{q})\, y_{\mathrm{F}}(t) = B(\mathbf{q}) u_{\mathrm{F}}(t) + e(t),$$

where $y_{\mathrm{F}}(t) = \kappa^{-1}(q) y(t)$ and $u_{\mathrm{F}}(t) = \kappa^{-1}(q) u(t)$. By pre-filtering the data to obtain $y_{\mathrm{F}}(t)$ and $u_{\mathrm{F}}(t)$, standard least-squares techniques can be applied to estimate the time-series model.

Alternatively, consider an errors-in-variables time-series model in which the input $u(t)$ and the output $y(t)$ are corrupted by coloured noise $\kappa_1(\mathbf{q})e_1(t)$ and $\kappa_2(\mathbf{q})e_2(t)$, respectively, where $e_1(t)$ and $e_2(t)$ are white and $\kappa_1(\mathbf{q})$ and $\kappa_2(\mathbf{q})$ are colouring filters. The time-series model can then be written as

$$A(q)\, y(t) = B(q)u(t) + A(q)\kappa_1(q)e_1(t) + B(q)\kappa_2(q)e_2(t).$$

In this case, it follows that the equation error is $A(q)\kappa_1(q)e_1(t) + B(q)\kappa_2(q)e_2(t)$. Now, even if the colouring filters $\kappa_1(\mathbf{q})$ and $\kappa_2(\mathbf{q})$ are known, the spectrum of the equation error remains unknown since $A(q)$ and $B(q)$ are unknown. Therefore the pre-filtering technique can no longer be applied. Further details on these cases are given in Section 5.

The issue of bias in time-series model identification is well known, and the relevant literature is extensive. An early, pivotal contribution is the Koopmans–Levin (KL) method (Levin 1964), in which knowledge of the noise statistics is used to reduce the bias in the estimates of the model coefficients. This technique hinges on the minimisation of a ratio of quadratic forms, which in turn leads to the solution of a generalised eigenvalue problem, for which the generalised eigenvector provides the parameter estimates. The KL method is further investigated in Smith and Hilton (1967), Aoki and Yue (1970), Sagara and Wada (1977), Zhdanov and Katsyuba (1979), Fernando and Nicholson (1985) and Zheng (2002b).

Since identification of a transfer function entails estimation of the coefficients of the numerator and denominator polynomials, scaling is needed to remove the inherent ambiguity due to a ratio of polynomials. The most obvious scaling is to scale both polynomials so that the denominator polynomial is monic. However, alternative constraints can be enforced. For example, a constraint on the norm of the coefficients can be used to scale the numerator and denominator coefficients (Furuta and Paquet 1970; Regalia 1994; Zhang and Feng 1995). If the norm constraint involves a quadratic form, then the resulting quadratically constrained least squares (QCLS) estimation problem can be solved using Lagrange multipliers, as pursued in Lemmerling and de Moor (2001), van Pelt and Bernstein (2001) and Yeredor and de Moor (2004). The resulting optimality conditions have the form of a generalised eigenvalue problem, in which the generalised eigenvector is an estimate of the transfer function coefficients.

The above discussion suggests that the KL method can be viewed as the consequence of a particular QCLS optimisation problem. A careful examination of the literature on unbiasedness and consistency shows that this linkage has been largely overlooked, resulting in a fragmented collection of approaches to this problem. Related techniques for removing the bias in least-squares identification include James, Souter, and Dixon (1972), Akashi (1975), Merhav (1975), Söderström (1981), Zheng and Feng (1989), de Moor, Gevers, and Goodwin (1994), Zheng (1998, 1999, 2002a) and Guo and Billings (2007). Finally, the instrumental variable method (Wong and Polak 1967; Söderström and Stoica 1981; Stoica and Soderstrom 1982) provides yet another approach to this problem. Recent works, such as Lemmerling and de Moor (2001), Söderström, Soverini, and Mahata (2002), Zheng (2002a), Yeredor and de Moor (2004), Chen (2007), Hong, Söderström, and Zheng (2007), Mahata (2007), Söderström (2007) and Agüero and Goodwin (2008) suggest continued interest in unbiased and consistent estimators.

Errors-in-variables identification problems can be classified as either *functional* or *structural* (Agüero and Goodwin 2008). When the true input to the system is an arbitrary deterministic signal, the identification problem is called a functional problem, whereas a structural problem arises when the true input to the system is a stochastic, usually white noise, signal. For the structural case, results on consistency and unbiasedness are available (see e.g. Söderström et al. 2002; Mahata 2007), but few proofs of consistency are available for the functional case. Moreover, in the structural case, the noise covariance can be estimated from the input data; however, in the functional case it is typically assumed that the noise covariance matrix is known. For instance, for the functional problem, Hong et al. (2007) show that the estimates are asymptotically Gaussian distributed (unbiased but not consistent), when the covariance matrix is known to within two unknown parameters. Additionally, Kukush, Markovsky, and Huffel (2005) show that when the covariance matrix is known to within a scalar multiple and the data matrices satisfy certain structural constraints, structured total least squares provides consistent estimates. A detailed discussion on all of these cases is provided in Söderström (2007).

In this article, we focus on the functional case, and we assume that the autocovariance matrix of the noise is known to within a scalar multiple. Moreover, the algorithm derived in this article is closely related to the KL method, bias-compensation methods and the Frisch scheme (Koopmans 1937; Levin 1964; Aoki and Yue 1970; Fernando and Nicholson 1985; Zheng 1998, 2002a; Hong, Söderström, and Zheng 2006; Mahata 2007; Söderström 2007). However, the novel features of this article include an alternative formulation of the above-mentioned methods, and, more importantly, this alternative formulation allows us to derive unbiasedness and consistency results for the functional case.

The first objective of this article is accomplished by the above discussion, that is, to point out that the KL method can be viewed as a special case of QCLS estimation. With this realisation in mind, our objective is then to characterise the solution set of an appropriate quadratically constrained quadratic programming problem. Because of space constraints, the details of this development are provided in Palanthandalam-Madapusi, van Pelt, and Bernstein (2009), with only the key results quoted for use herein. We note, however, that, unlike earlier work based on Lagrange multipliers and necessary conditions, we provide necessary and sufficient conditions for the existence of solutions along with a complete characterisation of the solution set directly in terms of the properties of matrix pencils.

Next, we consider an errors-in-variables SISO ARMAX identification problem with minimal assumptions on the structure of the model and the nature of the noise. For this problem, we show that there exists a possibly indefinite quadratic constraint on the system coefficients such that the resulting solution of the optimisation problem yields the true coefficients. This analysis, as well as all persistency conditions, is entirely deterministic.

The existence of a quadratic constraint that leads to the true parameter values provides a framework for understanding the success of the KL algorithm. However, the constraint matrix in the deterministic formulation depends on the sensor errors (noise) and thus is unknowable. We therefore assume the noise signal to be stochastic and ergodic, and replace the indefinite QCLS constraint with a positive-semidefinite constraint based on the noise covariance. Under the assumption that this noise covariance is known to within a multiplicative constant, we prove that solutions of this modified QCLS problem are both unbiased and consistent in the sense that the averaged problem and limiting problem produce, respectively, unbiased and true (with probability 1) estimators. In addition, we provide numerical results that suggest that the QCLS estimator is unbiased and consistent in the standard sense.

The main objective of this article is to provide the missing foundation for the KL method and its numerous variants in the literature, while providing a complete development of unbiasedness and consistency in a precise sense. The essential idea of estimating the transfer function coefficients by means of constrained optimisation is given an intuitively compelling foundation, thus linking, clarifying and extending the prior literature.

1.1 *Notation*

$\mathbf{q}^{-1}$ is the backward shift operator, 0_n is the $n \times n$ zero matrix, I_n is the $n \times n$ identity matrix, $\mathbb{R}^n$ is the set of real $n \times 1$ column vectors and $\mathbb{R}^{n \times m}$ is the set of real $n \times m$ matrices. For $A \in \mathbb{R}^{n \times m}$, rank A is the rank of A, $\mathcal{R}(A)$ is the range of A, $\mathcal{N}(A)$ is the null space of A, def A is the defect (nullity) of A, $\lambda_{\max}(A)$ is the largest eigenvalue of A, $\lambda_{\min}(A)$ is the smallest eigenvalue of A and A^+ is the Moore–Penrose generalised inverse of A. Furthermore, diag(a_i) is the diagonal matrix with diagonal entries a_i. For symmetric matrices A and B, $A > B$ means that $A - B$ is positive definite, and $A \geq B$ means that $A - B$ is positive semidefinite.

2. Null-space condition

Consider the single-input, single-output system

$$a_0 y_0(k) + a_1 y_0(k-1) + \cdots + a_n y_0(k-n) = b_0 u_0(k) + b_1 u_0(k-1) + \cdots + b_n u_0(k-n), \quad (1)$$

where $u_0(k)$ is the *system input* and $y_0(k)$ is the *system output*. By defining the backward shift operator $\mathbf{q}^{-1}$ and the polynomials

$$A(\mathbf{q}^{-1}; \vartheta) \triangleq a_0 + a_1 \mathbf{q}^{-1} + \cdots + a_n \mathbf{q}^{-n} \quad (2)$$

and

$$B(\mathbf{q}^{-1}; \vartheta) \triangleq b_0 + b_1 \mathbf{q}^{-1} + \cdots + b_n \mathbf{q}^{-n}, \quad (3)$$

(1) can be rewritten as

$$A(\mathbf{q}^{-1}; \vartheta)\, y_0(k) = B(\mathbf{q}^{-1}; \vartheta) u_0(k), \quad (4)$$

where the *system parameter vector* $\vartheta \in \mathbb{R}^{2n+2}$ is

$$\vartheta \triangleq \left[a_0 \cdots a_n \quad b_0 \cdots b_n \right]^{\mathrm{T}}. \quad (5)$$

We assume that the polynomials $A(\mathbf{q}^{-1}; \vartheta)$ and $B(\mathbf{q}^{-1}; \vartheta)$ are coprime, that is, (1) is a minimal representation. Furthermore, we assume that $a_0 \neq 0$, which is equivalent to the assumption that (1) is causal. Hence $\vartheta \neq 0$.

Next, defining the n-th-order proper transfer function $G(\mathbf{q}^{-1};\vartheta)$ by

$$G(\mathbf{q}^{-1};\vartheta) \triangleq \frac{B(\mathbf{q}^{-1};\vartheta)}{A(\mathbf{q}^{-1};\vartheta)}, \tag{6}$$

we have

$$y_0(k) = G(\mathbf{q}^{-1};\vartheta)u_0(k). \tag{7}$$

Remark 2.1: Note that $G(\mathbf{q}^{-1};\vartheta) = G(\mathbf{q}^{-1};\eta\vartheta)$ for all nonzero $\eta \in \mathbb{R}$. This nonuniqueness of the system parametrisation can be removed by choosing $\eta = 1/a_0$ so that the first component of $\eta\vartheta$ is unity. Although this normalisation yields a unique system parametrisation, we choose not to do this in order to facilitate the following analysis.

Next, assuming the system input $u_0(k)$ and the system output $y_0(k)$ are available for $k = 0, \ldots, l$, where $l \geq n$, we define the *noise-free regression matrix* $\Phi_0 \in \mathbb{R}^{(l-n+1)\times(2n+2)}$ by

$$\Phi_0 \triangleq \begin{bmatrix} \phi_0^{\mathrm{T}}(n) \\ \vdots \\ \phi_0^{\mathrm{T}}(l) \end{bmatrix}, \tag{8}$$

where the *noise-free regression vector* $\phi_0(k) \in \mathbb{R}^{2n+2}$ is

$$\phi_0(k) \triangleq \begin{bmatrix} y_0(k) & \cdots & y_0(k-n) & -u_0(k) & \cdots & -u_0(k-n) \end{bmatrix}^{\mathrm{T}}. \tag{9}$$

With this notation, (1) can be written as

$$\Phi_0\vartheta = 0, \tag{10}$$

which is equivalent to the *null-space condition*

$$\vartheta \in \mathcal{N}(\Phi_0). \tag{11}$$

Finally, define the positive-semidefinite matrix $M_0 \in \mathbb{R}^{(2n+2)\times(2n+2)}$ by

$$M_0 \triangleq \frac{1}{l}\Phi_0^{\mathrm{T}}\Phi_0. \tag{12}$$

Note that $\operatorname{rank} M_0 = \operatorname{rank} \Phi_0$, $\mathcal{N}(M_0) = \mathcal{N}(\Phi_0)$ and $\operatorname{def} M_0 = \operatorname{def} \Phi_0$ (Bernstein 2005). Therefore, the null-space condition (11) is equivalent to

$$\vartheta \in \mathcal{N}(M_0). \tag{13}$$

Next, partition Φ_0 as

$$\Phi_0 = \begin{bmatrix} \Phi_{01} & \Phi_{02} \end{bmatrix}, \tag{14}$$

where $\Phi_{01} \in \mathbb{R}^{l-n+1}$ and $\Phi_{02} \in \mathbb{R}^{(l-n+1)\times(2n+1)}$, and partition ϑ as

$$\vartheta = \begin{bmatrix} a_0 \\ \tilde{\vartheta} \end{bmatrix}, \tag{15}$$

where $\tilde{\vartheta} \in \mathbb{R}^{2n+1}$. Then the null-space condition (11) is equivalent to

$$a_0\Phi_{01} + \Phi_{02}\tilde{\vartheta} = 0, \tag{16}$$

which implies that $\Phi_{01} = -a_0^{-1}\Phi_{02}\tilde{\vartheta} \in \mathcal{R}(\Phi_{02}) = \mathcal{R}(\Phi_0)$, and thus $\operatorname{rank}\Phi_{02} = \operatorname{rank}\Phi_0$. Hence, $\Phi_{01} = \Phi_{02}\Phi_{02}^{+}\Phi_{01} = \Phi_0\Phi_0^{+}\Phi_{01}$ and

$$\begin{bmatrix} 1 \\ -\Phi_{02}^{+}\Phi_{01} \end{bmatrix} \in \mathcal{N}(\Phi_0). \tag{17}$$

The following result characterises $\mathcal{N}(\Phi_0)$.

Proposition 2.2: *Let* $\{u_0(k)\}_{k=0}^{l}$ *and* $\{y_0(k)\}_{k=0}^{l}$ *satisfy* (7). *Then*

$$\Phi_0^{+} = \begin{bmatrix} \beta^{-1}\Phi_{01}^{\mathrm{T}}\left(\Phi_{02}\Phi_{02}^{\mathrm{T}}\right)^{+} \\ \Phi_{02}^{+} - \beta^{-1}\Phi_{02}^{+}\Phi_{01}\Phi_{01}^{\mathrm{T}}\left(\Phi_{02}\Phi_{02}^{\mathrm{T}}\right)^{+} \end{bmatrix}, \tag{18}$$

where

$$\beta \triangleq 1 + \Phi_{01}^{\mathrm{T}}\left(\Phi_{02}\Phi_{02}^{\mathrm{T}}\right)^{+}\Phi_{01}. \tag{19}$$

Furthermore,

$$\mathcal{N}(\Phi_0) = \mathcal{R}\left(I_{2n+2} - \Phi_0^{+}\Phi_0\right) = \mathcal{R}\left(\begin{bmatrix} \beta^{-1} & -\beta^{-1}\left(\Phi_{02}^{+}\Phi_{01}\right)^{\mathrm{T}} \\ -\beta^{-1}\Phi_{02}^{+}\Phi_{01} & \left\{\begin{matrix} I_{2n+1} - \Phi_{02}^{+}\Phi_{02} \\ +\beta^{-1}\Phi_{02}^{+}\Phi_{01} \\ \times\left(\Phi_{02}^{+}\Phi_{01}\right)^{\mathrm{T}} \end{matrix}\right\} \end{bmatrix}\right). \tag{20}$$

Proof: See Bernstein (2005, Fact 6.4.26). □

Next, note that the null-space condition (11) uniquely determines ϑ up to a scalar factor if $\mathcal{N}(\Phi_0)$ is one dimensional, that is, if $\operatorname{def}\Phi_0 = 1$. Noting that $\operatorname{rank}\Phi_0 + \operatorname{def}\Phi_0 = 2n+2$, the following definition considers this case.

Definition 2.3: The input sequence $\{u_0(k)\}_{k=0}^{l}$ is *persistently exciting* for $G(\mathbf{q}^{-1};\vartheta)$ if $\operatorname{rank}\Phi_0 = 2n+1$.

Note that, given $\{u_0(k)\}_{k=1}^{\infty}$ such that $\{u_0(k)\}_{k=1}^{\hat{l}}$ is persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$, it follows that, for all $l \geq \hat{l}$, $\{u_0(k)\}_{k=1}^{l}$ is persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$.

Proposition 2.4: *Let* $\{u_0(k)\}_{k=0}^{l}$ *and* $\{y_0(k)\}_{k=0}^{l}$ *satisfy* (7). *Then the following statements are equivalent*:

(i) $\{u_0(k)\}_{k=0}^{l}$ is persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$.
(ii) $\operatorname{rank} M_0 = 2n+1$.
(iii) $\operatorname{def} M_0 = 1$.
(iv) $\operatorname{rank}\Phi_0 = 2n+1$.
(v) $\operatorname{def}\Phi_0 = 1$.
(vi) $\operatorname{rank}\Phi_{02} = 2n+1$.
(vii) $\operatorname{def}\Phi_{02} = 0$.
(viii) $\Phi_{02}\Phi_{02}^{\mathrm{T}} > 0$.

In this case, it follows that $l \geq 3n$, $\Phi_{02}^{+} = (\Phi_{02}^{\mathrm{T}}\Phi_{02})^{-1}\Phi_{02}^{\mathrm{T}}$, $\Phi_{02}^{+}\Phi_{02} = I_{2n+1}$ and

$$\mathcal{N}(\Phi_0) = \mathcal{N}(M_0) = \mathcal{R}\left(\begin{bmatrix} 1 \\ -\Phi_{02}^{+}\Phi_{01} \end{bmatrix}\right) = \mathcal{R}(\vartheta). \quad (21)$$

Note that $\mathcal{R}(\vartheta) = \{c\vartheta : c \in \mathbb{R}\}$.

Suppose that $\{u_0(k)\}_{k=0}^{l}$ is persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$ so that $\mathcal{N}(\Phi_0) = \mathcal{R}(\vartheta)$. Then $G(\mathbf{q}^{-1};\vartheta) = G(\mathbf{q}^{-1};\theta)$ for all $\theta \in \mathcal{N}(\Phi_0)\setminus\{0\} = \mathcal{R}(\vartheta)\setminus\{0\}$, and $G(\mathbf{q}^{-1};\vartheta)$ is determined uniquely up to an arbitrary scaling by the null-space condition (11), which is characterised by (21).

3. Errors-in-variables formulation

Next, we assume that the system output $y_0(k)$ is corrupted by *output noise* $w(k)$ so that the *measured output* $y(k)$ is given by

$$y(k) = y_0(k) + w(k). \quad (22)$$

Hence

$$y(k) = G(\mathbf{q}^{-1};\vartheta)u_0(k) + w(k). \quad (23)$$

Furthermore, suppose that the system input $u_0(k)$ is uncertain so that the *measured input* $u(k)$ is given by

$$u(k) = u_0(k) + v(k), \quad (24)$$

where $v(k)$ is *input noise*. Then (23) can be written as the errors-in-variables model

$$y(k) = G(\mathbf{q}^{-1};\vartheta)u(k) - G(\mathbf{q}^{-1};\vartheta)v(k) + w(k). \quad (25)$$

Remark 3.1: We make no assumptions on $w(k)$ and $v(k)$ until Section 10. Since $w(k)$ need not be white, (25) can represent a Box–Jenkins model. By setting $v(k) \equiv 0$ and $w(k) = H(\mathbf{q}^{-1})\tilde{w}(k)$, where $H(\mathbf{q}^{-1})$ is a stable transfer function and $\tilde{w}(k)$ is zero-mean and white, it follows that (25) becomes the Box–Jenkins model

$$y(k) = G(\mathbf{q}^{-1};\vartheta)u(k) + H(\mathbf{q}^{-1})\tilde{w}(k). \quad (26)$$

We write (25) in regression form as

$$\begin{aligned} A(\mathbf{q}^{-1};\vartheta)\,y(k) - B(\mathbf{q}^{-1};\vartheta)u(k) \\ = A(\mathbf{q}^{-1};\vartheta)w(k) - B(\mathbf{q}^{-1};\vartheta)v(k), \end{aligned} \quad (27)$$

which can be rewritten as

$$\phi^{\mathrm{T}}(k)\vartheta = \psi^{\mathrm{T}}(k)\vartheta, \quad (28)$$

where the *regression vector* $\phi(k) \in \mathbb{R}^{2n+2}$ is

$$\phi(k) \triangleq \left[\, y(k)\cdots y(k-n) \quad -u(k)\cdots -u(k-n)\,\right]^{\mathrm{T}} \quad (29)$$

and the *noise vector* $\psi(k) \in \mathbb{R}^{2n+2}$ is defined as

$$\psi(k) \triangleq \left[\, w(k)\cdots w(k-n) \quad -v(k)\cdots -v(k-n)\,\right]^{\mathrm{T}}. \quad (30)$$

Furthermore, note that

$$\phi(k) = \phi_0(k) + \psi(k). \quad (31)$$

Henceforth, we consider a finite measured output sequence $\{y(k)\}_{k=0}^{l}$ generated by (25) with measured input sequence $\{u(k)\}_{k=0}^{l}$ and noise sequences $\{w(k)\}_{k=0}^{l}$ and $\{v(k)\}_{k=0}^{l}$. Assuming $l \geq n$, we define the *noisy regression matrix* $\Phi \in \mathbb{R}^{(l-n+1)\times(2n+2)}$ by

$$\Phi \triangleq \begin{bmatrix} \phi^{\mathrm{T}}(n) \\ \vdots \\ \phi^{\mathrm{T}}(l) \end{bmatrix} \quad (32)$$

and the *noise matrix* $\Psi \in \mathbb{R}^{(l-n+1)\times(2n+2)}$ by

$$\Psi \triangleq \begin{bmatrix} \psi^{\mathrm{T}}(n) \\ \vdots \\ \psi^{\mathrm{T}}(l) \end{bmatrix}. \quad (33)$$

It then follows from (28) that

$$\Phi\vartheta = \Psi\vartheta. \quad (34)$$

Since

$$\Phi - \Psi = \Phi_0, \quad (35)$$

it follows that (34) is equivalent to (10). However, note that

$$\vartheta \notin \mathcal{N}(\Phi) \quad (36)$$

if and only if $\Psi\vartheta \neq 0$. Now, partition $\Phi = [\Phi_1 \ \Phi_2]$, where $\Phi_1 \in \mathbb{R}^{l-n+1}$ and $\Phi_2 \in \mathbb{R}^{(l-n+1)\times(2n+1)}$, and define $M \in \mathbb{R}^{(2n+2)\times(2n+2)}$ by

$$M \triangleq \frac{1}{l}\Phi^{\mathrm{T}}\Phi. \quad (37)$$

Next, we consider the following definition.

Definition 3.2: The input sequence $\{u_0(k)\}_{k=0}^{l}$ and the noise sequences $\{w(k)\}_{k=0}^{l}$ and $\{v(k)\}_{k=0}^{l}$ are *jointly persistently exciting* for $G(\mathbf{q}^{-1};\vartheta)$ if rank $\Phi = 2n+2$.

The assumption that $\{u_0(k)\}_{k=0}^{l}$ is persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$ and the assumption that $\{u_0(k)\}_{k=0}^{l}$, $\{w(k)\}_{k=0}^{l}$ and $\{v(k)\}_{k=0}^{l}$ are jointly persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$ are independent, that is, one does not imply the other.

Proposition 3.3: *Let* $\{u(k)\}_{k=0}^{l}$ *and* $\{y(k)\}_{k=0}^{l}$ *satisfy* (25). *Then the following statements are equivalent*:

(i) $\{u_0(k)\}_{k=0}^{l}$, $\{w(k)\}_{k=0}^{l}$, *and* $\{v(k)\}_{k=0}^{l}$ *are jointly persistently exciting for* $G(\mathbf{q}^{-1};\vartheta)$.

(ii) rank $M = 2n+2$.
(iii) def $M = 0$.
(iv) $M > 0$.
(v) rank $\Phi = 2n+2$.
(vi) def $\Phi = 0$.

In this case, $l \geq 3n+1$.

4. Errors-in-variables least squares

In this section, we review the standard least-squares problem to provide a framework for QCLS identification. Consider the model $G(\mathbf{q}^{-1};\theta)$ of $G(\mathbf{q}^{-1};\vartheta)$, where the *model parameter vector* $\theta \in \mathbb{R}^{2n+2}$ is

$$\theta \triangleq \begin{bmatrix} \theta_0 \cdots \theta_n & \theta_{n+1} \cdots \theta_{2n+1} \end{bmatrix}^{\mathrm{T}}, \tag{38}$$

and where Remark 2.1 is also valid for θ. Note that the definition of θ assumes that the system order n is known. We partition θ as

$$\theta = \begin{bmatrix} \theta_0 \\ \tilde{\theta} \end{bmatrix}, \tag{39}$$

where $\tilde{\theta} \in \mathbb{R}^{2n+1}$.

In the noise-free case, ϑ satisfies $\Phi_0\vartheta = 0$ (see (10)). Hence, when noise is present, we seek θ that minimises $\|\Phi\theta\|_2$. Here we fix θ_0 and consider the *standard least-squares cost*

$$J(\tilde{\theta}) \triangleq \|\Phi\theta\|_2^2 = \|\theta_0\Phi_1 + \Phi_2\tilde{\theta}\|_2^2. \tag{40}$$

The *standard least-squares problem* is thus

$$\min_{\tilde{\theta}\in\mathbb{R}^{2n+1}} J(\tilde{\theta}). \tag{41}$$

All solutions to (41) are given by the following result.

Proposition 4.1: *Assume that* $\{u(k)\}_{k=0}^{l}$ *and* $\{y(k)\}_{k=0}^{l}$ *satisfy* (25). *Then* $\hat{\tilde{\theta}}$ *is a global minimiser of* $J(\tilde{\theta})$ *if and only if there exists* $\xi \in \mathbb{R}^{2n+1}$ *such that*

$$\hat{\tilde{\theta}} = -\theta_0\Phi_2^{+}\Phi_1 + \left(I_{2n+1} - \Phi_2^{+}\Phi_2\right)\xi. \tag{42}$$

In this case,

$$J(\hat{\tilde{\theta}}) = \theta_0^2\Phi_1^{\mathrm{T}}\left(I_{l-n+1} - \Phi_2\Phi_2^{+}\right)\Phi_1. \tag{43}$$

If, in addition, $\{u_0(k)\}_{k=0}^{l}$, $\{w(k)\}_{k=0}^{l}$ *and* $\{v(k)\}_{k=0}^{l}$ *are jointly persistently exciting for* $G(\mathbf{q}^{-1};\vartheta)$, *then*

$$\hat{\tilde{\theta}} = -\theta_0\Phi_2^{+}\Phi_1 = -\theta_0\left(\Phi_2^{\mathrm{T}}\Phi_2\right)^{-1}\Phi_2^{\mathrm{T}}\Phi_1 \tag{44}$$

is the unique minimiser of (40).

Proof: Note that

$$\begin{aligned} J(\tilde{\theta}) &= \|\theta_0\Phi_1 + \Phi_2\tilde{\theta}\|_2^2 \\ &= \|\theta_0\Phi_1 + \Phi_2\tilde{\theta}\|_2^2 + \theta_0^2\Phi_1^{\mathrm{T}}\Phi_2\Phi_2^{+}\Phi_1 - \theta_0^2\Phi_1^{\mathrm{T}}\Phi_2\Phi_2^{+}\Phi_1 \\ &= \|\Phi_2(\tilde{\theta} + \theta_0\Phi_2^{+}\Phi_1)\|_2^2 + \theta_0^2\Phi_1^{\mathrm{T}}\left(I_{l-n+1} - \Phi_2\Phi_2^{+}\right)\Phi_1. \end{aligned}$$

Since $\theta_0^2\Phi_1^{\mathrm{T}}\left(I_{l-n+1} - \Phi_2\Phi_2^{+}\right)\Phi_1$ is independent of $\tilde{\theta}$ and $\|\Phi_2\tilde{\theta} + \theta_0\Phi_2\Phi_2^{+}\Phi_1\|_2^2$ is nonnegative, all global minimisers of $J(\tilde{\theta})$ satisfy $\|\Phi_2\tilde{\theta} + \theta_0\Phi_2\Phi_2^{+}\Phi_1\|_2^2 = 0$. Thus $\hat{\tilde{\theta}}$ is a global minimiser of $J(\tilde{\theta})$ if and only if $\hat{\tilde{\theta}} = -\theta_0\Phi_2^{+}\Phi_1 + \nu$, where $\nu \in \mathcal{N}(\Phi_2)$. Since $\mathcal{N}(\Phi_2) = \mathcal{R}(I_{2n+1} - \Phi_2^{+}\Phi_2)$, all global minimisers of $J(\tilde{\theta})$ are of the form (42) with minimum value given by (43). Finally, assume that $\{u_0(k)\}_{k=0}^{l}$ and $\{w(k)\}_{k=0}^{l}$ are jointly persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$. Then, since rank $\Phi_2 = 2n+1$, it follows that $\Phi_2^{+}\Phi_2 = I_{2n+1}$, and (42) specialises to (44). □

5. Correlation between the regressor and noise vectors

First, partition $\Psi = [\Psi_1 \;\; \Psi_2]$, where $\Psi_1 \in \mathbb{R}^{l-n+1}$ and $\Psi_2 \in \mathbb{R}^{(l-n+1)\times(2n+1)}$. Then, as noted in Ljung (1999, p. 205), the asymptotic error in the least-squares estimate is due to correlation between Φ_2 and Ψ_2. Since, for all k, $y(k)$ and $w(k)$ are correlated, and $u(k)$ and $v(k)$ correlated, it follows that Φ_2 and Ψ_2 are generally correlated for the system (25). However, under the two special cases discussed in the following subsections, this correlation disappears. As noted in Söderström and Stoica (1989, pp. 66, 187), both of these cases are extremely specialised.

5.1 *White equation error*

Consider the white equation-error model

$$y(k) = G(\mathbf{q}^{-1};\vartheta)u(k) + \frac{1}{A(\mathbf{q}^{-1};\vartheta)}w(k), \tag{45}$$

where $w(k)$ is white. The model (45) is obtained as a special case of (25) by setting $v(k) \equiv 0$ and replacing $w(k)$ by $\frac{1}{A(\mathbf{q}^{-1};\vartheta)}w(k)$. Writing (45) in the time-series form yields

$$\begin{aligned} &a_0y(k) + a_1y(k-1) + \cdots + a_ny(k-n) \\ &\quad = b_0u(k) + b_1u(k-1) + \cdots + b_nu(k-n) + w(k), \end{aligned} \tag{46}$$

and thus it follows that, $\Psi_2 = 0$ and hence Φ_2 and Ψ_2 are uncorrelated.

5.2 *Finite impulse response*

A related case is the finite impulse-response model

$$y(k) = \frac{B(\mathbf{q}^{-1};\vartheta)}{a_0}u(k) + w(k), \tag{47}$$

which is a special case of (25) with $v(k) \equiv 0$ and $A(\mathbf{q}^{-1};\vartheta) = 1$. Again, writing (47) in the time-series form yields

$$a_0y(k)=b_0u(k)+b_1u(k-1)+\cdots+b_nu(k-n)+a_0w(k), \tag{48}$$

and thus it follows that $\Psi_2=0$ and hence Φ_2 and Ψ_2 are uncorrelated. Note that $w(k)$ need not be white. However, if $w(k)$ is white, then (48) is obtained as a special case of (46) by setting $a_1=a_2=\cdots=a_n=0$.

6. Quadratically constrained least squares

Since the standard least-squares solutions may be biased, we now develop an alternative approach to estimating ϑ. Instead of fixing a_0 in (38), we minimise the least-squares cost (40) subject to a quadratic constraint on θ. Hence, we consider the *generalised least-squares cost*

$$\mathcal{J}(\theta) \overset{\Delta}{=} \frac{1}{l}\|\Phi\theta\|_2^2 = \frac{1}{l}\|\theta_0\Phi_1 + \Phi_2\tilde{\theta}\|_2^2, \tag{49}$$

which is identical to (40) except that in (49) there is an additional factor of $1/l$ and the argument is $\theta\in\mathbb{R}^{2n+2}$. Using (37), it follows that (49) can be rewritten as

$$\mathcal{J}(\theta) = \theta^{\mathrm{T}}M\theta. \tag{50}$$

Since M is positive semidefinite it follows that $\mathcal{J}$ is convex on $\mathbb{R}^{2n+2}$. Furthermore, $\mathcal{J}$ is strictly convex on $\mathbb{R}^{2n+2}$ if and only if M is positive definite (see Bernstein 2005, p. 320).

Next, let $N\in\mathbb{R}^{(2n+2)\times(2n+2)}$ be symmetric, and define the *parameter constraint set* $\mathcal{D}(N)$ by

$$\mathcal{D}(N) \overset{\Delta}{=} \{\theta\in\mathbb{R}^{2n+2} : \theta^{\mathrm{T}}N\theta = 1\}. \tag{51}$$

The QCLS *problem* is then given by

$$\min_{\theta\in\mathcal{D}(N)} \mathcal{J}(\theta). \tag{52}$$

Note that $\mathcal{D}(N)$ is closed and symmetric, that is, $\theta\in\mathcal{D}(N)$ if and only if $-\theta\in\mathcal{D}(N)$. Furthermore, $\mathcal{D}(N)\neq\varnothing$ if and only if $\lambda_{\max}(N)>0$. Next, define the *solution set* $\mathcal{W}(N)$ by

$$\mathcal{W}(N) \overset{\Delta}{=} \{\theta\in\mathcal{D}(N) : \mathcal{J}(\theta) = \min_{\theta'\in\mathcal{D}(N)} \mathcal{J}(\theta')\}. \tag{53}$$

$\mathcal{W}(N)$ is closed and symmetric. Furthermore, If $\mathcal{W}(N)\neq\varnothing$ then the QCLS problem (52) has at least two solutions. In particular, $\theta\in\mathbb{R}^{2n+2}$ solves the QCLS problem (52) if and only if $-\theta$ does.

Define $N_{\mathrm{LS}}\in\mathbb{R}^{(2n+2)\times(2n+2)}$ by

$$N_{\mathrm{LS}} \overset{\Delta}{=} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}. \tag{54}$$

With $N=N_{\mathrm{LS}}$ it can be seen that solutions of the QCLS problem (52) are solutions of the standard least-squares problem (41) with $a_0=\pm1$. Although N_{LS} is positive semidefinite, we do not require that N be positive semidefinite.

7. Existence and uniqueness of QCLS problem

Let $A,\ B\in\mathbb{R}^{p\times p}$. Then the *matrix pencil* $P_{A,B}(s)$ is defined by

$$P_{A,B}(s) \overset{\Delta}{=} A - sB. \tag{55}$$

Furthermore, define the *characteristic polynomial* $\chi_{A,B}(s)$ by

$$\chi_{A,B}(s) \overset{\Delta}{=} \det(A - sB). \tag{56}$$

The pair (A,B) is *regular* if $\chi_{A,B}(s)$ is not the zero polynomial. The roots of $\chi_{A,B}(s)$ are the *generalised eigenvalues* of (A,B). Furthermore, if $A>0$, B is symmetric, and (A,B) is regular, then all generalised eigenvalues of (A,B) are real (Palanthandalam-Madapusi et al. 2009).

Next, define

$$\mathcal{S} \overset{\Delta}{=} \{\alpha \geq 0 : \alpha N \leq M\} \tag{57}$$

and $\alpha_{\max} \overset{\Delta}{=} \max\,\mathcal{S}$, and note that $\mathcal{S}=[0,\alpha_{\max}]$. The following result concerns properties of $\alpha_{\max}$.

Proposition 7.1: *Assume* $\lambda_{\max}(N)>0$. *Then the following statements hold*:

(i) *If* $N\leq M$, *then* $\alpha_{\max}\geq 1$.
(ii) $\alpha_{\max}$ *is a generalised eigenvalue of* (M,N).

Furthermore, assume that $\alpha_{\max}>0$ *and* (M,N) *is regular. Then the following statements hold*:

(iii) *Let* $\alpha_1,\alpha_2\in(0,\alpha_{\max})$. *Then*

$$0 = \operatorname{def}(M-\alpha_1N) = \operatorname{def}(M-\alpha_2N) < \operatorname{def}(M-\alpha_{\max}N). \tag{58}$$

(iv) $\alpha_{\max}$ *is the smallest positive generalised eigenvalue of* (M,N).

Proof: See Palanthandalam-Madapusi et al. (2009). □

Thus $M-\alpha_{\max}N$ has a nontrivial-null space. The following result considers existence of solutions to (52).

Theorem 7.2: *Assume* $\lambda_{\max}(N)>0$. *Then, for all* $\theta\in\mathcal{D}(N)$,

$$\mathcal{J}(\theta) \geq \alpha_{\max}. \tag{59}$$

Furthermore, $\theta\in\mathcal{D}(N)$ *satisfies*

$$\mathcal{J}(\theta) = \alpha_{\max} \tag{60}$$

if and only if $\theta \in \mathcal{N}(M - \alpha_{\max} N) \cap \mathcal{D}(N)$. *Finally*,

$$\mathcal{W}(N) = \mathcal{N}(M - \alpha_{\max} N) \cap \mathcal{D}(N). \tag{61}$$

Proof: See Palanthandalam-Madapusi et al. (2009). □

Corollary 7.3: *Assume* $M > 0$ *and* $\lambda_{\max}(N) > 0$. *Then*

$$\min_{\theta \in \mathcal{D}(N)} \mathcal{J}(\theta) = \alpha_{\max}, \tag{62}$$

and thus (52) *has a solution.*

8. Choosing N in QCLS

Define $\Delta M \in \mathbb{R}^{(2n+2)\times(2n+2)}$ by

$$\Delta M \triangleq M - M_0. \tag{63}$$

Note that $\Delta M \le M$ and

$$\Delta M = \frac{1}{l}\left(\Phi^{\mathrm{T}}\Phi - \Phi_0^{\mathrm{T}}\Phi_0\right), \tag{64}$$

which may be indefinite. In the noise-free case $\Delta M = 0$, and thus $\mathcal{D}(\Delta M) = \varnothing$. In the noisy case, however, ΔM may be nonzero. For the rest of this subsection we consider the case $N = \Delta M$.

Proposition 8.1: *Assume* $\lambda_{\max}(\Delta M) > 0$. *Then* $\alpha_{\max} \ge 1$. *Furthermore, if* $\{u_0(k)\}_{k=0}^{l}$ *is persistently exciting, and* $\{u_0(k)\}_{k=0}^{l}$, $\{w(k)\}_{k=0}^{l}$ *and* $\{v(k)\}_{k=0}^{l}$ *are jointly persistently exciting, then* $\alpha_{\max} = 1$ *is the smallest positive generalised eigenvalue of* $(M, \Delta M)$.

Proof: Since $\Delta M \le M$, it follows from Proposition 7.1 that $\alpha_{\max} \ge 1$. Next, since $\mathcal{N}(M) \cap \mathcal{N}(\Delta M) = \{0\}$ and $\operatorname{rank} M_0 = \operatorname{rank}(M - \Delta M) = 2n+1 < 2n+2 = \operatorname{rank} M$ it follows that $\operatorname{def}(M) = 0 < 1 = \operatorname{def}(M_0) = \operatorname{def}(M - \Delta M)$ and thus Proposition 7.1 with $N = \Delta M$ in (58) implies that $\alpha_{\max} = 1$. □

The following result gives conditions under which the QCLS problem correctly identifies $G(\mathbf{q}^{-1}; \vartheta)$.

Theorem 8.2: *Assume that* $\lambda_{\max}(\Delta M) > 0$, $\{u_0(k)\}_{k=0}^{l}$ *is persistently exciting for* $G(\mathbf{q}^{-1}; \vartheta)$, *and* $\{u_0(k)\}_{k=0}^{l}$, $\{w(k)\}_{k=0}^{l}$ *and* $\{v(k)\}_{k=0}^{l}$ *are jointly persistently exciting for* $G(\mathbf{q}^{-1}; \vartheta)$. *Then*

$$\mathcal{W}(\Delta M) = \left\{ \pm\sqrt{\frac{1}{\vartheta^{\mathrm{T}}\Delta M \vartheta}}\,\vartheta \right\}. \tag{65}$$

Proof: From Proposition 8.1, it follows that $\alpha_{\max} = 1$. Next, since $M > 0$, it follows from Corollary 7.3 that $\mathcal{W}(\Delta M) \ne \varnothing$. Furthermore, it follows from Theorem 7.2 that $\mathcal{W}(\Delta M) = \mathcal{N}(M_0) \cap \mathcal{D}(\Delta M) \ne \varnothing$. Furthermore, since $M > 0$, $M = M_0 + \Delta M$ and $\vartheta^{\mathrm{T}} M_0 \vartheta = 0$, it follows that $\vartheta^{\mathrm{T}} \Delta M \vartheta > 0$. Hence, since $\operatorname{def} M_0 = 1$, it follows from (13) that (65) holds. □

Note that if $M > 0$, then $\mathcal{N}(M) \cap \mathcal{N}(\Delta M) = \{0\}$. Thus, if $\operatorname{rank} M_0 = 2n+1$ and $M > 0$, then Theorem 8.2 implies that $G(\mathbf{q}^{-1}; \vartheta) = G(\mathbf{q}^{-1}; \theta)$ for all $\theta \in \mathcal{W}(\Delta M)$.

However, the choice $N = \Delta M$ is not possible in practice since ΔM is not known. In the next section, we consider the QCLS problem with an approximation $N \approx \Delta M$.

9. Unbiasedness of QCLS

It was shown in Section 8 that, if $N = \Delta M$ and $M > 0$, then the QCLS problem has exactly two solutions θ, both of which satisfy $G(\mathbf{q}^{-1}; \theta) = G(\mathbf{q}^{-1}; \vartheta)$. However, since ΔM is not known, we cannot set $N = \Delta M$ in practice. In this section, we choose an alternative but closely related value of N and show that the resulting QCLS solution is unbiased.

For the remainder of the article, we assume that $\{w(k)\}_{k=0}^{\infty}$ and $\{v(k)\}_{k=0}^{\infty}$ are stationary and have zero mean and finite second moments. Furthermore, we assume that $\{w(k)\}_{k=0}^{\infty}$ and $\{v(k)\}_{k=0}^{\infty}$ are jointly ergodic random processes in the sense that, for all ρ, $\sigma \in \{0, 1, 2\}$ and for all i, $\mathbb{E}[w^{\rho}(i)v^{\sigma}(i)] = \lim_{l\to\infty} \frac{1}{l} \times \sum_{k=0}^{l} w^{\rho}(k)v^{\sigma}(k)$ wp1.

For convenience, define the positive-semidefinite matrix $R \in \mathbb{R}^{(2n+2)\times(2n+2)}$ by

$$R \triangleq \mathbb{E}\left[\psi^{\mathrm{T}}(k)\psi(k)\right],$$

where $n \le k \le l$. However, note that since $w(k)$ and $v(k)$ are stationary, R is independent of k. Next, partition R as

$$R = \begin{bmatrix} R_{ww} & -R_{wv} \\ -R_{vw} & R_{vv} \end{bmatrix}, \tag{66}$$

where $R_{ww} \in \mathbb{R}^{(n+1)\times(n+1)}$ is given by

$$R_{ww} \triangleq \mathbb{E}\begin{bmatrix} w^2(k) & w(k)w(k-1) & \cdots & w(k)w(k-n) \\ w(k)w(k-1) & w^2(k) & \cdots & w(k)w(k-n+1) \\ \vdots & \vdots & \ddots & \vdots \\ w(k)w(k-n) & w(k)w(k-n+1) & \cdots & w^2(k) \end{bmatrix}, \tag{67}$$

while $R_{vv} \in \mathbb{R}^{(n+1)\times(n+1)}$ and $R_{wv} \in \mathbb{R}^{(n+1)\times(n+1)}$ are defined analogously.

Next, using $\Phi = \Phi_0 + \Psi$, ΔM has the form

$$\begin{aligned}\Delta M &= \frac{1}{l}\left(\Phi^{\mathrm{T}}\Phi - \Phi_0^{\mathrm{T}}\Phi_0\right)\\ &= \frac{1}{l}\left((\Phi_0+\Psi)^{\mathrm{T}}(\Phi_0+\Psi) - \Phi_0^{\mathrm{T}}\Phi_0\right)\\ &= \frac{1}{l}\left(\Phi_0^{\mathrm{T}}\Psi + \Psi^{\mathrm{T}}\Phi_0 + \Psi^{\mathrm{T}}\Psi\right). \qquad (68)\end{aligned}$$

Proposition 9.1: *Assume that $\{y_0(k)\}_{k=0}^{l}$ and $\{u_0(k)\}_{k=0}^{l}$ are bounded. Then*

$$\mathbb{E}[\Delta M] = R \qquad (69)$$

and

$$\mathbb{E}[M] = M_0 + R. \qquad (70)$$

Proof: The result follows from (68) and the fact that, for all continuous functions $f: \mathbb{R} \to \mathbb{R}$, $\mathbb{E}\left[\frac{1}{l}\sum_{k=0}^{l} f(w(k))\right] = \mathbb{E}[f(w(k))]$. □

Let $\eta > 0$. Then, define $\bar{\alpha}_{\max} \stackrel{\Delta}{=} \max\{\alpha \geq 0 : \alpha\eta R \leq \mathbb{E}[M]\}$.

Definition 9.2: Let $N = \eta R$. Then QCLS is an *unbiased generator* of ϑ if

$$\mathcal{N}(\mathbb{E}[M] - \bar{\alpha}_{\max}\eta R = \mathcal{R}(\vartheta). \qquad (71)$$

Definition 9.2 states that QCLS with $N = \eta R$ is an unbiased generator of ϑ if QCLS with M replaced by $\mathbb{E}[M]$ yields the estimated parameter vector $\hat{\theta} \in \mathcal{R}(\vartheta)$. That is, the averaged QCLS problem with the constraint matrix set to a scalar multiple of the covariance matrix yields the true parameter vector. We have the following unbiasedness result.

Proposition 9.3: *Assume that $\{y_0(k)\}_{k=0}^{l}$ and $\{u_0(k)\}_{k=0}^{l}$ are bounded and satisfy* (25). *Furthermore, assume that $\{u_0(k)\}_{k=0}^{l}$ is persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$, assume that $\{u_0(k)\}_{k=0}^{l}$, $\{w(k)\}_{k=0}^{l}$ and $\{v(k)\}_{k=0}^{l}$ are jointly persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$, and assume $M_0 + R > 0$. Then, for all $\eta > 0$, QCLS with $N = \eta R$ is an unbiased generator of ϑ.*

Proof: From Proposition 9.1, it follows that $\mathbb{E}[M] = M_0 + R$. Next, since $M_0 \geq 0$, $R \geq 0$ and $M_0 + R > 0$, it follows from cogredient simultaneous diagonalisation of M_0 and R (Rao and Mitra 1971, Theorem 6.2.3, p. 122) that there exists $S \in \mathbb{R}^{(2n+2)\times(2n+2)}$ such that SM_0S^{T} and SRS^{T} are diagonal. Next, for $i = 1, \ldots, 2n+2$, let $m_i \geq 0$ and $r_i \geq 0$ denote the diagonal entries of SM_0S^{T} and SRS^{T}, respectively. Hence $S(M_0+R)S^{\mathrm{T}} = \mathrm{diag}(m_i + r_i)$. Since $M_0 + R$ is positive definite, it follows that $m_i + r_i > 0$ for all $i = 1, \ldots, 2n+2$. Furthermore, since $\{u_0\}_{k=0}^{l}$ is persistently exciting, it follows from (21) that $\mathcal{N}(M_0) = \mathcal{R}(\vartheta)$ and $\mathrm{def}\, M_0 = 1$, and thus exactly one m_i is zero. For convenience, assume $m_1 = 0$. Next, since $S\mathbb{E}[M]S^{\mathrm{T}} = \mathrm{diag}(m_i + r_i)$, $\eta SRS^{\mathrm{T}} = \mathrm{diag}(\eta r_i)$, and $m_1 = 0$, it follows from Proposition 7.1 that $\bar{\alpha}_{\max} = 1/\eta$. Finally, $\mathcal{N}(\mathbb{E}[M] - \bar{\alpha}_{\max}\eta R) = \mathcal{N}(M_0 + R - R) = \mathcal{N}(M_0) = \mathcal{R}(\vartheta)$. □

Corollary 9.4: *Assume that $\{y_0(k)\}_{k=0}^{l}$ and $\{u_0(k)\}_{k=0}^{l}$ are bounded and satisfy* (25). *Furthermore, assume that $\{u_0(k)\}_{k=0}^{l}$ is persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$, assume that $\{u_0(k)\}_{k=0}^{l}$, $\{w(k)\}_{k=0}^{l}$, and $\{v(k)\}_{k=0}^{l}$ are jointly persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$, let $\eta > 0$, and assume $M_0 + R > 0$. Then $\bar{\alpha}_{\max} = 1/\eta$.*

Let $\hat{\theta}$ be a solution of the QCLS problem with M and $N = \eta R$, where $\eta > 0$. Since M is a random variable, it follows that $\hat{\theta}$ is a random variable. We note that the traditional notion of unbiasedness is defined in terms of $\hat{\theta}$ and states that

$$\mathbb{E}[\hat{\theta}] = \pm\frac{1}{\sqrt{\eta\vartheta^{\mathrm{T}}R\vartheta}}\vartheta. \qquad (72)$$

10. Consistency of QCLS

In this section, we write $M_{0,l}$, M_l, ΔM_l, $\hat{\theta}_l$ and $\alpha_{\max,l}$ for M_0, M, ΔM, $\hat{\theta}$ and $\alpha_{\max}$, respectively, to indicate the dependence on l. We then let l tend to infinity and show that the resulting QCLS solution is consistent.

Lemma 10.1: *Let $\{x(k)\}_{k=n}^{\infty} \subset \mathbb{R}$, assume there exists $\beta > 0$ such that, for all $k \geq n$, $0 \leq x(k) \leq \beta$, let $\mathcal{K} \subseteq \{n, n+1, \ldots\}$, and define $\mathcal{K}_l \stackrel{\Delta}{=} \mathcal{K} \cap \{n, \ldots, l\}$. Then,*

$$\lim_{l\to\infty}\frac{1}{l}\sum_{k\in\mathcal{K}_l} w(k)x(k) = 0 \quad \text{wp1}. \qquad (73)$$

Proof: Write $\mathcal{K} = \{k_1, k_2, \ldots\}$. Next, suppose that $\mathcal{K}$ has a finite number of elements. Then, since $w(k)$ has a finite second moment, it follows that the $\sum_{k\in\mathcal{K}_l} w(k)x(k)$ is finite and hence (73) holds. Next, suppose that $\mathcal{K}$ has an infinite number of elements. Letting $\hat{l}$ be the number of elements in $\mathcal{K}_l$, it follows that

$$\begin{aligned}-\frac{\beta}{\hat{l}}\left|\sum_{i=1}^{\hat{l}} w(k_i)\right| &\leq -\frac{1}{l}\left|\sum_{i=1}^{\hat{l}} w(k_i)x(k_i)\right| \leq \frac{1}{l}\sum_{k\in\mathcal{K}_l} w(k)x(k)\\ &\leq \frac{1}{l}\left|\sum_{i=1}^{\hat{l}} w(k_i)x(k_i)\right| \leq \frac{\beta}{\hat{l}}\left|\sum_{i=1}^{\hat{l}} w(k_i)\right|.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}-\beta\left|\lim_{\hat{l}\to\infty}\frac{1}{\hat{l}}\sum_{i=1}^{\hat{l}} w(k_i)\right| &\leq \lim_{l\to\infty}\frac{1}{l}\sum_{k\in\mathcal{K}_l} w(k)x(k)\\ &\leq \beta\left|\lim_{\hat{l}\to\infty}\frac{1}{\hat{l}}\sum_{i=1}^{\hat{l}} w(k_i)\right|.\end{aligned}$$

Furthermore, since $\{w(k)\}_{k=0}^{\infty}$ is stationary and ergodic, it follows that $\{w(k_i)\}_{i=1}^{\infty}$ is stationary and ergodic. Hence

$$0 = -\beta\big|\mathbb{E}[w(k)]\big| \le \lim_{l\to\infty}\frac{1}{l}\sum_{k\in\mathcal{K}_l} w(k)x(k) \le \beta\big|\mathbb{E}[w(k)]\big| = 0$$

□

Lemma 10.2: *Assume that* $\{y_0(k)\}_{k=0}^{\infty}$ *and* $\{u_0(k)\}_{k=0}^{\infty}$ *are bounded. Then*

$$\lim_{l\to\infty}\frac{1}{l}\Psi\Phi_0^{\mathrm{T}} = 0 \quad \text{wp1.} \tag{74}$$

Proof: The (1, 1) entry $\Psi\Phi_0^{\mathrm{T}}$ is $\sum_{k=n}^{l} w(k)\,y_0(k)$. Next, let

$$\mathcal{K}_+ \triangleq \{k : k \ge n \text{ and } \operatorname{sign}(y_0(k)) = 1\}$$

and

$$\mathcal{K}_- \triangleq \{k : k \ge n \text{ and } \operatorname{sign}(y_0(k)) = -1\}.$$

Furthermore, define $\mathcal{K}_{l,+} \triangleq \mathcal{K}_+ \cap \{n,\ldots,l\}$ and $\mathcal{K}_{l,-} \triangleq \mathcal{K}_- \cap \{n,\ldots,l\}$. Then, it follows that

$$\sum_{k=n}^{l} w(k)\,y_0(k) = \sum_{k\in\mathcal{K}_{l,+}} w(k)\,y_0(k) - \sum_{k\in\mathcal{K}_{l,-}} w(k)|\,y_0(k)|.$$

Using Lemma 10.1, it follows that

$$\lim_{l\to\infty}\frac{1}{l}\sum_{k=n}^{l} w(k)\,y_0(k) = \lim_{l\to\infty}\frac{1}{l}\sum_{k\in\mathcal{K}_{l,+}} w(k)\,y_0(k) - \lim_{l\to\infty}\frac{1}{l}\sum_{k\in\mathcal{K}_{l,-}} w(k)|\,y_0(k)| = 0 \quad \text{wp1.}$$

A similar argument holds for the remaining entries of $\Psi\Phi_0^{\mathrm{T}}$. □

Proposition 10.3: *Assume that* $\{y_0(k)\}_{k=0}^{\infty}$ *and* $\{u_0(k)\}_{k=0}^{\infty}$ *are bounded. Then*

$$\lim_{l\to\infty}\Delta M_l = R \quad \text{wp1.} \tag{75}$$

Proof: It follows from (68), (66), Lemma 10.2 and the fact that $w(k)$ and $v(k)$ are jointly ergodic that

$$\lim_{l\to\infty}\Delta M_l = \lim_{l\to\infty}\frac{1}{l}\left[\Psi^{\mathrm{T}}\Psi + \Phi_0^{\mathrm{T}}\Psi + \Psi^{\mathrm{T}}\Phi_0\right] = \lim_{l\to\infty}\frac{1}{l}\left[\Psi^{\mathrm{T}}\Psi\right] = \mathbb{E}\left[\psi(n)^{\mathrm{T}}\psi(n)\right] = R \quad \text{wp1.}$$ □

For the following development we consider $\mathcal{M}_0 \in \mathbb{R}^{(2n+2)\times(2n+2)}$ defined by

$$\mathcal{M}_0 \triangleq \lim_{l\to\infty} M_{0,l}, \tag{76}$$

when the limit exists.

Proposition 10.4: $\mathcal{M}_0$ *exists if and only if, for all* $0 \le \kappa \le n$, $\lim_{l\to\infty}\frac{1}{l}\sum_{k=0}^{l} u_0(k)u_0(k+\kappa)$, $\lim_{l\to\infty}\frac{1}{l}\sum_{k=0}^{l} y_0(k)\,y_0(k+\kappa)$, $\lim_{l\to\infty}\frac{1}{l}\sum_{k=0}^{l} y_0(k)u_0(k+\kappa)$ *and* $\lim_{l\to\infty}\frac{1}{l}\sum_{k=0}^{l} u_0(k)\,y_0(k+\kappa)$ *exist.*

Proposition 10.4 shows that $\mathcal{M}_0$ exists if and only if the given autocorrelations exist.

Lemma 10.5: *Given* $\{u_0(k)\}_{k=0}^{\infty}$, *assume that there exists* $\hat{l} \ge 3n$ *such that* $\{u_0(k)\}_{k=0}^{l}$ *is persistently exciting for* $G(\mathbf{q}^{-1};\vartheta)$. *Then, for all* $l \ge \hat{l}$, $\{u_0(k)\}_{k=0}^{l}$ *is persistently exciting for* $G(\mathbf{q}^{-1};\vartheta)$.

Next, let $\{u_0(k)\}_{k=0}^{\hat{l}}$ be persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$. Then using Lemma 10.5, it follows that $\{u_0(0),\ldots,u_0(\hat{l}),\,0,\,0,\ldots\}$ is persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$ and thus it follows from (21) that, for all $l \ge \hat{l}$, $\mathcal{N}(M_{0,l}) = \mathcal{R}(\vartheta)$. Furthermore, for the input $\{u_0(0),\ldots,u_0(\hat{l}),\,0,\,0,\ldots\}$, the stability of $G(\mathbf{q}^{-1};\vartheta)$ implies that $y_0(k)\to 0$ as $k\to\infty$. It thus follows that $\mathcal{M}_0 = 0$ and hence $\mathcal{N}(\mathcal{M}_0) = \mathbb{R}^{2n+2} \ne \mathcal{R}(\vartheta) = \mathcal{N}(M_{0,l})$ for all $l \ge \hat{l}$. Hence, in this case, for all $l \ge \hat{l}$, $\mathcal{N}(M_{0,l})$ is a proper subset of $\mathcal{N}(\mathcal{M}_0)$.

More generally, if $\mathcal{M}_0$ exists, then, for all $l \ge 3n$, $\mathcal{N}(M_{0,l}) \subseteq \mathcal{N}(\mathcal{M}_0)$. However, the above example shows that $\mathcal{N}(\mathcal{M}_0) = \mathcal{N}(M_{0,l})$ is not true in general. Thus we have the following definition. For convenience, for $\kappa \ge n$, let $\sigma_{2n+1,\kappa}$ be the second smallest singular value of $\begin{bmatrix}\phi_0(\kappa)\\ \vdots \\ \phi_0(\kappa+3n)\end{bmatrix}$.

Definition 10.6: The input sequence $\{u_0(k)\}_{k=0}^{\infty}$ is *infinitely persistently exciting* for $G(\mathbf{q}^{-1};\vartheta)$ if there exists $\varepsilon > 0$ such that, for all $\kappa \ge n$, $\sigma_{2n+1,\kappa} > \varepsilon$.

The following result is immediate.

Proposition 10.7: If $\{u_0(k)\}_{k=0}^{\infty}$ is infinitely persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$, then the following statements hold:

(i) For all $\kappa \ge n$, $\{u_0(\kappa+k)\}_{k=0}^{3n}$ is persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$.

(ii) For all $\theta \in \mathbb{R}^{2n+2}$ such that $\theta \notin \mathcal{R}(\vartheta)$, there exists $\varepsilon_\theta > 0$ such that, for all $\kappa \ge n$, $\left\|\begin{bmatrix}\phi_0(\kappa)\\ \vdots \\ \phi_0(\kappa+3n)\end{bmatrix}\theta\right\|_2 > \varepsilon_\theta$.

Lemma 10.8: *Assume that* $\mathcal{M}_0$ *exists and that* $\{u_0(k)\}_{k=0}^{\infty}$ *is infinitely persistently exciting for* $G(\mathbf{q}^{-1};\vartheta)$. *Then, for all* $l \ge 3n$,

$$\mathcal{N}(\mathcal{M}_0) = \mathcal{N}\big(M_{0,l}\big), \tag{77}$$

and thus

$$\mathcal{N}(\mathcal{M}_0) = \mathcal{R}(\vartheta). \tag{78}$$

Proof: Since $\vartheta \in \mathcal{N}(M_{0,l})$ for all $l \geq 3n$, it follows that $\vartheta \in \mathcal{N}(\mathcal{M}_0)$. Hence, for all $l \geq 3n$, $\mathcal{N}(\mathcal{M}_0) \supseteqq \mathcal{N}(M_{0,l})$. Conversely, since $\{u_0(k)\}_{k=0}^{\infty}$ is infinitely persistently exciting, it follows that, for all $l \geq 3n$ and $\kappa \geq n$,

$$\mathcal{N}\big(M_{0,l}\big) = \mathcal{N}\left(\begin{bmatrix} \phi_0(\kappa) \\ \vdots \\ \phi_0(\kappa+3n) \end{bmatrix}\right) = \mathcal{R}(\vartheta). \tag{79}$$

Next, let $\theta \in \mathbb{R}^{2n+2}$ be such that $\theta \notin \mathcal{N}(M_{0,l})$ for all $l \geq 3n$. Then it follows from (79) that, for all $\kappa \geq 0$, $\begin{bmatrix} \phi_0(\kappa) \\ \vdots \\ \phi_0(\kappa+3n) \end{bmatrix}\theta \neq 0$. Next, for all $l \geq 3n$, we have $\theta^{\mathrm{T}} M_{0,l}\theta = \theta^{\mathrm{T}} \frac{1}{l}\left(\sum_{k=0}^{l} \phi_0(k)^{\mathrm{T}}\phi_0(k)\right)\theta = \frac{1}{l}\sum_{k=0}^{l} r_k^2$, where $r_k \triangleq \phi_0(k)\theta$. Since $\{u_0(k)\}_{k=0}^{\infty}$ is infinitely persistently exciting and $\theta \notin \mathcal{R}(\vartheta)$, it follows from Proposition 10.7 that there exists $\varepsilon_\theta > 0$ such that, for all $\kappa \geq n$, $\sum_{k=\kappa}^{\kappa+3n} r_k^2 = \left\| \begin{bmatrix} \phi_0(\kappa) \\ \vdots \\ \phi_0(\kappa+3n) \end{bmatrix}\theta \right\|_2^2 > \varepsilon_\theta^2$, and thus, for all $l_0 > 0$, $\frac{1}{3nl_0}\sum_{k=0}^{3nl_0} r_k^2 > \frac{\varepsilon_\theta^2}{3n}$. It then follows that $\theta^{\mathrm{T}}\mathcal{M}_0\theta = \lim_{l\to\infty}\theta^{\mathrm{T}} M_{0,l}\theta = \lim_{l\to\infty}\frac{1}{l}\sum_{k=0}^{l} r_k^2 \geq \frac{\varepsilon_\theta^2}{3n} > 0$. Since $\mathcal{M}_0 \geq 0$, it follows that $\theta \notin \mathcal{N}(\mathcal{M}_0)$. □

Definition 10.9: QCLS with $N = \eta R$ is a *consistent generator* of ϑ if the following three conditions are satisfied:

(i) $\mathcal{M}_\infty \triangleq \lim_{l\to\infty} M_l$ wp1 exists.
(ii) $\mathcal{W}_\infty(\eta R) \triangleq \mathcal{N}(\mathcal{M}_\infty - \alpha_{\max,\infty}\eta R) \cap \mathcal{D}(\eta R) \neq \varnothing$, where $\alpha_{\max,\infty} \triangleq \max\{\alpha \geq 0 : \alpha\eta R \leq \mathcal{M}_\infty\}$.
(iii) $\mathcal{N}(\mathcal{M}_\infty - \alpha_{\max,\infty}\eta R) = \mathcal{R}(\vartheta)$.

Definition states that QCLS with $N = \eta R$ is a consistent generator of ϑ, if the limiting problem exists, has a solution and yields the estimated parameter vector $\hat{\theta}_\infty \in \mathcal{R}(\vartheta)$.

Lemma 10.10: *Assume $\mathcal{M}_0$ exists and that $\{y_0(k)\}_{k=0}^{\infty}$ and $\{u_0(k)\}_{k=0}^{\infty}$ are bounded and satisfy (25). Furthermore, for all $l \geq 3n$, assume that $\{u_0(k)\}_{k=0}^{l}$ is persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$, and $\{u_0(k)\}_{k=0}^{l}$, $\{w(k)\}_{k=0}^{l}$ and $\{v(k)\}_{k=0}^{l}$ are jointly persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$. Then, $\mathcal{M}_\infty$ exists and is given by*

$$\mathcal{M}_\infty = \mathcal{M}_0 + R. \tag{80}$$

Proof: Proposition 10.3 implies that $\lim_{l\to\infty} M_l = \lim_{l\to\infty} M_{0,l} + \lim_{l\to\infty}\Delta M_l = \mathcal{M}_0 + R$ wp1. □

Next, we have the following consistency result.

Theorem 10.11: *Assume $\mathcal{M}_0$ exists and that $\{y_0(k)\}_{k=0}^{\infty}$ and $\{u_0(k)\}_{k=0}^{\infty}$ are bounded and satisfy (25). Furthermore, for all $l \geq 3n$, assume that $\{u_0(k)\}_{k=0}^{l}$ is infinitely persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$, assume $\{u_0(k)\}_{k=0}^{l}$, $\{w(k)\}_{k=0}^{l}$ and $\{v(k)\}_{k=0}^{l}$ are jointly persistently exciting for $G(\mathbf{q}^{-1};\vartheta)$, let $\eta > 0$, and assume that $\mathcal{M}_\infty > 0$. Then QCLS with $N = \eta R$ is a consistent generator of ϑ.*

Proof: From Theorem 7.2, it follows that $\hat{\theta}_l$ satisfies

$$\big(M_l - \alpha_{\max,l}\eta R\big)\hat{\theta}_l = 0. \tag{81}$$

Next, using Lemma 10.10 and Corollary 7.3, it follows that $\mathcal{M}_\infty = \mathcal{M}_0 + R > 0$ and $\mathcal{W}_\infty \neq \varnothing$. Furthermore, since $\mathcal{M}_0$ and R are positive semidefinite, it follows by cogredient simultaneous diagonalisation of $\mathcal{M}_0$ and R (Rao and Mitra 1971, Theorem 6.2.3, p. 122) that there exists $S \in \mathbb{R}^{(2n+2)\times(2n+2)}$ such that $S\mathcal{M}_0S^{\mathrm{T}}$ and SRS^{T} are diagonal. Next, for $i = 1, \ldots, 2n+2$, let $m_i \geq 0$ and $r_i \geq 0$ denote the diagonal entries of $S\mathcal{M}_0S^{\mathrm{T}}$ and SRS^{T}, respectively. Hence $S(\mathcal{M}_0 + R)S^{\mathrm{T}} = \operatorname{diag}(m_i + r_i)$. Since $\mathcal{M}_0 + R$ is positive definite, it follows that $m_i + r_i > 0$ for all $i = 1, \ldots, 2n+2$. Furthermore, since $\{u_0\}_{k=0}^{\infty}$ is infinitely persistently exciting, it follows from Lemma 10.8 that $\mathcal{N}(\mathcal{M}_0) = \mathcal{R}(\vartheta)$ and $\operatorname{def}\mathcal{M}_0 = 1$, and thus exactly one m_i is zero. For convenience, assume $m_1 = 0$. Next, since $S\mathcal{M}_\infty S^{\mathrm{T}} = \operatorname{diag}(m_i + r_i)$, $\eta SRS^{\mathrm{T}} = \operatorname{diag}(\eta r_i)$, and $m_1 = 0$, it follows from Proposition 7.1 that $\alpha_{\max,\infty} = 1/\eta$. Finally, since $\mathcal{N}(\mathcal{M}_\infty - \alpha_{\max,\infty}\ \eta R) = \mathcal{N}(\mathcal{M}_0 + R - R) = \mathcal{N}(\mathcal{M}_0)$, it follows from Lemma 10.8 that $\mathcal{N}(\mathcal{M}_\infty - \alpha_{\max,\infty}\eta R) = \mathcal{R}(\vartheta)$ and thus QCLS with $N = \eta R$ is consistent. □

We note that Theorem 10.11 holds for arbitrary $\eta > 0$. Hence, in practice, R need only be known to within a scalar multiple.

The traditional notion of consistency states that

$$\hat{\theta}_l \to \pm\frac{\vartheta}{\sqrt{\eta\vartheta^{\mathrm{T}}R\vartheta}} \quad \text{as} \quad l \to \infty. \tag{82}$$

Numerical simulations given below suggest that QCLS with $N = \eta R$ is consistent in this sense.

11. An analytical example illustrating consistency

In this section, we illustrate that when the noise statistics are known, QCLS with $N = \eta R$ is a consistent generator of ϑ, whereas the standard least-squares estimator is generally not consistent.

Consider the first-order strictly proper stable system

$$G(\mathbf{q}^{-1};\vartheta) \triangleq \frac{b_1\mathbf{q}^{-1}}{a_0 + a_1\mathbf{q}^{-1}}, \tag{83}$$

where, in accordance with Remark 2.1, $\vartheta = [a_0\ a_1\ b_1]^{\mathrm{T}}$, $\theta = [\theta_0\ \ \theta_1\ \ \theta_2]^{\mathrm{T}}$ and $\phi(k) = [y(k)\ \ y(k-1)\ \ -u(k-1)]^{\mathrm{T}}$. Furthermore, assume that the noise sequence $w(k)$ is white, $v(k) \equiv 0$ and let the input sequence $\{u_0(k)\}_{k=0}^{l}$ be a realisation of a zero-mean white noise sequence with variance σ_u^2. The measured output $\{y(k)\}_{k=0}^{l}$ is given by (22).

Next, note that the standard least-squares error can be written as

$$\delta\tilde{\theta}_l \overset{\Delta}{=} (1/\theta_0)\hat{\tilde{\theta}}_l - (1/a_0)\tilde{\vartheta} = \left(\frac{1}{l}\Phi_2^{\mathrm{T}}\Phi_2\right)^{-1}\frac{1}{l}\Phi_2^{\mathrm{T}}\Psi\vartheta, \quad (84)$$

so that the asymptotic standard least-squares bias $\delta\tilde{\theta}_\infty$ is given by

$$\delta\tilde{\theta}_\infty \overset{\Delta}{=} \lim_{l\to\infty}\delta\tilde{\theta}_l = R^{-1}q, \quad (85)$$

where R is defined by (66) and

$$q = \lim_{l\to\infty}\frac{1}{l}\Phi_2^{\mathrm{T}}\Psi\vartheta = \begin{bmatrix} a_0\mathbb{E}[y(k-1)w(k)] + a_1\mathbb{E}[y(k-1)w(k-1)] \\ -a_0\mathbb{E}[u(k-1)w(k)] - a_1\mathbb{E}[u(k-1)w(k-1)] \end{bmatrix}. \quad (86)$$

Furthermore, computing the expectations in R and (86) we have

$$\mathbb{E}[y^2(k)] = \frac{b_1^2\sigma_u^2}{a_0^2 - a_1^2} + \sigma_w^2, \quad (87)$$

$$\mathbb{E}[y(k-1)u(k-1)] = 0, \quad (88)$$

$$\lim_{l\to\infty} M = \begin{bmatrix} \mathbb{E}[y^2(k)] & \mathbb{E}[y(k)y(k-1)] & -\mathbb{E}[y(k)u(k-1)] \\ \mathbb{E}[y(k)y(k-1)] & \mathbb{E}[y^2(k-1)] & -\mathbb{E}[y(k-1)u(k-1)] \\ -\mathbb{E}[y(k)u(k-1)] & -\mathbb{E}[y(k-1)u(k-1)] & \mathbb{E}[u^2(k-1)] \end{bmatrix}. \quad (100)$$

$$\mathbb{E}[u^2(k)] = \sigma_u^2, \quad (89)$$

$$\mathbb{E}[y(k-1)w(k)] = 0, \quad (90)$$

$$\mathbb{E}[y(k-1)w(k-1)] = \sigma_w^2, \quad (91)$$

$$\mathbb{E}[u(k-1)w(k)] = 0, \quad (92)$$

$$\mathbb{E}[u(k-1)w(k-1)] = 0. \quad (93)$$

Substituting (87)–(93) into R and (86), it follows that (85) becomes

$$\delta\tilde{\theta}_\infty = \begin{bmatrix} \frac{b_1^2\sigma_u^2}{a_0^2-a_1^2}+\sigma_w^2 & 0 \\ 0 & \sigma_u^2 \end{bmatrix}^{-1}\begin{bmatrix} a_1\sigma_w^2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{a_1\sigma_w^2(a_0^2-a_1^2)}{(b_1^2\sigma_u^2+(a_0^2-a_1^2)\sigma_w^2)} \\ 0 \end{bmatrix}. \quad (94)$$

Furthermore, we define the *signal-to-noise ratio* ς by

$$\varsigma \overset{\Delta}{=} \sqrt{\sum_{k=0}^{l} y^2(k)/\sum_{k=0}^{l} w^2(k)}, \quad (95)$$

so that

$$\lim_{l\to\infty}\varsigma = \sqrt{\frac{\mathbb{E}[y^2(k)]}{\mathbb{E}[w^2(k)]}} = \sqrt{\frac{b_1^2\sigma_u^2}{(a_0^2-a_1^2)\sigma_w^2}+1}. \quad (96)$$

Therefore, using (96), (94) becomes

$$\delta\tilde{\theta}_\infty = \begin{bmatrix} \frac{1}{\varsigma^2} \\ 0 \end{bmatrix}. \quad (97)$$

Note that the estimate of b_1 is consistent, while the estimate of a_1 is biased.

Next, using QCLS identification, we assume that R_{ww} is known to within a scalar multiple. Since $R_{ww} = \sigma_w^2 I_{n+1}$ in this example, and we choose

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (98)$$

Using Theorem 7.2 and Corollary 7.3 we write

$$(M - \alpha_{\max}N)\theta = 0. \quad (99)$$

We note that

Noting $\mathbb{E}[y^2(k-1)] = \mathbb{E}[y^2(k)]$ and

$$\mathbb{E}[y(k)y(k-1)] = \frac{-a_1b_1^2\sigma_u^2}{a_0(a_0^2-a_1^2)}, \quad (101)$$

$$\mathbb{E}[y(k)u(k-1)] = \frac{b_1^2\sigma_u^2}{a_0}, \quad (102)$$

we substitute (87)–(93), (101) and (102) into (100) and (99). It then follows that $\hat{\theta}_\infty$ satisfies

$$\begin{bmatrix} \frac{b_1^2\sigma_u^2}{(a_0^2-a_1^2)}+\sigma_w^2-\alpha_{\max} & \frac{-a_1b_1^2\sigma_u^2}{a_0(a_0^2-a_1^2)} & \frac{-b_1\sigma_u^2}{a_0} \\ \frac{-a_1b_1^2\sigma_u^2}{a_0(a_0^2-a_1^2)} & \frac{b_1^2\sigma_u^2}{(a_0^2-a_1^2)}+\sigma_w^2-\alpha_{\max} & 0 \\ \frac{-b_1\sigma_u^2}{a_0} & 0 & \sigma_u^2 \end{bmatrix}\hat{\theta}_\infty = 0. \quad (103)$$

Solving (103) for $\alpha_{\max}$, we obtain

$$\alpha_{\max} = \sigma_w^2, \quad \text{or} \quad \alpha_{\max} = \sigma_w^2 + \frac{b_1^2\sigma_u^2(a_0^2+a_1^2)}{a_0^2(a_0^2-a_1^2)}. \quad (104)$$

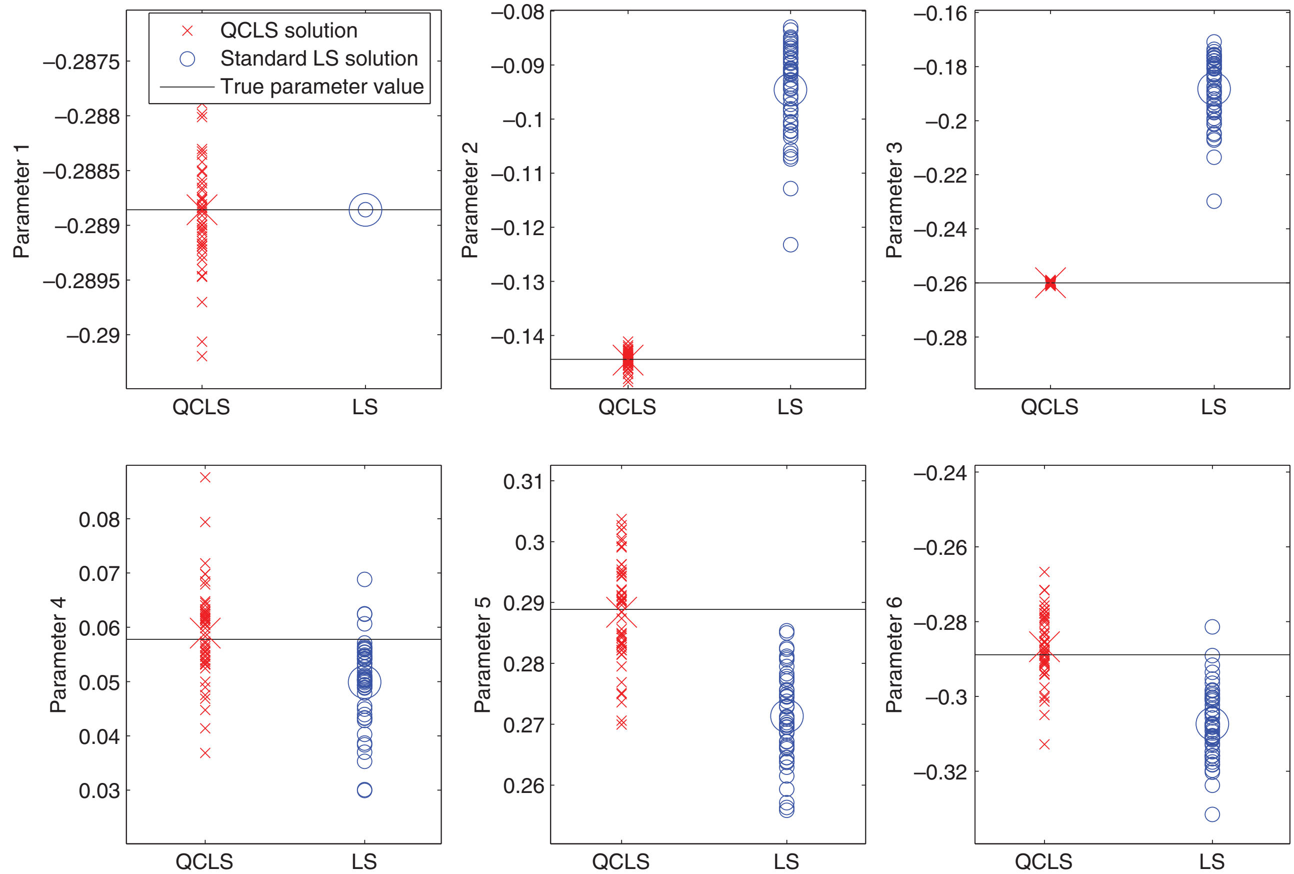

Figure 1. Numerical example. Comparison of estimates of system parameters obtained using standard least squares and using QCLS, for $l=1000$. The large $\times$ and $\bigcirc$ represent the average of the estimates over 50 runs for the QCLS and standard least-squares solutions, respectively. The averages of the QCLS estimates over 50 runs are close to the true parameter values, which suggests that the QCLS solutions are unbiased.

Furthermore, since the system is stable, $a_1 < a_0$, and thus $\frac{b_1^2\sigma_u^2(a_0^2+a_1^2)}{a_0^2(a_0^2-a_1^2)} > 0$. Hence $\alpha_{\max} = \sigma_w^2$. By substituting $\alpha_{\max}$ into (103) and solving for $\hat{\theta}_\infty$ we obtain

$$\hat{\theta}_\infty = \frac{\theta_0}{a_0}\vartheta, \tag{105}$$

where a_0 is a nonzero scalar. Therefore, using Remark 2.1, it follows that $G(\mathbf{q}^{-1};\vartheta) = G(\mathbf{q}^{-1};\hat{\theta}_\infty)$. Hence, in accordance with Theorem 10.11, QCLS with $N=\eta R$ is a consistent generator of ϑ.

12. A numerical example illustrating unbiasedness and consistency

Consider the stable transfer function

$$G(\mathbf{q}^{-1};\vartheta) = \frac{-0.2 + \mathbf{q}^{-1} - \mathbf{q}^{-2}}{-1 - 0.5\mathbf{q}^{-1} - 0.9\mathbf{q}^{-2}}, \tag{106}$$

where $\vartheta = [-1\ -0.5\ -0.9\ 0.2\ 1\ -1]^{\mathrm{T}}$ and $\theta = [\theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5\ \theta_6]^{\mathrm{T}}$. We construct $\{u_0(k)\}_{k=0}^{l}$ to the sum of two sinusoids at frequencies 2π rad/s and 1.5π rad/s as well as a realisation of a zero-mean white noise sequence with standard deviation 1. The input $\{u_0(k)\}_{k=0}^{l}$ is corrupted by white noise $v(k)$ with standard deviation 0.04, while the output $\{y_0(k)\}_{k=0}^{l}$ is corrupted by white noise $w(k)$ with standard deviation 0.4. Thus the noise covariance matrix R is

$$R = \begin{bmatrix} 0.16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0016 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0016 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0016 \end{bmatrix}. \tag{107}$$

Since we use $N=R$ in the QCLS problem, the normalised parameter vector is

$$\begin{aligned}\vartheta_0 &= \sqrt{\frac{1}{\vartheta^{\mathrm{T}} R \vartheta}}\vartheta \\ &= [-0.2888\ -0.1444\ -0.2599\ 0.0577\ 0.2888\ -0.2888]^{\mathrm{T}}. \end{aligned} \tag{108}$$

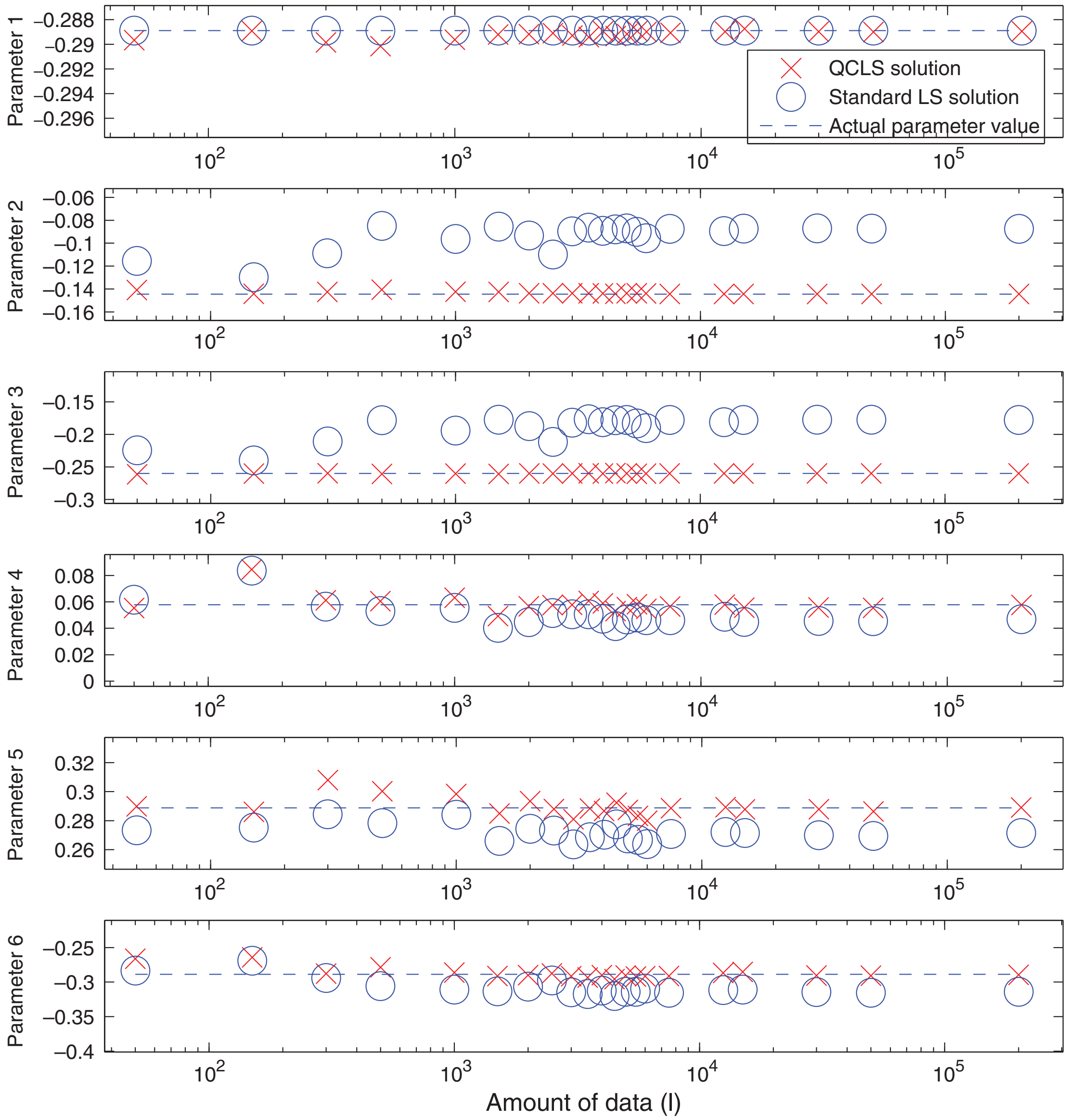

Figure 2. Numerical example. Comparison of estimates of system parameters obtained using standard least squares and using QCLS, as a function of the amount of data. It is seen that the QCLS estimates converge to the true parameters as l becomes large.

We set $l = 1000$, perform 50 runs with different realisations of $w(k)$ and $v(k)$ and compute the QCLS estimates as determined by Theorem 7.2 along with the standard least-squares estimates. Since standard least squares is equivalent to normalising θ_1 to ± 1, we scale the least-squares estimates so that θ_1 matches the first component of ϑ_0. Figure 1 shows that the averages of QCLS estimates over the 50 runs are close to the true parameter, indicating that the QCLS estimates are unbiased.

Next, we vary l from 50 to 8×10^5 and compute the QCLS solutions as determined by Theorem 7.2. We compare these estimates to parameter estimates obtained by standard least-squares. Figure 2 shows a comparison between the QCLS estimates and the standard least-squares estimates of all six parameters for increasing l. It is seen that the QCLS estimates converge to the true parameters as l becomes large.

13. Conclusions

In this article, we investigated the consistency of parameter estimates obtained from least-squares identification with a quadratic parameter constraint. For generality, we considered infinite impulse-response systems with coloured and possibly correlated input and output noise. In the case of finite data, we showed that there always exists a possibly indefinite quadratic constraint depending on the noise realisation that results in a generator that yields the true parameters of the system when a persistency condition is satisfied. When the noise covariance matrix is known to within a scalar multiple, we showed that the QCLSs estimator with a semidefinite constraint matrix yields is unbiased and consistent in the sense that the averaged problem and limiting problem produce, respectively, unbiased and true (with probability 1) estimators. We thus provided the missing foundation for the KL method and its numerous variants in the literature, while providing a complete development of unbiasedness and consistency in a precise sense. Future work will investigate whether the QCLS estimator is unbiased and consistent in the traditional sense.

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