Asymptotic Smooth Stabilization of the Inverted 3D Pendulum

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***Abstract*—The 3D pendulum consists of a rigid body, supported at a fixed pivot, with three rotational degrees of freedom; it is acted on by gravity and it is fully actuated by control forces. The 3D pendulum has two disjoint equilibrium manifolds, namely a hanging equilibrium manifold and an inverted equilibrium manifold. The contribution of this paper is that two fundamental stabilization problems for the inverted 3D pendulum are posed and solved. The first problem, asymptotic stabilization of a specified equilibrium in the inverted equilibrium manifold, is solved using smooth and globally defined feedback of angular velocity and attitude of the 3D pendulum. The second problem, asymptotic stabilization of the inverted equilibrium manifold, is solved using smooth and globally defined feedback of angular velocity and a reduced attitude vector of the 3D pendulum. These control problems for the 3D pendulum exemplify attitude stabilization problems on the configuration manifold $\mathrm{SO}(3)$ in the presence of potential forces. Lyapunov analysis and nonlinear geometric methods are used to assess global closed-loop properties, yielding a characterization of the almost global domain of attraction for each case.**

***Index Terms*—3D pendulum, equilibrium manifold, attitude control, gravity potential, almost global stabilization.**

I. Introduction

PENDULUM models have provided a rich source of examples in nonlinear dynamics and, in recent decades, in nonlinear control. The most common rigid pendulum model consists of a mass particle attached to one end of a massless, rigid link; the other end of the link is fixed to a pivot point that provides a rotational joint for the link and mass particle. If the link and mass particle are constrained to move within a fixed plane, the system is a planar 1D pendulum. If the link and mass particle are unconstrained, the system is a spherical 2D pendulum. Control problems for planar and spherical pendulum models have been studied in [1]–[11].

Numerous extensions of simple pendulum models have been proposed. These include various elastic pendulum models and multi-body pendulum models. Interesting examples of multi-body pendulum models are: a pendulum on a cart, an acrobot, a pendubot, a pendulum actuated by a reaction wheel, the Furuta pendulum, and pendula consisting of multiple coupled bodies. Dynamics and control problems for these multi-body pendulum models have been studied in [10], [12], [13], [14], [15], [16], [17], [18].

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Pendulum models are useful for both pedagogical and research reasons. They represent simplified versions of mechanical systems arising in robotics and spacecraft. In addition to their role in demonstrating the foundations of nonlinear dynamics and control, pendulum models have motivated research in nonlinear dynamics and nonlinear control. In [19], controllers for pendulum problems with applications to control of oscillations have been presented using the speed-gradient method.

The 3D pendulum is a rigid body supported at a fixed pivot point with three rotational degrees of freedom. It is acted on by a uniform gravity force and, perhaps, by control and disturbance forces. The 3D pendulum was introduced in [20], and preliminary stabilization results were presented in [21]. The 3D pendulum has two equilibrium manifolds, namely, the hanging and inverted equilibrium manifolds. This paper treats two stabilization problems for stabilization of the inverted 3D pendulum. The first part of the paper studies stabilization of a specified inverted equilibrium in the inverted equilibrium manifold using angular velocity and attitude feedback. The second part of the paper studies stabilization of the inverted equilibrium manifold using angular velocity and *reduced* attitude feedback. These control problems for the 3D pendulum exemplify attitude stabilization problems on the configuration manifold $\mathrm{SO}(3)$ in the presence of potential forces. Stabilization of the inverted equilibrium manifold is distinct, in terms of the physical meaning and in terms of the nonlinear control details, from the problem of stabilizing an equilibrium that lies in the inverted equilibrium manifold.

The controllers designed for each case provide asymptotic stabilization with local exponential convergence. Analysis of the closed-loops shows that the domains of attraction are *almost global*. By *almost global*, we mean that the domain of attraction of the equilibrium is open and dense. In [18] almost global stability of the inverted equilibrium of a simple planar pendulum was studied. The proposed controller involved a switching strategy; however unlike previous results, the nonlinear controller renders the inverted equilibrium not only attractive, but also stable. As mentioned in [18], this could only be carried out for a simple planar pendulum because of its simple phase space given by $\mathbb{R} \times S^1$. In this paper, we present a single smooth nonlinear controller, which involves no switching, that achieves almost global asymptotic stability of the 3D inverted pendulum.

It may be noted that the work in [22], [23], wherein controllers for systems evolving on Riemannian manifolds were proposed, does not directly apply to the stabilization of an equilibrium manifold of the 3D pendulum. Furthermore, in contrast with the controllers in [22], [23] that generally give a conservative estimate of the domain of attraction, we provide almost global asymptotic stabilization results for the 3D pendulum. Indeed, one of the aims of this paper is to give a complete picture of the global dynamics for the closed-loop 3D pendulum.

This paper arose out of our continuing research on a laboratory facility, namely, the Triaxial Attitude Control Testbed (TACT). The TACT provides a testbed for physical experiments on attitude dynamics and attitude control. This device is supported by a spherical air bearing that serves as an ideal frictionless pivot, allowing nearly unrestricted motion in three degrees of freedom. Issues of nonlinear dynamics for the TACT have been treated in [24], [25], [26], and stability and control issues have been treated in [27], [28], [29], [30], [31], [32]. The present paper is partly motivated by the realization that the TACT is, in fact, a physical implementation of a 3D pendulum.

II. Mathematical Models of the 3D Pendulum

The 3D pendulum is a rigid body supported by a fixed, frictionless pivot, acted on by constant uniform gravity as well as control forces. Two Euclidean reference frames are introduced. An inertial frame has its origin at the pivot; the first two axes lie in the horizontal plane while the third axis is vertical and points in the direction of gravity. A second frame with origin at the pivot point is fixed to the pendulum body. In this body fixed frame, the moment of inertia matrix of the pendulum is constant.

Rotation matrices, which provide global representations of attitude, are used to describe the attitude of the 3D pendulum. In this paper, we follow the convention in which a rotation matrix maps representations of vectors resolved in the body-fixed frame to representations resolved in the inertial frame. Although attitude representations such as exponential coordinates, quaternions, and Euler angles can be used, each of these representations has a disadvantage due to an ambiguity or singularity [33]. Therefore, the attitude of the 3D pendulum is represented by a rotation matrix R, viewed as an element of the special orthogonal group $\mathrm{SO}(3)$. The angular velocity of the 3D pendulum with respect to the inertial frame, resolved in the body-fixed frame, is denoted by ω in $\mathbb{R}^3$. Although global representations are used, the feedback controllers proposed in this paper could be expressed in terms of feedback using any other attitude representation, such as Euler angle or quaternions.

The constant inertia matrix, resolved in the body-fixed frame, is denoted by J. The vector from the pivot to the center of mass of the 3D pendulum, resolved in the body-fixed frame, is denoted by ρ. The symbol g denotes the constant acceleration due to gravity.

Standard techniques yield the equations of motion for the 3D pendulum. The dynamics are given by the Euler-Poincaré equation which includes the moment due to gravity and a control moment $u \in \mathbb{R}^3$ which represents the control torque applied to the 3D pendulum, resolved in the body-fixed frame,

$$J\dot{\omega} = J\omega \times \omega + mg\rho \times R^{\mathrm{T}} e_3 + u, \tag{1}$$

where $e_3 = [0\ 0\ 1]^{\mathrm{T}}$. The rotational kinematics equation is

$$\dot{R} = R\widehat{\omega}, \tag{2}$$

where $R \in \mathrm{SO}(3)$, $\omega \in \mathbb{R}^3$ and

$$\widehat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{3}$$

Note that $a \times b = \widehat{a}b$.

The equations of motion (1) and (2) for the 3D pendulum model has dynamics that evolve on the tangent bundle $T\mathrm{SO}(3)$ [35]. Note that since $e_3 = [0\ 0\ 1]^{\mathrm{T}}$ denotes the unit vector in the direction of gravity in the specified inertial frame, $R^{\mathrm{T}} e_3$ in (1) denotes the dimensionless unit vector in the direction of gravity resolved in the body-fixed frame.

In the case where the center of mass of the 3D pendulum is located at the pivot, $\rho = 0$, equation (1) simplifies to the Euler equation with no gravity terms. In the context of the 3D pendulum, this is referred to as the balanced case. In this paper we focus on the more interesting unbalanced case, where $\rho \neq 0$.

III. Equilibrium Structure of the Uncontrolled 3D Pendulum

In this section, we set $u = 0$ and obtain two integrals of motion for the uncontrolled 3D pendulum. These integrals expose the unforced dynamics of the 3D pendulum and can be used to construct control-Lyapunov functions.

There are two conserved quantities for the 3D pendulum. First, the total energy, which is the sum of the rotational kinetic energy and the gravitational potential energy, is conserved. The other conserved quantity is the component of angular momentum about the vertical axis through the pivot.

Proposition 1 ([20], [26]): Let $u = 0$ in (1). The total energy $E = \frac{1}{2}\omega^{\mathrm{T}} J\omega - mg\rho^{\mathrm{T}} R^{\mathrm{T}} e_3$ and the component of the angular momentum vector about the vertical axis through the pivot given by $h = \omega^{\mathrm{T}} J R^{\mathrm{T}} e_3$ are constant along motions of the 3D pendulum given by (1) and (2).

To further understand the dynamics of the 3D pendulum, we study the equilibria of (1) and (2). Equating the RHS of (1) and (2) to zero with $u = 0$ yields

$$J\omega_{\mathrm{e}} \times \omega_{\mathrm{e}} + mg\rho \times R_{\mathrm{e}}^{\mathrm{T}} e_3 = 0, \tag{4}$$

$$R_{\mathrm{e}}\widehat{\omega}_{\mathrm{e}} = 0. \tag{5}$$

Now $R_{\mathrm{e}}\widehat{\omega}_{\mathrm{e}} = 0$ if and only if $\omega_{\mathrm{e}} = 0$. Substituting $\omega_{\mathrm{e}} = 0$ in (4), we obtain

$$\rho \times R_{\mathrm{e}}^{\mathrm{T}} e_3 = 0. \tag{6}$$

Hence,

$$R_{\mathrm{e}}^{\mathrm{T}} e_3 = \frac{\rho}{\|\rho\|}, \tag{7}$$

or

$$R_{\mathrm{e}}^{\mathrm{T}} e_3 = -\frac{\rho}{\|\rho\|}. \tag{8}$$

Hence an attitude R_{e} is an equilibrium attitude if and only if the direction of gravity resolved in the body-fixed frame, $R_{\mathrm{e}}^{\mathrm{T}} e_3$, is collinear with the vector ρ. If $R_{\mathrm{e}}^{\mathrm{T}} e_3$ is in the same direction as the vector ρ, then $(0, R_\mathrm{e})$ where R_e satisfies (7), is a hanging equilibrium of the 3D pendulum; if $R_\mathrm{e}^\mathrm{T} e_3$ is in the opposite direction as the vector ρ, then $(0, R_\mathrm{e})$ where R_e satisfies (8), is an inverted equilibrium of the 3D pendulum.

According to (7) and (8), there is a smooth manifold of hanging equilibria and a smooth manifold of inverted equilibria, and these two equilibrium manifolds are clearly distinct. The former is the hanging equilibrium manifold; the latter is the inverted equilibrium manifold.

IV. Asymptotic Stabilization of a Specified Inverted Equilibrium

Let $(0, R_\mathrm{d})$ denote a specified equilibrium in the inverted equilibrium manifold of the 3D pendulum given by (1) and (2). In this section we present controllers that stabilize this specified equilibrium $(0, R_\mathrm{d})$.

Let $\Phi : [0,\infty) \to [0,\infty)$ be a C^2 function such that

$$\Phi(0) = 0 \;\text{ and }\; \Phi'(x) > 0 \text{ for all } x \in [0,\infty). \tag{9}$$

Let $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 function satisfying

$$\left.\begin{array}{l}\Psi'(0) \text{ is positive definite,}\\ \mathscr{P}(x) \le x^\mathrm{T}\Psi(x) \le \alpha(\|x\|) \text{ for all } x \in \mathbb{R}^3,\end{array}\right\} \tag{10}$$

where $\mathscr{P} : \mathbb{R}^3 \to \mathbb{R}$ is a positive definite function and $\alpha(\cdot)$ is a class-$\mathcal{K}$ function [36]. Given $\mathbf{a} = [a_1\ a_2\ a_3]^\mathrm{T} \in \mathbb{R}^3$, denote

$$\Omega_\mathbf{a}(R) \triangleq \sum_{i=1}^{3} a_i \Big[(R_\mathrm{d}^\mathrm{T} e_i) \times (R^\mathrm{T} e_i)\Big]. \tag{11}$$

Further, let $A \in \mathbb{R}^{3\times 3}$, be a diagonal matrix defined as

$$A \triangleq \mathrm{diag}(\mathbf{a}). \tag{12}$$

We study controllers of the form

$$u = -\Psi(\omega) + \kappa\Big((R_\mathrm{d}^\mathrm{T} e_3) \times (R^\mathrm{T} e_3)\Big) + \Phi'\Big(\mathrm{trace}(A - AR_\mathrm{d}R^\mathrm{T})\Big)\Omega_\mathbf{a}(R), \tag{13}$$

where $\kappa \ge mg\|\rho\|$.

The controller (13) requires measurements of the angular velocity and attitude, in the form of the rotation matrix R, of the 3D pendulum. The angular velocity dependent term $\Psi(\omega)$ in (13) provides damping, while the attitude dependent term in (13) can be viewed as a modification or shaping of the gravity potential. For the control law (13), no knowledge is required of the moment of inertia or of the location of the center of mass of the 3D pendulum relative to the pivot. However, the constant κ is an upper bound on the gravity moment about the pivot. Hence a bound on $mg\|\rho\|$ must be known.

We subsequently show that $(\omega, R) = (0, R_\mathrm{d})$ is an equilibrium of the closed-loop consisting of (1), (2) and (13), and it is almost globally asymptotically stable with locally exponential convergence.

A. Equilibrium Structure of the Closed-Loop

In this section, we study the equilibria in $T\mathrm{SO}(3)$ of the closed-loop system consisting of (1), (2) and (13). Define

$$\bar{\mathbf{a}} \triangleq [a_1\ a_2\ \bar{a}_3]^\mathrm{T}, \tag{14}$$

where

$$\bar{a}_3 \triangleq a_3 + \frac{\kappa - mg\|\rho\|}{\Phi'\Big(\mathrm{trace}(A - AR_\mathrm{d}R^\mathrm{T})\Big)} \ge a_3. \tag{15}$$

Since $(0, R_\mathrm{d})$ lies in the inverted equilibrium manifold, it follows from (8) that $R_\mathrm{d}^\mathrm{T} e_3 = -\dfrac{\rho}{\|\rho\|}$. Then, substituting (13) in (1) and (2), and simplifying, we can express the closed-loop system as

$$\left.\begin{array}{l}J\dot{\omega} = J\omega \times \omega - \Psi(\omega) + \Phi'\Big(\mathrm{trace}(A - AR_\mathrm{d}R^\mathrm{T})\Big)\Omega_{\bar{\mathbf{a}}}(R),\\ \dot{R} = R\widehat{\omega}.\end{array}\right\} \tag{16}$$

Lemma 1: Consider the closed-loop system (16) of a 3D pendulum given by (1) and (2), with controller (13), where the functions Φ and Ψ satisfy (9) and (10), $\kappa \ge mg\|\rho\|$ and A defined in (12) satisfies $a_1, a_2 < a_3$ where a_1 and a_2 are distinct positive numbers. Then, the closed-loop system (16) has four equilibrium solutions given by

$$\mathscr{E} = \Big\{(\omega, R) \in T\mathrm{SO}(3) : \omega = 0, R = MR_\mathrm{d},\ M \in \mathcal{M}_c\Big\}, \tag{17}$$

where

$$\mathcal{M}_c \triangleq \Big\{\mathrm{diag}(1,1,1),\ \mathrm{diag}(-1,1,-1), \mathrm{diag}(1,-1,-1), \mathrm{diag}(-1,-1,1)\Big\}. \tag{18}$$

Proof: To obtain the equilibria of the closed-loop system, equate the RHS of (16) to zero, which yields $\omega = 0$ and

$$\Omega_{\bar{\mathbf{a}}}(R) = a_1 \widehat{R_\mathrm{d}^\mathrm{T} e_1} R^\mathrm{T} e_1 + a_2 \widehat{R_\mathrm{d}^\mathrm{T} e_2} R^\mathrm{T} e_2 + \bar{a}_3 \widehat{R_\mathrm{d}^\mathrm{T} e_3} R^\mathrm{T} e_3 = 0. \tag{19}$$

Next multiplying both sides by R_d and using the equality $\widehat{Re_i} = R\widehat{e_i}R^\mathrm{T}$ [37], we obtain

$$a_1 \widehat{e_1} R_\mathrm{d} R^\mathrm{T} e_1 + a_2 \widehat{e_2} R_\mathrm{d} R^\mathrm{T} e_2 + \bar{a}_3 \widehat{e_3} R_\mathrm{d} R^\mathrm{T} e_3 = 0. \tag{20}$$

Writing $R_\mathrm{d}R^\mathrm{T} = [r_{i,j}]_{i,j\in\{1,2,3\}}$, equation (20) can be expressed as $a_2 r_{32} = \bar{a}_3 r_{23},\ a_1 r_{31} = \bar{a}_3 r_{13},\ a_1 r_{21} = a_2 r_{12}$. Then, since a_1, a_2 and $\bar{a}_3$ are positive, the rotation matrix $R_\mathrm{d}R^\mathrm{T}$ can be expressed as

$$R_\mathrm{d}R^\mathrm{T} = \begin{bmatrix} r_{11} & r_{12} & r_{13}\\ \frac{a_2}{a_1} r_{12} & r_{22} & r_{23}\\ \frac{\bar{a}_3}{a_1} r_{13} & \frac{\bar{a}_3}{a_2} r_{23} & r_{33}\end{bmatrix}, \tag{21}$$

Then it follows from orthogonality of rotation matrices and algebraic manipulations that

$$\left.\begin{array}{l}\left(1 - \dfrac{a_2^2}{a_1^2}\right) r_{12}^2 + \left(1 - \dfrac{\bar{a}_3^2}{a_1^2}\right) r_{13}^2 = 0,\\ \left(1 - \dfrac{\bar{a}_3^2}{a_1^2}\right) r_{13}^2 + \left(1 - \dfrac{\bar{a}_3^2}{a_2^2}\right) r_{23}^2 = 0,\\ \left(1 - \dfrac{a_2^2}{a_1^2}\right) r_{12}^2 - \left(1 - \dfrac{\bar{a}_3^2}{a_2^2}\right) r_{23}^2 = 0.\end{array}\right\} \tag{22}$$

Since $\bar{a}_3 \geq a_3 > a_1, a_2$, therefore a_1, a_2 and $\bar{a}_3$ are distinct positive integers and hence it can be easily show that the solution to (22) is given by $r_{12} = r_{13} = r_{23} = 0$. Hence from (21), $R_{\rm d}R^{\rm T}$ is one of the four matrices given in (18). ■

Remark 1: Note that the desired equilibrium $(0, R_{\rm d}) \in \mathscr{E}$. Each of the other three equilibrium solutions in $\mathscr{E}$ corresponds to an attitude configuration formed by the desired attitude R_d, followed by a rotation about one of the three body fixed axes by 180 degrees.

B. Local Analysis of the Closed-Loop

Consider a perturbation of the initial conditions about an equilibrium $(0, R_{\rm e}) \in \mathscr{E}$ given in (17) in terms of a perturbation parameter $\varepsilon \in \mathbb{R}$. We express the perturbation in the rotation matrix using exponential coordinates [35], [37], [38]. Let the perturbation in the initial condition for attitude be given as $R(0,\varepsilon) = R_{\rm e} e^{\varepsilon \widehat{\Theta}_0}$, where $R_{\rm e}R_{\rm d}^{\rm T} \in \mathcal{M}_c$ and $\Theta_0 \in \mathbb{R}^3$ is a constant vector. The perturbation in the initial condition for angular velocity is given as $\omega(0,\varepsilon) = \varepsilon\omega_0$, where $\omega_0 \in \mathbb{R}^3$ is a constant vector. Note that if $\varepsilon = 0$ then, $(\omega(0,0), R(0,0)) = (0, R_{\rm e})$ and hence

$$(\omega(t,0), R(t,0)) \equiv (0, R_{\rm e}) \tag{23}$$

for all time $t \in \mathbb{R}$. This simply represents the unperturbed equilibrium solution.

Next, consider the solution to the perturbed equations of motion for the closed-loop 3D pendulum given by (16). These are given by

$$\begin{aligned} J\dot{\omega}(t,\varepsilon) &= J\omega(t,\varepsilon) \times \omega(t,\varepsilon) - \Psi(\omega(t,\varepsilon)) \\ &\quad + \Phi'\Big(\mathrm{trace}(A - AR_{\rm d}R^{\rm T}(t,\varepsilon))\Big)\Omega_{\bar{\mathbf{a}}}(R(t,\varepsilon)), \quad (24) \\ \dot{R}(t,\varepsilon) &= R(t,\varepsilon)\widehat{\omega}(t,\varepsilon). \quad (25) \end{aligned}$$

Differentiating both sides with respect to ε and substituting $\varepsilon = 0$, yields

$$\begin{aligned} J\dot{\omega}_\varepsilon(t,0) &= -\Psi'(0)\omega_\varepsilon(t,0) \\ &\quad + \Phi'(\mathrm{trace}(A - AR_{\rm d}R_{\rm e}^{\rm T}))\Omega_{\bar{\mathbf{a}}}(R_\varepsilon(t,0)), \quad (26) \\ \dot{R}_\varepsilon(t,0) &= R_{\rm e}\widehat{\omega}_\varepsilon(t,0), \quad (27) \end{aligned}$$

where $\omega_\varepsilon(t,0) \triangleq \left.\dfrac{\partial\,\omega(t,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}$ and $R_\varepsilon(t,0) \triangleq \left.\dfrac{\partial R(t,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}$.

Define linearization variables $\Delta\omega, \Delta\Theta \in \mathbb{R}^3$ as $\Delta\omega(t) \triangleq \omega_\varepsilon(t,0)$ and $\widehat{\Delta\Theta}(t) \triangleq R_{\rm e}^{\rm T} R_\varepsilon(t,0)$. Then from (27) we obtain

$$\Delta\dot{\Theta} = \Delta\omega, \tag{28}$$

and from (26), we obtain

$$J\Delta\dot{\omega} = -\Psi'(0)\Delta\omega + \Phi'(\mathrm{trace}(A - AR_{\rm d}R_{\rm e}^{\rm T}))\Omega_{\bar{\mathbf{a}}}(R_{\rm e}\widehat{\Delta\Theta}). \tag{29}$$

Combining (28) and (29) and simplifying, we obtain the linearization of (16) as

$$J\Delta\ddot{\Theta} + \Psi'(0)\,\Delta\dot{\Theta} + \mathsf{K}\Delta\Theta = 0, \tag{30}$$

where

$$\begin{aligned} \mathsf{K} = \Phi'\Big(\mathrm{trace}(A - AR_{\rm d}R_{\rm e}^{\rm T})\Big)\Bigg[&- a_1\widehat{R_{\rm d}^{\rm T}e_1}\,\widehat{R_{\rm e}^{\rm T}e_1} \\ &- a_2\widehat{R_{\rm d}^{\rm T}e_2}\,\widehat{R_{\rm e}^{\rm T}e_2} - \bar{a}_3\widehat{R_{\rm d}^{\rm T}e_3}\,\widehat{R_{\rm e}^{\rm T}e_3}\Bigg], \end{aligned} \tag{31}$$

and $\bar{a}_3$ is given in (15) with $R = R_{\rm e}$. Since $M = R_{\rm e}R_{\rm d}^{\rm T} \in \mathcal{M}_c$, the equality $\widehat{Re_i} = R\widehat{e}_iR^{\rm T}$, where $R \in \mathrm{SO}(3)$ [37], yields

$$\widehat{R_{\rm d}^{\rm T}e_i}\,\widehat{R_{\rm e}^{\rm T}e_i} = \widehat{R_{\rm e}^{\rm T}Me_i}\,\widehat{R_{\rm e}^{\rm T}e_i} = R_{\rm e}^{\rm T}\widehat{Me_i}\,\widehat{e}_iR_{\rm e},$$

for $i \in \{1,2,3\}$. Using the above, the expression for K in (31) can be written as

$$\mathsf{K} = \Phi'\Big(\mathrm{trace}(A - AR_{\rm d}R_{\rm e}^{\rm T})\Big)R_{\rm e}^{\rm T}\,\mathcal{Q}\,R_{\rm e}, \tag{32}$$

where

$$\mathcal{Q} = -a_1\widehat{Me_1}\,\widehat{e}_1 - a_2\widehat{Me_2}\,\widehat{e}_2 - \bar{a}_3\widehat{Me_3}\,\widehat{e}_3 \tag{33}$$

and $M = R_{\rm e}R_{\rm d}^{\rm T} \in \mathcal{M}_c$, as in (18).

Lemma 2: Consider the closed-loop model of a 3D pendulum given by (1) and (2), with controller (13), where the functions Φ and Ψ satisfy (9) and (10), $\kappa \geq mg\|\rho\|$ and A defined in (12) satisfies $a_1, a_2 < a_3$ where a_1, a_2 and a_3 are distinct positive numbers. Then the closed-loop equilibrium $(0, R_{\rm d}) \in \mathscr{E}$ is asymptotically stable and the convergence is locally exponential.

Proof: Combining equations (1), (2) and (13), we obtain the closed-loop system given by (16). Next, we linearize the dynamics of (16) about the equilibrium $(0, R_{\rm d})$ yielding equation (30) where $R_{\rm e} = R_{\rm d}$.

Now $\Psi'(0)$ is positive definite and M is the identity matrix. Hence, from (33)

$$\mathcal{Q} = -a_1\,\widehat{e}_1^{\,2} - a_2\,\widehat{e}_2^{\,2} - \bar{a}_3\,\widehat{e}_3^{\,2}$$

is positive definite. Next, since $\Phi'(\cdot)$ is positive and K is a similarity transform of $\mathcal{Q}$, K in (32) is positive definite. Thus, since K and $\Psi'(0)$ are positive definite, linear theory guarantees that the linearized system given by (30) is asymptotically stable. Hence, the equilibrium $(0, R_{\rm d})$ of (16) is locally asymptotically stable with local exponential convergence. ■

Consider the equilibria $(0, R_{\rm e})$ of the closed-loop (16) such that $R_{\rm e} \neq R_{\rm d}$. From Lemma 1, we express the three equilibria $(0, R_{\rm e}) \in \mathscr{E}$ such that $R_{\rm e} \neq R_{\rm d}$ as $R_{{\rm e},i} = M_iR_{\rm d}$, $i \in \{1,2,3\}$, where M_1, M_2 and M_3 are

$$\mathrm{diag}(1,-1,-1), \quad \mathrm{diag}(-1,1,-1), \quad \mathrm{diag}(-1,-1,1), \tag{34}$$

respectively. We next show that the above three equilibria $(0, R_{{\rm e},i})$, $i \in \{1,2,3\}$ of the closed-loop (16) are unstable and present local properties of the closed-loop trajectories.

Lemma 3: Consider the closed-loop model of a 3D pendulum given by (1) and (2), with controller (13), where the functions Φ and Ψ satisfy (9) and (10), $\kappa \geq mg\|\rho\|$ and A defined in (12) satisfies $a_1, a_2 < a_3$ where a_1 and a_2 are distinct positive numbers. Consider an equilibrium $(0, R_{{\rm e},i}) \in \mathscr{E}$, such that $R_{{\rm e},i} \neq R_{\rm d}$, $i \in \{1,2,3\}$. Then, $(0, R_{{\rm e},i})$ is unstable. Furthermore, there exist an invariant 3-dimensional submanifold and an invariant 4-dimensional submanifold ($\mathcal{M}_1$

and $\mathcal{M}_2$), and an invariant 5-dimensional submanifold $\mathcal{M}_3$ in $T\mathrm{SO}(3)$ with zero Lebesgue measure, such that **(a)** for all initial conditions $(\omega(0), R(0)) \in \mathcal{M}_i$, $i \in \{1, 2, 3\}$, the closed-loop solutions converge to the equilibrium $(0, R_{\mathrm{e},i})$ and **(b)** for all initial conditions $(\omega(0), R(0)) \in T\mathrm{SO}(3)\backslash\mathcal{M}_i$, the closed-loop solutions do not converge to the equilibrium $(0, R_{\mathrm{e},i})$, $i \in \{1, 2, 3\}$.

Proof: Combining equations (1), (2) and (13), we obtain the closed-loop system given by (16). Next, we linearize the dynamics of (16) about the equilibrium $(0, R_{\mathrm{e},i})$, $i \in \{1, 2, 3\}$ yielding equation (30). Since, $R_{\mathrm{e},i} \neq R_{\mathrm{d}}$, the three equilibria are given by $(0, R_{\mathrm{e},i}) = (0, M_i R_{\mathrm{d}})$, where M_i, $i \in \{1, 2, 3\}$ is as given in (34).

Next, we compute the matrix $\mathcal{Q}_i$ using (33) corresponding to the three equilibria $(0, M_i R_{\mathrm{d}})$, $i \in \{1, 2, 3\}$. This yields

$$\begin{aligned}
\mathcal{Q}_1 &= \mathrm{diag}(-a_2 - \bar{a}_3, a_1 - \bar{a}_3, a_1 - a_2),\\
\mathcal{Q}_2 &= \mathrm{diag}(a_2 - \bar{a}_3, -a_1 - \bar{a}_3, -a_1 + a_2),\\
\mathcal{Q}_3 &= \mathrm{diag}(-a_2 + \bar{a}_3, -a_1 + \bar{a}_3, -a_1 - a_2).
\end{aligned}$$

Since $0a_1, a_2 < a_3 \leq \bar{a}_3$ and a_1 and a_2 are distinct positive numbers, all eigenvalues of $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ lie in $\mathbb{R}\backslash\{0\}$ and each of $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ has a negative eigenvalue.

Since $R_{\mathrm{e},i} \in \mathrm{SO}(3)$, it follows from (32) that corresponding to $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$, all eigenvalues of the matrices $\mathsf{K}_1, \mathsf{K}_2$ and K_3 lie in $\mathbb{R}\backslash\{0\}$ and each of $\mathsf{K}_1, \mathsf{K}_2$ and K_3 has a negative eigenvalue. Now, it follows from [38] that (30) is unstable. Hence, each equilibrium $(0, R_{\mathrm{e},i})$, $i \in \{1, 2, 3\}$ of the closed-loop (16) is unstable.

Next, since, $\Psi'(0)$ is positive definite and all eigenvalues of the matrices $\mathsf{K}_1, \mathsf{K}_2$ and K_3 lie in $\mathbb{R}\backslash\{0\}$, it follows that each equilibrium $(0, R_{\mathrm{e},i}) \in \mathscr{E}$, $i \in \{1, 2, 3\}$ of (16) is *hyperbolic*. Theorem 3.2.1 in [39] guarantees that each equilibrium $(0, R_{\mathrm{e},i}) \in \mathscr{E}$ of (16) has a nontrivial unstable manifold W_i^u. Let W_i^s denote its corresponding stable manifold. The tangent space to the stable manifold W_i^s at the equilibrium $(0, R_{\mathrm{e},i})$ is tangent to the stable eigenspace of the linearized system (30), and hence is 3-dimensional and 4-dimensional for either W_1^s or W_2^s, and is 5-dimensional for W_3^s.

Since, the equilibria are hyperbolic, there are no center manifolds. Then, all trajectories near $(0, R_{\mathrm{e},i})$ other than those in W_i^s diverge from that equilibrium. Since the dimension of the submanifold W_i^s is less than the dimension of the tangent bundle $T\mathrm{SO}(3)$, the Lebesgue measure of the global invariant submanifold W_i^s is zero [40]. Denoting $\mathcal{M}_i \triangleq W_i^s$, $i \in \{1, 2, 3\}$, the result follows. ■

C. Global Analysis of the Closed-Loop

In the last subsection, we presented results for local properties of the closed-loop (16) near each of its equilibria. In this subsection, we describe the global convergence properties of closed-loop trajectories.

Theorem 1: Consider the closed-loop model of a 3D pendulum given by (1) and (2), with controller (13), where the functions Φ and Ψ satisfy (9) and (10), $\kappa \geq mg\|\rho\|$ and A defined in (12) satisfies $a_1, a_2 < a_3$ where a_1 and a_2 are distinct positive numbers. Then, $(0, R_{\mathrm{d}})$ is an asymptotically stable equilibrium of the closed-loop (16) with local exponential convergence. Furthermore, there exist an invariant manifold $\mathsf{M} \subset T\mathrm{SO}(3)$, whose Lebesgue measure is zero and whose complement is open and dense such that for all initial conditions $(\omega(0), R(0)) \in T\mathrm{SO}(3)\backslash\mathsf{M}$, the solutions of the closed-loop system given by (16) satisfy $\lim_{t\to\infty} \omega(t) = 0$ and $\lim_{t\to\infty} R(t) = R_{\mathrm{d}}$. For all other initial conditions $(\omega(0), R(0)) \in \mathsf{M}$, the solutions of the closed-loop system given by (16) satisfy $\lim_{t\to\infty} (\omega(t), R(t)) \in \mathscr{E}\backslash\{(0, R_{\mathrm{d}})\}$.

Proof: Consider the closed-loop system consisting of (1), (2) and (13) given by (16). Then, it immediately follows from Lemma 2 that $(0, R_{\mathrm{d}})$ is an asymptotically stable equilibrium of the closed-loop (16) with local exponential convergence.

Next, we propose the following candidate Lyapunov function.

$$\begin{aligned}
V(\omega, R) = \frac{1}{2}\omega^{\mathrm{T}} J\omega + (\kappa - mg\|\rho\|)(1 - e_3^{\mathrm{T}} R_{\mathrm{d}} R^{\mathrm{T}} e_3)&\\
+ \Phi\Big(\mathrm{trace}(A - AR_{\mathrm{d}} R^{\mathrm{T}})\Big).& \qquad (35)
\end{aligned}$$

Note that $V(\omega, R) \geq 0$ for all $(\omega, R) \in T\mathrm{SO}(3)$ and $V(\omega, R) = 0$ if and only if $(\omega, R) = (0, R_{\mathrm{d}})$. Thus $V(\omega, R)$ is a positive definite function on $T\mathrm{SO}(3)$.

We show that the Lie derivative of the Lyapunov function along any solution of the closed-loop vector field of (16) is negative semidefinite. Denote the closed-loop vector field of (16) by Z. Then,

$$\begin{aligned}
&\mathscr{L}_Z \Phi\Big(\mathrm{trace}(A - AR_{\mathrm{d}} R^{\mathrm{T}})\Big)\\
&\quad = -\Phi'\Big(\mathrm{trace}(A - AR_{\mathrm{d}} R^{\mathrm{T}})\Big)[\mathrm{trace}(AR_{\mathrm{d}}(R\widehat{\omega})^{\mathrm{T}})],\\
&\quad = -\Phi'\Big(\mathrm{trace}(A - AR_{\mathrm{d}} R^{\mathrm{T}})\Big)\omega^{\mathrm{T}}\Omega_{\mathbf{a}}(R).
\end{aligned}$$

The derivative of the Lyapunov function along a solution of the closed-loop is

$$\begin{aligned}
\dot{V}(\omega, R) &= \omega^{\mathrm{T}} J\dot{\omega} - (\kappa - mg\|\rho\|)e_3^{\mathrm{T}} R_{\mathrm{d}} \dot{R}^{\mathrm{T}} e_3\\
&\qquad + \mathscr{L}_Z \Phi\Big(\mathrm{trace}(A - AR_{\mathrm{d}} R^{\mathrm{T}})\Big),\\
&= \omega^{\mathrm{T}}\left\{J\omega \times \omega + mg\rho \times R^{\mathrm{T}} e_3 + u\right\} + (\kappa - mg\|\rho\|)e_3^{\mathrm{T}} R_{\mathrm{d}} \widehat{\omega} R^{\mathrm{T}} e_3\\
&\qquad + \mathscr{L}_Z \Phi\Big(\mathrm{trace}(A - AR_{\mathrm{d}} R^{\mathrm{T}})\Big),\\
&= \omega^{\mathrm{T}}\left\{u - \kappa(R_{\mathrm{d}}^{\mathrm{T}} e_3 \times R^{\mathrm{T}} e_3) - \Phi'\Big(\mathrm{trace}(A - AR_{\mathrm{d}} R^{\mathrm{T}})\Big)\Omega_{\mathbf{a}}(R)\right\}.
\end{aligned} \qquad (36)$$

Substituting (13) into (36), we obtain $\dot{V}(\omega, R) = -\omega^{\mathrm{T}}\Psi(\omega) \leq -\mathscr{P}(\|\omega\|)$. Thus, the derivative of the Lyapunov function along a solution of the closed-loop system is negative semidefinite.

Recall that $\Phi(\cdot)$ is a strictly increasing monotone function and $\mathrm{SO}(3)$ is compact. Hence, for any $(\omega(0), R(0)) \in T\mathrm{SO}(3)$, the set

$$\mathscr{K} = \Big\{(\omega, R) \in T\mathrm{SO}(3) : V(\omega, R) \leq V(\omega(0), R(0))\Big\},$$

is a compact, positively invariant set of the closed-loop.

By the invariant set theorem, it follows that all solutions that begin in $\mathscr{K}$ converge to the largest invariant set in $\dot{V}^{-1}(0)$ contained in $\mathscr{K}$. Now, since $\mathscr{P}$ is a positive definite function,

$\dot{V}(\omega, R) \equiv 0$ implies $\omega \equiv 0$. Substituting this into the closed-loop system (16), it can be shown that

$$\dot{V}^{-1}(0) = \Big\{(\omega, R) \in T\mathrm{SO}(3) : \omega \equiv 0, \quad \Omega_{\bar{\mathbf{a}}}(R) \equiv 0\Big\},$$

where $\Omega_{\bar{\mathbf{a}}}(\cdot)$ is as given in (11). Thus, following the same arguments as in Lemma 1, it can be shown that the largest invariant set in $\dot{V}^{-1}(0)$ is given by (17). Note that each of the four points given in (17) correspond to an equilibrium of the closed-loop system in $T\mathrm{SO}(3)$. Hence, all solutions of the closed-loop system converge to one of the equilibrium solutions in $\mathscr{E} \cap \mathscr{K}$, where $\mathscr{E}$ is given in (17).

Next, consider an equilibrium $(0, R_{\mathrm{e},i}) \in \mathscr{E}$ such that $R_{\mathrm{e},i} \neq R_{\mathrm{d}}$, $i \in \{1,2,3\}$. Lemma 3 implies that the solutions of the closed-loop system except for solutions in the invariant submanifolds $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$, whose Lebesgue measure is zero, diverge from the equilibria $(0, R_{\mathrm{e},i})$, $i \in \{1,2,3\}$. Thus, solutions of the closed-loop system for initial conditions that do not lie in $\mathsf{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ must converge to the equilibrium $(0, R_{\mathrm{d}})$. Thus, since the domain of attraction of an asymptotically stable equilibrium is open, M is closed and hence it follows that it is nowhere dense. This follows from the fact that M is a closed subset of $T\mathrm{SO}(3)$ of Lebesgue measure zero. Solutions of the the closed-loop system (16) for initial conditions that lie in $\mathsf{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ converge to one of the equilibrium solutions in $\mathscr{E} \backslash \{(0, R_{\mathrm{d}})\}$. ■

Theorem 1 is the main result on asymptotic stabilization of a specified inverted equilibrium of the 3D pendulum. Under the indicated assumptions, almost global asymptotic stabilization is achieved. This is the best possible result for this stabilization problem, in the sense that global stabilization using smooth feedback is not achievable [33].

The controller expression (13) is quite useful in control design as it allows freedom to arbitrarily design the local dynamics of the closed-loop near the desired inverted equilibrium. It also provides some freedom in shaping the manifold M of solutions that do not converge to the desired inverted equilibrium. In this way, control design can be carried out to achieve both local and global control objectives.

Simulation studies that demonstrate the validity of Theorem 1 and illustrate its use in a control design context are reported in [34]. Due to space limitations, we do not provide the results of those studies in this paper.

V. Asymptotic Stabilization of the Inverted Equilibrium Manifold of the 3D Pendulum

In the last section, we presented a control law that almost globally asymptotically stabilizes a specified inverted equilibrium in the inverted equilibrium manifold. The specific focus of this section is to develop stabilizing controllers for the inverted equilibrium manifold of the 3D pendulum described by (1) and (2).

For the purpose of stabilization of the inverted equilibrium manifold, it is advantageous to study a lower dimensional reduced model for the 3D pendulum. This model is obtained by noting that the dynamics and kinematics equations can be written in terms of the reduced attitude vector $\Gamma = R^{\mathrm{T}} e_3 \in S^2$, which is the unit vector that expresses the direction of gravity in the body-fixed coordinate frame.

Specifically, let Π_ψ denote the S^1-group action $\Pi_\psi : \mathrm{SO}(3) \to \mathrm{SO}(3)$ as $\Pi_\psi(R) = Re^{\widehat{\Gamma}\psi}$, where $\Gamma = R^{\mathrm{T}} e_3$ and $\psi \in [-\pi, \pi)$. Then, the *orbit space* $\mathrm{SO}(3)/S^1$ is the equivalent set of rotations

$$[R] \triangleq \Big\{R' \in \mathrm{SO}(3) : R' = Re^{\widehat{\Gamma}\,\psi},\ \Gamma = R^{\mathrm{T}} e_3,\ \psi \in \mathbb{R}\Big\}. \tag{37}$$

The equivalence relation in (37) is that $R_1 \sim R_2$ if and only if $R_1^{\mathrm{T}} e_3 = R_2^{\mathrm{T}} e_3$ and hence the equivalence relation in (37) can be alternately expressed as

$$[R] \triangleq \Big\{R_s \in \mathrm{SO}(3) : R_s^{\mathrm{T}} e_3 = R^{\mathrm{T}} e_3\Big\}. \tag{38}$$

Thus, for each $R \in \mathrm{SO}(3)$, $[R]$ can be identified with $\Gamma = R^{\mathrm{T}} e_3 \in S^2$ and hence $\mathrm{SO}(3)/S^1 \cong S^2$. Since $T\mathrm{SO}(3) \cong \mathrm{SO}(3) \times \mathbb{R}^3$, Π_ψ induces a projection $\pi : T\mathrm{SO}(3) \to T\mathrm{SO}(3)/S^1$ given as $\pi : (R, \omega) \mapsto ([R], \omega)$, where $[R]$ is as given in (38).

Proposition 2 ([41]): The dynamics of the 3D pendulum given by (1) and (2) induce a flow on the quotient space $T\mathrm{SO}(3)/S^1$ through the projection $\pi : T\mathrm{SO}(3) \to T\mathrm{SO}(3)/S^1$, given by the dynamics

$$J\dot{\omega} = J\omega \times \omega + mg\rho \times \Gamma + u, \tag{39}$$

and the kinematics for the reduced attitude

$$\dot{\Gamma} = \Gamma \times \omega. \tag{40}$$

Furthermore, $T\mathrm{SO}(3)/S^1 \cong S^2 \times \mathbb{R}^3$.

Equations (39) and (40) are expressed in a non-canonical form; they are referred to as the *reduced* attitude dynamics of the 3D pendulum on $T\mathrm{SO}(3)/S^1$.

Let R_{e} denote an attitude rotation that satisfies (6) and define $\Gamma_{\mathrm{e}} = R_{\mathrm{e}}^{\mathrm{T}} e_3$. Then, every attitude in the configuration manifold given by

$$\Big\{R \in \mathrm{SO}(3) \ : \ R = R_{\mathrm{e}}\, e^{\widehat{\Gamma}_e \bar{\psi}},\ \bar{\psi}\ \in \mathbb{R}\Big\} = [R_{\mathrm{e}}], \tag{41}$$

satisfies (6) and hence, defines an equilibrium attitude corresponding to $\omega = 0$. We can use Rodrigues's formula to write

$$e^{\widehat{\Gamma}_e \bar{\psi}} = I_3 + \sin \bar{\psi}\, \widehat{\Gamma}_e + (1 - \cos \bar{\psi}) \widehat{\Gamma}_e^2.$$

Thus, if the attitude R_{e} of a 3D pendulum satisfies (6), then starting from R_{e}, a rotation of the 3D pendulum about the gravity vector by an arbitrary angle is also an equilibrium. As mentioned before, the manifold corresponding to the case where the center of mass is below the pivot for each attitude in the manifold is referred to as the *hanging* equilibrium manifold, and the manifold corresponding to the case where the center of mass is above the pivot for each attitude in the manifold is referred to as the *inverted* equilibrium manifold. Note that the invariant solutions in each of the equilibrium manifolds are the equilibrium solutions.

Now, if R_{e} satisfies (6), then $(0, \Gamma_{\mathrm{e}})$ is an equilibrium of (39) and (40). Thus, corresponding to the hanging equilibrium manifold and the inverted equilibrium manifold of (1) and (2), there exist two isolated equilibrium solutions of the reduced attitude equations (39) and (40). These are given by the hanging equilibrium $(0, \Gamma_h)$ and the inverted equilibrium $(0, \Gamma_i)$, where

$$\Gamma_h = \frac{\rho}{\|\rho\|},\ \Gamma_i = -\frac{\rho}{\|\rho\|}.$$

Proposition 3 ([41]): The hanging and inverted equilibrium manifolds of the 3D pendulum given by (1) and (2) are identified with the hanging and the inverted equilibrium solutions of the reduced attitude equations given by (39) and (40).

From Proposition 3, it follows that the stabilization of the inverted equilibrium of the reduced attitude dynamics (39) and (40) guarantees stabilization of the inverted equilibrium manifold of the 3D pendulum dynamics (1) and (2).

We next present controllers that stabilize the inverted equilibrium manifold. The controllers use angular velocity and reduced attitude feedback for stabilization. Thus, to stabilize the inverted equilibrium manifold of (1) and (2), we do not require complete knowledge of the attitude $R \in \mathrm{SO}(3)$, but only the direction of gravity resolved in the body-fixed coordinate frame given by $\Gamma \in S^2$.

Let $\Phi : [0, \infty) \to [0, \infty)$ be a C^2 function such that

$$\Phi(0) = 0 \quad \text{and} \quad \Phi'(x) \geq 0 \text{ for all } x \in [0, \infty). \tag{42}$$

Let $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth function satisfying

$$\begin{cases} \Psi'(0)^{\mathrm{T}} = \Psi'(0),\ \Psi'(0) \text{ is positive definite}, \\ \mathcal{P}(x) \leq x^{\mathrm{T}}\Psi(x) \leq \alpha(\|x\|), \qquad \forall x \in \mathbb{R}^3, \end{cases} \tag{43}$$

where $\mathcal{P} : \mathbb{R}^3 \to \mathbb{R}$ is a positive definite function, and $\alpha(\cdot)$ is a class-$\mathcal{K}$ function. We propose controllers given by

$$u = K(\Gamma)(\Gamma_i \times \Gamma) - \Psi(\omega), \tag{44}$$

where $K(\Gamma) = \left[\Phi'\left(\frac{1}{2}(1 - \Gamma_i^{\mathrm{T}}\Gamma)\right) + \kappa\right]$ and κ is a positive number satisfying $\kappa > mg\|\rho\|$.

Again, we do not require knowledge of the moment of inertia or the vector ρ, and the only parameter needed to construct (44) is an upper bound on the maximum moment due to gravity given by $mg\|\rho\|$.

Expressing $\Gamma_i = -\rho/\|\rho\|$, the closed-loop attitude dynamics based on (1), (2) and the controller (44) are

$$\left.\begin{aligned} &J\dot{\omega} = J\omega \times \omega - \Psi(\omega) \\ &\quad + \left[mg\|\rho\| - \kappa - \Phi'\left(\tfrac{1 - \Gamma_i^{\mathrm{T}} R^{\mathrm{T}} e_3}{2}\right)\right] \frac{\rho}{\|\rho\|} \times R^{\mathrm{T}} e_3, \\ &\dot{R} = R\widehat{\omega}, \end{aligned}\right\} \tag{45}$$

and the closed-loop *reduced* attitude dynamics based on (39), (40) and the controller (44) are

$$\left.\begin{aligned} &J\dot{\omega} = J\omega \times \omega - \Psi(\omega) \\ &\quad + \left[mg\|\rho\| - \kappa - \Phi'\left(\tfrac{(1 - \Gamma_i^{\mathrm{T}}\Gamma)}{2}\right)\right] \frac{\rho}{\|\rho\|} \times \Gamma, \\ &\dot{\Gamma} = \Gamma \times \omega. \end{aligned}\right\} \tag{46}$$

The controller (44) can be interpreted as modifying the potential of the 3D pendulum through the attitude dependent term. The term $\Psi(\omega)$ induces energy dissipation in the closed-loop. The function Φ in $K(\Gamma)$ provides freedom to shape the potential.

Note that since Φ is non-decreasing and $\kappa > mg\|\rho\|$, the coefficient multiplying $(\rho \times \Gamma)$ in (46) is strictly negative. Hence, it can be shown that the closed-loop (46) has exactly two equilibrium solutions corresponding to the hanging equilibrium and the inverted equilibrium. As expected, it can be shown that the closed-loop (45) has two equilibrium manifolds corresponding to the hanging equilibrium manifold and the inverted equilibrium manifold. Thus, all equilibria of (45) are given by $(0, R_{\mathrm{e}}) \in T\mathrm{SO}(3)$, where R_{e} satisfies (6).

A. Local Analysis of the Closed-Loop

We begin the analysis of the closed-loop (45) by studying the eigenstructure of its linearization about an arbitrary equilibrium in one of the two equilibrium manifolds. Consider an equilibrium $(0, R_{\mathrm{e}}) \in T\mathrm{SO}(3)$ where $R_{\mathrm{e}} \in \mathrm{SO}(3)$ satisfies (6). To linearize the closed-loop (45), consider a perturbation of the initial conditions about an equilibrium $(0, R_{\mathrm{e}})$. Then, as in Section IV, one can show that the linearization of the closed-loop (45) is given by

$$J\Delta\ddot{\Theta} + \Psi'(0)\Delta\dot{\Theta} - k\,\frac{mg}{\|\rho\|}\widehat{\rho}^{\,2}\Delta\Theta = 0, \tag{47}$$

where $k \in \mathbb{R}$ is given by

$$k = \begin{cases} k_i \triangleq \dfrac{1}{mg\|\rho\|}\left(\kappa + \Phi'(0) - mg\|\rho\|\right), \\ k_h \triangleq -\dfrac{1}{mg\|\rho\|}\left(\kappa + \Phi'(1) - mg\|\rho\|\right), \end{cases} \tag{48}$$

where k_i and k_h correspond to $(0, R_{\mathrm{e}})$ being an inverted or hanging equilibrium, respectively. Since $\kappa > mg\|\rho\|$ and Φ is a C^1 non-decreasing function, it follows that $k_i > 0$ and $k_h < 0$.

Now note that $\widehat{\rho}^{\,2}$ is a rank 2, symmetric, negative-semidefinite matrix. Thus, it follows from [42], [43] that one can simultaneously diagonalize J and $\widehat{\rho}^{\,2}$. Thus, there exists a non-singular matrix M such that $J = MM^{\mathrm{T}}$ and $-\frac{mg}{\|\rho\|}\widehat{\rho}^{\,2} = M\Lambda M^{\mathrm{T}}$, where Λ is a diagonal matrix. Denote $\Lambda \triangleq \mathrm{diag}(mgl_1, mgl_2, 0)$, where l_1 and l_2 are positive. Define $x \triangleq M^{\mathrm{T}}\Delta\Theta$ and denote $D \triangleq M^{-1}\Psi'(0)\,M^{-\mathrm{T}}$. Since $\Psi'(0)$ is symmetric and positive definite, $D^{\mathrm{T}} = D$ and D is positive definite.

From (47), the linearization of (45) at $(0, R_{\mathrm{e}})$ can be expressed using $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ as

$$\ddot{x} + D\dot{x} + k\,\Lambda x = 0. \tag{49}$$

Equation (49) consists of three coupled second order linear differential equations.

We next study linearization of (46) about an equilibrium $(0, \Gamma_{\mathrm{e}}) = (0, R_{\mathrm{e}}^{\mathrm{T}} e_3)$, where $(0, R_{\mathrm{e}})$ is an equilibrium of (45). Since $\dim\left[T\mathrm{SO}(3)/S^1\right] = 5$, the linearization of (46) evolves on $\mathbb{R}^5$.

Proposition 4: The linearization of the reduced attitude dynamics of the 3D pendulum at the equilibrium $(0, \Gamma_{\mathrm{e}}) = (0, R_{\mathrm{e}}^{\mathrm{T}} e_3)$ described by equations (46) can be expressed using $(x_1, x_2, \dot{x}_1, \dot{x}_2, \dot{x}_3) \in \mathbb{R}^5$ according to (49).

Proof: Consider a perturbation $(\omega(t, \varepsilon), R(t, \varepsilon))$ of the closed-loop (45) in terms of the perturbation parameter $\varepsilon \in \mathbb{R}$. Then, since $\Gamma = R^{\mathrm{T}} e_3$, the perturbed solution of the closed-loop (46) is given by $(\omega(t,\varepsilon), \Gamma(t,\varepsilon))$ where $\Gamma(t,\varepsilon) = R^{\mathrm{T}}(t,\varepsilon)e_3$. Define the linearization variables of (46) as $\Delta\omega(t) \triangleq \omega_\varepsilon(t,0)$ and $\Delta\Gamma(t) \triangleq \Gamma_\varepsilon(t,0) = R_\varepsilon^{\mathrm{T}}(t,0)e_3$. From definition of $\Delta\Theta$ in Section IV, note that $\Delta\Gamma = -\widehat{\Delta\Theta} R_{\mathrm{e}}^{\mathrm{T}} e_3 = \widehat{\Gamma}_{\mathrm{e}}\Delta\Theta \in T_{\Gamma_{\mathrm{e}}} S^2$. Then from (47) and the definition of $\Delta\Gamma$, it can be easily shown that the linearization of (46) is given as

$$J\Delta\dot{\omega} = -\Psi'(0)\Delta\omega - \|k\| \frac{\widehat{\rho}}{\|\rho\|}\, \Delta\Gamma, \tag{50}$$

$$\Delta\dot{\Gamma} = -\mathrm{sign}(k)\, \frac{\widehat{\rho}}{\|\rho\|}\, \Delta\omega. \tag{51}$$

We next, express (50) and (51) in terms of $(x, \dot{x})$. Specifically, we show that $(\Delta\Gamma, \Delta\omega) \in T_{\Gamma_{\mathrm{e}}} S^2 \times \mathbb{R}^3$ can be expressed using $(x_1, x_2, \dot{x}_1, \dot{x}_2, \dot{x}_3) \in \mathbb{R}^5$.

Since $x = M^{\mathrm{T}}\Delta\Theta$, and M is nonsingular, $\Delta\omega = M^{-\mathrm{T}}\dot{x}$ and $\Delta\Gamma = -\mathrm{sign}(k)\, \dfrac{\widehat{\rho}}{\|\rho\|} M^{-\mathrm{T}} x$. We now give an orthogonal decomposition of the vector $\Delta\Theta = M^{-\mathrm{T}}x$ into a component along the vector ρ and a component normal to the vector ρ. This decomposition is

$$M^{-\mathrm{T}}x = -\frac{\widehat{\rho}^{\,2}}{\|\rho\|^2}(M^{-\mathrm{T}}x) + \frac{1}{\|\rho\|^2}\left[\rho^{\mathrm{T}}(M^{-\mathrm{T}}x)\right]\rho,$$

where $\dfrac{1}{\|\rho\|^2}\left[\rho^{\mathrm{T}}(M^{-\mathrm{T}}x)\right]\rho \in \mathrm{span}\{\rho\}$ and $-\dfrac{\widehat{\rho}^{\,2}}{\|\rho\|^2}(M^{-\mathrm{T}}x) \in \mathrm{span}\{\rho\}^{\perp}$.

Thus, $\Delta\Gamma = -\mathrm{sign}(k)\,\frac{\widehat{\rho}}{\|\rho\|}\Delta\Theta = -\mathrm{sign}(k)\,\frac{\widehat{\rho}}{\|\rho\|} M^{-\mathrm{T}}x = \frac{-\mathrm{sign}(k)}{mg\|\rho\|^2}\widehat{\rho} M \Lambda\, x$, does not depend on x_3 since $\Lambda = \mathrm{diag}(mgl_1, mgl_2, 0)$. Thus, we can express the linearization of (46) at $(0, \Gamma_{\mathrm{e}}) = (0, R_{\mathrm{e}}^{\mathrm{T}} e_3)$ in terms of the variables $(x_1, x_2, \dot{x}_1, \dot{x}_2, \dot{x}_3)$ according to (49). ■

Remark 2: It is easily seen from the structure of the matrix Λ that (49) is not asymptotically stable for $k = k_i > 0$. This is due to the fact that the inverted equilibrium manifold constitutes a 1D center submanifold in $T\mathrm{SO}(3)$. However, due to our careful choice of variables, one can discard x_3 from (49) to study the stability property of the inverted equilibrium manifold. Thus, x_3 corresponds to a component of the perturbation in the attitude that is tangential to the inverted equilibrium manifold.

The following Lemmas are needed.

Lemma 4: Consider the linear model (49), representing linearization of (45) at an inverted equilibrium $(0, R_{\mathrm{e}})$ expressed in first order form as

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I \\ -k\,\Lambda & -D \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} \triangleq A_i \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \tag{52}$$

where $x = (x_1, x_2, x_3)$. Then, A_i has one zero eigenvalue and all other eigenvalues have negative real part and at least one of them is negative real.

Proof: In the prior notation, $k = k_i > 0$ in (49) for an inverted equilibrium. Let $v = [v_1^{\mathrm{T}}\ v_2^{\mathrm{T}}]^{\mathrm{T}}$ be an eigenvector of A_i corresponding to the eigenvalue λ, where $v_1, v_2 \in \mathbb{C}^3$. Then, $A_i v = \lambda v$ yields $v_2 = \lambda v_1$ and $Dv_2 + k_i\Lambda v_1 = -\lambda v_2$. Combining these equations yields $\lambda^2 v_1 + D\lambda v_1 + \Lambda v_1 = 0$. Thus, every eigenvalue-eigenvector pair $(\lambda, [v_1^{\mathrm{T}}\ v_2^{\mathrm{T}}]^{\mathrm{T}})$ of A_i satisfies $v_2 = \lambda v_1$ and $\lambda^2 v_1 + \lambda D v_1 + k_i \Lambda v_1 = 0$.

Next, taking the inner product of the above equation with respect to the complex conjugate of v_1 yields

$$a\lambda^2 + b\lambda + c = 0, \tag{53}$$

where $a = \bar{v}_1^{\mathrm{T}} v_1$, $b = \bar{v}_1^{\mathrm{T}} D v_1$ and $c = k_i \bar{v}_1^{\mathrm{T}}\Lambda v_1$. Since D is symmetric and positive definite, Λ is diagonal and positive semidefinite, $k_i > 0$, and $v_1 \neq 0$, it follows that $a, b, c \in \mathbb{R}$ satisfy $a > 0$, $b > 0$ and $c \geq 0$. Furthermore, since $\Lambda = \mathrm{diag}(mgl_1, mgl_2, 0)$, $c = 0$ if and only if $v_1 = \beta e_3$, where $e_3 = [0\ 0\ 1]^{\mathrm{T}}$ and $\beta \in \mathbb{C}\backslash\{0\}$.

Now, since (53) has two solutions and $v = [v_1^{\mathrm{T}}\ v_2^{\mathrm{T}}]^{\mathrm{T}} = [v_1^{\mathrm{T}}\ \lambda v_1^{\mathrm{T}}]^{\mathrm{T}}$, it is clear that the eigenvalue-eigenvector pair (λ, v) of A_i can be written as

$$\left\{\left(\lambda_j, \begin{bmatrix} v_{1j} \\ \lambda_j\, v_{1j} \end{bmatrix}\right), \left(\lambda_j^*, \begin{bmatrix} v_{1j} \\ \lambda_j^*\, v_{1j} \end{bmatrix}\right)\right\},\ j \in \{1,2,3\}, \tag{54}$$

where λ_j and λ_j^* are the two solutions to the quadratic equation (53) corresponding to $a = a_j = \bar{v}_{1j}^{\mathrm{T}} v_{1j}$, $b = b_j = \bar{v}_{1j}^{\mathrm{T}} D v_{1j}$ and $c = c_j = k_i \bar{v}_{1j}^{\mathrm{T}} \Lambda v_{1j}$, $j \in \{1,2,3\}$.

Now choose $v_{11} = \beta e_3$. Then, $a_1 = \bar{\beta}\beta = |\beta|^2$, $b_1 = |\beta|^2 e_3^{\mathrm{T}} D e_3 > 0$ and $c_1 = 0$. Therefore, the roots of (53) are given by $\lambda_1 = 0$ and $\lambda_1^* = -e_3^{\mathrm{T}} D e_3$. Hence, 0 and $-e_3^{\mathrm{T}} D e_3$ are two of the six eigenvalues of A_i in (52). Thus, A_i has a zero and a negative real eigenvalue.

Now for each of v_{12} and v_{13}, we obtain a corresponding quadratic equation as given in (53). First, note that since $v_{11} = \beta e_3$ yields a zero eigenvalue, neither v_{12} nor v_{13} is equal to γe_3, $\gamma \in \mathbb{C}\backslash\{0\}$. This follows since if not, then there is a repeated zero eigenvalue which implies that A_i has rank less than or equal to four. However, it is easy to see that all columns of A_i except the third column, which is identically zero, are linearly independent. Since both v_{12} and v_{13} are not equal to γe_3, it follows that a_j, b_j and c_j, $j \in \{2,3\}$ are positive. Then the corresponding roots of (53) are given by

$$\lambda_j = -\frac{b_j}{2a_j} + \frac{\sqrt{b_j^2 - 4a_j c_j}}{2a_j}, \text{ and } \lambda_j^* = -\frac{b_j}{2a_j} - \frac{\sqrt{b_j^2 - 4a_j c_j}}{2a_j},$$

where $j \in \{2,3\}$. Thus, since $4a_j c_j > 0$, it follows that if $b_j^2 - 4a_j c_j < 0$, then λ_j and λ_j^* are complex with negative real part given by $-\dfrac{b_j}{2a_j}$, and if $b_j^2 - 4a_j c_j \geq 0$, then λ_j and λ_j^* are real negative since $b_j^2 > b_j^2 - 4a_j c_j$. Thus, the real part of λ_j and λ_j^* is negative for $j \in \{2,3\}$. ■

Lemma 5: Consider the reduced attitude dynamics of the 3D pendulum given by (39) and (40) with the controller (44). Assume that $\Phi : [0,\infty) \to [0,\infty)$ is a C^2 function satisfying (42), $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth function satisfying (43), and $\kappa > mg\|\rho\|$. Then, the inverted equilibrium of the closed-loop reduced attitude dynamics (46) is asymptotically stable and the convergence is locally exponential.

Proof: Consider the linearization of the closed-loop system (46) about the inverted equilibrium, given by (49) written in terms of the state variable $z = (x_1, x_2, \dot{x}_1, \dot{x}_2, \dot{x}_3) \in \mathbb{R}^5$. Writing (49) in terms of z yields $\dot{z} = \bar{A}_i z$, where $\bar{A}_i$ is obtained by deleting the third row and third column from A_i given in (52). Let $\mathrm{Spec}(M)$ denote the eigenvalues of the matrix M. Then, it is easy to see that $\det(A_i - \lambda I_6) = -\lambda \det(\bar{A}_i - \lambda I_5)$.

Hence, $\mathrm{Spec}(\bar{A}_i) = \mathrm{Spec}(A_i)\backslash\{0\}$. Then, from Lemma 4, it follows that all eigenvalues of $\bar{A}_i$ have negative real parts and at least one eigenvalue is negative real.

Hence, it follows that all eigenvalues of the linearization of the closed-loop system (46) about the inverted equilibrium have negative real parts. Thus, the inverted equilibrium of the nonlinear system (46) is asymptotically stable with local exponential convergence. ∎

Remark 3: Let $(0, R_\mathrm{e})$ be an inverted equilibrium. Suppose $D = \mathrm{diag}(d_1, d_2, d_3)$ is diagonal and $d_i > 0$, $i \in \{1, 2, 3\}$. Recall that $k_i > 0$. Then the eigenvalues of the linearized closed-loop reduced attitude dynamics (46) at $(0, R_\mathrm{e}^\mathrm{T} e_3)$ are the roots of the polynomial

$$(s^2 + d_1 s + k_i mgl_1)(s^2 + d_2 s + k_i mgl_2)(s + d_3) = 0.$$

Next, we study the linearization of the closed-loop at the hanging equilibrium. This yields the local structure of trajectories of the closed-loop (46) near the hanging equilibrium.

Lemma 6: Consider the linear model (49), representing linearization of (45) at a hanging equilibrium $(0, R_\mathrm{e})$ expressed in first order form as

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I \\ -k\,\Lambda & -D \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} \triangleq A_\mathrm{hng}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \tag{55}$$

where $x = (x_1, x_2, x_3)$. Then, A_hng has one zero, three negative and two positive real eigenvalues.

Proof: In the prior notation $k = k_h < 0$ in (49) for a hanging equilibrium. Let $v = [v_1^\mathrm{T}\ v_2^\mathrm{T}]^\mathrm{T}$ be an eigenvector corresponding to the eigenvalue λ of A_hng, where $v_1, v_2 \in \mathbb{C}^3$. Then, as in Lemma 4, one can show that all eigenvalues of A_hng satisfy

$$a\lambda^2 + b\lambda - c = 0,$$

where $a = \bar{v}_1^\mathrm{T} v_1$, $b = \bar{v}_1^\mathrm{T} D v_1$ and $c = |k_h|\bar{v}_1^\mathrm{T}\Lambda v_1$. Arguing as in Lemma 4, one can show that $\lambda_1 = 0$ and $\lambda_1^* = -e_3^\mathrm{T} D e_3$ are two of the six eigenvalues of A_hng in (55) and the other four eigenvalues are of the form

$$\lambda_i = -\frac{b_i}{2a_i} + \frac{\sqrt{b_i^2 + 4a_i c_i}}{2a_i}, \text{ and } \lambda_i^* = -\frac{b_i}{2a_i} - \frac{\sqrt{b_i^2 + 4a_i c_i}}{2a_i},$$

where a_i, b_i and c_i, $i \in \{2, 3\}$ are positive. Thus, since $4a_i c_i > 0$, it follows that λ_i is positive and λ_i^* is negative for $i \in \{2, 3\}$. Hence, A_hng has one zero eigenvalue, three negative eigenvalues, and two positive eigenvalues. ∎

Lemma 7: Consider the reduced attitude dynamics of the 3D pendulum given by (39) and (40) with the controller (44). Assume that $\Phi : [0, \infty) \to [0, \infty)$ is a C^2 function satisfying (42), $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth function satisfying (43), and $\kappa > mg\|\rho\|$. Then, the hanging equilibrium of the closed-loop reduced attitude dynamics (46) is unstable. Furthermore, the set of closed-loop trajectories that converge to the hanging equilibrium is a 3-dimensional invariant manifold $\mathcal{M}_h$.

Proof: Consider the linearization of the closed-loop system (46) about the hanging equilibrium, given by (49) written in terms of the state variable $z = (x_1, x_2, \dot{x}_1, \dot{x}_2, \dot{x}_3) \in \mathbb{R}^5$. Writing (49) in terms of z yields $\dot{z} = \bar{A}_\mathrm{hng} z$, where $\bar{A}_\mathrm{hng}$ is obtained by deleting the third row and third column from A_hng given in (55). Then, it is easy to see that $\det(A_\mathrm{hng} - \lambda I_6) = -\lambda\det(\bar{A}_\mathrm{hng} - \lambda I_5)$. Hence, $\mathrm{Spec}(\bar{A}_\mathrm{hng}) = \mathrm{Spec}(A_\mathrm{hng})\backslash\{0\}$. Then, from Lemma 6, it follows that $\bar{A}_\mathrm{hng}$ has three negative and two positive eigenvalues. Hence, the inverted equilibrium of (46) is unstable.

Furthermore, there exists a 3-dimensional stable invariant manifold $\mathcal{M}_h$ of the closed-loop (46) such that all solutions that start in $\mathcal{M}_h$ converge to the hanging equilibrium [39]. The tangent space to this manifold at the hanging equilibrium is the stable eigenspace corresponding to the negative eigenvalues. Since there are no eigenvalues on the imaginary axis, the closed-loop (46) has no center manifold and every closed-loop trajectory that converges to the hanging equilibrium lies in the stable manifold $\mathcal{M}_h$. ∎

Remark 4: Let $(0, R_\mathrm{e})$ be a hanging equilibrium. Suppose $D = \mathrm{diag}(d_1, d_2, d_3)$ is diagonal and $d_i > 0$, $i \in \{1, 2, 3\}$. Recall that $k_h < 0$. Then the eigenvalues of the linearized closed-loop reduced attitude dynamics at $(0, R_\mathrm{e}^\mathrm{T} e_3)$ are the roots of the polynomial

$$(s^2 + d_1 s + k_h mgl_1)(s^2 + d_2 s + k_h mgl_2)(s + d_3) = 0.$$

In summary, we have shown that the inverted equilibrium of the closed-loop (46) is locally exponentially stable and the hanging equilibrium of (46) is unstable. Furthermore, the set of all closed-loop trajectories that converge to the hanging equilibrium form a 3-dimensional, invariant manifold $\mathcal{M}_h$.

B. Global Analysis of the Closed-Loop

In this section, we study the global behavior of trajectories of the closed-loop system (46) using Lyapunov analysis.

Theorem 2: Consider the reduced attitude dynamics of the 3D pendulum given by (39) and (40) with the controller (44). Assume that $\Phi : [0, \infty) \to [0, \infty)$ is a C^2 function satisfying (42), $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth function satisfying (43), and $\kappa > mg\|\rho\|$. Let $\mathcal{M}_h$ denote the 3-dimensional invariant manifold as in Lemma 7. Then all solutions of the closed-loop given by (46), such that $(\omega(0), \Gamma(0)) \in (T\mathrm{SO}(3)/S^1)\backslash\mathcal{M}_h$, satisfy $\lim_{t\to\infty}\omega(t) = 0$ and $\lim_{t\to\infty}\Gamma(t) = \Gamma_i$. Furthermore, all solutions of the closed-loop (46), such that $(\omega(0), \Gamma(0)) \in \mathcal{M}_h$, satisfy $\lim_{t\to\infty}\omega(t) = 0$ and $\lim_{t\to\infty}\Gamma(t) = \Gamma_h$.

Proof: Consider the closed-loop reduced attitude dynamics given by (46), and the Lyapunov function given as

$$V(\omega, \Gamma) = \frac{1}{2}\,\omega^\mathrm{T} J\omega + \Big(\kappa - mg\|\rho\|\Big)(1 - \Gamma_i^\mathrm{T}\Gamma) + 2\Phi\left(\tfrac{(1-\Gamma_i^\mathrm{T}\Gamma)}{2}\right). \tag{56}$$

Note that $V(\omega, \Gamma)$ is positive definite on $T\mathrm{SO}(3)/S^1 \cong S^2 \times \mathbb{R}^3$ and $V(0, \Gamma_i) = 0$. Furthermore, since S^2 is compact and the Lyapunov function $V(\omega, \Gamma)$ is quadratic in ω, each sublevel set of $V(\omega, \Gamma)$ is compact. Next, we compute the derivative of V along a trajectory of the closed-loop. Thus,

$$\dot{V}(\omega, \Gamma) = \omega^\mathrm{T} J\dot{\omega} - \Big(\kappa - mg\|\rho\|\Big)\Gamma_i^\mathrm{T}\dot{\Gamma} - \Phi'\left(\tfrac{(1-\Gamma_i^\mathrm{T}\Gamma)}{2}\right)\Gamma_i^\mathrm{T}\dot{\Gamma}.$$

Since, $\Gamma_i = -\dfrac{\rho}{\|\rho\|}$, it follows that

$$\dot{V}(\omega, \Gamma) = -\omega^\mathrm{T}\Psi(\omega) \leq -\mathcal{P}(\omega). \tag{57}$$

Thus, $\dot{V}(\omega, \Gamma)$ is a negative semi-definite function and hence,

$$\mathcal{K} \triangleq \Big\{(\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : V(\omega, \Gamma) \leq V(\omega(0), \Gamma(0))\Big\}$$

is a compact, positively invariant sublevel set. Hence, by the invariant set theorem, all solutions converge to the largest invariant set in $\left\{(\omega, \Gamma) \in \mathcal{K} : \dot{V}(\omega, \Gamma) = 0\right\}$. Since $\mathcal{P}$ is a positive definite function, $\dot{V}(\omega, \Gamma) = 0$ implies $\omega \equiv 0$.

Substituting $\omega \equiv 0$ in (46), it can be shown that the largest such invariant set is given by $\{(0, \Gamma_h)\} \bigcup \{(0, \Gamma_i)\}$. However, from Lemma 7, we know that all trajectories that converge to the hanging equilibrium are contained in the 3-dimensional manifold $\mathcal{M}_h$. Therefore, all solutions of the closed-loop given by (46), such that $(\omega(0), \Gamma(0)) \in (T\mathrm{SO}(3)/S^1)\backslash\mathcal{M}_h$, converge to the inverted equilibrium and hence, satisfy $\lim_{t\to\infty} \omega(t) = 0$ and $\lim_{t\to\infty} \Gamma(t) = \Gamma_i$. ∎

Remark 5: Since $\mathcal{M}_h$ is a 3-dimensional invariant manifold, its Lebesgue measure is zero [40]. Furthermore, following arguments as in Theorem 1, it can be shown that the complement of $\mathcal{M}_h$ in $T\mathrm{SO}(3)/S^1$ is open and dense. Thus, from Theorem 2 it follows that the domain of attraction of the inverted equilibrium for the closed-loop (46) is *almost global*.

The result presented in Theorem 2 applies to the solution of the closed-loop reduced attitude dynamics of the 3D pendulum given by (46). Thus, the almost global result holds on $T\mathrm{SO}(3)/S^1$. We now study the implication for the 3D pendulum dynamics given by (45). Specifically, we show that the controller (44) almost globally asymptotically stabilizes the inverted equilibrium manifold in $T\mathrm{SO}(3)$. The following Lemma is needed.

Lemma 8 ([41]): Let $\pi : T\mathrm{SO}(3) \to T\mathrm{SO}(3)/S^1$ denote the projection, where $T\mathrm{SO}(3)/S^1$ is endowed with the quotient topology. Let $\mathcal{U} \subseteq T\mathrm{SO}(3)/S^1$ be a set whose complement is open and dense in $T\mathrm{SO}(3)/S^1$. Then, $\pi^{-1}(\mathcal{U}) \subseteq T\mathrm{SO}(3)$ is a set whose complement is open and dense in $T\mathrm{SO}(3)$.

We now show that the controller (44) almost globally asymptotically stabilizes the inverted equilibrium manifold according to the closed-loop equations (45).

Theorem 3: Consider the dynamics of the 3D pendulum given by (1) and (2) with the controller (44). Assume that $\Phi : [0, \infty) \to [0, \infty)$ is a C^2 function satisfying (42), $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth function satisfying (43), and $\kappa > mg\|\rho\|$. Then, the inverted equilibrium manifold is asymptotically stable with local exponential convergence. Furthermore, there exists an invariant set $\mathsf{M}_h \subset T\mathrm{SO}(3)$ with Lebesgue measure zero such that $T\mathrm{SO}(3)\backslash\mathsf{M}_h$ is open and dense and all solutions of the closed-loop given by (1), (2) and (44) such that $(\omega(0), R(0)) \in T\mathrm{SO}(3)\backslash\mathsf{M}_h$ converge to the inverted equilibrium manifold. All solutions of the closed-loop such that $(\omega(0), R(0)) \in \mathsf{M}_h$ converge to the hanging equilibrium manifold.

Proof: Asymptotic stability of the inverted equilibrium manifold and local exponential convergence of closed-loop trajectories follows immediately from Lemma 5 and Proposition 3. We next show almost global convergence property of the closed-loop trajectories.

Consider the projection $\pi : T\mathrm{SO}(3) \to T\mathrm{SO}(3)/S^1$ which yields the reduced attitude dynamics as in Proposition 2. Let $\mathcal{M}_h$ be the set as in Theorem 2 and denote $\mathsf{M}_h = \pi^{-1}(\mathcal{M}_h) \subset T\mathrm{SO}(3)$. It follows from Lemma 8 that the complement of M_h in $T\mathrm{SO}(3)$ is open and dense. Furthermore, from Proposition 3 and Theorem 2, it follows that for the closed-loop (1), (2) and (44), all trajectories contained in $T\mathrm{SO}(3)\backslash\mathsf{M}_h$ converge to the inverted equilibrium manifold and all trajectories in the set M_h converge to the hanging equilibrium manifold. Since the dimension of the equilibrium manifold is one and the Lebesgue measure of $\mathcal{M}_h$ is zero, it follows that the Lebesgue measure of $\mathsf{M}_h = \pi^{-1}(\mathcal{M}_h)$ is zero [40]. ∎

Remark 6: Since $T\mathrm{SO}(3)\backslash\mathsf{M}_h$ is open and dense, it follows from Theorem 3 that the domain of attraction of the inverted equilibrium manifold for the closed-loop (1), (2) and (44) is *almost global*.

The controllers in (44) provide a combination of potential shaping and damping injection. It may be noted that the argument of the potential function $\Phi(\cdot)$ is proportional to the cosine of the angle between Γ and Γ_i. This yields the following closed-loop property: if $\omega(0) = 0$, then for all $t \geq 0$, the angle between $\Gamma(t)$ and Γ_i is bounded above by the angle between $\Gamma(0)$ and Γ_i.

Corollary 1: Consider the reduced attitude dynamics of the 3D pendulum given by (39) and (40) with controller as in (44). Furthermore, let $\omega(0) = 0$ and $\Gamma(0) \neq \Gamma_h$. Then, for all $t \geq 0$, trajectories of the closed-loop reduced attitude dynamics (46) satisfy

$$\measuredangle(\Gamma_i, \Gamma(t)) \leq \measuredangle(\Gamma_i, \Gamma(0)).$$

Proof: Consider the candidate Lyapunov function (56), for the closed-loop (46). As already shown in Theorem 2, $\dot{V}(\omega, \Gamma) = -\omega^{\mathrm{T}}\Psi(\omega)$. Thus, since $\dot{V}(\omega, \Gamma)$ is negative semidefinite, $V(\omega(t), \Gamma(t)) \leq V(\omega(0), \Gamma(0))$. Thus substituting $\omega(0) = 0$ in (56), we obtain the result that for all $t \geq 0$,

$$\frac{1}{2}\,\omega(t)^{\mathrm{T}} J \omega(t) + [\kappa - mg\|\rho\|](1 - \Gamma_i^{\mathrm{T}}\Gamma(t)) + 2\Phi\left(\tfrac{(1-\Gamma_i^{\mathrm{T}}\Gamma(t))}{2}\right) \leq \Big(\kappa - mg\|\rho\|\Big)(1 - \Gamma_i^{\mathrm{T}}\Gamma(0)) + 2\Phi\left(\frac{1}{2}(1 - \Gamma_i^{\mathrm{T}}\Gamma(0))\right).$$

Since, the kinetic energy term is strictly non-negative, $\kappa > mg\|\rho\|$ and $\Phi(\cdot)$ is a non-decreasing function, we obtain $(1 - \Gamma_i^{\mathrm{T}}\Gamma(t)) \leq (1 - \Gamma_i^{\mathrm{T}}\Gamma(0))$, and hence, $\Gamma_i^{\mathrm{T}}\Gamma(t) \geq \Gamma_i^{\mathrm{T}}\Gamma(0)$, for all $t \geq 0$. Since, $\|\Gamma(t)\| \equiv 1$ for all $t \geq 0$, it follows that $\cos\left(\measuredangle(\Gamma_i, \Gamma(t))\right) \geq \cos\left(\measuredangle(\Gamma_i, \Gamma(0))\right),\ t \geq 0$. Since $\measuredangle(\Gamma_i, \Gamma(t)) \in [0, \pi)$ and $\cos(\cdot)$ is non-increasing in $[0, \pi)$, the result follows. ∎

A trivial choice for the function Φ is given by $\Phi(x) \equiv 0$. Then, the controller (44) simplifies to

$$u = \kappa(\Gamma_i \times \Gamma) - \Psi(\omega), \tag{58}$$

where Ψ is chosen as in (43) and κ is a positive number satisfying $\kappa > mg\|\rho\|$. Thus, a nontrivial Φ function in (44) is not essential for stability of the inverted equilibrium manifold. However, it can be used to modify the domain of attraction or the amount of control effort required as a function of the reduced attitude. Although the domain of attraction of the inverted equilibrium is almost global, the domain of attraction is hard to compute explicitly.

Theorem 3 is the main result on asymptotic stabilization of the inverted equilibrium manifold of the 3D pendulum. Under the indicated assumptions, almost global asymptotic stabilization is achieved. This is the best possible result for this stabilization problem, in the sense that global stabilization using smooth feedback is not achievable.

The controller expression (44) is quite useful in control design as it allows freedom to arbitrarily design the local dynamics of the closed-loop near the inverted equilibrium manifold. It also provides some freedom in shaping the manifold of solutions that do not converge to the inverted equilibrium manifold. In this way, control design can be carried out to achieve both local and global control objectives.

Simulation studies that demonstrate the validity of Theorem 3 and illustrate its use in a control design context are reported in [21], [44], [45]. In addition, experimental studies have been reported in [46] that demonstrate the value of this theory when used to asymptotically stabilize the inverted equilibrium manifold of the TACT. Due to space limitations, we do not provide the results of those studies in this paper.

VI. Closed-Loop Performance Limitations

We presented two families of controllers that stabilize a specified equilibrium or the inverted equilibrium manifold, respectively. Consider the controller (13) for stabilization of a specified equilibrium in the inverted equilibrium manifold. It was shown that there exists a nowhere dense set M of Lebesgue measure zero such that all solutions that do not start in this set converge to the specified inverted equilibrium.

The existence of the set M is related to the topological obstruction that exists for any continuous time-invariant, stabilizing controller defined on a compact configuration space. However, there also exists a performance constraint for such controllers. This arises since the set M influences the dynamics of nearby closed loop trajectories. Note that M is composed of the union of stable manifolds of the unstable equilibria; hence solutions starting in M converge to one of the unstable equilibria $(0, R_{\mathrm{e},i})$. Thus, from continuity of solutions with respect to initial conditions, it follows that solutions that start close to M remain near M for an extended period of time before they converge to the specified equilibrium. The closer a trajectory lies to M, the longer it takes to converge to the specified equilibrium. This property is due to the saddle character of the unstable closed loop equilibria. Computation of the set M is difficult. Although, one can easily obtain linear approximations to this manifold using the stable subspace of the linearized equations about any of the unstable equilibria, this provides information about M only near the unstable equilibrium solutions. The non-local properties of the set M are not well understood.

It is important to emphasize that the presence of the set M, whose existence is asserted in Theorem 1, is not a consequence of the specific smooth controller that is proposed. Rather, such a set M necessarily exists for any continuous time-invariant feedback controller. Indeed, there is a topological obstruction to global attitude stabilization [33], and we have shown that there is also a performance limitation that arises from this fact in the sense that there always exist initial conditions in the domain of attraction that converge arbitrarily slowly to the desired equilibrium attitude. A similar result holds for stabilization of the inverted equilibrium manifold.

VII. Conclusions

This paper has presented a complete analysis for two problems on stabilization of the inverted 3D pendulum. In the first case, we treat the problem of asymptotically stabilizing a specified equilibrium solution in the inverted equilibrium manifold. In the second case, we treat the problem of asymptotically stabilizing the inverted equilibrium manifold. In each case, we have developed feedback expressions, based on feedback of the angular velocity and attitude, or reduced attitude, of the 3D pendulum. The emphasis throughout the paper has been on global definition of the 3D pendulum models, global description of the controllers, and global geometric analysis of the closed-loops. The control problems for the 3D pendulum exemplify attitude stabilization problems on the compact configuration manifold $\mathrm{SO}(3)$ in the presence of potential forces. The results obtained in the paper demonstrate the complexity of nonlinear control problems for the 3D pendulum.

In a related paper [41], we have treated asymptotic stabilization of the hanging equilibrium manifold of the 3D pendulum using feedback of angular velocity only. That paper and this one that treats asymptotic stabilization of the inverted 3D pendulum can be considered as providing the basic stabilization theory for the 3D pendulum.

A number of interesting extensions can be suggested. This would include extensions assuming underactuation of the control inputs, partial or incomplete feedback, and control saturation. Attention has already been given to some of these topics [44], [47]. Extensions to problems involving multibody 3D pendulum problems can also be formulated; for example, we mention the challenging problem of stabilization of an inverted 3D pendulum, mounted on a cart that can be controlled to move in a plane.

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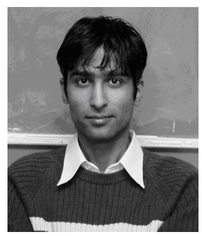

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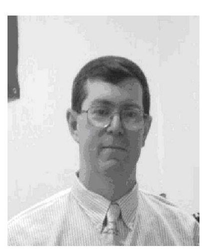

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