

## Extreme Points of Riccati Inequalities

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**Abstract**—Relations between solutions of the algebraic Riccati equation and the associated quadratic matrix inequalities are discussed and explained.

The purpose of this note is to clarify some of the relations that exist between solutions of the algebraic Riccati equation and the associated quadratic matrix inequalities. In particular, the main result of the note is to establish that there are *extreme points* in the solutions set of quadratic matrix inequalities that are *not* solutions of the algebraic Riccati equation. The history of this result is typical of many results involving Riccati equations in the engineering literature—total confusion. It has been part of the folklore for many years that the solutions of the algebraic Riccati equation are extreme points of the above-mentioned quadratic matrix inequalities. In Badawi's thesis [1] a very elegant proof is given, however, in review it was discovered that there had in fact appeared a proof in the literature. In Faurre *et al.* [3] there is indeed a proof and a footnote to the effect that there are extreme points other than the solutions of the algebraic Riccati equation. However, it has evolved as part of the folklore that the two sets coincide, even though it seems to be known that there are extreme points that are not solutions of the algebraic Riccati equation. However, we have been unable to find a proof. In this note we present a class of examples that establishes that there are other extreme points.

The example is based on the simple analysis presented in [2]. Following the notation of [1] we let  $F$ ,  $G$  and  $H$  and  $R$  be matrices such that  $F$  is  $2 \times 2$ ,  $G$  is  $2 \times 1$ , and  $H$  is  $1 \times 2$ . We define then the function  $W(P) = FP + PF' + (G - PH)R^{-1}(G - PH)'$ .  $W(P)$  is the Riccati operator. The matrix Riccati inequality referred to above is of course the inequality  $W(P) \leq 0$  in the sense of positive definite matrices. The Riccati equation is the equation  $W(P) = 0$ . Now we choose  $F$ ,  $G$ , and  $H$  such that the Hamiltonian associated with the Riccati equation has complex eigenvalues. The Hamiltonian is constructed by transforming the above equation to the more standard form (for the purposes of geometric analysis)

$$W(P) = (F - GR^{-1}H)P + P(F - GR^{-1}H)' + GR^{-1}G' + PH'R^{-1}HP$$

and writing the Hamiltonian  $\mathcal{H}$

$$\mathcal{H} = \begin{pmatrix} A & D \\ -Q & -A' \end{pmatrix}$$

where we let  $A = F - GR^{-1}H$ ,  $D = -H'R^{-1}G'$ , and  $Q = GR^{-1}G'$ . This matrix has four complex eigenvalues which can be denoted by  $r, r, -r, -r$ . (A standard result about infinitesimal symplectic matrices.) It is trivial to establish that such matrices exist. For example, let  $A$  have complex eigenvalues and let  $Q = sI$ . Since  $\mathcal{H}$  has complex eigenvalues with  $s = 0$ , it follows (from continuity) that when  $s$  is sufficiently small, the eigenvalues of the Hamiltonian are also. Choose the matrices  $A$ ,  $B$ ,  $Q$ , and  $D$  such that  $(A, H)$  is controllable and  $Q$  and  $D$  are positive semidefinite. Then using the results of [2] there exist exactly two real solutions of the associated algebraic Riccati equation. Thus, if the set of extreme points of the quadratic matrix inequalities consists only of these two solutions then the solution set is a linear segment. This is not the case and hence there must exist other extreme points.

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## The Optimal Projection Equations for Fixed-Order Dynamic Compensation

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**Abstract**—First-order necessary conditions for quadratically optimal, steady-state, fixed-order dynamic compensation of a linear, time-invariant plant in the presence of disturbance and observation noise are derived in a new and highly simplified form. In contrast to the pair of matrix Riccati equations for the full-order LQG case, the optimal steady-state fixed-order dynamic compensator is characterized by four matrix equations (two modified Riccati equations and two modified Lyapunov equations) coupled by a projection whose rank is precisely equal to the order of the compensator and which determines the optimal compensator gains. The coupling represents a graphic portrayal of the demise of the classical separation principle for the reduced-order controller case.

## I. INTRODUCTION

Because of constraints imposed by on-line computations, dynamic controllers for high-order systems such as flexible spacecraft must be of relatively modest order. Hence, this paper is concerned with the design of quadratically optimal, fixed-order (i.e., reduced-order) dynamic compensation for a plant subject to stochastic disturbances and nonsingular measurement noise. Since white noise in all measurement channels precludes direct output feedback (see Section II), only purely dynamic controllers are considered. The requirements for resolution of this optimization problem include the following.

1) Conditions for the existence of an optimal, stabilizing compensator of the prescribed order. (In the full-order case these are the usual stabilizability and detectability conditions of LQG theory.)

2) Stationary conditions, i.e., first-order necessary conditions, rendered in a tractable form to facilitate developments in items 3) and 4) below. (In the full-order case these conditions are precisely the LQG gain relations together with the regulator and observer Riccati equations.)

3) Sufficiency conditions, i.e., additional restrictions on solutions of the first-order necessary conditions which characterize local minima and single out the *global* minimum. (In the full-order case the global minimum is distinguished by the unique nonnegative-definite solutions to the LQG Riccati equations.)

4) Convergent numerical algorithms for simultaneous satisfaction of the necessary and sufficient conditions. (In the full-order case numerical algorithms have been devised which take full advantage of the highly structured form of the Riccati equations.)

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The present paper deals exclusively with item 2). Although the stationary conditions for the fixed-order compensation problem *have* been written down (see [1]–[12], for example), full exploitation has undoubtedly been impeded by their extreme complexity [see (3.3)–(3.11)]. What has been lacking, to quote the insightful remarks of [9], “is a deeper understanding of the structural coherence of these equations.” The contribution of the present paper is to show how the originally *very* complex stationary conditions can be transformed without loss of generality to much simpler and more tractable forms. The resulting equations (2.10)–(2.17) preserve the simple form of LQG relations for the gains in terms of covariance and cost matrices which, in turn, are determined by a coupled system of two modified Riccati equations and two modified Lyapunov equations. This coupling, by means of a projection (idempotent matrix) whose rank is precisely equal to the order of the compensator, represents a graphic portrayal of the demise of the classical separation principle for the reduced-order controller case. When, as a special case, the order of the compensator is required to be equal to the order of the plant, the modified Riccati equations reduce to the standard LQG Riccati equations and the modified Lyapunov equations express the proviso that the compensator be minimal, i.e., controllable and observable. Since the LQG Riccati equations as such are nothing more than the necessary conditions for full-order compensation, we believe that the “optimal projection equations” provide a clear and simple generalization of standard LQG theory.

Since we are concerned with *optimal* fixed-order compensator design, our approach does not represent yet another model- or controller-reduction scheme along the lines of [13]–[17]. Indeed, the optimal projection equations, by virtue of their relatively transparent structure, can reveal the extent to which the design equations of a given ad hoc reduction scheme conform to the necessary conditions for optimality. For example, the oblique projection which arises in the present formulation may not be of the form  $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$  even in the basis corresponding to the “balanced” realization [13]–[16]. These issues are discussed in [18] where the results of [19] are simplified by means of the approach of the present paper and where the balancing method of [13] is reinterpreted in the context of optimality theory.

The fact that the optimal projection equations consist of four coupled matrix equations, i.e., two modified Riccati equations and two modified Lyapunov equations, should not be at all surprising for the following simple reason. Reduced-order control-design methods often involve either LQG applied to a reduced-order model or model reduction applied to a full-order LQG design. Both approaches, then, involve the solution of precisely four equations: two Riccati equations (for LQG) plus two Lyapunov equations (for model reduction via balancing, as in [13]). The *coupled* form of the optimal projection equations is thus a strong reminder that the LQG and order-reduction operations *cannot* be iterated but must, in a certain sense, be performed simultaneously.

## II. PROBLEM STATEMENT AND THE MAIN THEOREM

Given the control system

$$\dot{x}(t) = Ax(t) + Bu(t) + w_1(t), \quad (2.1)$$

$$y(t) = Cx(t) + w_2(t) \quad (2.2)$$

design a fixed-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \quad (2.3)$$

$$u(t) = C_c x_c(t) \quad (2.4)$$

which minimizes the steady-state performance criterion

$$J(A_c, B_c, C_c) \triangleq \lim_{t \rightarrow \infty} \mathbb{E}[x(t)^T R_1 x(t) + u(t)^T R_2 u(t)] \quad (2.5)$$

where:  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^l$ ,  $x_c \in \mathbb{R}^{n_c}$ ,  $n_c \leq n$ ,  $A, B, C, A_c, B_c, C_c, R_1$ , and  $R_2$  are matrices of appropriate dimension with  $R_1$  (symmetric) nonnegative definite and  $R_2$  (symmetric) positive definite;  $w_1$  is white disturbance noise with  $n \times n$  nonnegative-definite intensity  $V_1$  and  $w_2$  is

white observation noise with  $l \times l$  positive-definite intensity  $V_2$ ;  $w_1$  and  $w_2$  are uncorrelated and have zero mean. We note that the assumptions of nonsingular control weighting and nonsingular observation noise preclude the use of direct output feedback as in

$$u(t) = C_c x_c(t) + D_c y(t) \quad (2.6)$$

since  $J$  is undefined unless (see [7])

$$\text{tr}[D_c^T R_2 D_c V_2] = 0 \quad (\Leftrightarrow) \quad R_2 D_c V_2 = 0. \quad (2.7)$$

To guarantee that  $J$  is finite and independent of initial conditions we restrict our attention to the set of admissible stabilizing compensators

$$\mathcal{G} \triangleq \left\{ (A_c, B_c, C_c): \tilde{A} \triangleq \begin{bmatrix} A & B C_c \\ B_c C & A_c \end{bmatrix} \text{ is asymptotically stable} \right\}$$

where  $\tilde{A}$  is the closed-loop dynamics matrix. Since the value of  $J$  is independent of the internal realization of the compensator, we can further restrict our attention to

$$\mathcal{G}_+ \triangleq \{ (A_c, B_c, C_c) \in \mathcal{G}:$$

$$(A_c, B_c) \text{ is controllable and } (C_c, A_c) \text{ is observable} \}.$$

For the following lemma call a square matrix nonnegative (respectively, positive) semisimple if it has a diagonal Jordan form and nonnegative (respectively, positive) eigenvalues. Let  $I_r$  denote the  $r \times r$  identity matrix.

**Lemma 2.1:** Suppose  $\hat{Q}, \hat{P} \in \mathbb{R}^{n \times n}$  are nonnegative definite. Then  $\hat{Q}\hat{P}$  is nonnegative semisimple. Furthermore, if  $\text{rank } \hat{Q}\hat{P} = n_c$  then there exist  $G, \Gamma \in \mathbb{R}^{n_c \times n}$  and positive-semisimple  $M \in \mathbb{R}^{n_c \times n_c}$  such that

$$\hat{Q}\hat{P} = G^T M \Gamma, \quad (2.8)$$

$$\Gamma G^T = I_{n_c}. \quad (2.9)$$

*Proof.* The result is an immediate consequence of [20, Theorem 6.2.5, p. 123].  $\square$

For convenience in stating the Main Theorem, define

$$\Sigma \triangleq B R_2^{-1} B^T, \quad \tilde{\Sigma} \triangleq C^T V_2^{-1} C$$

and call  $G, M$ , and  $\Gamma$  satisfying (2.8) and (2.9) a  $(G, M, \Gamma)$ -factorization of  $\hat{Q}\hat{P}$ .

**Main Theorem:** Suppose  $(A_c, B_c, C_c) \in \mathcal{G}_+$  solves the steady-state fixed-order dynamic-compensation problem. Then there exist  $n \times n$  nonnegative-definite matrices  $Q, P, \hat{Q}$ , and  $\hat{P}$  such that  $A_c, B_c$ , and  $C_c$  are given by

$$A_c = \Gamma(A - Q\tilde{\Sigma} - \Sigma P)G^T, \quad (2.10)$$

$$B_c = \Gamma Q C^T V_2^{-1}, \quad (2.11)$$

$$C_c = -R_2^{-1} B^T P G^T \quad (2.12)$$

for some  $(G, M, \Gamma)$ -factorization of  $\hat{Q}\hat{P}$ , and such that with  $\tau \triangleq G^T \Gamma$  the following conditions are satisfied:

$$0 = (A - \tau Q\tilde{\Sigma})Q + Q(A - \tau Q\tilde{\Sigma})^T + V_1 + \tau Q\tilde{\Sigma}Q\tau^T, \quad (2.13)$$

$$0 = (A - \Sigma P\tau)^T P + P(A - \Sigma P\tau) + R_1 + \tau^T P \Sigma P \tau, \quad (2.14)$$

$$0 = \tau[(A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^T + Q\tilde{\Sigma}Q], \quad (2.15)$$

$$0 = [(A - Q\tilde{\Sigma})^T \hat{P} + \hat{P}(A - Q\tilde{\Sigma}) + P\Sigma P]\tau, \quad (2.16)$$

$$\text{rank } \hat{Q} = \text{rank } \hat{P} = \text{rank } \hat{Q}\hat{P} = n_c. \quad (2.17)$$

**Remark 2.1:** Because of (2.9) the  $n \times n$  matrix  $\tau$  which couples the four equations (2.13)–(2.16) is idempotent, i.e.,  $\tau^2 = \tau$ . In general this “optimal projection” is an *oblique* projection (as opposed to an orthogonal projection) since it is not necessarily symmetric. Note that Sylvester’s inequality and (2.9) imply that  $\text{rank } \tau = n_c$ .

**Remark 2.2:** Using the relations  $\hat{Q} = \tau\hat{Q}$  and  $\hat{P} = \hat{P}\tau$  [see (3.12)],

the optimal projection equations (2.13)–(2.16) can be written in the equivalent form

$$0 = A\bar{Q} + \bar{Q}A^T + V_1 - \bar{Q}\bar{\Sigma}\bar{Q} + \tau_\perp \bar{Q}\bar{\Sigma}^T \bar{Q}\tau_\perp^T, \quad (2.18)$$

$$0 = A^T \bar{P} + \bar{P}A + R_1 - \bar{P}\bar{\Sigma}P + \tau_\perp^T \bar{P}\bar{\Sigma}P\tau_\perp, \quad (2.19)$$

$$0 = (A - \bar{\Sigma}P)\bar{Q} + \bar{Q}(A - \bar{\Sigma}P)^T + \bar{Q}\bar{\Sigma}\bar{Q} - \tau_\perp \bar{Q}\bar{\Sigma}^T \bar{Q}\tau_\perp^T, \quad (2.20)$$

$$0 = (A - \bar{Q}\bar{\Sigma})^T \bar{P} + \bar{P}(A - \bar{Q}\bar{\Sigma}) + \bar{P}\bar{\Sigma}P - \tau_\perp^T \bar{P}\bar{\Sigma}P\tau_\perp, \quad (2.21)$$

where  $\tau_\perp \triangleq I_n - \tau$ . Note that in the full-order case  $n_c = n$ ,  $\tau = G = \Gamma = I_n$  and thus (2.18) and (2.19) reduce to the standard observer and regulator Riccati equations and (2.10)–(2.12) yield the usual LQG expressions. Furthermore, it can be shown that (2.20), (2.21), and (2.17) are equivalent to the assumption that  $(A_c, B_c, C_c)$  is controllable and observable.

*Remark 2.3:* Since  $\hat{Q}\hat{P}$  is nonnegative semisimple it has a group generalized inverse  $(\hat{Q}\hat{P})^\#$  given by  $G^T M^{-1} \Gamma$  (see e.g., [21, p. 124]). Hence, by (2.9) the optimal projection  $\tau$  is given by

$$\tau = \hat{Q}\hat{P}(\hat{Q}\hat{P})^\#. \quad (2.22)$$

*Remark 2.4:* The modified Riccati equations (2.13) and (2.14) are similar to the (single) “extended algebraic Riccati equation” which arises in the static output feedback problem (see, e.g., [22]).

*Remark 2.5:* Replacing  $x_c$  by  $Sx_c$ , where  $S$  is invertible, yields the “equivalent” compensator  $(SA_c S^{-1}, SB_c, C_c S^{-1})$ . Since  $J(A_c, B_c, C_c) = J(SA_c S^{-1}, SB_c, C_c S^{-1})$  one would expect the Main Theorem to apply also to  $(SA_c S^{-1}, SB_c, C_c S^{-1})$ . This is indeed the case since transformation of the compensator state basis corresponds to the alternative factorization  $\hat{Q}\hat{P} = (S^{-T}G)^T (SMS^{-1}) (S\Gamma)$ . See [10] for related remarks.

*Remark 2.6:* By introducing the quasi-full-state estimate  $\hat{x} \triangleq G^T x_c \in \mathbb{R}^n$  so that  $\tau\hat{x} = \hat{x}$  and  $x_c = \Gamma\hat{x} \in \mathbb{R}^{n_c}$ , (2.1)–(2.4) can be written as

$$\dot{\hat{x}} = A\hat{x} + B\hat{C}_c\tau\hat{x} + w_1,$$

$$\hat{x} = \tau(A - \hat{B}_c C + B\hat{C}_c)\tau\hat{x} + \tau\hat{B}_c(Cx + w_2)$$

where  $\hat{B}_c \triangleq QC^T V_2^{-1}$  and  $\hat{C}_c \triangleq -R_2^{-1} B^T P$ . Although the implemented compensator has the state  $x_c \in \mathbb{R}^{n_c}$ , it can be viewed as a quasi-full-order compensator whose geometric structure is entirely dictated by the projection  $\tau$ . Sensor inputs  $\hat{B}_c y$  are annihilated unless they are contained in  $[\mathfrak{U}(\tau)]^\perp = \mathfrak{R}(\tau^T)$ , where  $\mathfrak{U}$  and  $\mathfrak{R}$  denote null space and range. Furthermore, the quasi-full-order state estimate  $\tau\hat{x}$  employed in the control input is contained in  $\mathfrak{R}(\tau)$ . Thus,  $\mathfrak{R}(\tau)$  and  $\mathfrak{R}(\tau^T)$  are the control and observation subspaces of the compensator.

### III. PROOF OF THE MAIN THEOREM

The proof given here considerably simplifies the original derivation given in [23] and [24]. Using the fact that  $\mathfrak{G}_+$  is open, the Fritz John version of the Lagrange multiplier theorem can be used to rigorously derive the first-order necessary conditions ([7], see also [25])

$$0 = \bar{A}\bar{Q} + \bar{Q}\bar{A}^T + \bar{V}, \quad (3.1)$$

$$0 = \bar{A}^T \bar{P} + \bar{P}\bar{A} + \bar{R}, \quad (3.2)$$

$$0 = P_{12}^T Q_{12} + P_2 Q_{22}, \quad (3.3)$$

$$B_c = -(P_2^{-1} P_{12}^T Q_{12} + Q_{12}^T) C^T V_2^{-1}, \quad (3.4)$$

$$C_c = -R_2^{-1} B^T (P_1 Q_{12} Q_2^{-1} + P_{12}), \quad (3.5)$$

where

$$\bar{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix}, \quad \bar{R} \triangleq \begin{bmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{bmatrix}$$

and  $(n + n_c) \times (n + n_c)$   $\bar{Q}, \bar{P}$  are partitioned into  $n \times n$ ,  $n \times n_c$ , and  $n_c \times n_c$  subblocks as

$$\bar{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \quad \bar{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}.$$

Expanding (3.1) and (3.2) yields

$$0 = A Q_1 + Q_1 A^T + B C_c Q_{12}^T + Q_{12} (B C_c)^T + V_1, \quad (3.6)$$

$$0 = A Q_{12} + Q_{12} A^T + B C_c Q_2 + Q_1 (B C_c)^T, \quad (3.7)$$

$$0 = A_c Q_2 + Q_2 A_c^T + B_c C Q_{12} + Q_{12}^T (B_c C)^T + B_c V_2 B_c^T, \quad (3.8)$$

$$0 = A^T P_1 + P_1 A + (B C)^T P_{12}^T + P_{12} B_c C + R_1, \quad (3.9)$$

$$0 = P_{12} A_c + A^T P_{12} + (B C)^T P_{12} + P_{12} B_c C, \quad (3.10)$$

$$0 = A_c^T P_2 + P_2 A_c + (B C_c)^T P_{12} + P_{12}^T B C_c + C_c^T R_2 C_c. \quad (3.11)$$

Writing (3.8) as (see [26], [27])

$$0 = (A_c + B_c C Q_{12} Q_2^{-1}) Q_2 + Q_2 (A_c + B_c C Q_{12} Q_2^{-1})^T + B_c V_2 B_c^T$$

where  $Q_2^{-1}$  is the Moore–Penrose or Drazin generalized inverse of  $Q_2$ , it follows from [28, Lemmas 2.1 and 12.2] that  $Q_2$  is positive definite. Similarly, (3.11) implies that  $P_2$  is positive definite. This justifies (3.4) and (3.5).

Now define the  $n \times n$  nonnegative-definite matrices (see [26], [27])

$$Q \triangleq Q_1 - Q_{12} Q_2^{-1} Q_{12}^T, \quad P \triangleq P_1 - P_{12} P_2^{-1} P_{12}^T,$$

$$\hat{Q} \triangleq Q_{12} Q_2^{-1} Q_{12}^T, \quad \hat{P} \triangleq P_{12} P_2^{-1} P_{12}^T$$

and note that (3.3) implies (2.8) and (2.9) with

$$G \triangleq Q_2^{-1} Q_{12}^T, \quad M \triangleq Q_2 P_2, \quad \Gamma \triangleq -P_2^{-1} P_{12}^T.$$

Since  $Q_2 P_2 = P_2^{-1/2} (P_2^{1/2} Q_2 P_2^{1/2}) P_2^{1/2}$ ,  $M$  is positive semisimple. Sylvester’s inequality yields (2.17). Note also that

$$\hat{Q} = \tau \hat{Q}, \quad \hat{P} = \hat{P} \tau. \quad (3.12)$$

Next (2.11) and (2.12) follow from (3.4) and (3.5) by using the identities

$$Q_1 = Q + \hat{Q}, \quad P_1 = P + \hat{P}, \quad (3.13)$$

$$Q_{12} = \hat{Q} \Gamma^T, \quad P_{12} = -\hat{P} G^T, \quad (3.14)$$

$$Q_2 = \Gamma \hat{Q} \Gamma^T, \quad P_2 = G \hat{P} G^T. \quad (3.15)$$

Now substitute (2.11), (2.12), and (3.13)–(3.15) into (3.6)–(3.11) and use the relations

$$B_c C = \Gamma Q \bar{\Sigma}, \quad B C_c = -\Sigma P G^T,$$

$$B_c V_2 B_c^T = \Gamma Q \bar{\Sigma} Q \Gamma^T, \quad C_c^T R_2 C_c = G P \bar{\Sigma} P G^T.$$

Then (2.10) follows from (3.8)–(3.7). Substituting (2.10) into (3.7), (3.8), (3.10), and (3.11) shows that  $(\Gamma(3.7)G)^T$  and  $-(3.10)\Gamma$  are precisely (2.15) and (2.16). Since  $G^T(3.8)G = (2.15)\tau$  and  $\Gamma^T(3.11)\Gamma = \tau(2.16)$ , (3.8) and (3.11) can be omitted. Finally, using (3.12) it follows that (2.13) = (3.6) + (2.15) $\tau$  – (2.15)–(2.15) $\tau^T$  and similarly for (2.14).  $\square$

### IV. DIRECTIONS FOR FURTHER RESEARCH

With regard to the existence of a stabilizing compensator, known results (e.g., [28]–[34]) can be exploited to a great extent. A numerical algorithm for solving the optimal projection equations has been developed in [24] and [35]. The proposed computational scheme is philosophically quite different from gradient search algorithms [2], [3], [6], [7], [9], [11], [36], [37] in that it operates through direct solution of the optimal projection equations by iterative refinement of the optimal projection. Methods for eliminating local extrema are being investigated by applying component cost analysis [17]. Generalizations of the optimal projection equations can arise by considering the following extensions of the fixed-order dynamic-compensation problem.

1) *Discrete-Time System/Discrete-Time Compensator:* Digital implementation can be modeled by a discrete-time compensator with control of a continuous-time system facilitated by sampling and reconstruction devices.

2) *Cross Weighting/Correlated Disturbance and Observation Noise*: This extension is straightforward and entirely analogous to the LQG case (see, e.g., [3, p. 351]).

3) *Singular Observation Noise/Singular Control Weighting*: With due attention to (2.7), direct output feedback can be used in the singular case. The nature of the problem forebodes all of the difficulties associated with the singular LQG problem. Note that the output feedback problem [22], [38], when viewed in this context, is *highly singular*.

4) *Infinite-Dimensional Systems*: The optimal projection equations have been extended in [39] and [40] to the case in which (2.1) is a distributed parameter system, for example, a partial or functional differential equation.

5) *Decentralized Fixed-Order Controller*: The optimal projection equations can be derived for the case in which the dynamic controller has a fixed decentralized structure.

6) *Parameter Uncertainties*: The original derivation in [23] treated a Stratonovich state-dependent noise model representing parameter uncertainties in the plant. Further consideration of control- and measurement-dependent noise raises the possibility of directly including the impact of parameter uncertainties in the design of robust, implementable compensation for large-order systems.

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## Optimal Control of Systems Possessing Symmetries

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**Abstract**—It is shown that a symmetry in an optimization problem induces a decomposition of the optimal feedback control law into factors. One factor can be calculated algebraically and depends only on the symmetry; the other factor corresponds to a lower dimensional optimization problem. This gives *a priori* information about the structure of the optimal feedback control law and indicates a possibly more efficient method for optimizing such systems.

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