Optimal Rejection of Stochastic and Deterministic Disturbances[1]

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The problem of optimal $\mathcal{H}_2$ rejection of noisy disturbances while asymptotically rejecting constant or sinusoidal disturbances is considered. The internal model principle is used to ensure that the expected value of the output approaches zero asymptotically in the presence of persistent deterministic disturbances. Necessary conditions are given for dynamic output feedback controllers that minimize an $\mathcal{H}_2$ disturbance rejection cost plus an upper bound on the integral square output cost for transient performance. The necessary conditions provide expressions for the gradients of the cost with respect to each of the control gains. These expressions are then used in a quasi-Newton gradient search algorithm to find the optimal feedback gains.

1 Introduction

Asymptotic rejection of deterministic disturbances with known frequency content is a central problem in feedback control theory. A state space approach to this problem was developed by Johnson (1971), where the disturbances are characterized by an exogenous system with unknown initial conditions. The controllers given by Johnson (1971) provide asymptotic disturbance rejection under the restrictive assumption that the range of the disturbance input matrix is a subspace of the range of the control input matrix.

An alternative approach to this problem is based on asymptotic tracking of reference commands. Davison and Goldenberg (1975) showed that for a system to achieve asymptotic disturbance rejection, the controller must contain an internal model of the exogenous dynamics that produce the disturbance. Furthermore, asymptotic rejection of the disturbances requires that the exogenous dynamics be replicated in each feedback loop. A compensator is then used to stabilize the augmented system consisting of the plant and the internal model. For this approach, no condition on the range space of the disturbance input matrix is required.

Because a controller that achieves disturbance rejection consists of both an internal model and a stabilizing controller, there is considerable freedom in the design of such controllers. This design freedom can be used to meet additional objectives such as pole placement, time and frequency response criteria (Davison and Ferguson, 1981), or the optimization of a performance criterion such as disturbance rejection via minimization of the $\mathcal{H}_2$ norm (Iftar and Ozguner, 1986).

Unfortunately, the problem of minimizing the $\mathcal{H}_2$ norm of a closed-loop system while achieving asymptotic disturbance rejection is not straightforward. Since the internal models for disturbances such as steps, ramps, and sinusoids have imaginary axis eigenvalues, these modes are not observable by the performance variables used in the $\mathcal{H}_2$ cost functional. Hence, there does not exist a stabilizing solution to the Riccati equation for the augmented system so that standard $\mathcal{H}_2$ techniques cannot be applied.

Although the use of an internal model addresses the steady-state disturbance rejection problem, transient behavior is also of interest. This behavior can be quantified by means of the integral square of the mean output, which provides a measure of the effectiveness of the controller in rejecting disturbances. To this end, the approach of Abedor, et al. (1992) yields a family of controllers that achieve asymptotic disturbance rejection and stabilize the augmented plant with transient performance determined by the scalar parameter α. However, the parameterization of Abedor et al. (1992) does not necessarily yield an optimal tradeoff between these competing objectives.

The goal of the present paper is to determine controllers that not only provide asymptotic disturbance rejection but also achieve better transient performance for the same $\mathcal{H}_2$ cost. To do this, necessary conditions are given for the problem of minimizing a cost function consisting of an $\mathcal{H}_2$ cost plus an upper bound on the integral square output cost. These necessary conditions provide analytical expressions for the gradients of the cost with respect to each of the control gains and can then be used by a gradient optimization algorithm to find control gains that minimize the cost function. Tradeoffs between transient performance and $\mathcal{H}_2$ disturbance rejection can be obtained by varying the weights in the cost function. The results in the present paper complement those of Sparks and Bernstein (1995), where necessary conditions are given for the related problem of asymptotic tracking.

Iftar (1990) addressed the problem of rejecting stochastic and deterministic disturbances, giving conditions for the existence of the optimal $\mathcal{H}_2$ controller that asymptotically rejects deterministic disturbances, as well as showing robustness of the controller. The present paper expands the work of Iftar (1990) by considering transient performance as well as $\mathcal{H}_2$ disturbance rejection by providing analytical expressions for gradients of the cost with respect to the controller gains.

2 Problem Formulation

Consider the plant model

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t) + D_{d1}w_d(t), \qquad (1)$$

$$y(t) = Cx(t) + D_2w(t) + D_{d2}w_d(t), \qquad (2)$$

$$z(t) = E_1x(t) + E_2u(t), \qquad (3)$$

where $x(t) \in \mathbb{R}^n$ is the plant state, $u(t) \in \mathbb{R}^m$ is the control, $y(t) \in \mathbb{R}^l$ is the measurement, $w(t) \in \mathbb{R}^q$ is a stochastic disturbance, $w_d(t) \in \mathbb{R}^d$ is a deterministic disturbance, $z(t) \in \mathbb{R}^p$ is the performance output, (A, B) is controllable, and (C, A) is observable.

The control objective is to have $\mathbb{E}[y(t)]$ approach zero asymptotically so that in the absence of a stochastic signal, $y(t)$ approaches zero asymptotically. In addition, we wish to minimize the $\mathcal{H}_2$ norm of the closed-loop transfer function between $w(t)$ and $z(t)$ as well as the integral square output $\int_0^\infty \mathbb{E}[y(t)]^T M \mathbb{E}[y(t)]dt$, where M is a nonnegative definite matrix.

In this paper, two types of disturbances $w_d(t)$ will be considered, namely, constant disturbances and sinusoidal disturbances. For the case of constant disturbances, we assume that each element w_{d_i} of the vector $w_d(t)$ is uncertain, that is,

$$w_d(t) = \begin{bmatrix} w_{d_1} \\ w_{d_2} \\ \vdots \\ w_{d_l} \end{bmatrix}, \qquad (4)$$

where the elements w_{d_i} are uncertain. For the case of sinusoidal disturbances, we assume that each element $w_{d_i}(t)$ of the vector

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$w_d(t)$ consists of a sinusoid whose frequency ω is known, but whose amplitude and phase are uncertain, that is,

$$w_d(t) = \begin{bmatrix} w_{d_1} \sin(\omega t + \phi_1) \\ w_{d_2} \sin(\omega t + \phi_2) \\ \vdots \\ w_{d_l} \sin(\omega t + \phi_l) \end{bmatrix},$$

where the amplitudes w_{d_i} and the phases ϕ_i are uncertain.

We represent the disturbance $w_d(t)$ by means of an exogenous system of the form

$$\dot{x}_d(t) = A_d x_d(t), \quad x_d(0) = x_{d0} \tag{5}$$

$$w_d(t) = C_d x_d(t), \tag{6}$$

where $x_d(t) \in \mathbb{R}^{n_d}$. For the case of constant disturbances, let $n_d = 1$, $A_d = 0$, and $x_{d0} = 1$, so that $w_d(t) = C_d$, and thus the elements of C_d determine the magnitudes of the disturbance components. Similarly, for the case of sinusoidal disturbances, let $n_d = 2$,

$$A_d = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \quad x_{d0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and let $C_d \in \mathbb{R}^{l \times 2}$ be an uncertain matrix. Then, $w_{d_i}(t) = C_{d1i} \sin \omega t + C_{d2i} \cos \omega t$, where C_{d1i} and C_{d2i} are the i^{th} elements of the first and second columns of C_d. Equivalently, $w_{d_i}(t)$ can be rewritten as $w_{d_i}(t) = w_{d_i} \sin(\omega t + \phi_i)$, where $w_{d_i} = \sqrt{C_{d1i}^2 + C_{d2i}^2}$ and $\phi_i = \tan^{-1} C_{d2i}/C_{d1i}$. Conversely, $C_{d1i} = (w_{d_i}/\sqrt{\tan^2 \phi_i + 1})$ and $C_{d2i} = (w_{d_i} \tan \phi_i/\sqrt{\tan^2 \phi_i + 1})$. Hence, each component $w_{d_i}(t)$ of the disturbance has uncertain amplitude and phase.

To guarantee that the expected value of the measurement $\mathbb{E}[y(t)]$ approaches zero asymptotically, the feedback loop must contain an internal model, which is a replicated version of the exogenous dynamics (5) (Davison and Goldenberg, 1975). The internal model is given in state space form by

$$\dot{x}_{sc}(t) = A_{sc} x_{sc}(t) + B_{sc} y(t), \tag{7}$$

where $x_{sc}(t) \in \mathbb{R}^{n_{sc}}$ is the servocompensator state and where A_{sc} is comprised of l replications of the matrix A_d. For a constant disturbance $w_d(t) = C_d$, the matrices A_{sc} and B_{sc} are given by

$$A_{sc} = 0_{l \times l}, \quad B_{sc} = I_l, \tag{8}$$

where $0_{i \times j}$ is the $i \times j$ zero matrix and I_i is the $i \times i$ identity matrix. Analogously, for a sinusoidal disturbance $w_d = C_{d1} \sin \omega t + C_{d2} \cos \omega t$, the matrices A_{sc} and B_{sc} are given by

$$A_{sc} = \begin{bmatrix} 0_{l \times l} & \omega I_l \\ -\omega I_l & 0_{l \times l} \end{bmatrix}, \quad B_{sc} = \begin{bmatrix} 0_{l \times l} \\ I_l \end{bmatrix}. \tag{9}$$

We can now form the augmented system as

$$\dot{x}_a(t) = A_a x_a(t) + B_a u(t) + D_a w(t) + D_{da} w_d(t), \tag{10}$$

where

$$x_a(t) \triangleq \begin{bmatrix} x(t) \\ x_{sc}(t) \end{bmatrix}, \quad A_a \triangleq \begin{bmatrix} A & 0_{n \times n_{sc}} \\ B_{sc} C & A_{sc} \end{bmatrix},$$

$$B_a \triangleq \begin{bmatrix} B \\ 0_{n_{sc} \times m} \end{bmatrix}, \quad D_a \triangleq \begin{bmatrix} D_1 \\ B_{sc} D_2 \end{bmatrix}, \quad D_{da} \triangleq \begin{bmatrix} D_{d1} \\ B_{sc} D_{d2} \end{bmatrix}.$$

The following lemma gives sufficient conditions under which the pair (A_a, B_a) of the augmented system (10) is controllable.

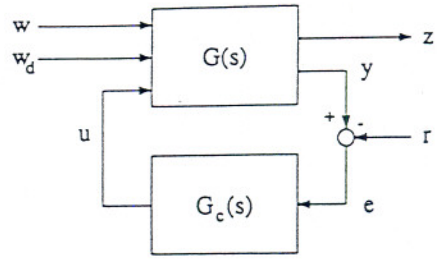

Fig. 1 Block diagram of the closed-loop system

Lemma 2.1. If

$$\text{rank} \begin{bmatrix} j\omega I - A & B \\ -C & 0 \end{bmatrix} = n + l, \tag{11}$$

then the pair (A_a, B_a) is controllable.

Remark 2.1. The rank condition in (11) ensures that there are no pole-zero cancellations in the cascaded realization of the plant model and the internal model. This rank condition is a requirement for asymptotic disturbance rejection. Lemma 2.1 specializes to the case of a constant disturbance by letting $\omega = 0$.

Now consider a dynamic compensator of the form

$$\dot{x}_c(t) = A_c x_c(t) + A_{csc} x_{sc}(t) + B_c y(t), \tag{12}$$

$$u(t) = C_c x_c(t) + C_{sc} x_{sc}(t), \tag{13}$$

where $x_c(t) \in \mathbb{R}^{n_c}$, so that the controller consisting of the servocompensator (7) and the dynamic compensator (12), (13) has the realization

$$G_c(s) \sim \left[\begin{array}{cc|c} A_c & A_{csc} & B_c \\ 0 & A_{sc} & B_{sc} \\ \hline C_c & C_{sc} & 0 \end{array}\right]. \tag{14}$$

The closed-loop system, (1)–(3), (7), (12), and (13) thus has the form

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t) + D_d w_d(t), \tag{15}$$

$$y(t) = \tilde{C}\tilde{x}(t), \tag{16}$$

$$z(t) = \tilde{E}\tilde{x}(t), \tag{17}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_{sc}(t) \\ x_c(t) \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A & BC_{sc} & BC_c \\ B_{sc}C & A_{sc} & 0_{n_{sc} \times n_c} \\ B_c C & A_{csc} & A_c \end{bmatrix},$$

$$\tilde{C} \triangleq [C \quad 0_{l \times (n_c + n_{sc})}], \quad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_{sc} D_2 \\ B_c D_2 \end{bmatrix},$$

$$D_d \triangleq \begin{bmatrix} D_{d1} \\ B_{sc} D_{d2} \\ B_c D_{d2} \end{bmatrix}, \quad \tilde{E} \triangleq [E_1 \quad E_2 C_{sc} \quad E_2 C_c].$$

If (A_a, B_a) is stabilizable, then a stabilizing controller exists so that the closed-loop augmented system is stable. A block diagram of the closed-loop system is shown in Fig. 1.

The following two lemmas are special cases of Theorem 1 of Iftar (1990).

Lemma 2.2. Suppose the disturbance $w_d(t) = C_d$, and assume the augmented matrix $\tilde{A}$ in (15) with internal model (8) is asymptotically stable. Then $\mathbb{E}[y(t)] \to 0$ as $t \to \infty$.

Lemma 2.3. Suppose the disturbance $w_d(t) = C_{d1} \sin \omega t + C_{d2} \cos \omega t$ and assume the augmented matrix $\tilde{A}$ in (15) with internal model (9) is asymptotically stable. Then $\mathbb{E}[y(t)] \to 0$ as $t \to \infty$.

Remark 2.2. The internal model ensures that the expected value of each output decays to zero. It is essential that the exogenous dynamics be replicated l times in the internal model, since a single copy of the exogenous system dynamics is not sufficient to ensure that the expected value of each output decays to zero individually. If a single copy of the exogenous system dynamics were used in the internal model, then only a linear combination of the expected value of the outputs would decay to zero, that is, $B_{sc}\mathbb{E}[y(t)] \to 0$.

The following propositions provide expressions for the integral square error.

Proposition 2.1. Let $w_d(t) = C_d$ and suppose $\tilde{A}$ is asymptotically stable. Then, the integral square output is given by

$$\int_0^\infty \mathbb{E}[y(t)]^T M \mathbb{E}[y(t)]dt = C_d^T D_d^T T D_d C_d, \tag{18}$$

where T satisfies

$$0 = \tilde{A}^T T + T\tilde{A} + \tilde{A}^{-T}\tilde{C}^T M \tilde{C}\tilde{A}^{-1}. \tag{19}$$

Proof. It follows from (15) that

$$\mathbb{E}[\tilde{x}(t)] = \tilde{A}^{-1}e^{\tilde{A}t}D_d C_d - \tilde{A}^{-1}D_d C_d.$$

Thus, $\mathbb{E}[y(t)] = \tilde{C}\tilde{A}^{-1}e^{\tilde{A}t}D_d C_d - \tilde{C}\tilde{A}^{-1}D_d C_d$. Next, using (15) and since $\tilde{A}$ is asymptotically stable, $\lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)] = -\tilde{A}^{-1}D_d C_d$. It follows from (2) that $\lim_{t\to\infty} \mathbb{E}[y(t)] = -\tilde{C}\tilde{A}^{-1}D_d C_d$. Since, be Lemma 2.2, $\mathbb{E}[y(t)] \to 0$ as $t \to \infty$, it follows that $\tilde{C}\tilde{A}^{-1}D_d C_d = 0$, hence $\mathbb{E}[y(t)] = \tilde{C}\tilde{A}^{-1}e^{\tilde{A}t}D_d C_d$, which yields (18), where T satisfies (19). □

By Proposition 2.1, the minimum value of the integral square output depends on C_d, which is uncertain. For constant disturbances, we assume that C_d belongs to the set $\mathcal{C}_d$, defined by

$$\mathcal{C}_d \triangleq \{C_d \in \mathbb{R}^l : C_d C_d^T \le V\},$$

where $V \ge 0$ is a given uncertainty bound. Thus, if $C_d \in \mathcal{C}_d$, it follows that

$$\int_0^\infty \mathbb{E}[y(t)]^T M \mathbb{E}[y(t)]dt \le \operatorname{tr} D_d^T T D_d V. \tag{20}$$

Proposition 2.2. Let $w_d(t) = C_{d1} \sin \omega t + C_{d2} \cos \omega t$ and let $\tilde{A}$ be asymptotically stable. Then, the integral square output is

$$\int_0^\infty \mathbb{E}[y(t)]^T M \mathbb{E}[y(t)]dt$$
$$= (\omega C_{d1}^T D_d^T + C_{d2}^T D_d^T \tilde{A}^T) T (\omega D_d C_{d1} + \tilde{A} D_d C_{d2}), \tag{21}$$

where T satisfies

$$0 = \tilde{A}^T T + T\tilde{A} + (\tilde{A}^2 + \omega^2 I)^{-T}\tilde{C}^T M \tilde{C}(\tilde{A}^2 + \omega^2 I)^{-1}. \tag{22}$$

Proof. It follows from (2) and some simple manipulation that

$$\mathbb{E}[y(t)] = \tilde{C}(\tilde{A}^2 + \omega^2 I)^{-1}e^{\tilde{A}t}(\omega D_d C_{d1} + \tilde{A}D_d C_{d2})$$
$$- \tilde{C}(\tilde{A}^2 + \omega^2 I)^{-1}[(\tilde{A}\sin \omega t + \omega \cos \omega t I)D_d C_{d1}$$
$$+ (\tilde{A}\cos \omega t - \omega \sin \omega t I)D_d C_{d2}]. \tag{23}$$

Since by Lemma 2.3 $\mathbb{E}[y(t)] \to 0$ as $t \to \infty$, and since $\tilde{A}$ is asymptotically stable, it follows that $e^{\tilde{A}t} \to 0$ as $t \to \infty$. Taking the limit of both sides of (23), it follows that the terms involving $\sin \omega t$ and $\cos \omega t$ are zero. Hence the expected value of the output is

$$\mathbb{E}[y(t)] = \tilde{C}(\tilde{A}^2 + \omega^2 I)^{-1}e^{\tilde{A}t}(\omega D_d C_{d1} + \tilde{A}D_d C_{d2}).$$

The integral square output can be written as (21), where T satisfies (22). □

By Proposition 2.2, the minimum value of the integral square output depends on C_d, which is uncertain. For sinusoidal disturbances, we assume that C_d belongs to the set $\mathcal{C}_d$ defined by

$$\mathcal{C}_d \triangleq \left\{ C_d \in \mathbb{R}^{l\times 2} : \begin{bmatrix} C_{d1} \\ C_{d2} \end{bmatrix} \begin{bmatrix} C_{d1} \\ C_{d2} \end{bmatrix}^T \le \begin{bmatrix} V_1 & V_{12} \\ V_{12}^T & V_2 \end{bmatrix} = V \right\}, \tag{24}$$

where $V \ge 0$ is a given uncertainty bound. Thus, if $C_d \in \mathcal{C}_d$, it follows that

$$\int_0^\infty \mathbb{E}[y(t)]^T M \mathbb{E}[y(t)]dt \le \omega^2 \operatorname{tr} D_d^T T D_d V_1$$
$$+ \operatorname{tr} D_d^T \tilde{A}^T T \tilde{A} D_d V_2 + 2\omega \operatorname{tr} D_d^T T \tilde{A} D_d V_{12}^T. \tag{25}$$

We can now state the optimal control problem.

Optimal Robust Disturbance Rejection Problem. Given the plant dynamics (1) and the internal model dynamics (7), find control gains A_c, B_c, C_c, A_{csc}, and C_{sc} that stabilize $\tilde{A}$ and minimize

$$J(A_c, B_c, C_c, A_{csc}, C_{sc}) \triangleq \|T_{zw}\|_2^2$$
$$+ \max_{C_d \in \mathcal{C}_d} \int_0^\infty \mathbb{E}[y(t)]^T M \mathbb{E}[y(t)]dt, \tag{26}$$

where T_{zw} is the transfer function from $w(t)$ to $z(t)$.

3 Servocompensator Problem Necessary Conditions

In this section we present necessary conditions for the Optimal Robust Disturbance Rejection Problem defined in the previous section, for which the disturbances are constant and sinusoidal. These necessary conditions provide the basis for numerically optimizing the controller. For convenience, let X_{ij} denote the ij^{th} block of X partitioned in the same manner as $\tilde{A}$.

Theorem 3.1. Suppose A_c, B_c, C_c, A_{csc}, and C_{sc} solve the Optimal Robust Disturbance Rejection Problem for constant disturbances. Then there exist nonnegative-definite matrices P, Q, T, S that satisfy

$$0 = \tilde{A}^T P + P\tilde{A} + \tilde{E}^T\tilde{E}, \tag{27}$$

$$0 = \tilde{A}Q + Q\tilde{A}^T + \tilde{D}\tilde{D}^T, \tag{28}$$

$$0 = \tilde{A}^{2T}T\tilde{A} + \tilde{A}^T T\tilde{A}^2 + \tilde{C}^T M\tilde{C}, \tag{29}$$

$$0 = \tilde{A}^2 S\tilde{A}^T + \tilde{A}S\tilde{A}^{2T} + D_d V D_d^T, \tag{30}$$

$$0 = \Omega_{33}^T + \Phi_{33}^T + \Theta_{33}^T + \Psi_{33}^T, \tag{31}$$

$$0 = (P_{31}D_1 + P_{32}B_{sc}D_2 + P_{33}B_c D_2)D_2^T$$
$$+ (\Omega_{13}^T + \Phi_{13}^T + \Theta_{13}^T + \Psi_{13}^T)C^T$$
$$+ (T_{31}D_{d1} + T_{32}B_{sc}D_{d2} + T_{33}B_c D_{d2})VD_{d2}^T \tag{32}$$

$$0 = E_2^T(E_1 Q_{13} + E_2 C_{sc} Q_{23} + E_2 C_c Q_{33})$$
$$+ B^T(\Omega_{31}^T + \Phi_{31}^T + \Theta_{31}^T + \Psi_{31}^T), \tag{33}$$

$$0 = \Omega_{23}^T + \Phi_{23}^T + \Theta_{23}^T + \Psi_{23}^T, \tag{34}$$

$$0 = E_2^T(E_1 Q_{12} + E_2 C_{sc} Q_{22} + E_2 C_c Q_{32})$$
$$+ B^T(\Omega_{21}^T + \Phi_{21}^T + \Theta_{21}^T + \Psi_{21}^T). \tag{35}$$

where $\Phi \triangleq S\tilde{A}^T T\tilde{A}$, $\Psi \triangleq S\tilde{A}^{2T}T$, $\Theta \triangleq \tilde{A}S\tilde{A}^T T$, and $\Omega \triangleq QP$.

The proof of Theorem 3.1 is similar to that of the following result and is hence omitted.

Theorem 3.2. Suppose A_c, B_c, C_c, A_{csc}, and C_{sc} solve the Optimal Robust Disturbance Rejection Problem for sinusoidal disturbances. Then there exist nonnegative-definite matrices P, Q, T, S that satisfy (27), (28),

$$0 = (\tilde{A}^2 + \omega^2 I)^T \tilde{A}^T T(\tilde{A}^2 + \omega^2 I) + (\tilde{A}^2 + \omega^2 I)^T T\tilde{A}(\tilde{A}^2 + \omega^2 I) + \tilde{C}^T M \tilde{C}, \quad (36)$$

$$0 = (\tilde{A}^2 + \omega^2 I)\tilde{A}S(\tilde{A}^2 + \omega^2 I)^T + (\tilde{A}^2 + \omega^2 I)S\tilde{A}^T(\tilde{A}^2 + \omega^2 I)^T + \omega^2 D_d V_1 D_d^T + \tilde{A}D_d V_2 D_d^T \tilde{A}^T + \omega \tilde{A} D_d V_{12}^T D_d^T + \omega D_d V_{12} D_d^T \tilde{A}^T, \quad (37)$$

$$0 = \Omega_{33}^T + \Phi_{33}^T + \Theta_{33}^T + \Gamma_{33}^T + \Psi_{33}^T + \Pi_{33}^T + \Delta_{33}^T + \Lambda_{33}^T, \quad (38)$$

$$\begin{aligned} 0 = {} & (P_{31}D_1 + P_{32}B_{sc}D_2 + P_{33}B_cD_2)D_2^T \\ & + \omega^2(T_{31}D_{d1} + T_{32}B_{sc}D_{d2} + T_{33}B_cD_{d2})V_1D_{d2}^T \\ & + \omega[(T\tilde{A})_{31}D_{d1} + (T\tilde{A})_{32}B_{sc}D_{d2} + (T\tilde{A})_{33}B_cD_{d2}]V_{12}^T D_{d2}^T \\ & + \omega[(\tilde{A}^T T)_{31}D_{d1} + (\tilde{A}^T T)_{32}B_{sc}D_{d2} \\ & + (\tilde{A}^T T)_{33}B_cD_{d2}]V_{12}D_{d2}^T + [(\tilde{A}^T T\tilde{A})_{31}D_{d1} \\ & + (\tilde{A}^T T\tilde{A})_{32}B_{sc}D_{d2} + (\tilde{A}^T T\tilde{A})_{33}B_cD_{d2}]V_2D_{d2}^T \\ & + (\Omega_{13}^T + \Phi_{13}^T + \Theta_{13}^T + \Gamma_{13}^T \\ & + \Psi_{13}^T + \Pi_{13}^T + \Delta_{13}^T + \Lambda_{13}^T)C^T, \end{aligned} \quad (39)$$

$$0 = E_2^T(E_1Q_{13} + E_2C_{sc}Q_{23} + E_2C_cQ_{33}) + B^T(\Omega_{31}^T + \Phi_{31}^T + \Theta_{31}^T + \Gamma_{31}^T + \Psi_{31}^T + \Pi_{31}^T + \Delta_{31}^T + \Lambda_{31}^T), \quad (40)$$

$$0 = \Omega_{23}^T + \Phi_{23}^T + \Theta_{23}^T + \Gamma_{23}^T + \Psi_{23}^T + \Pi_{23}^T + \Delta_{23}^T + \Lambda_{23}^T, \quad (41)$$

$$0 = E_2^T(E_1Q_{12} + E_2C_{sc}Q_{22} + E_2C_cQ_{32}) + B^T(\Omega_{21}^T + \Phi_{21}^T + \Theta_{21}^T + \Gamma_{21}^T + \Psi_{12}^T + \Pi_{12}^T + \Delta_{12}^T + \Lambda_{12}^T), \quad (42)$$

where $\Theta \triangleq \tilde{A}S(\tilde{A}^2 + \omega^2 I)^T T\tilde{A}$, $\Gamma \triangleq (\tilde{A}^2 + \omega^2 I)S(\tilde{A}^2 + \omega^2 I)^T T$, $\Psi \triangleq \tilde{A}S(\tilde{A}^2 + \omega^2 I)^T \tilde{A}^T T$, $\Phi \triangleq S(\tilde{A}^2 + \omega^2 I)^T T\tilde{A}^2$, $\Pi \triangleq S(\tilde{A}^2 + \omega^2 I)^T \tilde{A}^T T\tilde{A}$, $\Omega \triangleq QP$, and $\Delta \triangleq \omega D_d V_{12}^T D_d^T$.

Proof. To obtain the necessary conditions, first write the $\mathcal{H}_2$ cost in the form tr $P\tilde{D}\tilde{D}^T$. Next, write (26) as

$$J(A_c, B_c, C_c, A_{csc}, C_{sc}) \triangleq \|T_{\tilde{z}w}\|_2^2 + \omega^2 \operatorname{tr} D_d^T T D_d V_1 + \operatorname{tr} D_d^T \tilde{A}^T T\tilde{A} D_d V_2 + 2\omega \operatorname{tr} D_d^T T\tilde{A} D_d V_{12}^T, \quad (43)$$

and note that (22) can be rewritten as (36). Form the Lagrangian $\mathcal{L}$ by affixing (27) and (36) via Lagrange multipliers Q and S, respectively, to obtain

$$\begin{aligned} \mathcal{L} = {} & \operatorname{tr} P\tilde{D}\tilde{D}^T + \operatorname{tr} Q(\tilde{A}^T P + P\tilde{A} + \tilde{E}^T\tilde{E}) + \omega^2 \operatorname{tr} TD_dV_1D_d^T \\ & + \operatorname{tr} \tilde{A}^T T\tilde{A} D_d V_2 D_d^T + 2\omega \operatorname{tr} D_d^T T\tilde{A} D_d V_{12}^T \\ & + \operatorname{tr} S((\tilde{A}^2 + \omega^2 I)^T \tilde{A}^T T(\tilde{A}^2 + \omega^2 I) \\ & + (\tilde{A}^2 + \omega^2 I)^T T\tilde{A}(\tilde{A}^2 + \omega^2 I) + \tilde{C}^T M\tilde{C}). \end{aligned} \quad (44)$$

Setting $(1/2)(\partial\mathcal{L}/\partial A_c)$, $(1/2)(\partial\mathcal{L}/\partial B_c)$, $(1/2)(\partial\mathcal{L}/\partial C_c)$, $(1/2)(\partial\mathcal{L}/\partial A_{csc})$, and $(1/2)(\partial\mathcal{L}/\partial C_{sc})$ to zero gives the necessary conditions (38)–(42). Taking the derivatives $\partial\mathcal{L}/\partial Q$, $\partial\mathcal{L}/\partial P$, $\partial\mathcal{L}/\partial S$, and $\partial\mathcal{L}/\partial T$ and setting them equal to zero gives (27), (28), (36), and (37). □

Theorems 3.1 and 3.2 can be used with an optimization algorithm such as in Dennis and Schnabel (1983) to find controllers that solve the Optimal Robust Disturbance Rejection Problem. Equation (31)–(35) and (38)–(42) provide analytic expressions for the gradients of the cost with respect to each of the control gains. The reader is referred to Sparks and Bernstein (1995) for details on such an approach applied to a similar problem.

4 Summary and Conclusions

Necessary conditions were given for gains that minimize a cost consisting of two components, namely an $\mathcal{H}_2$ disturbance rejection cost and a deterministic disturbance rejection cost. Theorem 3.1 considered constant disturbances, while Theorem 3.2 treated sinusoidal disturbances. The necessary conditions were obtained by characterizing the cost in terms of scalar functions that depend on solutions to Lyapunov equations. The Lagrangian was formed by attaching the Lyapunov constraints to the scalar cost function via Lagrange multipliers. This technique gave gradients of the cost with respect to each of the control gains, which can be used within a gradient search algorithm to find optimal gains.

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