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Brief Paper

# An Exact Treatment of the Achievable Closed-loop $H_2$ Performance of Sampled-data Controllers: from Continuous-time to Open-loop*

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**Key Words**—Sampled-data systems; linear optimal control; optimal control; digital control; discrete time systems.

**Abstract**—In this paper we investigate the closed-loop performance of a sampled-data control system by utilizing exact discretization techniques. In particular, for an $H_2$ performance measure we give exact expressions for the closed-loop cost for a given sample interval $h$. After applying discrete-time LQG synthesis to the sampled-data system, the achievable performance is evaluated for fast sampling near continuous time, $h \rightarrow 0$, and slow sampling near open loop, $h \rightarrow \infty$. Connections between the continuous-time Riccati equation for the analog control system and the discrete-time Riccati equation for the sampled-data system are investigated. Finally, several numerical examples are given to illustrate the convergence from sampled-data control to continuous-time control and open-loop.

1. *Introduction*

One of the first design decisions a control engineer must make concerns the capabilities of the real-time feedback processor. Assuming that the controller will be implemented digitally, it is useful to understand how processor capabilities affect the stability and achievable performance of the closed-loop system, including intersample behavior (De Souza and Goodwin, 1984; Lennartson and Söderström 1989; Leung *et al.*, 1991). Although it seems reasonable to conjecture that closed-loop performance improves as processor speed increases, there exist relatively few results that rigorously document this fact.

The goal of this paper is to develop a sampled-data design formulation that accounts precisely for all sampling effects, including intersample behavior. A unique feature of our approach is its unified treatment of both continuous-time and discrete-time controllers. Thus, by appropriate choice of analog-to-digital (A/D) and digital-to-analog (D/A) devices, we expect to recover continuous-time controller performance as the sample interval $h$ approaches zero and open-loop performance as $h$ approaches infinity. To the best of our knowledge, this paper presents the first attempt to provide a 'seamless' treatment of these two extreme cases in the context of dynamic compensation with white measurement noise.

An immediate benefit of our approach is the ability to carefully examine the effect of increasing or decreasing the sample rate. For example, although specific choices of the sampling interval will result in the loss of controllability and thus degraded performance, by increasing $h$ above these values one can recover controllability and thereby improve performance. We believe that the quantification of this observation in terms of achievable closed-loop performance will be useful in applications such as the control of flexible structures that possess modes with frequencies above the Nyquist rate of any sampled-data controller. The results we obtain are developed for an LQG-type control problem.

The problem of exactly discretizing a sampled-data system has also been considered in Khargonekar and Sivashankar (1991) and Bamieh and Pearson (1992) through the use of a zeroth-order hold and impulse sampler. However, in Khargonekar and Sivashankar (1991), the continuous-time measurement noise is directly replaced by a discrete-time measurement noise because of the ill-posedness of impulsive sampling of white noise. In Bamieh and Pearson (1992) this difficulty is overcome through the assumption that the measurement is noiseless. In the present paper the white noise difficulty is overcome through the use of an averaging/resetting A/D device to allow an exact treatment of the sampled-data problem with measurement noise.

The contents of the paper are as follows. In Section 2 we state the sampled-data control problem along with all assumptions concerning A/D and D/A devices. Of special interest is the choice of sampling device as in Åström (1970), Shats and Shaked (1989) and Bernstein *et al.* (1986), which permits the unified treatment of analog and digital controllers without recourse to 'fictitious' discrete-time white measurement noise. In Section 3 we state the LQG control problem for the equivalent discrete-time problem. In Sections 4 and 5 we examine the dependence of the closed-loop performance on the sample interval $h$ as $h$ approaches both zero and infinity. Finally, in Section 6 we illustrate these results by means of several examples, including both open-loop stable and open-loop unstable plants. The main feature of interest here is the dependence of the achievable closed-loop performance on $h$.

2. *Derivation of the exact discrete-time model*

Consider the continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t), \tag{1}$$

$$y(t) = Cx(t) + Du(t) + D_2 w(t), \tag{2}$$

$$z(t) = E_1 x(t) + E_2 u(t), \tag{3}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $w \in \mathbb{R}^d$ and $z \in \mathbb{R}^p$ are the state, input, measurement, disturbance and performance

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respectively. The disturbance $w$ is a standard zero-mean white noise process. By forming $z^{\mathrm{T}}z$, the cost to be minimized is

$$J(G_c) = \lim_{t\to\infty} \mathscr{E}\left\{\frac{1}{t}\int_0^t [x^{\mathrm{T}}(s)R_1x(s) + 2x^{\mathrm{T}}(s)R_{12}u(s) + u^{\mathrm{T}}(s)R_2u(s)]\,\mathrm{d}s\right\}, \quad (4)$$

where $G_c$ denotes a feedback controller, $\mathscr{E}$ denotes expectation, and $R_1 \triangleq E_1^{\mathrm{T}}E_1$, $R_{12} \triangleq E_1^{\mathrm{T}}E_2$ and $R_2 \triangleq E_2^{\mathrm{T}}E_2$. Throughout this paper, we assume that $(A, B)$ and $(A - V_{12}V_2^{-1}C, (V_1 - V_{12}V_2^{-1}V_{12}^{\mathrm{T}})^{1/2})$ are stabilizable, and $(C, A)$ and $((R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}})^{1/2}, A - BR_2^{-1}R_{12}^{\mathrm{T}})$ are detectable.

In (4) $G_c$ denotes a continuous-time controller, whereas $G_{c,h}$ will represent a discrete-time controller with sampling interval $h$. For the sampled-data controller, the measurements are given by an averaging/resetting A/D device of the form

$$y'(k) \triangleq \frac{1}{h}\int_{(k-1)h}^{kh} y(s)\,\mathrm{d}s. \quad (5)$$

This device, which was studied by Åström (1970) and Shats and Shaked (1989), recognizes the fact that the A/D operation is not instantaneous. Moreover, (5) circumvents difficulties that arise from direct sampling of continuous-time white noise and, as will be seen, allows a smooth transition from continuous-time to sampled-data controllers. Finally, to obtain continuous-time control signals, we employ a D/A zeroth-order hold of the form

$$u(t) = u(kh), \quad kh \le t < (k+1)h. \quad (6)$$

The corresponding discretized state, measurement and cost expressions are thus given by (Bernstein *et al.*, 1986)

$$\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}u'(k) + \tilde{w}(k), \quad (7)$$

$$y'(k) = \tilde{C}\tilde{x}(k), \quad (8)$$

$$J(G_{c,h}) = \delta_h + \lim_{k\to\infty} \mathscr{E}[\tilde{x}^{\mathrm{T}}(k)\tilde{R}_1\tilde{x}(k) + 2\tilde{x}^{\mathrm{T}}(k)\tilde{R}_{12}u'(k) + u'^{\mathrm{T}}(k)R_2'u'(k)], \quad (9)$$

where

$$\tilde{x}(k) \triangleq \begin{bmatrix} y'(k) \\ y'(k) \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A' & 0 \\ C' & 0 \end{bmatrix}, \quad \tilde{B} \triangleq \begin{bmatrix} B' \\ D' \end{bmatrix},$$

$$\tilde{C} \triangleq \begin{bmatrix} 0 \\ I \end{bmatrix}^{\mathrm{T}}, \quad \tilde{R}_1 \triangleq \begin{bmatrix} R_1' & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{R}_{12} \triangleq \begin{bmatrix} R_{12}' \\ 0 \end{bmatrix},$$

$$x'(k) \triangleq x(kh), \quad u'(k) \triangleq u(kh), \quad A' \triangleq e^{Ah},$$

$$B' \triangleq H(h)B, \quad H(s) \triangleq \int_0^s e^{Ar}\,\mathrm{d}r,$$

$$C' \triangleq \frac{1}{h}CH(h), \quad D' \triangleq \frac{1}{h}C\int_0^h H(s)\,\mathrm{d}s\,B + D,$$

$$V_1' \triangleq \int_0^h e^{As}V_1e^{A^{\mathrm{T}}s}\,\mathrm{d}s,$$

$$V_{12}' \triangleq \frac{1}{h}\int_0^h e^{As}V_1H(s)\,\mathrm{d}s\,C^{\mathrm{T}} + \frac{1}{h}H(h)V_{12},$$

$$V_2' \triangleq \frac{1}{h^2}C\int_0^h H(s)V_1H^{\mathrm{T}}(s)\,\mathrm{d}s\,C^{\mathrm{T}} + \frac{1}{h^2}C\int_0^h H(s)\,\mathrm{d}s\,V_{12} + \frac{1}{h^2}V_{12}^{\mathrm{T}}\int_0^h H^{\mathrm{T}}(s)\,\mathrm{d}s\,C^{\mathrm{T}} + \frac{1}{h}V_2,$$

$$R_1' \triangleq \frac{1}{h}\int_0^h e^{A^{\mathrm{T}}s}R_1e^{As}\,\mathrm{d}s,$$

$$R_{12}' \triangleq \frac{1}{h}\int_0^h e^{A^{\mathrm{T}}s}R_1H(s)\,\mathrm{d}s\,B + \frac{1}{h}H^{\mathrm{T}}(h)R_{12},$$

$$R_2' \triangleq \frac{1}{h}B^{\mathrm{T}}\int_0^h H^{\mathrm{T}}(s)R_1H(s)\,\mathrm{d}s\,B + \frac{1}{h}B^{\mathrm{T}}\int_0^h H^{\mathrm{T}}(s)\,\mathrm{d}s\,R_{12} + \frac{1}{h}R_{12}^{\mathrm{T}}\int_0^h H(s)\,\mathrm{d}s\,B + R_2,$$

$$V_1 \triangleq D_1D_1^{\mathrm{T}}, \quad V_{12} \triangleq D_1D_2^{\mathrm{T}}, \quad V_2 \triangleq D_2D_2^{\mathrm{T}},$$

$$\delta_h \triangleq \mathrm{tr}\left[\frac{1}{h}\int_0^h\int_0^t e^{As}V_1e^{A^{\mathrm{T}}s}\,\mathrm{d}s\,\mathrm{d}t\,R_1\right].$$

For computational purposes, the method given by Van Loan (1978) can be used to evaluate these integrals. For details see Bernstein *et al.* (1986) and Osburn and Bernstein (1993).

### 3. *Continuous-time and sampled-data LQG control*

In this section we apply standard LQG theory to obtain optimal stabilizing controllers. Consider the $n$th-order strictly proper continuous-time dynamic compensator $G_c$ with realization

$$\dot{x}_c(t) = A_cx_c(t) + B_cy(t), \quad u(t) = C_cx_c(t).$$

The optimal LQG controller $G_c^{\mathrm{opt}}$ is given by

$$A_c = A + BC_c - B_cC, \quad B_c = (QC^{\mathrm{T}} + V_{12})V_2^{-1},$$
$$C_c = -R_2^{-1}(B^{\mathrm{T}}P + R_{12}^{\mathrm{T}}),$$

where $P$ and $Q$ are the unique nonnegative-definite matrices satisfying the continuous-time Riccati equations

$$0 = AQ + QA^{\mathrm{T}} + V_1 - (V_{12} + QC^{\mathrm{T}})V_2^{-1}(V_{12} + QC^{\mathrm{T}})^{\mathrm{T}}, \quad (10)$$

$$0 = A^{\mathrm{T}}P + PA + R_1 - (R_{12} + PB)R_2^{-1}(R_{12} + PB)^{\mathrm{T}}. \quad (11)$$

With the continuous-time LQG controller, the optimal cost is given by

$$J(G_c^{\mathrm{opt}}) = \mathrm{tr}\,[QR_1 + \hat{Q}(R_1 + 2C_c^{\mathrm{T}}R_{12} + C_c^{\mathrm{T}}R_2C_c)], \quad (12)$$

where $\hat{Q}$ satisfies

$$0 = (A + BC_c)\hat{Q} + \hat{Q}(A + BC_c)^{\mathrm{T}} + (V_{12} + QC^{\mathrm{T}})V_2^{-1}(V_{12} + QC^{\mathrm{T}})^{\mathrm{T}}. \quad (13)$$

Now we consider the problem of obtaining an $(n+l)$th-order strictly proper discrete-time dynamic compensator $G_{c,h}$ with realization

$$x_{c,h}(k+1) = A_{c,h}x_{c,h}(k) + B_{c,h}y'(k), \quad u'(k) = C_{c,h}x_{c,h}(k).$$

The optimal discrete-time LQG controller $G_{c,h}^{\mathrm{opt}}$ for the sampled-data system with sampling interval $h$ as given by Bernstein *et al.* (1986) and Dorato and Levis (1971) is

$$A_{c,h} = \tilde{A} + \tilde{B}C_{c,h} - B_{c,h}\tilde{C}, \quad B_{c,h} = \tilde{A}Q_h\tilde{C}^{\mathrm{T}}(\tilde{C}Q_h\tilde{C}^{\mathrm{T}})^{-1},$$
$$C_{c,h} = -(R_2' + \tilde{B}^{\mathrm{T}}P_h\tilde{B})^{-1}(\tilde{B}^{\mathrm{T}}P_h\tilde{A} + \tilde{R}_{12}^{\mathrm{T}}),$$

where the nonnegative-definite matrices $Q_h$ and $P_h$ satisfy the discrete-time Riccati equations

$$Q_h = \tilde{A}Q_h\tilde{A}^{\mathrm{T}} - \tilde{A}Q_h\tilde{C}^{\mathrm{T}}(\tilde{C}Q_h\tilde{C}^{\mathrm{T}})^{-1}\tilde{C}Q_h\tilde{A}^{\mathrm{T}} + \tilde{V}, \quad (14)$$

$$P_h = \tilde{A}^{\mathrm{T}}P_h\tilde{A} - (\tilde{B}^{\mathrm{T}}P_h\tilde{A} + \tilde{R}_{12}^{\mathrm{T}})^{\mathrm{T}}(R_2' + \tilde{B}^{\mathrm{T}}P_h\tilde{B})^{-1} \times (\tilde{B}^{\mathrm{T}}P_h\tilde{A} + \tilde{R}_{12}^{\mathrm{T}}) + \tilde{R}_1. \quad (15)$$

With the discrete-time LQG controller, the optimal cost is given by

$$J(G_{c,h}^{\mathrm{opt}}) = \delta_h + \mathrm{tr}\,[Q_h\tilde{R}_1 + \hat{Q}_h(\tilde{R}_1 + 2C_{c,h}^{\mathrm{T}}\tilde{R}_{12} + C_{c,h}^{\mathrm{T}}R_2'C_{c,h})], \quad (16)$$

where $\hat{Q}_h$ satisfies

$$\hat{Q}_h = (\tilde{A} + \tilde{B}C_{c,h})\hat{Q}_h(\tilde{A} + \tilde{B}C_{c,h})^{\mathrm{T}} + (\tilde{A}Q_h\tilde{C}^{\mathrm{T}}) \times (\tilde{C}Q_h\tilde{C}^{\mathrm{T}})^{-1}(\tilde{A}Q_h\tilde{C}^{\mathrm{T}})^{\mathrm{T}}. \quad (17)$$

4. *Analysis of optimal performance as $h \to 0$*

In this section we consider the case of fast sampling, that is, $h \to 0$. Ideally, one would expect the optimal cost for sampled-data control to approach the optimal cost for continuous-time control. Since the sampled-data problem involves an augmented plant of order $n + l$, in contrast to the continuous-time plant, which is of order $n$, it is not apparent that the optimal continuous-time control cost will be recovered in the limit. Nevertheless, in this section we shall show that, in fact, the optimal cost for sampled-data control converges to the optimal cost for continuous-time control. Before proceeding, it is useful to define

$$A_h \triangleq \frac{1}{h}(A' - I), \quad B_h \triangleq \frac{1}{h}B'$$

as in Salgado *et al.* (1988), and Middleton and Goodwin (1990). Note that $\lim_{h\to 0} A_h = A$ and $\lim_{h\to 0} B_h = B$.

*Theorem* 1. Let $Q$ be the unique nonnegative-definite solution to the continuous-time Riccati equation (10), and suppose there exists a unique nonnegative-definite solution

$$Q_h = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \in \mathbb{R}^{(n+l)\times(n+l)}$$

to the discrete-time Riccati equation (14). Assume $Q \triangleq \lim_{h\to 0} Q_1$ exists. Then

$$\lim_{h\to 0} \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & hQ_2 \end{bmatrix}$$

exists and is given by

$$\lim_{h\to 0} Q_{12} = V_{12} + QC^T, \quad \lim_{h\to 0} hQ_2 = V_2.$$

*Proof.* See Osburn and Bernstein (1993). □

*Theorem* 2. Let $P$ be the unqiue nonnegative-definite solution to the continuous-time Riccati equation (11), and suppose there exists a unique nonnegative-definite solution

$$P_h = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix} \in \mathbb{R}^{(n+l)\times(n+l)}$$

to the discrete-time Riccati equation (15). Then $P_{12} = 0$, $P_2 = 0$, and $\lim_{h\to 0} hP_1 = P$.

*Proof.* See Osburn and Bernstein (1993). □

Theorems 1 and 2 show that $Q_2 \to \infty$ and $P_1 \to \infty$, and thus the discrete-time Riccati equations become numerically ill-conditioned for fast sampling. Salgado *et al.* (1988), have shown that this problem can be overcome using normalizing methods.

*Corollary* 1. The discrete-time LQG gain $C_{c,h}$ has the form $C_{c,h} = [C_1 \ 0]$, where $C_1 \in \mathbb{R}^{l\times n}$ satisfies $\lim_{h\to 0} C_1 = C_c$.

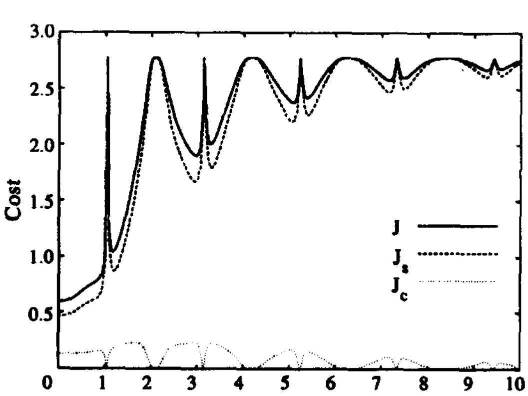


Fig. 1

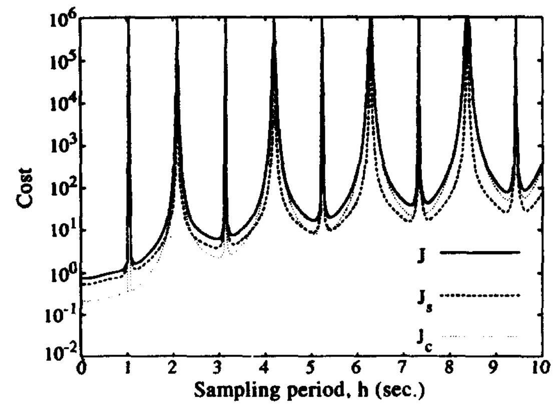


Fig. 2

*Theorem* 3. Let

$$\hat{Q}_h = \begin{bmatrix} \hat{Q}_1 & \hat{Q}_{12} \\ \hat{Q}_{12}^T & \hat{Q}_2 \end{bmatrix} \in \mathbb{R}^{(n+l)\times(n+l)}$$

satisfy the discrete-time Lyapunov equation (17). Then $\lim_{h\to 0} \hat{Q}_1 = \hat{Q}$, where $\hat{Q}$ satisfies the continuous-time Lyapunov equation (13).

*Corollary* 2. The discretization cost $\delta_h$ satisfies $\lim_{h\to 0} \delta_h = 0$.

*Corollary* 3. Consider the discrete-time optimal cost $J(G_{c,h}^{opt})$ given by (16) and the continuous-time optimal cost $J(G_c^{opt})$ given by (12). Then $\lim_{h\to 0} J(G_{c,h}^{opt}) = J(G_c^{opt})$.

*Proof.* See Osburn and Bernstein (1993). □

5. *Analysis of optimal performance as $h \to \infty$*

In this section we consider the asymptotic dependence of optimal performance on the sampling period $h$ for slow sampling, that is, for $h \to \infty$. As the plant approaches open-loop conditions, one would expect the cost to approach the open-loop cost.

*Proposition* 1. The discretization cost $\delta_h$ is a monotonically increasing function of $h$.

*Theorem* 4. Let $\delta_\infty \triangleq \lim_{h\to\infty} \delta_h$, which may or may not be finite, and assume that $\lim_{h\to\infty} Q_1$ and $\lim_{h\to\infty} \hat{Q}_1$ exist. Then

$$\delta_\infty = \lim_{h\to\infty} \operatorname{tr} \int_0^h e^{As} V_1 e^{A^T s}\, ds\, R_1.$$

Furthermore, $\delta_\infty$ is finite if and only if every eigenvalue of $A$ in the closed right half-plane is either uncontrollable by disturbance or unobservable by weighting. If, in particular, $A$ is Hurwitz them $\lim_{h\to\infty} J(G_{c,h}^{opt}) = \delta_\infty < \infty$.

*Proof.* See Osburn and Bernstein (1993). □

6. *Numerical examples*

*Example* 1. Consider the lightly damped system

$$A = \begin{bmatrix} 0 & 1 \\ -9.0001 & -0.02 \end{bmatrix},$$

$$B = E_2 = D_2^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and $D = 0$. The eigenvalues of $A$ are $-0.01 \pm 3j$, and $R_{12} = 0$ and $V_{12} = 0$. The continuous-time LQG controller yields the

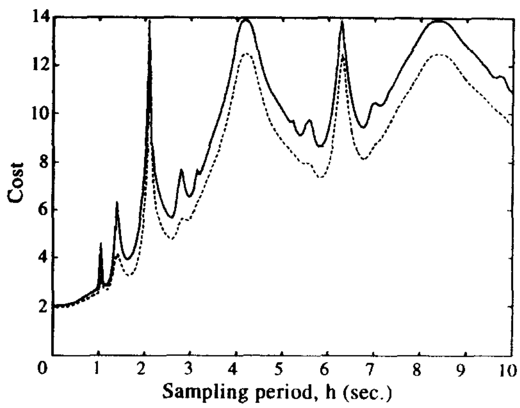


Fig. 3

cost $J(G_c^{opt}) = 0.5948$, while the open-loop cost for the plant is $J(0) = 2.7777$. The optimal cost as a function of $h$ is shown in Fig. 1, with the state and control costs $J_s$ and $J_c$ as described in Osburn and Bernstein (1993). It can be seen that as $h \rightarrow 0$, $J(G_{c,h}^{opt})$ approaches its continuous-time counterpart. Similarly, as $h \rightarrow \infty$, the cost approaches the open-loop cost $J(0)$. In Fig. 1 it can be seen that the cost $J(G_{c,h}^{opt})$ jumps to the open-loop cost at integer multiples of the Nyquist period. At these points the state cost rises to open loop and the control drops to zero.

*Example* 2. Next we turn our attention to an unstable plant. Consider the plant given in Example 1 with the eigenvalues reflected across the imaginary axis into the right half-plane. The optimal continuous-time LQG cost is $J(G_c^{opt}) = 0.7569$. Since the eigenvalues are reflected, it can be seen from Fig. 2 that the critical values of $h$ are the same as those in Example 1. At these critical values the cost is infinite, corresponding to an unstable system. For small $h$ the discretized costs approach the corresponding continuous-time costs, as expected.

*Example* 3. Consider the lightly damped system

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -9.0004 & -0.04 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2.2501 & -0.02 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$C = [1 \quad 0 \quad 1 \quad 0], \quad D = 0,$$

$$E_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_2 = D_2^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The eigenvaues of $A$ are $-0.02 \pm 3j$ and $-0.01 \pm 1.5j$. The locations of the maxima in Fig. 3 correspond to the loss of controllability predicted by Kalman *et al.* (1963), with the exception of additional maxima due to interactions of modes having approximately equal real part. Hence, although loss of controllability does not occur, the cost has a local maximum at the intermediate sampling intervals $h = \frac{4}{9}k\pi$. This point will be further explored in the next example, where all eigenvalues of the plant have equal real part.

*Example* 4. In Example 3 we observed local cost maxima at certain intermediate sampling periods. To explain this effect and further explore the modal interactions predicted by Kalman *et al.* (1963), we consider a system of the form given in Example 3 that has two lightly damped modes with eigenvalues $-0.01 \pm 3j$ and $-0.01 \pm 1.5j$. This case is similar to that in Example 3, except that the eigenvalues now have equal real part. As seen in Fig. 3, all of the maxima are located at $h = \frac{1}{3}k\pi$ and $h = \frac{4}{9}k\pi$, which is precisely where loss of controllability is predicted by Kalman *et al.* (1963). Comparing the cost plots in Fig. 3, it can be seen that the locations of the maxima agree, although some of the peaks are more pronounced, owing to loss of controllability.

When sampling faster than the Nyquist frequency, one might expect that performance gains can always be obtained by increasing the sampling rate. However, it can be shown that this is not true in general (Osburn and Bernstein, 1993). Thus, even below the Nyquist sampling rate, improved performance sometimes can be achieved by using a slower sample rate.

## 7. *Conclusion*

In this paper we have investigated the achievable performance of an exactly discretized sampled-data system with an LQG compensator for small and large sampling periods. We have shown that, with this exact conversion, the achievable performance of the sampled-data system approaches the continuous-time LQG cost when $h \rightarrow 0$ and the open-loop cost when $h \rightarrow \infty$. We have also shown by examples that the achievable performance is not necessarily monotonic with respect to the sampling interval, even below the Nyquist rate.

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