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The Optimal Projection Equations for Reduced-Order State Estimation: The Singular Measurement Noise Case

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Dedicated to the memory of Professor Violet B. Haas
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Abstract—The optimal projection equations for reduced-order state estimation are generalized to allow for singular (i.e., colored) measurement noise. The noisy and noise-free measurements serve as inputs to dynamic and static estimators, respectively. The optimal solution is characterized by necessary conditions which involve a pair of oblique projections corresponding to reduced estimator order and singular measurement noise intensity.

I. INTRODUCTION

It has recently been shown [1] that solutions to the steady-state reduced-order state-estimation problem can be characterized by means of a system of modified Riccati and Lyapunov equations coupled by an oblique projection. As in classical Kalman filter theory [2], however, this solution is based on the assumption that all measurements are corrupted by white noise. When the measurement noise is singular (i.e., colored), the optimal solution cannot be applied since the filter gains are given in terms of the

inverse of the noise intensity matrix. Hence, it is not surprising that a sizable body of literature has been devoted to the singular measurement noise problem in both continuous and discrete time [2]-[14]. For an overview of stochastic observer theory, see [15].

Much of the continuous-time singular estimation literature attempts to overcome the noise singularity by introducing new measurements obtained by differentiating noise-free measurements. The present note complements these results in the following way. For the available noisy and noise-free measurements we simultaneously design a reduced-order dynamic estimator for the noisy measurements and a static estimator for the noise-free measurements. We are not concerned here with the question of how the measurements are generated (e.g., via successive differentiation). Rather, our goal is to develop a unified dynamic/static estimation design theory which permits full utilization of both noisy and noise-free measurements. Application of these results to previously proposed approaches to singular estimation involving differentiation and transformation should be an interesting area for future research.

The results given herein directly generalize the results obtained in [1]. Specifically, the modified Riccati/Lyapunov equations are now coupled by a pair of oblique projections. As in [1] the requirement for reduced estimator order gives rise to the projection

$$\tau_2 = \hat{Q}\hat{P}(\hat{Q}\hat{P})^\# \quad (1.1)$$

where $(\)^\#$ denotes group generalized inverse and \hat{Q} and \hat{P} are rank-deficient nonnegative-definite matrices analogous to the controllability and observability Gramians of the estimator. In addition, the presence of noise-free measurements

$$y(t) = C_2 x(t) \quad (1.2)$$

leads to the projection

$$\tau_1 = QC_2^T(C_2QC_2^T)^{-1}C_2 \quad (1.3)$$

where Q is the steady-state error covariance. The contribution of the present note is a concise, unified statement of the optimality conditions in a form which clearly displays the role of the oblique projections τ_1 and τ_2 in explicitly characterizing optimal static/dynamic (nonstrictly proper) estimators. An additional feature of the present note is the presence of state- and measurement-dependent white noise in the plant model. This model has been studied in a state-estimator context in [16]-[18] and has been justified as an approach to robustness in [19]-[22].

In Section III of the note, we consider the case in which the noisy and noise-free measurements are fed to the dynamic and static estimators, respectively. In Section IV, we note that feeding the noisy measurements to the static estimator results in an ill-posed problem, and we consider the general case in which the noise-free measurements are fed to both the static and dynamic estimators. Optimality conditions now lead to the interesting disjointness condition

$$0 = \tau_2 \tau_1 \quad (1.4)$$

concerning the relationship between the static and dynamic estimators. The meaning of (1.4) for proposed singular estimation schemes will be explored in future papers.

The goal of this note is confined to a rigorous development of necessary conditions for the optimal estimation problem. In support of this aim it should be noted that the usefulness of necessary conditions in optimization and optimal control has been amply demonstrated by classical results such as the maximum principle and Euler-Lagrange theory. For practical purposes, necessary conditions are largely free from restrictive special assumptions which invariably accompany sufficiency theory. Most importantly, success in addressing the problems of existence, sufficiency and global optimality is far more likely after the full elucidation of the necessary conditions has been achieved. Indeed, sufficiency conditions are often obtained by strengthening necessary conditions by means of additional restrictive assumptions.

Even without a complete resolution of questions pertaining to existence

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and sufficiency, the necessary conditions fulfill several immediate needs. Specifically, the structure of these conditions provides insight into the properties of the solution arising from optimality considerations. This has been demonstrated for the closely related problem of reduced-order modeling for which local minima are characterized in terms of an eigensystem decomposition [23]. Potentially more useful than insight for practical applications are prospects for constructing novel computational algorithms which avoid traditional gradient search methods. Thus far, two distinct algorithms have been developed, namely, an iterative method which exploits the structure of the oblique projection [23] and a homotopy algorithm which eliminates the need for eigensystem calculations and provides the means for attaining global optimality [24]. For computational purposes it should also be noted that under an existence assumption the necessary conditions are guaranteed to possess a solution to the problem, while sufficient conditions may fail in this regard.

III. PROBLEM STATEMENT AND MAIN THEOREM

Reduced-Order State-Estimation Problem

Given the n th-order observed system

$$\dot{x}(t) = \left(A + \sum_{i=1}^p v_i(t) A_i \right) x(t) + w_0(t), \tag{3.1}$$

$$y_1(t) = \left(C_1 + \sum_{i=1}^p v_i(t) C_{1i} \right) x(t) + w_1(t), \tag{3.2}$$

$$y_2(t) = C_2 x(t), \tag{3.3}$$

II. NOTATION AND DEFINITIONS

$\mathbb{R}, \mathbb{R}^{r \times s}, \mathbb{R}^r, \mathbb{E}$	real numbers, $r \times s$ real matrices, $\mathbb{R}^{r \times 1}$, expectation
$I_n, ()^T, ()^\#$	$n \times n$ identity, transpose, group generalized inverse [25, p. 124]
\oplus, \otimes	Kronecker sum, Kronecker product [26]
$\text{tr } Z$	trace of a square matrix Z
τ_\perp	$I_n - \tau, \tau \in \mathbb{R}^{n \times n}$
n, l_1, l_2, n_e, p, q	positive integers, $1 \leq n_e \leq n$
\tilde{n}	$n + n_e$
x, x_e	n, n_e -dimensional vectors
y_1, y_2, y_e	l_1, l_2, q -dimensional vectors
$A, A_i; C_1, C_{1i}$	$n \times n$ matrices; $l_1 \times n$ matrices, $i = 1, \dots, p$
C_2	$l_2 \times n$ matrix
A_e, B_e, C_e, D_e	$n_e \times n_e, n_e \times l_1, q \times n_e, q \times l_2$ matrices
$v_i(t)$	unit variance white noise, $i = 1, \dots, p$
$w_0(t), w_1(t)$	n -dimensional, l_1 -dimensional white noise processes
V_0	$n \times n$ nonnegative-definite intensity of $w_0(t)$
V_1	$l_1 \times l_1$ positive-definite intensity of $w_1(t)$
V_{01}	$n \times l_1$ cross intensity of $w_0(t), w_1(t)$
R	$q \times q$ positive-definite matrix
L	$q \times n$ matrix
\tilde{A}, \tilde{A}_i	$\begin{bmatrix} A & 0 \\ B_e C_1 & A_e \end{bmatrix}, \begin{bmatrix} A_i & 0 \\ B_e C_{1i} & 0 \end{bmatrix}, i = 1, \dots, p$
K	$n_e \times l_2$ matrix
$\tilde{\tilde{A}}$	$\begin{bmatrix} A & 0 \\ B_e C_1 + K C_2 & A_e \end{bmatrix}$
$\tilde{w}(t)$	$\begin{bmatrix} w_0(t) \\ B_e w_1(t) \end{bmatrix}$
\tilde{V}	$\begin{bmatrix} V_0 & V_{01} B_e^T \\ B_e V_{01}^T & B_e V_1 B_e^T \end{bmatrix}$
\tilde{R}	$\begin{bmatrix} L^T R L - L^T R D_e C_2 - C_2^T D_e^T R L + C_2^T D_e^T R D_e C_2 & -L^T R C_e + C_2^T D_e^T R C_e \\ -C_2^T R L + C_2^T R D_e C_2 & C_e^T R C_e \end{bmatrix}$
asymptotically stable matrix	matrix with eigenvalues in open left-half plane
nonnegative-semisimple matrix	semisimple (nondefective) matrix with nonnegative eigenvalues
nonnegative-definite matrix	symmetric matrix with nonnegative eigenvalues
positive-definite matrix	symmetric matrix with positive eigenvalues.

For arbitrary $n \times n$ Q, \hat{Q} define:

$$V_{1s} \triangleq V_1 + \sum_{i=1}^p C_{1i} (Q + \hat{Q}) C_{1i}^T,$$

$$Q_s \triangleq Q C_1^T + V_{01} + \sum_{i=1}^p A_i (Q + \hat{Q}) C_{1i}^T,$$

$$A_Q \triangleq A - Q_s V_{1s}^{-1} C_1.$$

where $t \in [0, \infty)$, design an n_e th-order state estimator

$$\dot{x}_e(t) = A_e x_e(t) + B_e y_1(t), \tag{3.4}$$

$$y_e(t) = C_e x_e(t) + D_e y_2(t) \tag{3.5}$$

which minimizes the state-estimation error criterion

$$J(A_e, B_e, C_e, D_e) \triangleq \lim_{T \rightarrow \infty} \mathbb{E} [Lx(t) - y_e(t)]^T R [Lx(t) - y_e(t)]. \tag{3.6}$$

To guarantee that J is finite, assume that A is asymptotically stable and consider the set of asymptotically stable reduced-order (i.e., fixed-order) estimators

$$A \triangleq \{(A_e, B_e, C_e, D_e) : A_e \text{ is asymptotically stable}\}.$$

Since the value of J is independent of the internal realization of the transfer function corresponding to (3.4) and (3.5), without loss of generality we further restrict our attention to the set of admissible estimators

$$A^- \triangleq \{(A_e, B_e, C_e, D_e) \in A : (A_e, B_e) \text{ is controllable and } (A_e, C_e) \text{ is observable}\}.$$

An additional technical requirement is that (A_e, B_e, C_e, D_e) be confined to the set

$$\hat{A}^+ \triangleq \{(A_e, B_e, C_e, D_e) \in \hat{A}^+ : C_2(Q_{11} - Q_{12}Q_2^{-1}Q_{12}^T)C_2^T \text{ is positive definite}\},$$

where

$$\hat{Q} \triangleq \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \in \mathbb{R}^{\bar{n} \times \bar{n}}$$

satisfies

$$0 = \hat{A}\hat{Q} + \hat{Q}\hat{A}^T + \sum_{i=1}^p \hat{A}_i \hat{Q} \hat{A}_i^T + \hat{V}$$

and Q_2 is invertible since (A_e, B_e) is controllable. The positive definiteness condition holds when C_2 has full row rank and \hat{Q} is positive definite. As can be seen from the proof of Theorem 3.1, this condition implies the existence of the projection τ_1 defined below.

The following factorization lemma is needed for the statement of the main result.

Lemma 3.1: Suppose $n \times n$ \hat{Q} , \hat{P} are nonnegative definite. Then $\hat{Q}\hat{P}$ is nonnegative semisimple. If, in addition, $\text{rank } \hat{Q}\hat{P} = n_e$, then there exist $n_e \times n$ G , Γ and $n_e \times n_e$ invertible M such that

$$\hat{Q}\hat{P} = G^T M \Gamma, \quad (3.7a)$$

$$\Gamma G^T = I_{n_e}. \quad (3.7b)$$

Furthermore, G , M , and Γ are unique modulo a change of basis in \mathbb{R}^{n_e} .

Proof: The result follows from [27, Theorem 6.2.5]. \square

Since $\hat{Q}\hat{P}$ is semisimple (diagonalizable) it has a group generalized inverse $(\hat{Q}\hat{P})^\# = G^T M^{-1} \Gamma$ and

$$\tau_2 \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^\# = G^T \Gamma \quad (3.8)$$

is an oblique projection.

Theorem 3.1: Suppose A is asymptotically stable and $(A_e, B_e, C_e, D_e) \in \hat{A}^+$ solves the reduced-order state-estimation problem. Then there exist $n \times n$ nonnegative-definite matrices Q , \hat{Q} , and \hat{P} such that A_e , B_e , C_e , and D_e are given by

$$A_e = \Gamma(A - Q_s V_{1s}^{-1} C_1) G^T, \quad (3.9)$$

$$B_e = \Gamma Q_s V_{1s}^{-1}, \quad (3.10)$$

$$C_e = L \tau_{1\perp} G^T, \quad (3.11)$$

$$D_e = L Q C_2^T (C_2 Q C_2^T)^{-1} \quad (3.12)$$

and such that Q , \hat{Q} , and \hat{P} satisfy

$$0 = A Q + Q A^T + \sum_{i=1}^p A_i (Q + \hat{Q}) A_i^T + V_0 - Q_s V_{1s}^{-1} Q_s^T + \tau_{2\perp} Q_s V_{1s}^{-1} Q_s^T \tau_{2\perp}^T, \quad (3.13)$$

$$0 = A \hat{Q} + \hat{Q} A^T + Q_s V_{1s}^{-1} Q_s^T - \tau_{2\perp} Q_s V_{1s}^{-1} Q_s^T \tau_{2\perp}^T, \quad (3.14)$$

$$0 = A^T \hat{P} + \hat{P} A + \tau_{1\perp}^T L^T R L \tau_{1\perp} - \tau_{2\perp}^T \tau_{1\perp}^T L^T R L \tau_{1\perp} \tau_{2\perp}, \quad (3.15)$$

$$\text{rank } \hat{Q} = \text{rank } \hat{P} = \text{rank } \hat{Q}\hat{P} = n_e \quad (3.16)$$

where

$$\tau_1 \triangleq Q C_2^T (C_2 Q C_2^T)^{-1} C_2. \quad (3.17)$$

Remark 3.1: Several special cases can be recovered from Theorem 3.1. For example, when the observation noise is nonsingular, i.e., when y_2 is absent, delete (3.12) and set $\tau_1 = 0$ [22]. Deleting also the multiplicative noise terms yields the Main Theorem of [1].

Specializing Theorem 3.1 to the full-order case $n_e = n$ reveals that the Lyapunov equation for \hat{P} is superfluous. In this case $G = \Gamma^{-1}$ and thus $G = \Gamma = I_n$ without loss of generality.

Corollary 3.1: Assume $n_e = n$, A is asymptotically stable and $(A_e, B_e, C_e, D_e) \in \hat{A}^+$ solves the full-order state-estimation problem. Then there exist $n \times n$ nonnegative-definite matrices Q and \hat{Q} such that A_e , B_e , C_e , D_e are given by

$$A_e = A - Q_s V_{1s}^{-1} C_1, \quad (3.18)$$

$$B_e = Q_s V_{1s}^{-1}, \quad (3.19)$$

$$C_e = L \tau_{1\perp}, \quad (3.20)$$

$$D_e = L Q C_2^T (C_2 Q C_2^T)^{-1} \quad (3.21)$$

and such that Q and \hat{Q} satisfy

$$0 = A Q + Q A^T + \sum_{i=1}^p A_i (Q + \hat{Q}) A_i^T + V_0 - Q_s V_{1s}^{-1} Q_s^T, \quad (3.22)$$

$$0 = A \hat{Q} + \hat{Q} A^T + Q_s V_{1s}^{-1} Q_s^T. \quad (3.23)$$

Remark 3.2: Note that by setting $A_i = 0$, $C_i = 0$, $i = 1, \dots, p$, it follows that (3.22) and (3.23) are decoupled and (3.23) is superfluous. To recover the standard Kalman filter which involves nonsingular noise, set $C_2 = 0$, delete (3.21) and define $\tau_1 = 0$.

IV. ADDITIONAL ESTIMATOR PATHS

We now consider the more general estimator

$$\dot{x}_e(t) = A_e x_e(t) + B_e y_1(t) + K y_2(t), \quad (4.1)$$

$$y_e(t) = C_e x_e + D_e y_2(t) + \hat{K} y_1(t) \quad (4.2)$$

involving the additional gains K and \hat{K} .

Note that the additional path introduced in (4.2) implies that J is infinite and thus the problem is meaningless. Hence, set $\hat{K} = 0$, and consider the additional path introduced by (4.1), i.e., filtering the noise-free measurement.

Replacing (3.4) by (4.1) and optimizing with respect to K yields

$$0 = G \hat{P} Q C_2^T, \quad (4.3)$$

which implies

$$0 = \tau_2 \tau_1. \quad (4.4)$$

Using (4.3), $\hat{Q} = \tau_2 \hat{Q}$ and $\hat{P} = \hat{P} \tau_2$ [see (5.17)], the filter gains (3.9)–(3.15) become

$$A_e = \Gamma(A - Q_s V_{1s}^{-1} C_1) G^T - K C_2 G^T, \quad (4.5)$$

$$B_e = \Gamma Q_s V_{1s}^{-1}, \quad (4.6)$$

$$C_e = L\tau_{1\perp}G^T, \quad (4.7)$$

$$D_e = LQC_2^T(C_2QC_2^T)^{-1}, \quad (4.8)$$

$$0 = \hat{A}Q + Q\hat{A}^T + \sum_{i=1}^p A_i(Q + \hat{Q})A_i^T + V_0 - Q_s V_{1s}^{-1} Q_s^T + \tau_{2\perp} Q_s V_{1s}^{-1} Q_s^T \tau_{2\perp}^T, \quad (4.9)$$

$$0 = A\hat{Q} + \hat{Q}A^T + Q_s V_{1s}^{-1} Q_s^T - \tau_{2\perp} Q_s V_{1s}^{-1} Q_s^T \tau_{2\perp}^T - QC_2^T K^T G - G^T K C_2 Q, \quad (4.10)$$

$$0 = \hat{A}^T \hat{P} + \hat{P} \hat{A} + \tau_{1\perp}^T L^T R L \tau_{1\perp} - \tau_{2\perp}^T \tau_{1\perp}^T L^T R L \tau_{1\perp} \tau_{2\perp}, \quad (4.11)$$

where

$$\hat{A} \triangleq A - G^T K C_2, \quad \hat{A}_Q \triangleq A_Q - G^T K C_2.$$

V. PROOF OF THE MAIN THEOREM

Using the notation of Section II the augmented system (3.1)–(3.4) can be written as

$$\dot{\bar{x}}(t) = \left(\bar{A} + \sum_{i=1}^p v_i(t) \bar{A}_i \right) \bar{x}(t) + \bar{w}(t) \quad (5.1)$$

where

$$\bar{x}(t) \triangleq [x^T(t), x_e^T(t)]^T.$$

To analyze (5.1) define the second-moment matrix

$$\bar{Q}(t) = \mathbb{E}[\bar{x}(t)\bar{x}^T(t)]. \quad (5.2)$$

It follows from [28, Theorem 8.5.5, p. 142] that $\bar{Q}(t)$ satisfies

$$\dot{\bar{Q}}(t) = \bar{A}\bar{Q}(t) + \bar{Q}(t)\bar{A}^T + \sum_{i=1}^p \bar{A}_i \bar{Q}(t) \bar{A}_i^T + \bar{V}, \quad t \geq 0. \quad (5.3)$$

Lemma 5.1: $A_e \in \mathbb{A}$ if and only if

$$\mathbb{A} \triangleq \bar{A} \oplus \bar{A} + \sum_{i=1}^p \bar{A}_i \otimes \bar{A}_i$$

is asymptotically stable.

Proof: The result follows from properties of the Kronecker product applied to partitioned matrices. See [22], [26] for details. \square

Hence, \mathbb{A} stable assures

$$\bar{Q} \triangleq \lim_{t \rightarrow \infty} \mathbb{E}[\bar{x}(t)\bar{x}^T(t)]$$

exists. Furthermore, \bar{Q} and its nonnegative-definite dual \bar{P} are unique solutions of the modified Lyapunov equations

$$0 = \bar{A}\bar{Q} + \bar{Q}\bar{A}^T + \sum_{i=1}^p \bar{A}_i \bar{Q} \bar{A}_i^T + \bar{V}, \quad (5.4)$$

$$0 = \bar{A}^T \bar{P} + \bar{P} \bar{A} + \sum_{i=1}^p \bar{A}_i^T \bar{P} \bar{A}_i + \bar{R}. \quad (5.5)$$

Partition $\bar{n} \times \bar{n}$ \bar{Q} , \bar{P} into $n \times n$, $n \times n_e$, and $n_e \times n_e$ subblocks as

$$\bar{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \quad \bar{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix},$$

and define the $n \times n$ nonnegative-definite matrices

$$Q \triangleq Q_1 - Q_{12} Q_2^{-1} Q_{12}^T, \quad P \triangleq P_1 - P_{12} P_2^{-1} P_{12}^T, \\ \hat{Q} \triangleq Q_{12} Q_2^{-1} Q_{12}^T, \quad \hat{P} \triangleq P_{12} P_2^{-1} P_{12}^T$$

and the $n_e \times n$, $n_e \times n_e$, $n_e \times n$ matrices

$$G \triangleq Q_2^{-1} Q_{12}^T, \quad M \triangleq Q_2 P_2, \quad \Gamma \triangleq -P_2^{-1} P_{12}^T.$$

To optimize (3.6) subject to the constraint (5.4) over \hat{A}^+ , form the Lagrangian

$$L(A_e, B_e, C_e, D_e, \bar{Q}, \bar{P}, \lambda)$$

$$\triangleq \text{tr} \left[\lambda \bar{Q} \bar{R} + \left(\bar{A} \bar{Q} + \bar{Q} \bar{A}^T + \sum_{i=1}^p \bar{A}_i \bar{Q} \bar{A}_i^T + \bar{V} \right) \bar{P} \right]$$

where the Lagrange multipliers $\lambda \geq 0$ and $\bar{P} \in \mathbb{R}^{\bar{n} \times \bar{n}}$ are not both zero and \bar{Q} and \bar{P} are viewed as arbitrary $\bar{n} \times \bar{n}$ matrix variables. Setting $\partial L / \partial \bar{Q} = 0$, $\lambda = 0$ implies $\bar{P} = 0$ since $(A_e, B_e, C_e, D_e) \in \hat{A}^+$. Hence, without loss of generality, set $\lambda = 1$. Thus, the stationarity conditions are given by¹

$$\frac{\partial L}{\partial \bar{P}} = \bar{A} \bar{Q} + \bar{Q} \bar{A}^T + \sum_{i=1}^p \bar{A}_i \bar{Q} \bar{A}_i^T + \bar{V} = 0, \quad (5.6)$$

$$\frac{\partial L}{\partial \bar{Q}} = \bar{A}^T \bar{P} + \bar{P} \bar{A} + \sum_{i=1}^p \bar{A}_i^T \bar{P} \bar{A}_i + \bar{R} = 0, \quad (5.7)$$

$$\frac{\partial L}{\partial A_e} = P_{12}^T Q_{12} + P_2 Q_2 = 0, \quad (5.8)$$

$$\frac{\partial L}{\partial B_e} = (P_{12}^T Q_1 + P_2 Q_{12}^T) C_1^T + \sum_{i=1}^p (P_{12}^T A_i Q_1 C_{1i}^T + P_2 B_e C_{1i} Q_1 C_{1i}^T) + P_{12}^T V_{01} + P_2 B_e V_1 = 0, \quad (5.9)$$

$$\frac{\partial L}{\partial C_e} = -R L Q_{12} + R D_e C_2 Q_{12} + R C_e Q_2 = 0, \quad (5.10)$$

$$\frac{\partial L}{\partial D_e} = -R L Q_1 C_2^T + R D_e C_2 Q_1 C_2^T + R C_e Q_{12}^T C_2^T = 0. \quad (5.11)$$

Expanding (5.6) and (5.7) yields

$$0 = A Q_1 + Q_1 A^T + \sum_{i=1}^p A_i Q_1 A_i^T + V_0, \quad (5.12)$$

$$0 = A Q_{12} + Q_1 C_{11}^T B_e^T + Q_{12} A_e^T + \sum_{i=1}^p A_i Q_1 C_{1i}^T B_e^T + V_{01} B_e^T, \quad (5.13)$$

$$0 = B_e C_1 Q_{12}^T + A_e Q_2 + Q_{12}^T C_{11}^T B_e^T + Q_2 A_e^T + \sum_{i=1}^p B_e C_{1i} Q_1 C_{1i}^T B_e^T + B_e V_1 B_e^T, \quad (5.14)$$

$$0 = A^T P_{12} + C_{11}^T B^T P_2 + P_{12} A_e - L^T R C_e + C_{12}^T D_e^T R C_e, \quad (5.15)$$

$$0 = A_e^T P_2 + P_2 A_e + C_e^T R C_e. \quad (5.16)$$

Note that the (1, 1) subblock of (5.7) characterizing P_1 has been omitted from the above equations since the estimator gains are independent of P_1 .

Note that (5.8) implies (3.7a) and (3.7b). Since

$$Q_2 P_2 = P_2^{-1/2} (P_2^{1/2} Q_2 P_2^{1/2}) P_2^{1/2},$$

M is positive semisimple. Sylvester's inequality yields (3.16). Note also that

$$\hat{Q} = \tau_2 \bar{Q}, \quad \hat{P} = \bar{P} \tau_2. \quad (5.17)$$

Next (3.10), (3.11), and (3.12) follow from (5.9), (5.10), and (3.11) by

¹ As shown in [29], the formula for the derivative of a scalar function with respect to symmetric arguments \bar{Q} and \bar{P} entails a modification of (5.6) and (5.7). Since these gradients are being set to zero, however, the final result is identical. Alternatively, \bar{Q} and \bar{P} can be viewed (as we are doing here) as arbitrary matrix variables. Symmetry is imposed only *a posteriori* by the form of (5.4) and (5.5) and the stability of \bar{A} . Hence, mathematically, the result of [29] is not required.

using the identities

$$Q_1 = Q + \hat{Q}, \quad P_1 = P + \hat{P}, \quad (5.18)$$

$$Q_{12} = \hat{Q}\Gamma^T, \quad P_{12} = -\hat{P}G^T, \quad (5.19)$$

$$Q_2 = \Gamma\hat{Q}\Gamma^T, \quad P_2 = G\hat{P}G^T. \quad (5.20)$$

Substituting (3.10), (3.11), (3.12) and (5.18)–(5.20) into (5.12)–(5.16) and using (5.12) + $G^T\Gamma(5.13)G - (5.13)G - (5.13G)^T$ and $G^T\Gamma(5.13)G - (5.13)G - (5.13G)^T$ yields (3.13) and (3.14). Using $\Gamma^TG(5.15)\Gamma - (5.15)\Gamma - (5.15\Gamma)^T$ yields (3.15). Finally, $\Gamma(5.13) - (5.14)$ or $G(5.15) - (5.16)$ yields (3.9). \square

Remark 5.1. Equations (4.5)–(4.11) are derived in a similar manner with \tilde{A} replaced by \tilde{A} in (5.1).

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The Optimal Projection Equations for Static and Dynamic Output Feedback: The Singular Case

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Dedicated to the memory of Professor Violet B. Haas
November 23, 1926–January 21, 1986

Abstract—Oblique projections have been shown to arise naturally in both static and dynamic optimal design problems. For static controllers an oblique projection was inherent in the early work of Levine and Athans, while for dynamic controllers an oblique projection was developed by Hyland and Bernstein. This note is motivated by the following natural question: What is the relationship between the oblique projection arising in optimal static output feedback and the oblique projection arising in optimal fixed-order dynamic compensation? We show that in nonstrictly proper optimal output feedback there are, indeed, three distinct oblique projections corresponding to singular measurement noise, singular control weighting, and reduced compensator order. Moreover, we unify the Levine–Athans and Hyland–Bernstein approaches by rederiving the optimal projection equations for combined static/dynamic (nonstrictly proper) output feedback in a form which clearly illustrates the role of the three projections in characterizing the optimal feedback gains. Even when the dynamic component of the nonstrictly proper controller is of full order, the controller is characterized by four matrix equations which generalize the standard LQG result.

I. INTRODUCTION

The optimal static output-feedback problem [1], [2] and the optimal fixed-order dynamic-compensation problem [3], [4] have been extensively investigated. A salient feature of the necessary conditions for each of these problems is the presence of an oblique projection (idempotent matrix) which arises as a direct consequence of optimality. For the static problem with noise-free measurements (i.e., singular measurement noise) the necessary conditions involve the projection [2]

$$\tau_1 = QC^T(CQC^T)^{-1}C$$

where Q is the steady-state closed-loop state covariance. The dual projection

$$\tau_2 = B(B^T P B)^{-1} B^T P$$

arises analogously in the corresponding problem involving singular control weighting. Furthermore, for fixed-order dynamic compensation with noisy measurements, it has recently been shown [4] that the necessary conditions give rise to the projection

$$\tau_3 = \hat{Q}\hat{P}(\hat{Q}\hat{P})^\#$$

where $(\)^\#$ denotes group generalized inverse and \hat{Q} and \hat{P} are rank-deficient nonnegative-definite matrices analogous to the controllability

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