using the identities

$$Q_1 = Q + \hat{Q}, \quad P_1 = P + \hat{P}, \tag{5.18}$$

$$Q_{12} = \hat{Q}\Gamma^T, \quad P_{12} = -\hat{P}G^T, \tag{5.19}$$

$$Q_2 = \Gamma\hat{Q}\Gamma^T, \quad P_2 = G\hat{P}G^T. \tag{5.20}$$

Substituting (3.10), (3.11), (3.12) and (5.18)–(5.20) into (5.12)–(5.16) and using (5.12) + $G^T\Gamma$(5.13)G − (5.13)G − (5.13G)^T and $G^T\Gamma$(5.13)G − (5.13)G − (5.13G)^T yields (3.13) and (3.14). Using $\Gamma^T G$(5.15)Γ − (5.15)Γ − (5.15Γ)^T yields (3.15). Finally, Γ(5.13)–(5.14) or G(5.15)–(5.16) yields (3.9). □

Remark 5.1. Equations (4.5)–(4.11) are derived in a similar manner with $\tilde{A}$ replaced by $\tilde{\tilde{A}}$ in (5.1).

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The Optimal Projection Equations for Static and Dynamic Output Feedback: The Singular Case

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Dedicated to the memory of Professor Violet B. Haas
November 23, 1926–January 21, 1986

Abstract—**Oblique projections have been shown to arise naturally in both static and dynamic optimal design problems. For static controllers an oblique projection was inherent in the early work of Levine and Athans, while for dynamic controllers an oblique projection was developed by Hyland and Bernstein. This note is motivated by the following natural question: What is the relationship between the oblique projection arising in optimal static output feedback and the oblique projection arising in optimal fixed-order dynamic compensation? We show that in nonstrictly proper optimal output feedback there are, indeed, *three distinct oblique projections* corresponding to singular measurement noise, singular control weighting, and reduced compensator order. Moreover, we unify the Levine–Athans and Hyland–Bernstein approaches by rederiving the optimal projection equations for combined static/dynamic (nonstrictly proper) output feedback in a form which clearly illustrates the role of the three projections in characterizing the optimal feedback gains. Even when the dynamic component of the nonstrictly proper controller is of full order, the controller is characterized by four matrix equations which generalize the standard LQG result.**

I. INTRODUCTION

The optimal static output-feedback problem [1], [2] and the optimal fixed-order dynamic-compensation problem [3], [4] have been extensively investigated. A salient feature of the necessary conditions for each of these problems is the presence of an oblique projection (idempotent matrix) which arises as a direct consequence of optimality. For the static problem with noise-free measurements (i.e., singular measurement noise) the necessary conditions involve the projection [2]

$$\tau_1 = QC^T(CQC^T)^{-1}C$$

where Q is the steady-state closed-loop state covariance. The dual projection

$$\tau_2 = B(B^TPB)^{-1}B^TP$$

arises analogously in the corresponding problem involving singular control weighting. Furthermore, for fixed-order dynamic compensation with noisy measurements, it has recently been shown [4] that the necessary conditions give rise to the projection

$$\tau_3 = \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#}$$

where $(\ )^{\#}$ denotes group generalized inverse and $\hat{Q}$ and $\hat{P}$ are rank-deficient nonnegative-definite matrices analogous to the controllability

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and observability Gramians of the compensator. To understand the relationships among τ_1, τ_2, and τ_3, *the contribution of the present note is a unified treatment of the necessary conditions for optimal static/dynamic feedback compensation which clearly illustrates the role of the three projections in characterizing the optimal feedback gains.* Even in the full-order case in which τ_3 is the identity, the result provides a generalization of the standard LQG result to nonstrictly proper controllers in which case the separation principle does not hold.

To clarify the ramifications of noise and weighting singularities in optimal output feedback, consider the problem of minimizing

$$J=\lim_{t\to\infty} \mathbb{E}[x^T R_0 x + u^T R_1 u] \tag{1.1}$$

with plant dynamics

$$\dot{x}=Ax+Bu+w_0, \tag{1.2}$$

$$y=Cx+w_1, \tag{1.3}$$

and nonstrictly proper feedback compensator

$$\dot{x}_c=A_c x_c+B_c y, \tag{1.4}$$

$$u=C_c x_c+D_c y. \tag{1.5}$$

As pointed out in [3], J is finite only if

$$0=\mathrm{tr}\,[D_c^T R_1 D_c V_1] \Leftrightarrow 0=R_1 D_c V_1 \tag{1.6}$$

where V_1 denotes the intensity of w_1. Clearly, when R_1 and V_1 are nonsingular (1.6) implies $D_c=0$, and hence direct feedthrough is not permitted, i.e., the compensator must be strictly proper. Conversely, to utilize a static gain D_c, either R_1 or V_1 must be singular. By writing singular R_1 and V_1 without loss of generality as

$$R_1=\begin{bmatrix} \hat{R}_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad V_1=\begin{bmatrix} \hat{V}_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{1.7}$$

it follows that the static transmission between noisy measurements and weighted controls must be zero (see Fig. 1).

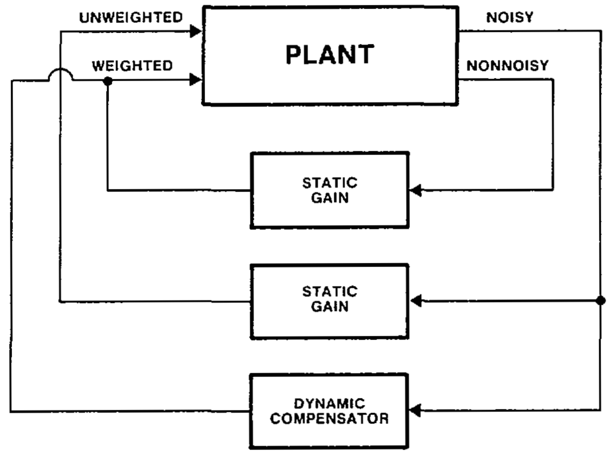

Fig. 1.

The reader will observe that three feedback paths which are not ruled out by (1.6) do not appear in Fig. 1. Specifically: 1) nonnoisy measurements can be fed back to unweighted controls; 2) dynamic-compensator outputs can be fed back to unweighted controls; and 3) nonnoisy measurements can serve as inputs to the dynamic compensator. The reason for considering the more limited configuration shown in Fig. 1 is that only these paths are explicitly characterized by the necessary conditions. Hence, for simplicity we first consider only the scheme of Fig. 1, and later introduce the remaining permissible paths. Interestingly, while these additional gains are not completely determined by the necessary conditions, they appear to play an important role in governing geometric interrelationships among the three projections.

Two final comments are in order. First, since our results are carried out in a multiplicative noise setting, we generalize previous results on state feedback [15]–[18] and dynamic compensation [9]–[11]. The motivation for using a multiplicative white noise model is to represent plant parameter uncertainties and thereby obtain robust controllers [12]. Also, the derivations of the necessary conditions are straightforward extensions of the Lagrange multiplier technique used in [4] and hence have been omitted.

II. Notation and Definitions

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$, $\mathbb{E}$	real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expectation
I_r, $(\ )^T$, $(\ )^\#$	$r \times r$ identity, transpose, group generalized inverse [13, p. 124]
$\oplus$, $\otimes$	Kronecker sum, Kronecker product
$\tau_\perp$	$I_n - \tau$, $\tau \in \mathbb{R}^{n\times n}$
asymptotically stable matrix	matrix with eigenvalues in open left-half plane
nonnegative-semisimple matrix	semisimple (nondefective) matrix with nonnegative eigenvalues
$n, m_1, m_2, l_1, l_2, n_c, p$	positive integers
$x, u_1, u_2, y_1, y_2, x_c$	$n, m_1, m_2, l_1, l_2, n_c$-dimensional vectors
$A, A_i; B_1, B_{1i}; C_1, C_{1i}$	$n \times n$ matrices, $n \times m_1$ matrices, $l_1 \times n$ matrices, $i = 1, \cdots, p$
B_2, C_2	$n \times m_2$ matrix, $l_2 \times n$ matrix
A_c, B_c, C_c, D_c, E_c	$n_c \times n_c$, $n_c \times l_1$, $m_1 \times n_c$, $m_1 \times l_2$, $m_2 \times l_1$ matrices
$v_i(t)$	unit variance white noise, $i = 1, \cdots, p$
$w_0(t)$, $w_1(t)$	n-dimensional, l_1-dimensional white noise
V_0, V_1	intensities of w_0, w_1; $V_0 \ge 0$, $V_1 > 0$
V_{01}	$n \times l_1$ cross intensity of w_0, w_1
R_0, R_1	state and control weightings; $R_0 \ge 0$, $R_1 > 0$
R_{01}	$n \times m_1$ cross weighting: $R_0 - R_{01}R_1^{-1}R_{01}^T \ge 0$
$\tilde{A}$, $\tilde{A}_i$	$A + B_1 D_c C_2 + B_2 E_c C_1$, $A_i + B_{1i} D_c C_2 + B_2 E_c C_{1i}$, $i = 1, \cdots, p$
$\tilde{V}$, $\tilde{R}$	$\begin{bmatrix} I_n \\ (B_2E_c)^T \end{bmatrix}^T \begin{bmatrix} V_0 & V_{01} \\ V_{01}^T & V_1 \end{bmatrix} \begin{bmatrix} I_n \\ (B_2E_c)^T \end{bmatrix}$, $\begin{bmatrix} I_n \\ D_cC_2 \end{bmatrix}^T \begin{bmatrix} R_0 & R_{01} \\ R_{01}^T & R_1 \end{bmatrix} \begin{bmatrix} I_n \\ D_cC_2 \end{bmatrix}$
$\tilde{\tilde{A}}$, $\tilde{\tilde{A}}_i$	$\begin{bmatrix} \tilde{A} & B_1C_c \\ B_cC_1 & A_c \end{bmatrix}$, $\begin{bmatrix} \tilde{A}_i & B_{1i}C_c \\ B_cC_{1i} & 0 \end{bmatrix}$

$$\tilde{\tilde{V}} \qquad \begin{bmatrix} \tilde{V} & V_{01}B_c^T+B_2E_cV_1B_c^T \\ B_cV_{01}^T+B_cV_1(B_2E_c)^T & B_cV_1B_c^T \end{bmatrix}$$

$$\tilde{\tilde{R}} \qquad \begin{bmatrix} \tilde{R} & R_{01}C_c+(D_cC_2)^TR_1C_c \\ C_c^TR_{01}^T+C_c^TR_1D_cC_2 & C_c^TR_1C_c \end{bmatrix}$$

For arbitrary $n \times n$ Q, P, $\hat{Q}$, $\hat{P}$, τ_1, τ_2 define:

$$R_{1s} \triangleq R_1+\sum_{i=1}^{p} B_{1i}^TPB_{1i}, \quad V_{1s} \triangleq V_1+\sum_{i=1}^{p} C_{1i}QC_{1i}^T,$$

$$\mathcal{Q}_s \triangleq QC_1^T+V_{01}+\sum_{i=1}^{p} A_iQC_{1i}^T, \quad \mathcal{P}_s \triangleq B_1^TP+R_{01}^T+\sum_{i=1}^{p} B_{1i}^TPA_i,$$

$$\hat{R}_{1s} \triangleq R_1+\sum_{i=1}^{p} B_{1i}^T(P+\hat{P})B_{1i}, \quad \hat{V}_{1s} \triangleq V_1+\sum_{i=1}^{p} C_{1i}(Q+\hat{Q})C_{1i}^T,$$

$$\hat{\mathcal{Q}}_s \triangleq QC_1^T+V_{01}+\sum_{i=1}^{p} A_i(Q+\hat{Q})C_{1i}^T,$$

$$\hat{\mathcal{P}}_s \triangleq B_1^TP+R_{01}^T+\sum_{i=1}^{p} B_{1i}^T(P+\hat{P})A_i,$$

$$\hat{A} \triangleq A-B_1\hat{R}_{1s}^{-1}(\hat{\mathcal{P}}_s+B_1^T\hat{P})\tau_1-\tau_2(\hat{\mathcal{Q}}_s+\hat{Q}C_1^T)\hat{V}_{1s}^{-1}C_1,$$

$$\hat{A}_i \triangleq A_i-B_{1i}\hat{R}_{1s}^{-1}(\hat{\mathcal{P}}_s+B_1^T\hat{P})\tau_1-\tau_2(\hat{\mathcal{Q}}_s+\hat{Q}C_1^T)\hat{V}_{1s}^{-1}C_{1i},$$

$$A_{Qs} \triangleq \hat{A}-(\tau_{2\perp}\hat{\mathcal{Q}}_s-\tau_2\hat{Q}C_1^T)\hat{V}_{1s}^{-1}C_1, A_{Ps} \triangleq \hat{A}-B_1\hat{R}_{1s}^{-1}(\hat{\mathcal{P}}_s\tau_{1\perp}-B_1^T\hat{P}\tau_1),$$

$$\hat{R}_0 \triangleq R_0-R_{01}\hat{R}_{1s}^{-1}(\hat{\mathcal{P}}_s+B_1^T\hat{P})\tau_1-\tau_1^T(\hat{\mathcal{P}}_s+B_1^T\hat{P})^T\hat{R}_{1s}^{-1}R_{01}^T$$
$$+\tau_1^T(\hat{\mathcal{P}}_s+B_1^T\hat{P})^T\hat{R}_{1s}^{-1}R_1\hat{R}_{1s}^{-1}(\hat{\mathcal{P}}_s+B_1^T\hat{P})\tau_1,$$

$$\hat{V}_0 \triangleq V_0-V_{01}\hat{V}_{1s}^{-1}(\hat{\mathcal{Q}}_s+\hat{Q}C_1^T)^T\tau_2^T-\tau_2(\hat{\mathcal{Q}}_s+\hat{Q}C_1^T)\hat{V}_{1s}^{-1}V_{01}^T$$
$$+\tau_2(\hat{\mathcal{Q}}_s+\hat{Q}C_1^T)\hat{V}_{1s}^{-1}V_1\hat{V}_{1s}^{-1}(\hat{\mathcal{Q}}_s+\hat{Q}C_1^T)^T\tau_2^T.$$

III. Static Output Feedback

Static Output Feedback Problem

Given the controlled system

$$\dot{x}(t)=\left(A+\sum_{i=1}^{p} v_i(t)A_i\right)x(t)$$
$$+\left(B_1+\sum_{i=1}^{p} v_i(t)B_{1i}\right)u_1(t)+B_2u_2(t)+w_0(t), \quad (3.1)$$

$$y_1(t)=\left(C_1+\sum_{i=1}^{p} v_i(t)C_{1i}\right)x(t)+w_1(t), \quad (3.2)$$

$$y_2(t)=C_2x(t) \quad (3.3)$$

where $t \in [0, \infty)$, determine D_c and E_c such that the static output feedback law

$$u_1(t)=D_cy_2(t), \quad (3.4)$$

$$u_2(t)=E_cy_1(t) \quad (3.5)$$

minimizes the performance criterion

$$J \triangleq \lim_{t\to\infty} \mathbb{E}[x(t)^TR_0x(t)+2x(t)^TR_{01}u_1(t)+u_1(t)^TR_1u_1(t)]. \quad (3.6)$$

To develop necessary conditions for this problem, D_c and E_c must be restricted to the set of second-moment-stabilizing gains

$$\mathcal{S} \triangleq \left\{(D_c, E_c) : \tilde{A} \oplus \tilde{A}+\sum_{i=1}^{p} \tilde{A}_i \otimes \tilde{A}_i \text{ is asymptotically stable}\right\}.$$

The requirement $(D_c, E_c) \in \mathcal{S}$ implies the existence of the steady-state closed-loop state covariance $Q \triangleq \lim_{t\to\infty} \mathbb{E}[x(t)x(t)^T]$. Furthermore, Q and its nonnegative-definite dual P are the unique solutions of the modified Lyapunov equations

$$0=\tilde{A}Q+Q\tilde{A}^T+\sum_{i=1}^{p} \tilde{A}_iQ\tilde{A}_i^T+\tilde{V}, \quad (3.7)$$

$$0=\tilde{A}^TP+P\tilde{A}+\sum_{i=1}^{p} \tilde{A}_i^TP\tilde{A}_i+\tilde{R}. \quad (3.8)$$

An additional technical assumption is that (D_c, E_c) be confined to the set

$$\mathcal{S}^+ \triangleq \{(D_c, E_c) \in \mathcal{S} : C_2QC_2^T>0 \text{ and } B_2^TPB_2>0\}.$$

In order to obtain closed-form expressions for the feedback gains we make the additional assumption here and in Section IV that

$$[B_{1i}\neq 0 \Rightarrow C_{1i}=0], \quad i=1, \cdots, p, \quad (3.9)$$

i.e., for each i, B_{1i} and C_{1i} are not both nonzero. By optimizing (3.6) with respect to D_c and E_c and manipulating (3.7) and (3.8), we obtain the following result.

Theorem 3.1: Suppose $(D_c, E_c) \in \mathcal{S}^+$ solves the static output feedback problem. Then there exist $n \times n$ nonnegative-definite Q, P such that

$$D_c=-R_{1s}^{-1}\mathcal{P}_sQC_2^T(C_2QC_2^T)^{-1}, \quad (3.10)$$

$$E_c=-(B_2^TPB_2)^{-1}B_2^TP\mathcal{Q}_sV_{1s}^{-1}, \quad (3.11)$$

and such that Q and P satisfy

$$0=(A-B_1R_{1s}^{-1}\mathcal{P}_s\tau_1)Q+Q(A-B_1R_{1s}^{-1}\mathcal{P}_s\tau_1)^T+V_0$$
$$+\sum_{i=1}^{p} (A_i-B_{1i}R_{1s}^{-1}\mathcal{P}_s\tau_1)Q(A_i-B_{1i}R_{1s}^{-1}\mathcal{P}_s\tau_1)^T$$
$$-\mathcal{Q}_sV_{1s}^{-1}\mathcal{Q}_s^T+\tau_{2\perp}\mathcal{Q}_sV_{1s}^{-1}\mathcal{Q}_s^T\tau_{2\perp}^T, \quad (3.12)$$

$$0=(A-\tau_2\mathcal{Q}_sV_{1s}^{-1}C_1)^TP+P(A-\tau_2\mathcal{Q}_sV_{1s}^{-1}C_1)+R_0$$
$$+\sum_{i=1}^{p} (A_i-\tau_2\mathcal{Q}_sV_{1s}^{-1}C_{1i})^TP(A_i-\tau_2\mathcal{Q}_sV_{1s}^{-1}C_{1i})$$
$$-\mathcal{P}_s^TR_{1s}^{-1}\mathcal{P}_s+\tau_{1\perp}^T\mathcal{P}_s^TR_{1s}^{-1}\mathcal{P}_s\tau_{1\perp}, \quad (3.13)$$

where

$$\tau_1 \triangleq QC_2^T(C_2QC_2^T)^{-1}C_2, \quad (3.14)$$

$$\tau_2 \triangleq B_2(B_2^TPB_2)^{-1}B_2^TP. \quad (3.15)$$

Remark 3.1: Several special cases can be recovered formally from Theorem 3.1. For example, when the control weighting is nonsingular and the measurement noise is zero, i.e., when u_2 and y_1 are absent, delete (3.11) and set $\tau_2=0$. Deleting also the multiplicative noise terms yields the usual static output feedback result [1], [2].

IV. Dynamic Output Feedback

We now expand the formulation of the static problem to include a purely dynamic (strictly proper) dynamic compensator.

Dynamic Output Feedback Problem

Given (3.1)–(3.3), determine A_c, B_c, C_c, D_c, E_c such that the static and dynamic output feedback law

$$\dot{x}_c(t)=A_cx_c(t)+B_cy_1(t), \tag{4.1}$$

$$u_1(t)=C_cx_c(t)+D_cy_2(t), \tag{4.2}$$

$$u_2(t)=E_cy_1(t) \tag{4.3}$$

minimizes the performance criterion (3.6).

We restrict our attention to second-moment-stabilizing controllers

$$\mathfrak{D} \triangleq \left\{(A_c, B_c, C_c, D_c, E_c): \tilde{\tilde{A}} \oplus \tilde{\tilde{A}}+\sum_{i=1}^{p} \tilde{\tilde{A}}_i \otimes \tilde{\tilde{A}}_i \text{ is asymptotically stable and } (A_c, B_c, C_c) \text{ is minimal}\right\},$$

which implies the existence of $\tilde{Q} \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)\tilde{x}(t)^T]$, where $\tilde{x}(t)\triangleq(x(t)^T, x_c(t)^T)^T$. Furthermore, $\tilde{Q}$ and its dual $\tilde{P}$ are the unique solutions of the modified Lyapunov equations

$$0=\tilde{\tilde{A}}\tilde{Q}+\tilde{Q}\tilde{\tilde{A}}^T+\sum_{i=1}^{p}\tilde{\tilde{A}}_i\tilde{Q}\tilde{\tilde{A}}_i^T+\tilde{\tilde{V}}, \tag{4.4}$$

$$0=\tilde{\tilde{A}}^T\tilde{P}+\tilde{P}\tilde{\tilde{A}}+\sum_{i=1}^{p}\tilde{\tilde{A}}_i^T\tilde{P}\tilde{\tilde{A}}_i+\tilde{\tilde{R}}. \tag{4.5}$$

Partitioning

$$\tilde{Q}=\begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \quad \tilde{P}=\begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix},$$

where Q_{12} and P_{12} are $n \times n_c$, we also require

$$\mathfrak{D}^+ \triangleq \{(A_c, B_c, C_c, D_c, E_c) \in \mathfrak{D} : C_2(Q_1-Q_{12}Q_2^{-1}Q_{12}^T)C_2^T>0 \text{ and } B_2^T(P_1-P_{12}P_2^{-1}P_{12}^T)B_2>0\}.$$

Optimizing (3.6) over $\mathfrak{D}^+$, introducing new variables

$$Q \triangleq Q_1-Q_{12}Q_2^{-1}Q_{12}^T, \quad P \triangleq P_1-P_{12}P_2^{-1}P_{12}^T, \tag{4.6}$$

$$\hat{Q} \triangleq Q_{12}Q_2^{-1}Q_{12}^T, \quad \hat{P} \triangleq P_{12}P_2^{-1}P_{12}^T, \tag{4.7}$$

and manipulating (4.4) and (4.5), we obtain the dynamic extension of Theorem 3.1. The following lemma is required for the statement of the result.

Lemma 4.1: Suppose $n \times n$ $\hat{Q}$, $\hat{P}$ are nonnegative definite. Then $\hat{Q}\hat{P}$ is nonnegative semisimple. If, in addition, rank $\hat{Q}\hat{P}=n_c$, then there exist $n_c \times n$ G, Γ and $n_c \times n_c$ invertible M such that

$$\hat{Q}\hat{P}=G^TM\Gamma, \quad \Gamma G^T=I_{n_c}. \tag{4.8a,b}$$

Proof: The result follows from [14, Theorem 6.2.5].

Since $\hat{Q}\hat{P}$ is semisimple it has a group inverse $(\hat{Q}\hat{P})^{\#} = G^TM^{-1}\Gamma$ and

$$\tau_3 \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#}=G^T\Gamma \tag{4.9}$$

is an oblique projection.

Theorem 4.1: Suppose $(A_c, B_c, C_c, D_c, E_c) \in \mathfrak{D}^+$ solves the dynamic output feedback problem. Then there exist $n \times n$ nonnegative-definite Q, P, $\hat{Q}$, $\hat{P}$ such that

$$A_c=\Gamma(A-B_1\hat{R}_{1s}^{-1}\hat{\mathcal{P}}_s-\hat{\mathcal{Q}}_s\hat{V}_{1s}^{-1}C_1)G^T, \tag{4.10}$$

$$B_c=\Gamma(\tau_{2\perp}\hat{\mathcal{Q}}_s-\tau_2\hat{Q}C_1^T)\hat{V}_{1s}^{-1}, \tag{4.11}$$

$$C_c=-\hat{R}_{1s}^{-1}(\hat{\mathcal{P}}_s\tau_{1\perp}-B_1^T\hat{P}\tau_1)G^T, \tag{4.12}$$

$$D_c=-\hat{R}_{1s}^{-1}(\hat{\mathcal{P}}_s+B_1^T\hat{P})QC_2^T(C_2QC_2^T)^{-1}, \tag{4.13}$$

$$E_c=-(B_2^TPB_2)^{-1}B_2^TP(\hat{\mathcal{Q}}_s+\hat{Q}C_1^T)\hat{V}_{1s}^{-1} \tag{4.14}$$

where τ_1 and τ_2 are given by (3.14) and (3.15), G, Γ satisfy (4.8a), (4.8b), and such that, with τ_3 given by (4.9), Q, P, $\hat{Q}$, and $\hat{P}$ satisfy

$$\begin{aligned}0=\hat{A}Q+Q\hat{A}^T+\hat{V}_0+\sum_{i=1}^{p}[\hat{A}_iQ\hat{A}_i^T&+(\hat{A}_i-B_{1i}\hat{R}_{1s}^{-1}[\hat{\mathcal{P}}_s\tau_{1\perp}-B_1^T\hat{P}\tau_1])\\ &\cdot\hat{Q}(\hat{A}_i-B_{1i}\hat{R}_{1s}^{-1}[\hat{\mathcal{P}}_s\tau_{1\perp}-B_1^T\hat{P}\tau_1])^T]-(\tau_{2\perp}\hat{\mathcal{Q}}_s-\tau_2\hat{Q}C_1^T)\hat{V}_{1s}^{-1}\\ &\cdot(\tau_{2\perp}\hat{\mathcal{Q}}_s-\tau_2\hat{Q}C_1^T)^T+\tau_{3\perp}(\tau_{2\perp}\hat{\mathcal{Q}}_s-\tau_2\hat{Q}C_1^T)\hat{V}_{1s}^{-1}(\tau_{2\perp}\hat{\mathcal{Q}}_s-\tau_2\hat{Q}C_1^T)^T\tau_{3\perp}^T,\end{aligned} \tag{4.15}$$

$$\begin{aligned}0=\hat{A}^TP+P\hat{A}+\hat{R}_0+\sum_{i=1}^{p}[\hat{A}_i^TP\hat{A}_i&+(\hat{A}_i-[\tau_{2\perp}\hat{\mathcal{Q}}_s-\tau_2\hat{Q}C_1^T]\hat{V}_{1s}^{-1}C_{1i})^T\\ &\cdot\hat{P}(\hat{A}_i-[\tau_{2\perp}\hat{\mathcal{Q}}_s-\tau_2\hat{Q}C_1^T]\hat{V}_{1s}^{-1}C_{1i})]-(\hat{\mathcal{P}}_s\tau_{1\perp}-B_1^T\hat{P}\tau_1)^T\hat{R}_{1s}^{-1}\\ &\cdot(\hat{\mathcal{P}}_s\tau_{1\perp}-B_1^T\hat{P}\tau_1)+\tau_{3\perp}^T(\hat{\mathcal{P}}_s\tau_{1\perp}-B_1^T\hat{P}\tau_1)^T\hat{R}_{1s}^{-1}(\hat{\mathcal{P}}_s\tau_{1\perp}-B_1^T\hat{P}\tau_1)\tau_{3\perp},\end{aligned} \tag{4.16}$$

$$\begin{aligned}0=A_{Ps}\hat{Q}+\hat{Q}A_{Ps}^T&+(\tau_{2\perp}\hat{\mathcal{Q}}_s-\tau_2\hat{Q}C_1^T)\hat{V}_{1s}^{-1}(\tau_{2\perp}\hat{\mathcal{Q}}_s-\tau_2\hat{Q}C_1^T)^T\\ &-\tau_{3\perp}(\tau_{2\perp}\hat{\mathcal{Q}}_s-\tau_2\hat{Q}C_1^T)\hat{V}_{1s}^{-1}(\tau_{2\perp}\hat{\mathcal{Q}}_s-\tau_2\hat{Q}C_1^T)^T\tau_{3\perp}^T,\end{aligned} \tag{4.17}$$

$$\begin{aligned}0=A_{Qs}^T\hat{P}+\hat{P}A_{Qs}&+(\hat{\mathcal{P}}_s\tau_{1\perp}-B_1^T\hat{P}\tau_1)^T\hat{R}_{1s}^{-1}(\hat{\mathcal{P}}_s\tau_{1\perp}-B_1^T\hat{P}\tau_1)\\ &-\tau_{3\perp}^T(\hat{\mathcal{P}}_s\tau_{1\perp}-B_1^T\hat{P}\tau_1)^T\hat{R}_{1s}^{-1}(\hat{\mathcal{P}}_s\tau_{1\perp}-B_1^T\hat{P}\tau_1)\tau_{3\perp},\end{aligned} \tag{4.18}$$

$$\text{rank } \hat{Q}=\text{rank } \hat{P}=\text{rank } \hat{Q}\hat{P}=n_c. \tag{4.19}$$

Remark 4.1: Setting $\tau_1 = \tau_2 = 0$, $D_c=0$, $E_c=0$ yields the results of [4], [11].

Remark 4.2: Suppose $n_c=n$ so that $\tau_3=I_n$. Then the resulting full-order nonstrictly proper controller is characterized by four matrix equations which generalize the standard LQG result. In this case the separation principle is no longer valid.

V. Additional Feedback Paths

We now introduce the feedback paths not shown in Fig. 1. For the static problem replace (3.5) by

$$u_2(t)=E_cy_1(t)+K_1y_2(t). \tag{5.1}$$

Optimizing with respect to K_1 yields the additional condition

$$0=C_2QPB_2 \tag{5.2}$$

which implies

$$0=\tau_2\tau_1. \tag{5.3}$$

This geometric condition holds when K_1 is optimally chosen. Although K_1

is not given explicitly, it does play a role in the necessary conditions since A is replaced by $A + B_2K_1C_2$.

For the dynamic problem replace (4.1) and (4.3) by

$$\dot{x}_c(t) = A_c x_c(t) + B_c y_1(t) + K_3 y_2(t), \tag{5.4}$$

$$u_2(t) = E_c y_1(t) + K_2 x_c(t) + K_1 y_2(t). \tag{5.5}$$

Optimizing with respect to K_1, K_2, K_3 yields

$$0 = C_2(QP + \hat{Q}P + Q\hat{P})B_2, \tag{5.6a}$$

$$0 = \hat{Q}PB_2, \tag{5.6b}$$

$$0 = C_2Q\hat{P}, \tag{5.6c}$$

which imply

$$0 = \tau_2\tau_1, \tag{5.7a}$$

$$0 = \tau_2\tau_3, \tag{5.7b}$$

$$0 = \tau_3\tau_1. \tag{5.7c}$$

Note that (5.7b) and (5.7c) imply

$$0 = \tau_2\tau_3\tau_1. \tag{5.8}$$

Using (5.7b), (5.7c), $\hat{Q} = \tau_3\hat{Q}$, $\hat{P} = \hat{P}\tau_3$ (see [4]), (4.10)–(4.18) become

$$A_c = \Gamma(A - B_1\hat{R}_{1s}^{-1}\hat{\mathcal{P}}_s - \hat{\mathcal{Q}}_s\hat{V}_{1s}^{-1}C_1 + B_2K_1C_2)G^T + \Gamma B_2K_2 - K_3C_2G^T, \tag{5.9}$$

$$B_c = \Gamma\tau_2\hat{\mathcal{Q}}_s\hat{V}_{1s}^{-1}, \tag{5.10}$$

$$C_c = -\hat{R}_{1s}^{-1}\hat{\mathcal{P}}_s\tau_{1\perp}G^T, \tag{5.11}$$

$$D_c = -\hat{R}_{1s}^{-1}\hat{\mathcal{P}}_sQC_2^T(C_2QC_2^T)^{-1}, \tag{5.12}$$

$$E_c = -(B_2^TPB_2)^{-1}B_2^TP\hat{\mathcal{Q}}_s\hat{V}_{1s}^{-1}, \tag{5.13}$$

$$0 = \hat{A}_3Q + Q\hat{A}_3^T + \hat{V}_0 + \sum_{i=1}^{p}[\hat{A}_iQ\hat{A}_i^T + (\hat{A}_i - B_{1i}\hat{R}_{1s}^{-1}\hat{\mathcal{P}}_s\tau_{1\perp}) \cdot \hat{Q}(\hat{A}_i - B_{1i}\hat{R}_{1s}^{-1}\hat{\mathcal{P}}_s\tau_{1\perp})^T] - \tau_{2\perp}\hat{\mathcal{Q}}_s\hat{V}_{1s}^{-1}\hat{\mathcal{Q}}_s^T\tau_{2\perp}^T + \tau_{3\perp}\tau_{2\perp}\hat{\mathcal{Q}}_s\hat{V}_{1s}^{-1}\hat{\mathcal{Q}}_s^T\tau_{2\perp}^T\tau_{3\perp}^T, \tag{5.14}$$

$$0 = \hat{A}_2^TP + P\hat{A}_2 + \hat{R}_0 + \sum_{i=1}^{p}[\hat{A}_i^TP\hat{A}_i + (\hat{A}_i + \tau_2\hat{\mathcal{Q}}_s\hat{V}_{1s}^{-1}C_{1i})^T \cdot \hat{P}(\hat{A}_i - \tau_2\hat{\mathcal{Q}}_s\hat{V}_{1s}^{-1}C_{1i})] - \tau_{1\perp}^T\hat{\mathcal{P}}_s^T\hat{R}_{1s}^{-1}\hat{\mathcal{P}}_s\tau_{1\perp} + \tau_{3\perp}^T\tau_{1\perp}^T\hat{\mathcal{P}}_s^T\hat{R}_{1s}^{-1}\hat{\mathcal{P}}_s\tau_{1\perp}\tau_{3\perp}, \tag{5.15}$$

$$0 = \hat{A}_{Ps}\hat{Q} + \hat{Q}\hat{A}_{Ps}^T + \tau_{2\perp}\hat{\mathcal{Q}}_s\hat{V}_{1s}^{-1}\hat{\mathcal{Q}}_s^T\tau_{2\perp}^T - \tau_{3\perp}\tau_{2\perp}\hat{\mathcal{Q}}_s\hat{V}_{1s}^{-1}\hat{\mathcal{Q}}_s^T\tau_{2\perp}^T\tau_{3\perp}^T - G^TK_3C_2Q - Q(K_3C_2)^TG, \tag{5.16}$$

$$0 = \hat{A}_{Qs}^T\hat{P} + \hat{P}\hat{A}_{Qs} + \tau_{1\perp}^T\hat{\mathcal{P}}_s^T\hat{R}_{1s}^{-1}\hat{\mathcal{P}}_s\tau_{1\perp} - \tau_{3\perp}^T\tau_{1\perp}^T\hat{\mathcal{P}}_s^T\hat{R}_{1s}^{-1}\hat{\mathcal{P}}_s)\tau_{1\perp}\tau_{3\perp} - \Gamma^TK_2B_2P - P(K_2B_2)^T\Gamma, \tag{5.17}$$

where

$$\hat{A}_3 = \hat{A} + B_2K_1C_2 - G^TK_3C_2, \quad \hat{A}_2 = \hat{A} + B_2K_1C_2 + B_2K_2\Gamma,$$

$$\hat{A}_{Qs} \triangleq A_{Qs} + B_2K_1C_2 - G^TK_3C_2, \quad \hat{A}_{Ps} \triangleq A_{Ps} + B_2K_1C_2 + B_2K_2\Gamma.$$

VI. Directions for Further Research

More general solutions can be obtained by incorporating singular estimation techniques [15] where noise-free measurements are repeatedly differentiated to enlarge the class of available outputs.

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