ELSEVIER

Systems & Control Letters 25 (1995) 175–183

SYSTEMS & CONTROL LETTERS

The octomorphic criterion for real parameter uncertainty: Real-μ bounds without circles and D, N-scales ☆

Wassim M. Haddad[a], Dennis S. Bernstein[b,*]

[a] *School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, USA*
[b] *Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2118, USA*

Received 2 November 1993; revised 6 June 1994

Abstract

In this paper we introduce new bounds for robust stability analysis with real parameter uncertainty. The approach is based on an absolute stability criterion that excludes the Nyquist plot from a paraboloidal region containing the point $-1 + \jmath 0$. Transformation of this criterion to the case of norm-bounded uncertainty leads to a stability criterion in terms of the octomorphic, or figure-eight shaped, region. The requirement that the Nyquist plot lie inside the octomorphic region thus yields a bound on the allowable real parameter uncertainty. This stability criterion is distinct from recent bounds for real-μ which involve frequency-dependent scales having a frequency-dependent, off-axis circle interpretation. Since the octomorphic region includes both upper and lower halves, it is able to encompass the entire Nyquist plot without using frequency-dependent scales.

Keywords: Absolute stability; Octomorphic region; Paraboloidal region; Real-μ bounds

1. Introduction

The problem of robust stability and performance of systems with structured real parameter uncertainty remains one of the most challenging problems in modern systems theory. A general framework for this problem is provided by mixed-μ theory [5, 24] as a refinement of the complex structured singular value. As discussed in [5, 24], the multiple-block-structured problem is NP complete and thus is inherently computationally complex.

In view of the computational complexity of the structured real parameter uncertainty problem, researchers have focussed on specialized problems, such as those addressed by Kharitonov-like theorems, as well as tractable upper and lower bounds for mixed-μ. One such upper bound, developed in [5, 13, 19, 24], involves complex frequency-dependent D, N-scales to account for the phase information inherent in the real parameter uncertainty. In recent work [9, 10, 13] it was shown that in the scalar uncertainty case this bound has the geometric interpretation of frequency-dependent off-axis circles, which are chosen to encompass the Nyquist plot at each frequency. This interpretation thus illustrates the relevance of classical absolute stability criteria which address robust stability with respect to specific classes of nonlinearities [17]. For example, the Popov

☆ This research was supported in part by the National Science Foundation under Grants ECS-9109558 and ECS-9350181 and the Air Force Office of Scientific Research under Grant F49620-92-J-0127.

* Corresponding author. Tel.: (313)764-3719. Fax.: (313)763-0578. E-mail: dsbaero@engin.umich.edu.

SSDI 0167-6911(94)00065-4

criterion, which applies to sector-bounded time-invariant nonlinearities [6, 8, 16], has long been known to have an off-axis circle interpretation [11], while absolute stability criteria for monotonic, odd-monotonic, and slope-bounded nonlinearities [2, 4, 7, 9, 14, 15, 18, 20, 22, 23, 25] have similar interpretations.

The purpose of this paper is to develop new bounds for real-μ that are fundamentally different from previous bounds. Specifically, the bounds developed herein are not based upon frequency-dependent circles in order to encompass the Nyquist plot. Rather, in the scalar case the new bounds are based upon a quartic plane curve that simultaneously encompasses the entire Nyquist plot *whether or not the Nyquist plot lies above or below the real axis.* A useful feature of this *octomorphic* (figure-eight shaped) region is the fact that, unlike off-axis circles, it need not change with frequency in order to encompass the Nyquist plot. Hence, the octomorphic bound for real-μ involves only constant scales, unlike prior bounds which invoke frequency-dependent scales.

The robust stability results derived in this paper are fundamentally different from prior results involving disks and half planes. This difference stems from the fact that our result is not based on either bounded real or positive real principles. Rather, the frequency domain condition of Theorem 3.1 is a *weak* positive real condition which, unlike bounded real and positive real theory, allows the transfer function to possess unstable poles. In this sense our results are in the spirit of the absolute stability criterion developed in [4, 23] which also involves a weak positive real condition. The results of [4, 23], however, have an off-axis circle interpretation. An additional connection between our results and those of [4, 23] as well as [7] is the fact that these absolute stability criteria involve plant-dependent multipliers which exploit knowledge of the plant.

Finally, we note that the octomorphic region arose naturally from the linear fractional transformation of a parabola. However, it does not appear that this curve has been studied in the classical literature [12].

2. Mathematical preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, let $(\ )^{\mathrm{T}}$ and $(\ )^*$ denote transpose and complex conjugate transpose, let "Re" and "Im" denote real and imaginary part, and let I_n or I denote the $n \times n$ identity matrix. Furthermore, we write $\|\cdot\|_2$ for the Euclidean norm, $\|\cdot\|_F$ for the Frobenius matrix norm, $\sigma_{\max}(\cdot)$ for the maximum singular value, "tr" for the trace operator, and $M \geqslant 0$ ($M > 0$) to denote the fact that the Hermitian matrix M is nonnegative (positive) definite. In this paper a *transfer function* is a real-rational matrix function each of whose elements is *proper,* i.e., finite at $s = \infty$, while a *strictly proper transfer function* is a transfer function that is zero at infinity. Finally, an *asymptotically stable transfer function* is a transfer function each of whose poles has negative real part. The space of asymptotically stable transfer functions is denoted by $\mathcal{RH}_\infty$, i.e., the real-rational subset of $\mathcal{H}_\infty$. Let

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

denote a state space realization of a transfer function $G(s)$, that is, $G(s) = C(sI - A)^{-1}B + D$. The notation $\overset{\min}{\sim}$ is used to denote a minimal realization. In addition, the parahermitian conjugate $G^{\sim}(s)$ of $G(s)$ has the realization

$$G^{\sim}(s) \sim \left[\begin{array}{c|c} -A^{\mathrm{T}} & C^{\mathrm{T}} \\ \hline -B^{\mathrm{T}} & D^{\mathrm{T}} \end{array}\right].$$

If $G = X + \jmath Y$, where $X = \operatorname{Re} G$ and $Y = \operatorname{Im} G$. Then $G^* = X^{\mathrm{T}} - \jmath Y^{\mathrm{T}}$, while the Hermitian part of G is given by $\operatorname{He} G \triangleq \frac{1}{2}(G + G^*) = \frac{1}{2}[X + X^{\mathrm{T}} + \jmath(Y - Y^{\mathrm{T}})]$.

A square transfer function $G(s)$ is called *positive real* [1, p. 216] if (1) all poles of $G(s)$ lie in the closed left half plane and (2) $\operatorname{He} G(s)$ is nonnegative definite for $\operatorname{Re} s > 0$. A square transfer function $G(s)$ is called *strictly positive real* [21] if (1) $G(s)$ is asymptotically stable, and (2) $\operatorname{He} G(\jmath\omega)$ is positive definite for all real ω. A square transfer function $G(s)$ is called *weakly positive real* if $G(s)$ has no imaginary poles and $\operatorname{He} G(\jmath\omega)$ is nonnegative definite for all real ω. A square transfer function $G(s)$ is called *strict weakly positive real* if $G(s)$ has no imaginary poles and $\operatorname{He} G(\jmath\omega)$ is positive definite for all real ω. Note that although a minimal realization of a positive real transfer function is stable in the sense of Lyapunov, a minimal realization of a weakly positive real transfer function may be unstable.

3. The paraboloidal absolute stability criterion

In this section we consider a robust stability problem involving an uncertain matrix of the form FI_m, where F is a real scalar, in a negative feedback interconnection with the $m \times m$ asymptotically stable transfer function

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right].$$

This interconnection has the form of the uncertain system

$$\dot{x}(t) = (A - FBC)x(t), \tag{3.1}$$

where $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$. The uncertain scalar F is assumed to satisfy $0 \leqslant F \leqslant M$, where M is real and positive.

For convenience in stating the main result define the notation

$$\hat{A} \triangleq \begin{bmatrix} A & 0 & 0 \\ BC & A & 0 \\ 0 & 0 & A \end{bmatrix}, \quad \hat{B} \triangleq \begin{bmatrix} B \\ 0 \\ NB \end{bmatrix}, \quad \hat{C} \triangleq [C \quad -NC \quad 0], \quad \hat{R} \triangleq \begin{bmatrix} 0 & 0 & C^{\mathrm{T}}C \\ 0 & 0 & 0 \\ C^{\mathrm{T}}C & 0 & 0 \end{bmatrix}.$$

Theorem 3.1. *Let $N \in \mathbb{R}$ and define*

$$\mathscr{G}(s) \triangleq \frac{1}{M}I + G(s) - N[G(s) - G^{\sim}(s)]G(s). \tag{3.2}$$

Then $\mathscr{G}(s)$ is weakly positive real if and only if there exist P, L, and W with $P = P^{\mathrm{T}}$ satisfying

$$0 = \hat{A}^{\mathrm{T}}P + P\hat{A} - \hat{R} + L^{\mathrm{T}}L, \tag{3.3}$$

$$0 = \hat{B}^{\mathrm{T}}P - \hat{C} + W^{\mathrm{T}}L, \tag{3.4}$$

$$0 = \frac{2}{M}I_m - W^{\mathrm{T}}W. \tag{3.5}$$

If, in addition, $\mathscr{G}(s)$ is strict weakly positive real, then the negative feedback interconnection of $G(s)$ and FI_m is asymptotically stable for all $0 \leqslant F \leqslant M$.

Proof. First we show that (3.3)–(3.5) imply that $\mathscr{G}(s)$ is weakly positive real. To do this, add and subtract $\jmath\omega P$ to and from (3.3) to obtain

$$0 = (-\jmath\omega I - \hat{A})^{\mathrm{T}}P + P(\jmath\omega I - \hat{A}) + \hat{R} - L^{\mathrm{T}}L. \tag{3.6}$$

Next, forming $\hat{B}^{\mathrm{T}}(-\jmath\omega I - \hat{A})^{-\mathrm{T}}$ (3.6) $(\jmath\omega I - \hat{A})^{-1}\hat{B}$ and using (3.4) we obtain

$$(\hat{C} - W^{\mathrm{T}}L)(\jmath\omega I - \hat{A})^{-1}\hat{B} + \hat{B}(-\jmath\omega I - \hat{A})^{-\mathrm{T}}(\hat{C} - W^{\mathrm{T}}L)^{\mathrm{T}} = \hat{B}^{\mathrm{T}}(-\jmath\omega I - \hat{A})^{-\mathrm{T}}[L^{\mathrm{T}}L - \hat{R}](\jmath\omega I - \hat{A})^{-1}\hat{B}. \tag{3.7}$$

Adding and subtracting $W^{\mathrm{T}}W$ to and from (3.7), using (3.5), and grouping terms yields

$$\frac{2}{M}I_m + \hat{C}(\jmath\omega I - \hat{A})^{-1}\hat{B} + \hat{B}(-\jmath\omega I - \hat{A})^{-\mathrm{T}}\hat{C}^{\mathrm{T}} + \hat{B}^{\mathrm{T}}(-\jmath\omega I - \hat{A})^{-\mathrm{T}}\hat{R}(\jmath\omega I - \hat{A})^{-1}\hat{B}$$
$$= [W + L(\jmath\omega I - \hat{A})^{-1}\hat{B}]^*[W + L(\jmath\omega I - \hat{A})^{-1}\hat{B}] \geqslant 0. \tag{3.8}$$

Next, using the identities

$$\hat{C}(\jmath\omega I - \hat{A})^{-1}\hat{B} = C(\jmath\omega I - A)^{-1}B - NC(\jmath\omega I - A)^{-1}BC(\jmath\omega I - A)^{-1}B$$

and

$$\hat{B}^{\mathrm{T}}(-\jmath\omega I - \hat{A})^{-\mathrm{T}}\hat{R}(\jmath\omega I - \hat{A})^{-1}\hat{B} = 2NB^{\mathrm{T}}(-\jmath\omega I - A)^{-\mathrm{T}}C^{\mathrm{T}}C(\jmath\omega I - A)^{-1}B,$$

it follows from (3.8) that $\mathrm{He}\,\mathcal{G}(\jmath\omega) \geqslant 0$. Hence, $\mathcal{G}(s)$ is weakly positive real.

Conversely, noting that $\mathcal{G}(s)$ has a minimal realization

$$\mathcal{G}(s) \overset{\min}{\sim} \left[\begin{array}{ccc|c} A & 0 & 0 & B \\ BC & A & 0 & 0 \\ C^{\mathrm{T}}C & 0 & -A^{\mathrm{T}} & 0 \\ \hline C & -NC & -NB^{\mathrm{T}} & (1/M)I_m \end{array}\right]$$

and assuming $\mathcal{G}(s)$ is weakly positive real, spectral factorization theory guarantees the existence of a proper stable spectral factor $\mathcal{N}(s)$ such that $\mathcal{G}(s) + \mathcal{G}^{\sim}(s) = \mathcal{N}^{\sim}(s)\mathcal{N}(s)$, where $\mathcal{N}^{\pm 1}(s) \in \mathcal{RH}_{\infty}$. The existence of P, L, and W with $P = P^{\mathrm{T}}$ satisfying (3.3)–(3.5) now follows from standard state space realization manipulations.

Next, suppose there exists $\omega \in \mathbb{R}$ and $F \in (0, M]$ such that $\det(I_m + FG(\jmath\omega)) = 0$. Then there exists nonzero $x \in \mathbb{C}^m$ such that $FG(\jmath\omega)x = -x$. Hence

$$2M^{-1}x^*x + x^*[G(\jmath\omega) + G^*(\jmath\omega)]x = (-2F/M)(M - F)x^*G^*(\jmath\omega)G(\jmath\omega)x \leqslant 0.$$

On the other hand, since $\mathcal{G}(s)$ is strict weakly positive real, it follows that

$$\begin{aligned} &2M^{-1}x^*x + x^*[G(\jmath\omega) + G^*(\jmath\omega)]x \\ &\quad > Nx^*[G(\jmath\omega) - G^*(\jmath\omega)]G(\jmath\omega)x + Nx^*G^*(\jmath\omega)[G^*(\jmath\omega)x - G(\jmath\omega)]x \\ &\quad = FNx^*G^*(\jmath\omega)[G^*(\jmath\omega) - G(\jmath\omega)]G(\jmath\omega)x - FNx^*G^*(\jmath\omega)[G(\jmath\omega) - G^*(\jmath\omega)]G(\jmath\omega)x \\ &\quad = 0. \end{aligned}$$

Thus $\det(I + FG(\jmath\omega)) \neq 0$ for all ω and for all $F \in [0, M]$. Since $G(s)$ is asymptotically stable, it follows that the negative feedback interconnection of $G(s)$ and FI_m is asymptotically stable for all $0 \leqslant F \leqslant M$. □

Remark 3.1. It can be seen from the minimal realization of $\mathcal{G}(s)$ given in the proof of Theorem 3.1 that $\mathcal{G}(s)$ is unstable. Thus, Theorem 3.1 invokes a weak positive real condition rather than a positive real condition as in standard absolute stability theory.

Note that $\mathcal{G}(s)$ can be written as

$$\mathcal{G}(s) = M^{-1}I + \mathcal{Z}(s)G(s), \tag{3.9}$$

where

$$\mathcal{Z}(s) \triangleq I - N[G(s) - G^{\sim}(s)]. \tag{3.10}$$

The form (3.9) is standard in classical absolute stability theory [14–16, 22, 23] for feedback systems involving a linear time-invariant system and a memoryless (possibly time-varying) nonlinearity. The transfer function $\mathcal{Z}(s)$ is a *stability multiplier* that distinguishes the class of allowable nonlinearities. Specific cases include memoryless time-invariant nonlinearities [16], monotonic and odd monotonic nonlinearities [14], and locally slope restricted nonlinearities [4, 22, 23, 25]. As in [4, 7, 23], the multiplier $\mathscr{Z}(s)$ in (3.10) depends upon the plant itself. As will be shown in Section 5, this provides a "tuning" effect that captures real parameter uncertainty while eliminating a large class of feedback nonlinearities.

Next, we give a geometric interpretation of the criterion developed in Theorem 3.1 in the scalar uncertainty case. To do this, let $G(\jmath\omega) = x + \jmath y$ and note that $\mathrm{He}\,\mathscr{G}(\jmath\omega) > 0$ is equivalent to

$$(1/M) + x + 2Ny^2 > 0\,. \tag{3.11}$$

Condition (3.11) is a frequency domain stability criterion with a graphical interpretation in the Nyquist plane in terms of a parabola which is symmetric about the real axis with vertex at $(-1/M, 0)$ and parameter N governing the curvature. It is important to note that unlike prior classical absolute stability criteria, Theorem 3.1 allows the Nyquist plot of $G(\jmath\omega)$ to enter all four quadrants of the complex plane while avoiding encirclements and crossings of the point $-1/M + \jmath 0$. Furthermore, as $N \to \infty$ this criterion allows the Nyquist plot to reside anywhere in the Nyquist plane except for the semi-infinite interval $(-\infty, -1/M)$ of the negative real axis.

Next, we extend Theorem 3.1 to the case of upper and lower uncertainty bounds. To do this, let $M_1, M_2 \in \mathbb{R}$ be such that $M = M_2 - M_1$ is positive and define the shifted uncertainty $F_s \in [M_1, M_2]$ along with the shifted transfer function

$$G_s(s) \triangleq (I + G(s)M_1)^{-1}G(s) \sim \left[\begin{array}{c|c} A - M_1BC & B \\ \hline C & 0 \end{array}\right]. \tag{3.12}$$

For upper and lower uncertainty bounds we have the following corollary to Theorem 3.1.

Corollary 3.1. *Let*

$$G_s(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right],$$

assume $G_s(s)$ is asymptotically stable, and let $N \in \mathbb{R}$. If

$$\mathscr{G}_s(s) \triangleq (M_2 - M_1)^{-1}I_m + G_s(s) - N[G_s(s) - G_s^{\sim}(s)]G_s(s) \tag{3.13}$$

is strict weakly positive real, then the negative feedback interconnection of $G(s)$ and F_sI_m is asymptotically stable for all $F_s \in [M_1, M_2]$.

Proof. Since $\mathscr{G}_s(s)$ is strict weakly positive real, it follows from Theorem 3.1 that the negative feedback interconnection of $G_s(s)$ and FI_m is asymptotically stable for all $F \in [0, M]$. By writing $F = F + M_1 - M_1$, it follows from standard loop shifting techniques [3] that the negative feedback interconnection of $G(s)$ and F_sI_m, where $F_s \triangleq F + M_1$, is asymptotically stable for all $F \in [0, M]$. Hence the negative feedback interconnection of $G(s)$ and F_sI_m is asymptotically stable for all $F_s \in [M_1, M_2]$. □

In the next section we provide a graphical interpretation of Corollary 3.1 as well as connections to real-μ.

4. Transformation to the octomorphic region with application to real-μ bounds

In this section we use Corollary 3.1 to obtain real-μ bounds that do not involve off-axis circles and dynamic D,N-scales. In the scalar case this result has an interesting geometric interpretation in terms of the octomorphic region.

To make connections with real-μ theory we set $-M_1 = M_2 = \gamma^{-1}$, where $\gamma > 0$, and consider the set of uncertain matrices

$$\boldsymbol{\Delta} \triangleq \{\Delta I_m \in \mathbb{R}^{m\times m} : -\gamma \leqslant \Delta \leqslant \gamma\}. \tag{4.1}$$

Next, recall that for real uncertainty $\Delta \in \mathbf{\Delta}$, the structured singular value $\mu(G(\jmath\omega))$ is defined by

$$\mu(G(\jmath\omega)) \triangleq \left(\min_{\Delta \in \mathbf{\Delta}}\{\sigma_{\max}(\Delta) : \det(I + G(\jmath\omega)\Delta) = 0\}\right)^{-1}, \tag{4.2}$$

while $\mu(G(\jmath\omega)) = 0$ if there exists no $\Delta \in \mathbf{\Delta}$ such that $\det(I + G(\jmath\omega)\Delta) = 0$. The following result provides a new upper bound for real-μ defined by (4.2).

Theorem 4.1. *For $\omega \in \mathbb{R}$ and $G(\jmath\omega) \in \mathbb{C}^{m\times m}$ with the uncertainty set $\mathbf{\Delta}$,*

$$\mu(G(\jmath\omega)) \leqslant \mu_{\text{octo}}(\omega), \tag{4.3}$$

where

$$\mu_{\text{octo}}(\omega) \triangleq \inf\{\gamma > 0 : \text{there exists } N \in \mathbb{R} \text{ such that } \operatorname{He}[\tfrac{1}{2}\gamma I + G_s(\jmath\omega) - N[G_s(\jmath\omega) - G_s^*(\jmath\omega)]G_s(\jmath\omega) > 0\}, \tag{4.4}$$

where

$$G_s(\jmath\omega) \triangleq [I - \gamma^{-1}G(\jmath\omega)]^{-1}G(\jmath\omega).$$

Proof. First note that with $-M_1 = M_2 = \gamma^{-1}$, the inequality in (4.4) is equivalent to $\operatorname{He}\mathcal{G}_s(\jmath\omega) \geqslant 0$. Thus, as in the proof of Theorem 3.1, it follows that $\det(I + G(\jmath\omega)\Delta) \neq 0, \Delta I_m \in \mathbf{\Delta}$. Hence, since $\mu_{\text{octo}}(\omega)$ is defined as the infimum over all γ such that the inequality in (4.4) is satisfied and since for each such γ it follows that $\det(I + G(\jmath\omega)\Delta) \neq 0, \Delta I_m \in \mathbf{\Delta}$, it follows that $\mu_{\text{octo}}(\omega)$ is an upper bound for $\mu(G(\jmath\omega))$. □

Next, we provide a graphical interpretation of the mixed-μ bound given in Theorem 4.1 in the scalar case $m = 1$. Letting $G = x + \jmath y$, it follows from the inequality in (4.4) that

$$\gamma\left[(1-\gamma^{-1}x)^2 + (\gamma^{-1}y)^2\right]^2 + 2[(1-\gamma^{-1}x)^2 + (\gamma^{-1}y)^2][(1-\gamma^{-1}x)x - \gamma^{-1}y^2] + 4Ny^2 > 0, \tag{4.5}$$

which, after algebraic manipulation, is equivalent to

$$(\gamma^{-2}x^2 - \gamma^{-1}x + \gamma^{-2}y^2)^2 - (\gamma^{-1}x - 1)^2 - 4N\gamma^{-1}y^2 < 0. \tag{4.6}$$

Condition (4.6) requires that the Nyquist plot of $G(\jmath\omega)$ lie inside the octomorphic region in the Nyquist plane shown in Fig. 1. By setting $y = 0$ in (4.6) it can be seen that this region has real-axis intercepts $\pm\gamma$. Note that for a given choice of the *static* scale N, minimizing over γ corresponds to minimizing the length of the real axis segment contained within the octomorph and hence maximizes the robustness boundary γ^{-1}. Thus minimizing γ as in Theorem 4.1 provides a real-μ upper bound.

Next we contrast our result with recent results on real-μ bounds [5, 13, 24]. Specifically, it was shown in [5, 13, 24] that, for $\omega \in \mathbb{R}, G(\jmath\omega) \in \mathbb{C}^{m\times m}$, and multiple-block-structured uncertainty set $\tilde{\mathbf{\Delta}}$,

$$\mu(G(\jmath\omega)) \leqslant \mu_{\text{cir}}(\omega), \tag{4.7}$$

where

$$\mu_{\text{cir}}(\omega) \triangleq \inf_{D \in \mathcal{D}_{\tilde{\Delta}}, N \in \mathcal{N}_{\tilde{\Delta}}} \inf\{\gamma \geqslant 0 : G^*DG + \jmath\gamma(NG - G^*N) \leqslant \gamma^2 D\}, \tag{4.8}$$

where $\mathcal{D}_{\tilde{\Delta}}$ and $\mathcal{N}_{\tilde{\Delta}}$ denote sets of positive-definite and Hermitian scaling matrices, respectively, which are compatible with the uncertainty structure $\tilde{\mathbf{\Delta}}$.

As noted in [9, 10, 13], in the scalar case the inequality in (4.8) has a graphical interpretation involving frequency-dependent off-axis circles. Specifically, since $D > 0$ and N is real with $G = x + \jmath y$, the inequality in (4.8) is equivalent to

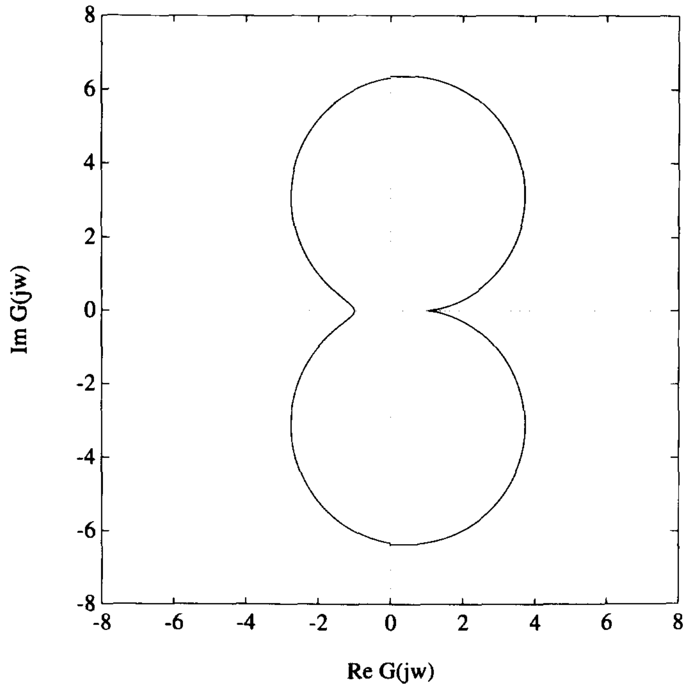

Fig. 1. Octomorphic region with x-axis intercepts $-1/M$ and $1/M$.

$$x^2 + \left(y^2 - \frac{N(\jmath\omega)}{D(\jmath\omega)}\right)^2 \leqslant \gamma^2 + \left(\frac{N(\jmath\omega)}{D(\jmath\omega)}\right)^2 . \tag{4.9}$$

This inequality corresponds to a circle in the Nyquist plane with a frequency-dependent center located at $(0, N(\jmath\omega)/D(\jmath\omega))$ and constant real-axis intercepts $\pm\gamma$. Furthermore, condition (4.9) requires that, at each frequency ω, the transfer function $G(\jmath\omega)$ lie inside the circle centered at $(0, N(\jmath\omega)/D(\jmath\omega))$ with radius $[\gamma^2 + (N(\jmath\omega)/D(\jmath\omega))^2]^{1/2}$. Note that this is fundamentally different from the octomorphic bound (4.3) which involves a *static* scale N and a *fixed* region in the Nyquist plane. The upper and lower halves of the octomorphic region eliminate the need to construct frequency-dependent scaling functions to enclose the Nyquist plot.

Finally, in the case of nonsymmetric uncertainty bounds $-M_1 \neq M_2 \neq \gamma^{-1}$ it follows from (3.23) in the scalar case that

$$[(1+M_1x)^2 + (M_1y)^2]^2 + (M_2 - M_1)[(1+M_1x)^2 + (M_1y)^2][(1+M_1x)x + M_1y^2] + 2N(M_2 - M_1)y^2 > 0, \tag{4.10}$$

which yields an octomorphic curve with real-axis intercepts satisfying

$$(1+M_1x)^3(1+M_2x) = 0, \tag{4.11}$$

i.e., $-1/M_2$ and $-1/M_1$.

5. Illustrative numerical example

To illustrate Theorem 3.1 and Corollary 3.1 consider the asymptotically stable plant

$$G(s) = \frac{-0.25s + 1}{3s^2 + s + 3} .$$

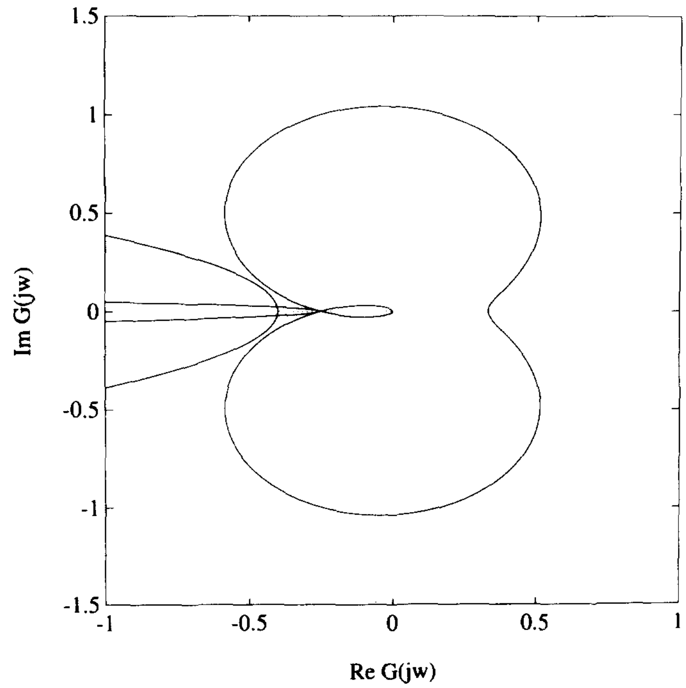

Fig. 2. Nyquist plot with two parabolas ($M = 3.9, N = 150$ and $M = 2.5, N = 2$).

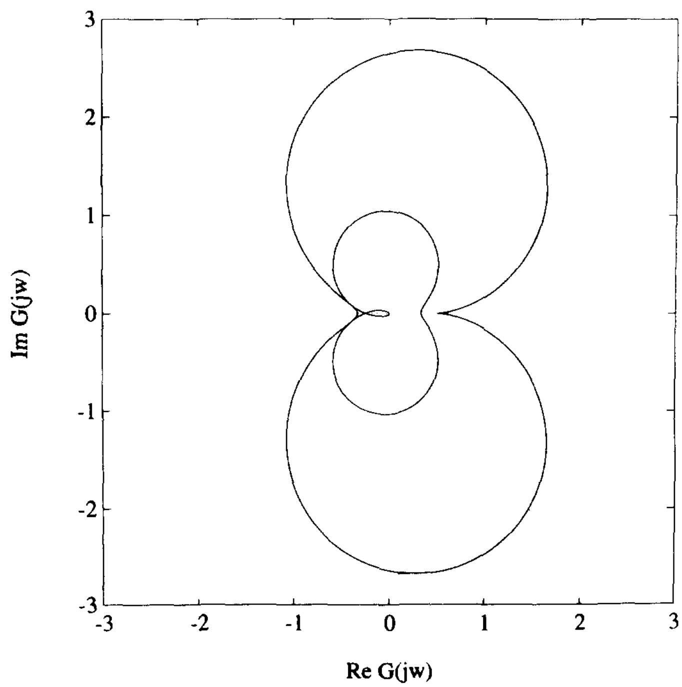

Fig. 3. Nyquist plot enclosed by octomorph ($M_1 = -2, M_2 = 3, N = 17$).

The closed-loop system (3.1) is asymptotically stable for $-2.99 \leqslant F \leqslant 3.99$. Setting $M = 2.5$ and $N = 2$ it follows that $\mathscr{G}(s)$ defined by (3.2) is strict weakly positive real. For $M = 3.9$ this condition is satisfied with $N = 150$. The parabolas corresponding to these stability conditions are shown in Fig. 2.

To illustrate Corollary 3.1, consider two-sided uncertainty $M_1 \leqslant F \leqslant M_2$. Letting $M_1 = -2.0$ and $M_2 = 3.0$, it follows that the Nyquist plot is contained in the octomorphic region with intercepts $-1/M_2 = -0.33$ and $-1/M_1 = 0.5$. This inclusion is shown in Fig. 3 where $N = 17$. For the full uncertainty range $M_1 = -2.99$ and $M_2 = 3.99$, this inclusion is obtained with $N = 4200$. For this example both the positive real and Popov criteria yield the conservative bound $0 \leqslant F \leqslant 1.66$.

References

[1] B.D.O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Theory Approach* (Prentice-Hall, Englewood Cliffs, NJ, 1973).

[2] Y.S. Cho and K.S. Narendra, An off-axis circle criterion for the stability of feedback systems with a monotonic nonlinearity, *IEEE Trans. Automat. Control* **AC-13** (1968) 413–416.

[3] C.A. Desoer and M. Vidyasagar, *Feedback Systems: Input–Output Properties* (Academic Press, New York, 1975).

[4] A.G. Dewey and E.I. Jury, A stability inequality for a class of nonlinear feedback systems, *IEEE Trans. Automat. Control* **AC-11** (1966) 54–62.

[5] M.K.H. Fan, A.L. Tits and J.C. Doyle, Robustness in the presence of mixed parametric uncertainty and unmodelled dynamics, *IEEE Trans. Automat. Control* **AC-36** (1991) 25–38.

[6] W.M. Haddad and D.S. Bernstein, Parameter-dependent Lyapunov functions, constant real parameter uncertainty, and the Popov criterion in robust analysis and synthesis, in: *Proc. IEEE Conf. Dec. Control*, Brighton, UK (1991) 2274–2279 (Part 1), 2632–2633 (Part 2).

[7] W.M. Haddad and D.S. Bernstein, Off-axis absolute stability criteria and μ-bounds involving non-positive-real plant-dependent multipliers for robust stability and performance with locally slope-restricted monotonic nonlinearities, in: *Proc. Amer. Control Conf.*, San Francisco, CA (June 1993) 2790–2794.

[8] W.M. Haddad and D.S. Bernstein, Parameter-dependent Lyapunov functions and the Popov criterion in robust analysis and synthesis, to appear in: *IEEE Trans. Automat. Control.*

[9] W.M. Haddad, J.P. How, S.R. Hall and D.S. Bernstein, Extensions of mixed-μ bounds to monotonic and odd monotonic nonlinearities using absolute stability theory, in: *Proc. IEEE Conf. Dec. Control*, Tucson, AZ (December 1992) 2813–2823; to appear in: *Internat. J. Control.*

[10] J.P. How and S.R. Hall, Connections between the Popov stability criterion and bounds for real parametric uncertainty, in: *Proc. Amer. Control Conf.*, San Francisco, CA (June 1993) 1084–1089.

[11] J.C. Hsu and A.V. Meyer, *Modern Control Principles and Applications* (McGraw-Hill, New York, 1968).

[12] J.D. Lawrence, *A Catalog of Special Plane Curves* (Dover, New York, 1972).

[13] L. Lee and A.L. Tits, Linear fractional transformations for the approximation of various uncertainty sets, in: S. Bhattacharyya and L. Keel, eds., *Control of Uncertain Dynamic Systems* (CRC Press, 1991) 53–62.

[14] K.S. Narendra and C.P. Neuman, Stability of a class of differential equations with a single monotone nonlinearity, *SIAM J. Control* **4** (1966) 295–308.

[15] K.S. Narendra and J.H. Taylor, *Frequency Domain Criteria for Absolute Stability* (Academic Press, New York, 1973).

[16] V.M. Popov, Absolute stability of nonlinear systems of automatic control, *Automat. Remote Control* **22** (1962) 857–875.

[17] M.G. Safonov, *Stability and Robustness of Multivariable Feedback Systems* (MIT Press, Cambridge, MA, 1980).

[18] M.G. Safonov, Stability of interconnected systems having slope-bounded nonlinearities, in: *Proc. 6th Internat. Conf. on Anal. Optimiz. of Systems*, Nice, France (June 1984) 275–287.

[19] M.G. Safonov and R.Y. Chiang, Real/complex K_m-synthesis without curve fitting, in: *Control and Dynamic Systems*, Vol. 56, Part 2 (Academic Press, New York, 1993) 303–324.

[20] M.A.L. Thathachar and M.D. Srinath, Some aspects of the Lur'e problem, *IEEE Trans. Automat. Control* **AC-12** (1967) 451–453.

[21] J.T. Wen, Time domain and frequency domain conditions for strict positive realness, *IEEE Trans. Automat. Control* **33** (1988) 988–992.

[22] V.A. Yakubovitch, The method of matrix inequalities in the theory of the stability of nonlinear control systems II: Absolute stability in a class of nonlinearities with a condition on the derivative, *Automat. Remote Control* **26** (1965) 577–592.

[23] V.A. Yakubovitch, Frequency conditions for the absolute stability and dissipativity of control systems with a single differentiable nonlinearity, *Soviet. Math.* **6** (1965) 98–101.

[24] P.M. Young, M.P. Newlin and J.C. Doyle, μ Analysis with real parametric uncertainty, in: *Proc. IEEE Conf. Dec. Control*, Brighton, UK, December 1991, 1251–1256.

[25] G. Zames and P.L. Falb, Stability conditions for systems with monotone and slope-restricted nonlinearities, *SIAM J. Control Optim.* **4** (1968) 89–108.