and that

$$\begin{bmatrix} I+D_1\Delta_2D_2 & D_1\Delta_2 \\ D_2\Delta_1 & 1+D_2\Delta_2D_1 \end{bmatrix}^{-1} = \begin{bmatrix} I & -\Delta_1^{-1}D_1\Delta_2 \\ +\Delta_2^{-1}D_2\Delta_1 & I \end{bmatrix}.$$

Now

$$A_\delta = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} B_1\Delta_2D_2 & B_1\Delta_2 \\ B_2\Delta_1 & B_2\Delta_1D_1 \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$$

so that (C_δ, A_δ) is detectable iff (C_i, A_i) is detectable, $i = 1, 2$. Therefore, $(A_\delta, B_\delta, C_\delta)$ is stabilizable and detectable iff (A_i, B_i, C_i) is stabilizable and detectable, $i = 1, 2$; since the latter condition holds by assumption, it follows that $\sigma(A_\delta) = \sigma(A_{cl}) \subset \mathbb{C}^-$ is equivalent to $H_\delta(s)$ stable.

III. Concluding Remarks

The proof given here shows that the injection of the virtual input vector δ renders the resulting system stabilizable from δ and detectable from y, provided the subsystems are individually stabilizable and detectable. Under these conditions, the transfer function relating y to δ naturally gives the correct stability information. The generalization of this result to a feedback system consisting of several subsystems is straightforward. It is hoped that the proof given here will provide further insight into the useful result established in [1].

References

[1] F. M. Callier and C. A. Desoer, *Multivariable Feedback Systems.* New York: Springer-Verlag, 1982.

[2] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach.* New York: Springer-Verlag, 1980.

The Optimal Projection Equations for Reduced-Order State Estimation

DENNIS S. BERNSTEIN AND DAVID C. HYLAND

Abstract—**First-order necessary conditions for optimal, steady-state, reduced-order state estimation for a linear, time-invariant plant in the presence of correlated disturbance and nonsingular measurement noise are derived in a new and highly simplified form. In contrast to the lone matrix Riccati equation arising in the full-order (Kalman filter) case, the optimal steady-state reduced-order estimator is characterized by *three* matrix equations (one modified Riccati equation and two modified Lyapunov equations) coupled by a projection whose rank is precisely equal to the order of the estimator and which determines the optimal estimator gains. This coupling is a graphic reminder of the suboptimality of proposed approaches involving either model reduction followed by "full-order" estimator design or full-order estimator design followed by estimator-reduction techniques. The results given here complement recently obtained results which characterize the optimal reduced-order model by means of a pair of coupled modified Lyapunov equations [7] and the optimal fixed-order dynamic compensator by means of a coupled system of two modified Riccati equations and two modified Lyapunov equations [6].**

I. Introduction

It has recently been shown (see [1]–[7]) that the first-order necessary conditions for the problems of optimal model reduction and optimal fixed-order dynamic compensation can be formulated in terms of an "optimal projection" matrix which arises as a direct consequence of optimality. These necessary conditions, by virtue of their remarkable simplicity, yield insight into the structure of the optimal design and permit the development of alternative numerical algorithms [2], [4], [7]. The purpose of this note is to develop analogous first-order necessary conditions for the reduced-order state-estimation problem. Since this problem falls midway between the problems of open-loop model reduction and closed-loop fixed-order dynamic compensation, it is not surprising that the necessary conditions for these problems are correspondingly related. Specifically, while the optimal projection equations for model reduction consist of a system of *two* matrix equations (a pair of modified Lyapunov equations) and the optimal projection equations for fixed-order dynamic compensation comprise a system of *four* matrix equations (a pair of modified Lyapunov equations plus a pair of modified Riccati equations), the optimal projection equations for reduced-order state estimation form a system of *three* matrix equations (a pair of modified Lyapunov equations along with a single modified Riccati equation). In each case the system of matrix equations is coupled by an *oblique* projection (idempotent matrix) which determines the gains of the optimal reduced-order system, whether it be a model, estimator, or compensator.

The need for designing an optimal reduced-order state estimator for a high-order dynamic system follows directly from real-world constraints on computing capability. A further motivation is the fact that although a system may have many degrees of freedom, it is often the case that estimates of only a small number of state variables are actually required. In the face of these practical motivations, numerous approaches to designing reduced-order state estimators have been proposed. See [8] for a recent review of previous results.

An important fact pointed out in [8] and [9] is that reduced-order estimators designed by means of either model reduction followed by "full-order" state estimation or full-order estimation followed by estimator reduction will not be optimal for the given order. In the present paper this point is graphically confirmed by the fact that the three matrix equations characterizing the optimal reduced-order state estimator reveal intrinsic coupling (via the optimal projection) between the "operations" of optimal estimation (the modified Riccati equation) and optimal model reduction (the pair of modified Lyapunov equations).

II. Problem Statement and Main Result

The following notation and definitions will be used throughout the paper:

n, l, n_e, p	positive integers, $1 \le n_e \le n$
x, y, x_e, y_e	n, l, n_e, p-dimensional vectors
A, C, L	$n \times n$, $l \times n$, $p \times n$ real matrices
A_e, B_e, C_e	$n_e \times n_e$, $n_e \times l$, $p \times n_e$ real matrices
$w_1(t), t \ge 0$	n-dimensional white noise with nonnegative-definite intensity V_1
$w_2(t), t \ge 0$	l-dimensional white noise with positive-definite intensity V_2
V_{12}	$n \times l$ matrix satisfying $\mathbb{E}[w_1(t)w_2(s)^T] = V_{12}\delta(t-s)$
R	$p \times p$ positive-definite matrix
I_r	$r \times r$ indentity matrix
Z^T	transpose of vector or matrix Z
Z^{-T}	$(Z^T)^{-1}$ or $(Z^{-1})^T$
$\mathfrak{N}(Z), \mathfrak{R}(Z), \rho(Z)$	null space, range, rank of matrix Z
$\mathbb{E}$	expected value
$\mathbb{R}, \mathbb{R}^{r\times s}$	real numbers, $r \times s$ real matrices
stable matrix	matrix with eigenvalues in open left half plane
nonnegative-definite matrix	symmetric matrix with nonnegative eigenvalues
positive-definite matrix	symmetric matrix with positive eigenvalues
nonnegative-semisimple matrix	matrix similar to a nonnegative-definite matrix

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The authors are with Harris Corporation, GASD, Melbourne, FL 32901.

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positive-semisimple matrix	matrix similar to a positive-definite matrix
positive-diagonal matrix	diagonal matrix with positive diagonal elements

We consider the following optimal reduced-order state-estimation problem. Given the system

$$\dot{x}=Ax+w_1, \tag{2.1}$$

$$y=Cx+w_2, \tag{2.2}$$

design a reduced-order state estimator

$$\dot{x}_e=A_ex_e+B_ey, \tag{2.3}$$

$$y_e=C_ex_e, \tag{2.4}$$

which minimizes the error criterion

$$J(A_e, B_e, C_e) \triangleq \lim_{t\to\infty} \mathbb{E}[(Lx-y_e)^TR(Lx-y_e)].$$

In this formulation the matrix L identifies the states, or linear combinations of states, whose estimates are desired. The order n_e of the estimator state x_e is determined by implementation constraints, i.e., by the computing capability available for realizing (2.3), (2.4) in real time. Hence, n_e is considered to be fixed in what follows and the problem is concerned with determining A_e, B_e, and C_e.

To guarantee that J is finite it is assumed that A is stable and we restrict our attention to the set of stable reduced-order estimators

$$\mathcal{A} \triangleq \{(A_e, B_e, C_e) : A_e \text{ is stable}\}.$$

Since the value of J is independent of the internal realization of the transfer function corresponding to (2.3) and (2.4), without loss of generality we further restrict our attention to the set of admissible estimators

$$\mathcal{A}_+ \triangleq \{(A_e, B_e, C_e)\in\mathcal{A}:$$
$$(A_e, B_e) \text{ is controllable and } (A_e, C_e) \text{ is observable}\}.$$

The following lemma, whose proof is given in [7], is needed for the statement of the main result.

Lemma 2.1: Suppose $\hat{Q}$, $\hat{P} \in \mathbb{R}^{n\times n}$ are nonnegative definite. Then $\hat{Q}\hat{P}$ is nonnegative semisimple. Furthermore, if $\rho(\hat{Q}\hat{P}) = n_e$, then there exist $G, \Gamma \in \mathbb{R}^{n_e\times n}$ and positive-semisimple $M \in \mathbb{R}^{n_e\times n_e}$ such that

$$\hat{Q}\hat{P}=G^TM\Gamma, \tag{2.5}$$

$$\Gamma G^T=I_{n_e}. \tag{2.6}$$

For convenience in stating the Main Theorem we shall refer to $G, \Gamma \in \mathbb{R}^{n_e\times n}$ and positive-semisimple $M \in \mathbb{R}^{n_e\times n_e}$ satisfying (2.5) and (2.6) as a (G, M, Γ)-factorization of $\hat{Q}\hat{P}$. Furthermore, define the notation

$$\tau \triangleq G^T\Gamma, \quad \tau_\perp \triangleq I_n-\tau$$

and

$$\mathcal{Q} \triangleq QC^T+V_{12},$$

where $Q \in \mathbb{R}^{n\times n}$.

Main Theorem: Suppose $(A_e, B_e, C_e) \in \mathcal{A}_+$ solves the optimal reduced-order state-estimation problem. Then there exist nonnegative-definite matrices Q, $\hat{Q}$, $\hat{P} \in \mathbb{R}^{n\times n}$ such that, for some (G, M, Γ)-factorization of $\hat{Q}\hat{P}$, A_e, B_e, and C_e are given by

$$A_e=\Gamma(A-\mathcal{Q}V_2^{-1}C)G^T, \tag{2.7}$$

$$B_e=\Gamma\mathcal{Q}V_2^{-1}, \tag{2.8}$$

$$C_e=LG^T \tag{2.9}$$

and such that the following conditions are satisfied:

$$0=AQ+QA^T+V_1-\mathcal{Q}V_2^{-1}\mathcal{Q}^T+\tau_\perp\mathcal{Q}V_2^{-1}\mathcal{Q}^T\tau_\perp^T, \tag{2.10}$$

$$0=A\hat{Q}+\hat{Q}A^T+\mathcal{Q}V_2^{-1}\mathcal{Q}^T-\tau_\perp\mathcal{Q}V_2^{-1}\mathcal{Q}^T\tau_\perp^T, \tag{2.11}$$

$$0=(A-\mathcal{Q}V_2^{-1}C)^T\hat{P}+\hat{P}(A-\mathcal{Q}V_2^{-1}C)+L^TRL-\tau_\perp^TL^TRL\tau_\perp, \tag{2.12}$$

$$\rho(\hat{Q})=\rho(\hat{P})=\rho(\hat{Q}\hat{P})=n_e. \tag{2.13}$$

Remark 2.1: It is useful to note that (2.7) can be replaced by

$$A_e=\Gamma AG^T-B_eCG^T. \tag{2.7$'$}$$

Remark 2.2: Because of (2.6) the $n \times n$ matrix τ which couples the three equations (2.10)–(2.12) is idempotent, i.e., $\tau^2 = \tau$. In general, this "optimal projection" is an *oblique* projection (as opposed to an orthogonal projection) since it is not necessarily symmetric. Note that from Sylvester's inequality and (2.6) it follows that $\rho(\tau) = n_e$. It should be stressed that the form of the optimal reduced-order estimator (2.7)–(2.9) is a direct consequence of optimality and not the result of an *a priori* assumption on the structure of the reduced-order estimator.

Remark 2.3: To obtain the standard steady-state Kalman filter result for the full-order case, set $p = n_e = n$ and $L = I_n$. Then $\tau = G = \Gamma = I_n$ and thus (2.10) reduces to the standard observer Riccati equation [10, p. 367] and (2.7) and (2.8) yield the usual expressions. Furthermore, it follows from (2.7)′ [11, Lemma 2.1] and standard results that (2.11)–(2.13) are equivalent to the assumption that (A_e, B_e, C_e) is controllable and observable.

Remark 2.4: Since $\hat{Q}\hat{P}$ is nonnegative semisimple it has a group generalized inverse $(\hat{Q}\hat{P})^\#$ given by $G^TM^{-1}\Gamma$ (see, e.g., [12, p. 124]). Hence, by (2.6) the optimal projection τ is given by

$$\tau=\hat{Q}\hat{P}(\hat{Q}\hat{P})^\#. \tag{2.14}$$

Remark 2.5: Replacing x_e by Sx_e, where S is invertible, yields the "equivalent" estimator $(SA_eS^{-1}, SB_e, C_eS^{-1})$. Since $J(A_e, B_e, C_e) = J(SA_eS^{-1}, SB_e, C_eS^{-1})$, one would expect the Main Theorem to apply also to $(SA_eS^{-1}, SB_e, C_eS^{-1})$. This is indeed the case since transformation of the estimator state basis corresponds to the alternative factorization $\hat{Q}\hat{P} = (S^{-T}G)^T(SMS^{-1})(S\Gamma)$.

Remark 2.6: Note that, for the optimal values of A_e, B_e, and C_e, (2.3) assumes the observer form

$$\dot{x}_e=\Gamma AG^Tx_e+\Gamma\mathcal{Q}V_2^{-1}(y-CG^Tx_e). \tag{2.15}$$

By introducing the quasi-full-state estimate $\hat{x} \triangleq G^Tx_e \in \mathbb{R}^n$ so that $\tau\hat{x} = \hat{x}$ and $x_e = \Gamma\hat{x} \in \mathbb{R}^{n_e}$, (2.15) can be written as

$$\dot{\hat{x}}=\tau A\tau\hat{x}+\tau\mathcal{Q}V_2^{-1}(y-C\hat{x}). \tag{2.16}$$

Note that although the *implemented* estimator (2.15) has the state $x_e \in \mathbb{R}^{n_e}$, (2.15) can be viewed as a quasi-full-order estimator whose geometric structure is entirely dictated by the projection τ. Specifically, error inputs $\mathcal{Q}V_2^{-1}(y - C\hat{x})$ are annihilated unless they are contained in $[\mathfrak{N}(\tau)]^\perp = \mathfrak{R}(\tau^T)$. Hence, the observation subspace of the estimator is precisely $\mathfrak{R}(\tau^T)$.

Remark 2.7: Although the form of (2.16) would lead one to surmise that the optimal reduced-order estimator is a projection of the optimal full-order estimator, this is not generally the case for the following simple reason. In the full-order case Q (which appears in $\mathcal{Q}$) is determined by solving a single Riccati equation, whereas in the reduced-order case Q must be found in conjunction with $\hat{Q}$ and $\hat{P}$ to satisfy all three matrix equations (2.10)–(2.12). Hence, the value of Q in the reduced-order case may be different from the value of Q in the full-order case. Thus, (2.16) may not be obtainable by simply projecting the full-order result.

To further clarify the relationship between $\hat{Q}$, $\hat{P}$, and τ, we now show that there exists a similarity transformation which simultaneously diagonalizes $\hat{Q}\hat{P}$ and τ.

Proposition 2.1: There exists invertible $\Phi \in \mathbb{R}^{n\times n}$ such that

$$\hat{Q}=\Phi^{-1}\begin{bmatrix}\Lambda_{\hat{Q}} & 0\\ 0 & 0\end{bmatrix}\Phi^{-T}, \quad \hat{P}=\Phi^T\begin{bmatrix}\Lambda_{\hat{P}} & 0\\ 0 & 0\end{bmatrix}\Phi, \tag{2.17}$$

$$\hat{Q}\hat{P}=\Phi^{-1}\begin{bmatrix}\Lambda & 0\\ 0 & 0\end{bmatrix}\Phi, \quad \tau=\Phi^{-1}\begin{bmatrix}I_{n_e} & 0\\ 0 & 0\end{bmatrix}\Phi, \tag{2.18a,b}$$

where $\Lambda_{\hat{Q}}, \Lambda_{\hat{P}} \in \mathbb{R}^{n_e \times n_e}$ are positive diagonal, $\Lambda \triangleq \Lambda_{\hat{Q}}\Lambda_{\hat{P}}$, and the diagonal elements of Λ are the eigenvalues of M. Consequently,

$$\hat{Q}=\tau\hat{Q}, \quad \hat{P}=\hat{P}\tau. \tag{2.19}$$

III. Proof of the Main Theorem

The proof proceeds exactly as in [6]. Using the fact that $\mathcal{Q}_+$ is open, the Fritz John version of the Lagrange multiplier theorem can be used to rigorously derive the first-order necessary conditions

$$0=\tilde{A}\tilde{Q}+\tilde{Q}\tilde{A}^T+\tilde{V}, \tag{3.1}$$

$$0=\tilde{A}^T\tilde{P}+\tilde{P}\tilde{A}+\tilde{R}, \tag{3.2}$$

$$0=P_{12}^TQ_{12}+P_2Q_2, \tag{3.3}$$

$$B_e=-[(P_2^{-1}P_{12}^TQ_1+Q_{12}^T)C^T+P_2^{-1}P_{12}^TV_{12}]V_2^{-1}, \tag{3.4}$$

$$C_e=LQ_{12}Q_2^{-1}, \tag{3.5}$$

where

$$\tilde{A}=\begin{bmatrix} A & 0 \\ B_eC & A_e \end{bmatrix}, \quad \tilde{V}=\begin{bmatrix} V_1 & V_{12}B_e^T \\ B_eV_{12} & B_eV_2B_e^T \end{bmatrix},$$

$$\tilde{R}=\begin{bmatrix} L^TRL & -L^TRC_e \\ -C_e^TRL & C_e^TR_2C_e \end{bmatrix}$$

and $(n+n_e)\times(n+n_e)\tilde{Q}, \tilde{P}$ are partitioned into $n\times n$, $n\times n_e$, and $n_e \times n_e$ subblocks as

$$\tilde{Q}=\begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \quad \tilde{P}=\begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}.$$

Expanding (3.1) and (3.2) yields

$$0=AQ_1+Q_1A^T+V_1, \tag{3.6}$$

$$0=AQ_{12}+Q_{12}A_e^T+Q_1(B_eC)^T+V_{12}B_e^T, \tag{3.7}$$

$$0=A_eQ_2+Q_2A_e^T+B_eCQ_{12}+Q_{12}^T(B_eC)^T+B_eV_2B_e^T, \tag{3.8}$$

$$0=A^TP_1+P_1A+(B_eC)^TP_{12}^T+P_{12}B_eC+L^TRL, \tag{3.9}$$

$$0=P_{12}A_e+A^TP_{12}+(B_eC)^TP_2-L^TRC_e, \tag{3.10}$$

$$0=A_e^TP_2+P_2A_e+C_e^TR_2C_e. \tag{3.11}$$

Note that (3.9) is superfluous and can be omitted. Writing (3.8) as (see [13], [14])

$$0=(A_e+B_eCQ_{12}Q_2^+)Q_2+Q_2(A_e+B_eCQ_{12}Q_2^+)^T+B_eV_2B_e^T,$$

where Q_2^+ is the Moore–Penrose or Drazin generalized inverse of Q_2, it follows from [11, Lemmas 2.1 and 12.2] that Q_2 is positive definite. Similarly, (3.11) implies that P_2 is positive definite. This justifies (3.4) and (3.5).

Now define the $n\times n$ nonnegative-definite matrices (see [13], [14])

$$Q=Q_1-Q_{12}Q_2^{-1}Q_{12}^T, \quad \hat{Q}=Q_{12}Q_2^{-1}Q_{12}^T, \quad \hat{P}=P_{12}P_2^{-1}P_{12}^T,$$

and note that (3.3) implies (2.5) and (2.6) with

$$G=Q_2^{-1}Q_{12}^T, \quad M=Q_2P_2, \quad \Gamma=-P_2^{-1}P_{12}^T.$$

Since $Q_2P_2 = P_2^{-1/2}(P_2^{1/2}Q_2P_2^{1/2})P_2^{1/2}$, M is positive semisimple. Sylvester's inequality yields (2.13). Note (2.19) and the identities

$$Q_1=Q+\hat{Q}, \tag{3.12}$$

$$Q_{12}=\hat{Q}\Gamma^T, \quad P_{12}=-\hat{P}G^T, \tag{3.13}$$

$$Q_2=\Gamma\hat{Q}\Gamma^T, \quad P_2=G\hat{P}G^T. \tag{3.14}$$

Using (3.12)–(3.14), (3.4) and (3.5) yield (2.8) and (2.9). Also, the right-hand sides of (3.8) and (3.7) yield (2.7). Substituting (2.7)–(2.9) into (3.6)–(3.8), (3.10) and (3.11), it can be seen that (3.8) and (3.11) are also superfluous. Finally, linear combinations of the remaining three equations (3.6), (3.7), and (3.10) yield (2.10)–(2.12).

IV. Concluding Remarks

The question of multiple local minima satisfying the optimal projection equations for reduced-order state estimation and the problem of constructing numerical methods for solving these equations are beyond the scope of this note. It should be pointed out, however, that promising numerical results for the model-reduction and fixed-order dynamic-compensation problems have been obtained by means of iterative algorithms that take full advantage of the presence and structure of the optimal projection [2], [4], [7].

Finally, the results of this paper can be extended to include the following related problems: 1) discrete-time system/discrete-time estimator; 2) infinite-dimensional system/finite-dimensional estimator [5]; and 3) parameter uncertainties [1], [15], [16].

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References

[1] D. C. Hyland, "Optimality conditions for fixed-order dynamic compensation of flexible spacecraft with uncertain parameters," AIAA 20th Aerosp. Sci. Meet., Orlando, FL, Jan. 1982, paper 82-0312.

[2] ——, "The optimal projection approach to fixed-order compensation: Numerical methods and illustrative results," AIAA 21st Aerosp. Sci. Meet., Reno, NV, Jan. 1983, paper 83-0303.

[3] D. C. Hyland and D. S. Bernstein, "Explicit optimality conditions for fixed-order dynamic compensation," in *Proc. 22nd IEEE Conf. Decision Contr.*, San Antonio, TX, Dec. 1983, pp. 161–165.

[4] D. C. Hyland, "Comparison of various controller-reduction methods: Suboptimal versus optimal projection," in *Proc. AIAA Dynam. Specialists Conf.*, Palm Springs, CA, May 1984, pp. 381–389.

[5] D. S. Bernstein and D. C. Hyland, "The optimal projection equations for fixed-order dynamic compensation of distributed parameter systems," presented at the AIAA Dynam. Specialists Conf., Palm Springs, CA, May 1984.

[6] D. C. Hyland and D. S. Bernstein, "The optimal projection equations for fixed-order dynamic compensation," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 1034–1037, 1984.

[7] ——, "The optimal projection approach to model reduction and the relationship between the methods of Wilson and Moore," in *Proc. 23rd Conf. Decision Contr.*, Las Vegas, NV, Dec. 1984.

[8] C. S. Sims, "Reduced-order modelling and filtering," in *Control and Dynamic Systems*, Vol. 18, C. T. Leondes, Ed., 1982, pp. 55–103.

[9] D. A. Wilson and R. N. Mishra, "Design of low order estimators using reduced models," *Int. J. Contr.*, vol. 23, pp. 447–456, 1979.

[10] K. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. New York: Wiley-Interscience, 1972.

[11] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*. New York: Springer-Verlag, 1974.

[12] S. L. Cambell and C. D. Meyer, Jr., *Generalized Inverses of Linear Transformations*. London: Pitman, 1979.

[13] A. Albert, "Conditions for positive and nonnegative definiteness in terms of pseudo inverse," *SIAM J. Appl. Math.*, vol. 17, pp. 434–440, 1969.

[14] E. Kreindler and A. Jameson, "Conditions for nonnegativeness of partitioned matrices," *IEEE Trans. Automat. Contr.*, vol. AC-17, pp. 147–148, 1972.

[15] P. J. McLane, "Optimal linear filtering for linear systems with state-dependent noise," *Int. J. Contr.*, vol. 10, pp. 42–51, 1969.

[16] D. S. Bernstein and D. C. Hyland, "Optimal projection/maximum entropy stochastic modeling and reduced-order design synthesis," IFAC Workshop on Model Error and Concepts and Compensation, Boston, MA, June 1985.

Legendre Series Approach to Identification and Analysis of Linear Systems

P. N. PARASKEVOPOULOS

Abstract—**In this paper the Legendre operational matrix of integration is introduced, and it is subsequently used for the identification and**

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The author was with the Department of Electrical Engineering, Democritus University of Thrace, Xanthi, Greece. He is now with the Division of Computer Science, Department of Electrical Engineering, National Technical University of Athens, Athens, Greece.