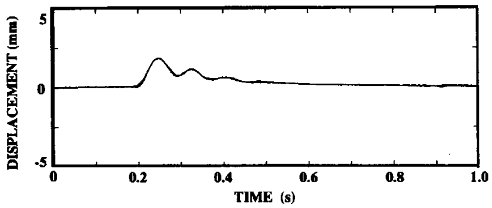

Fig. 10. Response to step disturbance with $\mathbf{K}_2$ (−34.30N).

- The closed-loop system with the μ controller K_2 achieves robust performance, while the closed-loop system with the H_∞controller K_1 does not.

VI. Conclusions

In this paper, we experimentally evaluated a controller designed by μ-synthesis methodology with an electromagnetic suspension system. We have obtained a nominal mathematical model as well as a set of plant models in which the real system is assumed to reside. With this set of the models we designed the control system to achieve robust performance objective utilizing μ-synthesis method.

First, four types of different model structures were derived based on the several idealizing assumptions for the real system. Second, for every model, the nominal value as well as the possible maximum and minimum values of each model parameter was determined by measurements and/or experiments. Third, a nominal model was naturally chosen. This model has the simplest model structure of all four models and makes use of nominal parameter values. Then, model perturbations were defined to account for additive unstructured uncertainties from such as neglected nonlinearities and model parameter errors. Fourth, we defined a family of plant models where the unstructured additive perturbation was employed. The method to model the plant as belonging to a family or set plays a key role for systematic robust control design. Fifth, we setup robust performance objective as a structured singular value test. Next, for the design, the D-K iteration approach was employed. Finally, the experimental results showed that the closed-loop system with the μ-controller achieves not only nominal performance and robust stability, but in addition robust performance.

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Parameter-Dependent Lyapunov Functions and the Popov Criterion in Robust Analysis and Synthesis

Wassim M. Haddad and Dennis S. Bernstein

***Abstract*— Many practical applications of robust feedback control involve constant real parameter uncertainty, whereas small gain or norm-bounding techniques guarantee robust stability against complex, frequency-dependent uncertainty, thus entailing undue conservatism. Since conventional Lyapunov bounding techniques guarantee stability with respect to time-varying perturbations, they possess a similar drawback. In this paper we develop a framework for parameter-dependent Lyapunov functions, a less conservative refinement of "fixed" Lyapunov functions. An immediate application of this framework is a reinterpretation of the classical Popov criterion as a parameter-dependent Lyapunov function. This result is then used for robust controller synthesis with full-order and reduced-order controllers.**

I. Introduction

The analysis and synthesis of robust feedback controllers entails a fundamental distinction between parametric and nonparametric uncertainty. Parametric uncertainty refers to plant uncertainty that is modeled as constant real parameters, whereas nonparametric uncertainty refers to uncertain transfer function gains modeled as complex frequency-dependent quantities. In the time domain, nonparametric uncertainty is manifested as time-varying uncertain real parameters.

The distinction between parametric and nonparametric uncertainty is critical to the achievable performance of feedback control systems. For example, in the problem of vibration suppression for flexible space structures, if stiffness matrix uncertainty is modeled as nonparametric uncertainty, then perturbations to the damping matrix will inadvertently be allowed. Predictions of stability and performance for given feedback gains will consequently be extremely conservative, thus limiting achievable performance [1]. Alternatively, this problem can be viewed by considering the classical analysis of Hill's equation (e.g., the Mathieu equation) which shows that time-varying parameter variations can destabilize a system even when the parameter variations are confined to a region in which constant variations are nondestabilizing. Consequently, a feedback controller designed for time-varying parameter variations will unnecessarily sacrifice performance when the uncertain real parameters are actually constant.

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The above distinction can also be illustrated by considering the central result of feedback control theory, namely, the small gain theorem, which guarantees robust stability by requiring that the loop gain (including desired weighting functions for loop shaping) be less than unity at all frequencies. The small gain theorem, however, does not make use of phase information in guaranteeing stability. In fact, the small gain theorem allows the loop transfer function to possess arbitrary phase at all frequencies, although in many applications at least some knowledge of phase is available [2]. Thus, small gain techniques such as H_∞ theory are generally conservative when phase information is available. More generally, since $|e^{j\phi}| = 1$ regardless of the phase angle ϕ, it can be expected that any robustness theory based upon norm bounds will suffer from the same shortcoming. Of course, every real parameter can be viewed as a complex parameter with phase angle $\phi = 0$ degrees or $\phi = 180$ degrees.

To some extent, phase information is accounted for by means of positivity theory [3]-[15] which is widely used to model passive systems such as flexible structures [16], [17]. In this theory, a positive real plant and strictly positive real uncertainty are both assumed to have phase less than 90 degrees so that the loop transfer function has less than 180 degrees of phase shift, hence guaranteeing robust stability in spite of gain uncertainty. Both gain and phase properties can be simultaneously accounted for by means of the circle criterion [15], [18]-[22] which yields the small gain theorem and positivity theorem as special cases. It is important to note that since positivity theory and the circle criterion can be obtained from small gain conditions by means of suitable transformations, they can be viewed as equivalent results from a mathematical point of view. The engineering ramifications of the ability to include phase information, however, can be significant [1].

The above discussion is further illuminated by means of Lyapunov function theory in [15]. Specifically, as pointed out in [15], a serious defect of conventional or fixed Lyapunov bounding theory is the fact that stability is guaranteed even if the plant uncertainty ΔA is a function of t. This observation follows from the fact that the Lyapunov derivative $\dot{V}(x(t)) = V_x(x(t))(A + \Delta A(t))x(t)$ need only be negative for each fixed value of t [15], [23]. Although this feature is desirable if ΔA is time varying, as discussed above, it leads to conservatism when ΔA is actually constant. This defect can be remedied, however, by utilizing an alternative approach, which is consistent with Lyapunov function bounding techniques, based upon parameter-dependent Lyapunov functions. The idea behind parameter-dependent Lyapunov functions is to allow the Lyapunov function to be a function of the uncertainty ΔA. In the usual case, $V(x) = x^{\mathrm{T}}Px$, P is a single, fixed matrix, whereas the parameter-dependent Lyapunov function $V_{\Delta A}(x) = x^{\mathrm{T}}P(\Delta A)x$ represents a family of Lyapunov functions.

The concept of a parameter-dependent Lyapunov functions is not new to this paper. Specifically, a parameter-dependent Lyapunov function of the form $V(x) = x^{\mathrm{T}}P(\lambda_1, \cdots, \lambda_r)x$, where $P(\lambda_1, \cdots, \lambda_r) = \sum_{i=1}^r \lambda_i P_i$, is considered in [24]. In this case the matrices P_i correspond to the vertices of a polytope of uncertain matrices with vertices $A_1, \cdots, A_r$. More recently, [25] considers a Lyapunov function with matrix $P(\sigma_1, \cdots, \sigma_r) = P_0 + \sum_{i=1}^r \sigma_i P_i$, where P_0 corresponds to the nominal system and the P_i are "first-order perturbations" of P_0. Numerical techniques are used to determine P_i and the range of robust stability. Both [24] and [25] discuss potential advantages of parameter-dependent Lyapunov functions over fixed Lyapunov functions.

The goal of the present paper is to develop robust analysis and synthesis techniques that exploit the fact that the classical Popov criterion [26] is based upon a parameter-dependent Lyapunov function. Indeed, recall that the Popov criterion is based upon the Lur'e-Postnikov Lyapunov function

$$V_\phi(x) = x^{\mathrm{T}}Px + N\int_0^y \phi(\sigma)\mathrm{d}\sigma \tag{1.22}$$

where $y = Cx$ and $\phi(\cdot)$ is a scalar memoryless time-invariant nonlinearity in the sector $[0, k]$, that is, $0 \leq \phi(y)y \leq ky^2$. Specializing to the linear uncertainty case $\phi(y) = Fy$, where $0 \leq F \leq k$, yields

$$\begin{aligned} V_F(x) &= x^{\mathrm{T}}Px + N\int_0^y F\sigma\mathrm{d}\sigma = x^{\mathrm{T}}Px + NF\frac{y^2}{2} \\ &= x^{\mathrm{T}}[P + \tfrac{1}{2}NFC^{\mathrm{T}}C]x = x^{\mathrm{T}}P(F)x. \end{aligned}$$

This form appears in [10, pp. 84-89] and was discussed in the context of robust analysis in [15].

For practical purposes the form of the parameter-dependent Lyapunov function $V_F(x)$ is useful since the presence of F restricts the allowable time-varying uncertain parameters [27]. That is, if $F(t)$ were permitted, then terms involving $\dot{F}(t)$ would arise and potentially subvert the negative definiteness of $\dot{V}_F(x)$.

This paper has four specific goals: 1) to provide a general framework for parameter-dependent Lyapunov functions; 2) to obtain a generalized multivariable version of the Popov criterion for linear matrix uncertainty ΔA (the classical Popov criterion is limited to scalar or diagonal nonlinearities) along with H_2 robust performance bounds; 3) to provide explicit uncertainty bounds for the multivariable Popov criterion in terms of a single Riccati equation that can be used for robust controller synthesis; and 4) to develop robust controller synthesis techniques based upon the multivariable Popov criterion with applications to full-order and reduced-order controllers.

Notation:

- $\mathbb{R}, \mathbb{R}^{r\times s}, \mathbb{R}^r$ —real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$
- $\mathbb{C}, \mathbb{C}^{r\times s}, \mathbb{C}^r$ —complex numbers, $r \times s$ complex matrices, $\mathbb{C}^{r\times 1}$
- $\mathbb{E}, \mathrm{tr}, 0_{r\times s}, \otimes$ —expectation, trace, $r \times s$ zero matrix, Kronecker product
- $I_r, ()^{\mathrm{T}}, ()^*$ —$r \times r$ identity, transpose, complex conjugate transpose
- $()^{-\mathrm{T}}, ()^{-*}$ —inverse transpose, complex conjugate inverse transpose
- $\mathbb{S}^r, \mathbb{N}^r, \mathbb{P}^r$ —$r \times r$ symmetric, nonnegative-definite, positive-definite matrices
- $Z_1 \leq Z_2, Z_1 < Z_2$ —$Z_2 - Z_1 \in \mathbb{N}^r$, $Z_2 - Z_1 \in \mathbb{P}^r$, $Z_1, Z_2 \in \mathbb{S}^r$
- $\|Z\|_{\mathrm{F}}, \|G(s)\|_2$ —$[\mathrm{tr}ZZ^*]^{1/2}$, $[(1/2\pi)\int_{-\infty}^{\infty} \|G(\jmath\omega)\|_{\mathrm{F}}^2 \mathrm{d}\omega]^{1/2}$

II. Robust Stability and Performance Problems: Analysis

Let $\mathcal{U} \subset \mathbb{R}^{n\times n}$ denote a set of perturbations ΔA of a given nominal dynamics matrix $A \in \mathbb{R}^{n\times n}$. Within the context of robustness analysis, it is assumed that A is asymptotically stable and $0 \in \mathcal{U}$. We begin by considering the stability of $A + \Delta A$ for all $\Delta A \in \mathcal{U}$.

Robust Stability Problem: Determine whether the linear system

$$\dot{x}(t) = (A + \Delta A)x(t), \quad t \in [0, \infty) \tag{2.1}$$

is asymptotically stable for all $\Delta A \in \mathcal{U}$.

To consider the problem of robust performance, we introduce an external disturbance model involving white noise signals as in standard LQG (H_2) theory. The robust performance problem concerns the worst-case H_2 norm, that is, the worst-case over $\mathcal{U}$ of the expected value of a quadratic form involving outputs $z(t) = Ex(t)$, where $E \in \mathbb{R}^{q\times n}$, when the system is subjected to a standard white noise disturbance $w(t) \in \mathbb{R}^d$ with weighting $D \in \mathbb{R}^{n\times d}$.

Robust Performance Problem: For the disturbed linear system

$$\dot{x}(t) = (A + \Delta A)x(t) + Dw(t),\ t \in [0, \infty), \tag{2.2}$$

$$z(t) = Ex(t) \tag{2.3}$$

where $w(\cdot)$ is a zero-mean d-dimensional white noise signal with intensity I_d, determine a performance bound β satisfying

$$J(\mathcal{U}) \triangleq \sup_{\Delta A \in \mathcal{U}} \limsup_{t\to\infty} \mathbb{E}\{\|z(t)\|_2^2\} \le \beta. \tag{2.4}$$

In Section VI, (2.2) will denote a control system in closed-loop configuration subjected to external white noise disturbances and for which $z(t)$ denotes the state and control regulation error.

Next, we express the H_2 performance measure in terms of the observability Gramian for the pair $(A + \Delta A, E)$. For convenience define the $n \times n$ nonnegative-definite matrices $R \triangleq E^{\mathrm{T}}E, V \triangleq DD^{\mathrm{T}}$.

Lemma 2.1: Suppose $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Then

$$J(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \operatorname{tr} P_{\Delta A} V = \sup_{\Delta A \in \mathcal{U}} \|G_{\Delta A}(s)\|_2^2 \tag{2.5}$$

where $P_{\Delta A} \in \mathbb{R}^{n\times n}$ is the unique, nonnegative-definite solution to

$$0 = (A + \Delta A)^{\mathrm{T}} P_{\Delta A} + P_{\Delta A}(A + \Delta A) + R \tag{2.6}$$

and $G_{\Delta A}(s) \triangleq E[sI - (A + \Delta A)]^{-1}D$.

III. Robust Stability and Performance via Parameter-Dependent Lyapunov Functions

The key step in obtaining robust stability and performance is to bound the uncertain terms $\Delta A^{\mathrm{T}} P_{\Delta A} + P_{\Delta A}\Delta A$ in the Lyapunov equation (2.6) by means of a parameter-dependent bounding function $\Omega(P, \Delta A)$ which guarantees robust stability by means of a family of Lyapunov functions. This procedure corresponds to the construction of a parameter-dependent Lyapunov function which constrains the class of allowable time-varying uncertainties. The following result forms the basis for all later developments.

Theorem 3.1: Let $\Omega_0\colon \mathbb{N}^n \to \mathbb{S}^n$ and $P_0\colon \mathcal{U} \to \mathbb{S}^n$ be such that

$$\Delta A^{\mathrm{T}} P + P\Delta A \le \Omega_0(P) - [(A + \Delta A)^{\mathrm{T}} P_0(\Delta A) + P_0(\Delta A)(A + \Delta A)],\ \Delta A \in \mathcal{U}, P \in \mathbb{N}^n \tag{3.1}$$

and suppose there exists $P \in \mathbb{N}^n$ satisfying

$$0 = A^{\mathrm{T}} P + PA + \Omega_0(P) + R \tag{3.2}$$

and such that $P + P_0(\Delta A)$ is nonnegative definite for all $\Delta A \in \mathcal{U}$. Then

$$(A + \Delta A, E) \text{ is detectable},\quad \Delta A \in \mathcal{U} \tag{3.3}$$

if and only if

$$A + \Delta A \text{ is asymptotically stable},\quad \Delta A \in \mathcal{U}. \tag{3.4}$$

In this case

$$P_{\Delta A} \le P + P_0(\Delta A),\quad \Delta A \in \mathcal{U} \tag{3.5}$$

where $P_{\Delta A}$ is given by (2.6). Therefore

$$J(\mathcal{U}) \le \operatorname{tr} PV + \sup_{\Delta A \in \mathcal{U}} \operatorname{tr} P_0(\Delta A)V. \tag{3.6}$$

If, in addition, there exists $\bar{P}_0 \in \mathbb{S}^n$ such that

$$P_0(\Delta A) \le \bar{P}_0,\quad \Delta A \in \mathcal{U} \tag{3.7}$$

then

$$J(\mathcal{U}) \le \operatorname{tr}[(P + \bar{P}_0)V]. \tag{3.8}$$

Proof: Note that in (3.1), P denotes an arbitrary element of $\mathbb{N}^n$, whereas in (3.2) P denotes a specific solution of the modified Lyapunov equation (3.2). This minor abuse of notation considerably simplifies the presentation. Now, note that for all $\Delta A \in \mathbb{R}^{n\times n}$, (3.2) is equivalent to

$$0 = (A+\Delta A)^{\mathrm{T}} P + P(A+\Delta A) + \Omega_0(P) - (\Delta A^{\mathrm{T}} P + P\Delta A) + R. \tag{3.9}$$

Adding and subtracting $(A + \Delta A)^{\mathrm{T}} P_0(\Delta A) + P_0(\Delta A)(A + \Delta A)$ to (3.9) yields

$$\begin{aligned}0 =&(A + \Delta A)^{\mathrm{T}}(P + P_0(\Delta A)) + (P + P_0(\Delta A))(A + \Delta A)\\ &+ \Omega_0(P) - [(A + \Delta A)^{\mathrm{T}} P_0(\Delta A) + P_0(\Delta A)(A + \Delta A)]\\ &- (\Delta A^{\mathrm{T}} P + P\Delta A) + R.\end{aligned} \tag{3.10}$$

Hence, by assumption, (3.10) has a solution $P \in \mathbb{N}^n$ for all $\Delta A \in \mathbb{R}^{n\times n}$. If ΔA is restricted to the set $\mathcal{U}$, then, by (3.1), $\Omega_0(P) - [(A + \Delta A)^{\mathrm{T}} P_0(\Delta A) + P_0(\Delta A)(A + \Delta A)] - (\Delta A^{\mathrm{T}} P + P\Delta A)$ is nonnegative definite. Thus if condition (3.3) holds for all $\Delta A \in \mathcal{U}$, then Theorem 3.6 of [28] implies $(A + \Delta A, [R + \Omega(P, \Delta A) - (\Delta A^{\mathrm{T}} P + P\Delta A)]^{1/2})$ is detectable for all $\Delta A \in \mathcal{U}$, where

$$\Omega(P, \Delta A) \triangleq \Omega_0(P) - [(A + \Delta A)^{\mathrm{T}} P_0(\Delta A) + P_0(\Delta A)(A + \Delta A)]. \tag{3.11}$$

It now follows from (3.10) and Lemma 12.2 of [28] that $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Conversely, if $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$, then (3.3) is immediate. Now, subtracting (2.6) from (3.10) yields

$$\begin{aligned}0 =&(A + \Delta A)^{\mathrm{T}}(P + P_0(\Delta A) - P_{\Delta A}) + (P + P_0(\Delta A) - P_{\Delta A})\\ &\times (A + \Delta A) + \Omega_0(P) - [(A + \Delta A)^{\mathrm{T}} P_0(\Delta A) + P_0(\Delta A)\\ &\times (A + \Delta A)] - (\Delta A^{\mathrm{T}} P + P\Delta A),\quad \Delta A \in \mathcal{U}\end{aligned} \tag{3.12}$$

or, equivalently, since $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$

$$\begin{aligned}P + P_0(\Delta A) - P_{\Delta A} =& \int_0^\infty e^{(A+\Delta A)^{\mathrm{T}} t}[\Omega(P, \Delta A)\\ &- (\Delta A^{\mathrm{T}} P + P\Delta A)]e^{(A+\Delta A)t}dt\\ &\ge 0, \Delta A \in \mathcal{U}\end{aligned} \tag{3.13}$$

which implies (3.5). The performance bounds (3.6) and (3.8) are now an immediate consequence of (2.5), (3.5), and (3.7). □

Note that with $\Omega(P, \Delta A)$ defined by (3.11) condition (3.1) can be written as

$$\Delta A^{\mathrm{T}} P + P\Delta A \le \Omega(P, \Delta A),\quad \Delta A \in \mathcal{U},\quad P \in \mathbb{N}^n \tag{3.1$'$}$$

where $\Omega(P, \Delta A)$ is a function of the uncertainty ΔA. For convenience we shall say that $\Omega(\cdot, \cdot)$ is a parameter-dependent Ω-bound, which is consistent with [29]. One can recover the standard guaranteed cost bound or parameter-independent Ω-bound by setting $P_0(\Delta A) \equiv 0$ so that $\Omega(P, \Delta A) \equiv \Omega_0(P)$ and therefore $\Delta A^{\mathrm{T}} P + P\Delta A \le \Omega_0(P)$ for all $\Delta A \in \mathcal{U}$. Finally, since we do not assume that $P_0(0) = 0$, it follows that $\Omega_0(P)$ need not be nonnegative definite. If, however, $P_0(0) = 0$, then it follows from (3.1) with $\Delta A = 0$ that $\Omega_0(P) \ge 0$ for all nonnegative-definite P. To apply Theorem 3.1, we first specify functions $\Omega_0(\cdot)$ and $P_0(\cdot)$ and an uncertainty set $\mathcal{U}$ such that (3.1)$'$ holds. If the existence of a nonnegative-definite solution P to (3.2) can be determined analytically or numerically and the detectability condition (3.3) is satisfied, then robust stability is guaranteed and the performance bound (3.8) can be computed.

Finally, we show that a parameter-dependent Ω-bound establishing robust stability is equivalent to the existence of a parameter-dependent Lyapunov function which also establishes robust stability. To show this, assume there exists a positive-definite solution to (3.2), let

$P_0: \mathcal{U} \to \mathbb{N}^n$, and define the parameter-dependent Lyapunov function $V_{\Delta A}(x) \triangleq x^{\mathrm{T}}(P + P_0(\Delta A))x$. Note that since P is positive definite and $P_0(\Delta A)$ is nonnegative definite, $V_{\Delta A}(x)$ is positive definite. The corresponding Lyapunov derivative is given by

$$\begin{aligned}\dot{V}_{\Delta A}(x) = &- x^{\mathrm{T}}[\Omega_0(P) - \{\Delta A^{\mathrm{T}} P + P\Delta A + (A+\Delta A)^{\mathrm{T}} \\ &\times P_0(\Delta A) + P_0(\Delta A)(A+\Delta A)\} + R]x. \end{aligned} \quad (3.14)$$

Thus, using (3.1) it follows that $\dot{V}_{\Delta A}(x) \leq 0$ so that $A + \Delta A$ is stable in the sense of Lyapunov. Asymptotic stability follows from the invariant set theorem.

IV. Construction of Parameter-Dependent Lyapunov Functions

We now assign explicit structure to the uncertainty set $\mathcal{U}$ and the parameter-dependent bounding function $\Omega(\cdot,\cdot)$. Specifically, let

$$\mathcal{U} \triangleq \{\Delta A \in \mathbb{R}^{n\times n} : \Delta A = B_0 F C_0, F \in \mathcal{F}\} \quad (4.1)$$

where $\mathcal{F}$ satisfies

$$\mathcal{F} \subseteq \hat{\mathcal{F}} \triangleq \{F \in \mathbb{R}^{m_0 \times m_0} : 0 \leq F \leq M\} \quad (4.2)$$

and where $B_0 \in \mathbb{R}^{n\times m_0}$, $C_0 \in \mathbb{R}^{m_0 \times n}$ are fixed matrices denoting the structure of the uncertainty, $F \in \mathbb{R}^{m_0\times m_0}$ is an uncertain symmetric matrix, and $M \in \mathbb{R}^{m_0 \times m_0}$ is a given positive-definite matrix. Note that $\mathcal{F}$ may be equal to $\hat{\mathcal{F}}$, although, for generality, $\mathcal{F}$ may be a specified proper subset of $\hat{\mathcal{F}}$. For example, $\hat{\mathcal{F}}$ may consist of block-structured matrices $F = \text{block-diag}(I_{\ell_1} \otimes F_1, I_{\ell_2} \otimes F_2, \cdots, I_{\ell_r} \otimes F_r)$. Note that if $F = \text{block-diag}(F_1, F_2, \cdots, F_{m_0})$ and $M = \text{block-diag}(M_1, \cdots, M_{m_0})$, then $0 \leq F_i \leq M_i$, $i = 1, \cdots, m_0$. Finally, we assume that $0 \in \mathcal{F}$ and $M \in \mathcal{F}$.

Next, we provide an equivalent characterization of the set $\hat{\mathcal{F}}$.

Lemma 4.1: Let $F \in \mathbb{S}^{m_0}$ and $M \in \mathbb{P}^{m_0}$. Then $FM^{-1}F \leq F$ if and only if $0 \leq F \leq M$.

For $\mathcal{U}$ given by (4.1), the parameter-dependent bound $\Omega(\cdot,\cdot)$ satisfying (3.12) can now be given a concrete form. Since the elements ΔA in $\mathcal{U}$ are parameterized by the elements F in $\mathcal{F}$, we shall write $P_0(F)$ in place of $P_0(\Delta A)$. Finally, we define the sets $\mathcal{N}_{\mathrm{s}}$ and $\mathcal{N}_{\mathrm{nd}}$ such that the product of the transpose of every matrix in $\mathcal{N}_{\mathrm{s}}$ (resp., $\mathcal{N}_{\mathrm{nd}}$) and every matrix in $\hat{\mathcal{F}}$ is symmetric (respectively, nonnegative definite) by

$$\mathcal{N}_{\mathrm{s}} \triangleq \{N \in \mathbb{R}^{m_0\times m_0} : FN = N^{\mathrm{T}}F, F \in \hat{\mathcal{F}}\}$$

and

$$\mathcal{N}_{\mathrm{nd}} \triangleq \{N \in \mathcal{N}_{\mathrm{s}} : FN \geq 0, F \in \hat{\mathcal{F}}\}.$$

Finally, Lemma 4.1 of [30] implies that there exists $\mu \in \mathbb{N}^{m_0}$ such that $FN \leq \mu$ for all $F \in \mathcal{F}$.

Proposition 4.1: Let $N \in \mathcal{N}_{\mathrm{s}}$ and

$$(M^{-1} - NC_0B_0) + (M^{-1} - NC_0B_0)^{\mathrm{T}} > 0. \quad (4.3)$$

Furthermore, let $\mathcal{U}$ be defined by (4.1) and define $\Omega_0(\cdot)$ and $P_0(\cdot)$ by

$$\begin{aligned}\Omega_0(P) =&(C_0 + NC_0A + B_0^{\mathrm{T}}P)^{\mathrm{T}}[(M^{-1} - NC_0B_0) \\ &+ (M^{-1} - NC_0B_0)^{\mathrm{T}}]^{-1}(C_0 + NC_0A + B_0^{\mathrm{T}}P) \end{aligned} \quad (4.4)$$

and

$$P_0(F) = C_0^{\mathrm{T}} F N C_0. \quad (4.5)$$

Then (3.1) is satisfied.

Proof: Since by (4.3) $(M^{-1} - NC_0B_0) + (M^{-1} - NC_0B_0)^{\mathrm{T}}$ is positive definite and by Lemma 4.1 $F - FM^{-1}F$ is nonnegative definite, it follows that

$$\begin{aligned}0 \leq&[(C_0 + NC_0A + B_0^{\mathrm{T}}P) - [(M^{-1} - NC_0B_0) \\ &+ (M^{-1} - NC_0B_0)^{\mathrm{T}}]FC_0]^{\mathrm{T}}[(M^{-1} - NC_0B_0) \\ &+ (M^{-1} - NC_0B_0)^{\mathrm{T}}]^{-1}[(C_0 + NC_0A + B_0^{\mathrm{T}}P) \\ &- [(M^{-1} - NC_0B_0) + (M^{-1} - NC_0B_0)^{\mathrm{T}}]FC_0] \\ &+ 2C_0^{\mathrm{T}}[F - FM^{-1}F]C_0 \\ =&\Omega_0(P) - C_0^{\mathrm{T}}F(C_0 + NC_0A + B_0^{\mathrm{T}}P) \\ &- (C_0^{\mathrm{T}} + A^{\mathrm{T}}C_0^{\mathrm{T}}N^{\mathrm{T}} + PB_0)FC_0 + C_0^{\mathrm{T}}F[(M^{-1} - NC_0B_0) \\ &+ (M^{-1} - NC_0B_0)^{\mathrm{T}}]FC_0 + 2C_0^{\mathrm{T}}[F - FM^{-1}F]C_0 \\ =&\Omega_0(P) - C_0^{\mathrm{T}}FB_0^{\mathrm{T}}P - PB_0FC_0 \\ &- C_0^{\mathrm{T}}FNC_0A - A^{\mathrm{T}}C_0^{\mathrm{T}}N^{\mathrm{T}}FC_0 \\ &- C_0^{\mathrm{T}}FNC_0B_0FC_0 - C_0^{\mathrm{T}}FB_0^{\mathrm{T}}C_0^{\mathrm{T}}N^{\mathrm{T}}FC_0 \\ =&\Omega_0(P) - [(A+\Delta A)^{\mathrm{T}}P_0(F) + P_0(F)(A+\Delta A)] \\ &- [\Delta A^{\mathrm{T}}P + P\Delta A]\end{aligned}$$

which proves (3.1) with $\mathcal{U}$ given by (4.1). □

Next, using Theorem 3.1 and Proposition 4.1 we have the following immediate result.

Theorem 4.1: Let $N \in \mathcal{N}_{\mathrm{nd}}$ and assume (4.3) is satisfied. Furthermore, suppose there exists a nonnegative-definite matrix P satisfying

$$\begin{aligned}0 =&A^{\mathrm{T}}P + PA + (C_0 + NC_0A + B_0^{\mathrm{T}}P)^{\mathrm{T}}[(M^{-1} - NC_0B_0) \\ &+ (M^{-1} - NC_0B_0)^{\mathrm{T}}]^{-1}(C_0 + NC_0A + B_0^{\mathrm{T}}P) + R. \end{aligned} \quad (4.6)$$

Then

$$(A + \Delta A, E) \text{ is detectable}, \quad \Delta A \in \mathcal{U} \quad (4.7)$$

if and only if

$$A + \Delta A \text{ is asymptotically stable}, \quad \Delta A \in \mathcal{U}. \quad (4.8)$$

In this case, if $\mu \in \mathbb{N}^{m_0}$ satisfies $FN \leq \mu$ for all $F \in \mathcal{F}$, then

$$J(\mathcal{U}) \leq \mathrm{tr}[(P + C_0^{\mathrm{T}}\mu C_0)V]. \quad (4.9)$$

Proof: The result is a direct specialization of Theorem 3.1 using Proposition 4.1 with $P_0(\Delta A) = C_0^{\mathrm{T}}FNC_0$. Since, by assumption, $FN \geq 0$ for all $F \in \mathcal{F}$, it follows that $P + P_0(F)$ is nonnegative definite for all $F \in \mathcal{F}$ as required by Theorem 3.1. □

Note that asymptotic stability in Theorem 4.1 is guaranteed by the parameter-dependent Lyapunov function $V_F(x) = x^{\mathrm{T}}(P + C_0^{\mathrm{T}}FNC_0)x$.

Remark 4.1: The condition $FN = N^{\mathrm{T}}F, F \in \mathcal{F}$ is analogous to the commuting assumption between the D-scales and Δ blocks in μ-analysis which accounts for structure in the uncertainty F. Note that there always exists such a matrix N even if $F \in \mathcal{F}$ is not diagonal. For example, if $F = F_1 I_{m_0}$, where F_1 is a scalar uncertainty, then N can be an arbitrary symmetric matrix. Alternatively, if F is nondiagonal, then one can choose $N = N_0 I_{m_0}$, where N_0 is a scalar. Finally, F and N may be block diagonal with commuting blocks situated on the diagonal. Characterization of the optimal multiplier N for robust controller analysis and synthesis is given in Section VI.

Remark 4.2 Standard loop-shifting techniques [31] can be used to consider uncertainties with upper and lower bounds of the form $M_1 \leq F \leq M_2$, where $F \in \hat{\mathcal{F}}$ and $M_1, M_2 \in \mathbb{S}^{m_0}$. In this case, Proposition 4.1 holds with F, A, and M replaced by $F - M_1, A + B_0M_1C_0$, and $M_2 - M_1$, respectively. Similar modifications can be made to Theorem 4.1.

Next, we use results from positivity theory to guarantee the existence of a positive-definite solution to (4.6). Let $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ denote a state space realization of a transfer function $G(s)$, that is, $G(s) = C(sI-A)^{-1}B+D$. The notation "$\overset{\min}{\sim}$" denotes a minimal realization.

Lemma 4.2. (Positive Real Lemma [4], [9]):

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

is positive real if and only if there exist matrices P, L, and W with P positive definite such that

$$0 = A^{\mathrm{T}}P + PA + L^{\mathrm{T}}L, \tag{4.10}$$
$$0 = PB - C^{\mathrm{T}} + L^{\mathrm{T}}W, \tag{4.11}$$
$$0 = D + D^{\mathrm{T}} - W^{\mathrm{T}}W. \tag{4.12}$$

Next, we show that if $D+D^{\mathrm{T}} > 0$ then (4.10)-(4.12) yield a single Riccati equation characterizing positive realness. For the statement of this result recall that a square transfer function $G(s)$ is strongly positive real if it is strictly positive real [15] and $D + D^{\mathrm{T}} > 0$.

Lemma 4.3 [14]: Let $G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Then $G(s)$ is strongly positive real if and only if $D + D^{\mathrm{T}} > 0$ and there exist positive-definite matrices P and R such that

$$0 = A^{\mathrm{T}}P+PA+(C-B^{\mathrm{T}}P)^{\mathrm{T}}(D+D^{\mathrm{T}})^{-1}(C-B^{\mathrm{T}}P)+R. \tag{4.13}$$

We now use Lemma 4.3 to obtain a sufficient condition for the existence of a solution to (4.6).

Theorem 4.2: Let

$$\mathcal{G}(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & -B_0 \\ \hline C_0 + NC_0A & M^{-1} - NC_0B_0 \end{array}\right].$$

Then $\mathcal{G}(s)$ is strongly positive real if and only if there exist positive definite matrices P and R satisfying (4.6).

Finally, we specialize Proposition 4.1 and Theorem 4.1 to the case in which $N = 0$ and $M = D_0^{-1}$, where $D_0 + D_0^{\mathrm{T}} > 0$. In this case we have the following result.

Proposition 4.2: Let $D_0 \in \mathbb{R}^{m_0 \times m_0}$ be such that $D_0 + D_0^{\mathrm{T}} > 0$. Furthermore, let $\mathcal{U}$ be defined by (4.1) with $M = D_0^{-1}$, let $P_0(F) = 0$, and define $\Omega_0(\cdot)$ by

$$\Omega_0(P) = (C_0 + B_0^{\mathrm{T}}P)^{\mathrm{T}}(D_0 + D_0^{\mathrm{T}})^{-1}(C_0 + B_0^{\mathrm{T}}P). \tag{4.14}$$

Then (3.1) is satisfied.

Since $P_0(F) = 0$, the case $N = 0$ corresponds to a parameter-independent Ω-bound. Hence, it follows from Theorem 3.1 that if there exists a nonnegative-definite matrix P satisfying

$$0 = A^{\mathrm{T}}P + PA + (C_0 + B_0^{\mathrm{T}}P)^{\mathrm{T}}(D_0 + D_0^{\mathrm{T}})^{-1}(C_0 + B_0^{\mathrm{T}}P) + R \tag{4.15}$$

then $(A + \Delta A, E)$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Furthermore, it follows from Lemma 4.3 that the existence of a positive-definite matrix P satisfying (4.15) implies that

$$G(s) \sim \left[\begin{array}{c|c} A & -B_0 \\ \hline C_0 & D_0 \end{array}\right]$$

is strongly positive real. Hence the parameter-independent Ω-bound (4.14) guarantees robust stability in the presence of positive real (but otherwise unknown) plant uncertainty. The situation is analogous to H_∞ bounded real theory, which also depends upon a parameter independent Ω-bound.

V. Connections to the Popov Criterion

In this section we demonstrate connections between the parameter-dependent Lyapunov function obtained in Section 4 and the classical multivariable Popov criterion. Traditionally, the Popov criterion is stated for component-decoupled time-invariant sector-bounded nonlinearities $\phi(y)$. We state the Popov criterion for this case and then specialize to the case of linear uncertainty. Hence let $M \in \mathbb{R}^{m_0 \times m_0}$ be a given positive-definite matrix and define

$$\Phi \triangleq \{\phi : \mathbb{R}^{m_0} \to \mathbb{R}^{m_0} : \ \phi^{\mathrm{T}}(y)[M^{-1}\phi(y) - y] \le 0, \ y \in \mathbb{R}^{m_0}, \text{ and } \phi(y) = [\phi_1(y_1), \phi_2(y_2), \cdots, \phi_{m_0}(y_{m_0})]^{\mathrm{T}}\}. \tag{5.1}$$

If $M = \operatorname{diag}(M_1, \cdots, M_{m_0})$ is diagonal, then the sector condition characterizing Φ is implied by the scalar sector conditions $0 \le \phi_i(y_i)y_i \le M_i y_i^2, y_i \in \mathbb{R}, i = 1, \cdots, m_0$.

Theorem 5.1. (The Popov Criterion) [15]: Suppose there exists a nonnegative-definite matrix $N = \operatorname{diag}(N_1, \cdots, N_{m_0})$ such that $M^{-1} + (I + Ns)G(s)$ is strongly positive real, where $G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$. Then, for all $\phi(\cdot) \in \Phi$, the negative feedback interconnection of $G(s)$ and $\phi(\cdot)$ is asymptotically stable with Lyapunov function

$$V_\phi(x) = x^{\mathrm{T}}Px + 2\sum_{i=1}^{m_0} \int_0^{y_i} \phi_i(\sigma)N_i \mathrm{d}\sigma. \tag{5.2}$$

Next, we specialize Theorem 5.1 to the case of constant linear parameter uncertainty. Specifically, consider the system $\dot{x}(t) = (A + \Delta A)x(t)$, where $\Delta A \in \mathcal{U}$ and $\mathcal{U}$ is defined by

$$\mathcal{U} \triangleq \{\Delta A : \Delta A = -BFC, \quad F = \operatorname{diag}(F_1, F_2, \cdots, F_{m_0}), \quad 0 \le F_i \le M_i, \quad i = 1, \cdots, m_0\}.$$

By setting $\phi(y) = Fy = FCx$ Theorem 5.1 guarantees that $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$.

It has thus been shown that in the special case that F and N are diagonal nonnegative-definite matrices, Theorem 4.1 (with B_0 replaced by $-B_0$) specializes to the Popov criterion when applied to linear parameter uncertainty. This is not surprising since the Lyapunov function (5.2) that establishes robust stability has the form

$$V_F(x) = x^{\mathrm{T}}Px + 2\sum_{i=1}^{m_0} \int_0^{y_i} F_i\sigma N_i \mathrm{d}\sigma, \quad y_i = (C_0x)_i \tag{5.3}$$

or, equivalently

$$V_F(x) = x^{\mathrm{T}}Px + x^{\mathrm{T}}C_0^{\mathrm{T}}FNC_0x = x^{\mathrm{T}}Px + \sum_{i=1}^{m_0} F_iN_i x^{\mathrm{T}}C_0^{\mathrm{T}}C_0x \tag{5.4}$$

which is a specialization of the parameter-dependent Lyapunov function considered in Section IV to the case of diagonal uncertainty F. The results of Section IV, however, allowed nondiagonal uncertain matrices F, which cannot be addressed by means of the nonlinear theory. Finally, note that the uncertain parameters F are not allowed to be arbitrarily time-varying, which is consistent with the fact that the Popov criterion is restricted to time-invariant nonlinearities.

VI. Robust Controller Synthesis via Parameter-Dependent Lyapunov Functions: Fixed-Order Dynamic Compensation

In this section we consider robust stability and performance with dynamic output-feedback controllers. For generality, the compensator dimension n_c may be less than the plant order n. Define $\tilde{n} \triangleq n + n_c$, where $n_c \le n$.

Dynamic Robust Stability and Performance Problem: Given the nth-order stabilizable and detectable plant with constant structured real-valued plant parameter variations

$$\dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + D_1 w(t), t \geq 0, \quad (6.1)$$
$$y(t) = Cx(t) + D_2 w(t) \quad (6.2)$$

where $u(t) \in \mathbb{R}^m, w(t) \in \mathbb{R}^d$, and $y(t) \in \mathbb{R}^\ell$, determine an n_cth-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \quad (6.3)$$
$$u(t) = C_c x_c(t) \quad (6.4)$$

that satisfies the following design criteria:

i) The closed-loop system (6.1)-(6.4) is asymptotically stable for all $\Delta A \in \mathcal{U}$ and
ii) The performance functional

$$J(A_c, B_c, C_c) \triangleq \sup_{\Delta A \in \mathcal{U}} \limsup_{t\to\infty} \frac{1}{t} \mathbb{E}\{\int_0^t [x^{\mathrm{T}}(s)R_1 x(s) + u^{\mathrm{T}}(s)R_2 u(s)]\mathrm{d}s\} \quad (6.5)$$

is minimized.

For each uncertain variation $\Delta A \in \mathcal{U}$, the closed-loop system (6.1)-(6.4) can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{D}w(t), \quad t \geq 0 \quad (6.6)$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}, \Delta\tilde{A} = \begin{bmatrix} \Delta A & 0_{n\times n_c} \\ 0_{n_c \times n} & 0_{n_c\times n_c} \end{bmatrix}$$

and where the closed-loop disturbance $\tilde{D}w(t)$ has intensity $\tilde{V} = \tilde{D}\tilde{D}^{\mathrm{T}}$, where $\tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix}, \tilde{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^{\mathrm{T}} \end{bmatrix}, V_1 = D_1 D_1^{\mathrm{T}}, V_{12} = D_1 D_2^{\mathrm{T}} = 0, V_2 = D_2 D_2^{\mathrm{T}}$. The closed-loop system uncertainty $\Delta\tilde{A}$ has the form $\Delta\tilde{A} = \tilde{B}_0 F \tilde{C}_0$, where $\tilde{B}_0 \triangleq \begin{bmatrix} B_0 \\ 0_{n_c\times m_0} \end{bmatrix}, \tilde{C}_0 \triangleq [C_0 \; 0_{m_0\times n_c}]$. Finally, if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$ for a given compensator (A_c, B_c, C_c), then it follows from Lemma 2.1 that the performance functional (6.5) is given by

$$J(A_c, B_c, C_c) = \sup_{\Delta A\in\mathcal{U}} \mathrm{tr}\tilde{P}_{\Delta\tilde{A}}\tilde{V} \quad (6.7)$$

where $P_{\Delta\tilde{A}}$ satisfies the $\tilde{n} \times \tilde{n}$ Lyapunov equation

$$0 = (\tilde{A} + \Delta\tilde{A})^{\mathrm{T}}\tilde{P}_{\Delta\tilde{A}} + \tilde{P}_{\Delta\tilde{A}}(\tilde{A} + \Delta\tilde{A}) + \tilde{R} \quad (6.8)$$

where

$$\tilde{E} = [E_1 \; E_2 C_c], \; \tilde{R} = \tilde{E}^{\mathrm{T}}\tilde{E} = \begin{bmatrix} R_1 & 0 \\ 0 & C_c^{\mathrm{T}} R_2 C_c \end{bmatrix}.$$

Next, we apply Theorem 4.1 to controller synthesis. Specifically, we replace the Lyapunov equation (6.8) for the dynamic problem with a Riccati equation that guarantees that the closed-loop system is robustly stable. Thus, for the dynamic output feedback problem, Theorem 4.1 holds with A, R, V replaced by $\tilde{A}, \tilde{R}, \tilde{V}$. This leads to the following problem.

Dynamic Auxiliary Minimization Problem: Determine $N \in \mathcal{N}_{\mathrm{nd}}$ and controllable and observable (A_c, B_c, C_c) that minimize

$$\mathcal{J}(A_c, B_c, C_c, N) \triangleq \mathrm{tr}(\tilde{P} + \tilde{C}_0^{\mathrm{T}}\mu\tilde{C}_0)\tilde{V} \quad (6.9)$$

where $\tilde{P} \in \mathbb{N}^{\tilde{n}}$ satisfies

$$\begin{aligned} 0 =& \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + (\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^{\mathrm{T}}\tilde{P})^{\mathrm{T}}[(M^{-1} - N\tilde{C}_0\tilde{B}_0) \\ &+ (M^{-1} - N\tilde{C}_0\tilde{B}_0)^{\mathrm{T}}]^{-1}(\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^{\mathrm{T}}\tilde{P}) + \tilde{R}. \end{aligned} \quad (6.10)$$

Necessary conditions for the dynamic auxiliary minimization problem will provide fixed-order dynamic output feedback controllers with guaranteed robust stability and performance. The following result is required for the statement of the main theorem.

Lemma 6.1 [32]: Let $\hat{Q}, \hat{P}$ be $n \times n$ nonnegative-definite matrices and suppose that rank $\hat{Q}\hat{P} = n_c$. Then there exist $n_c \times n$ matrices G, Γ and an $n_c \times n_c$ invertible matrix $\hat{M}$, unique except for a change of basis in $\mathbb{R}^{n_c}$, such that

$$\hat{Q}\hat{P} = G^{\mathrm{T}}\hat{M}\Gamma, \quad \Gamma G^{\mathrm{T}} = I_{n_c}. \quad (6.11)$$

Furthermore, the $n \times n$ matrices $\tau \triangleq G^{\mathrm{T}}\Gamma$ and $\tau_\perp \triangleq I_n - \tau$ are idempotent and have rank n_c and $n - n_c$, respectively.

To state the main result of this section let $P, Q \in \mathbb{R}^{n\times n}$ and define the notation

$$\begin{aligned} R_0 &\triangleq (M^{-1} - NC_0 B_0) + (M^{-1} - NC_0 B_0)^{\mathrm{T}}, \\ \tilde{C} &\triangleq C_0 + NC_0 A, \quad A_P \triangleq A + B_0 R_0^{-1}\tilde{C}, \\ R_{2a} &\triangleq R_2 + B^{\mathrm{T}} C_0^{\mathrm{T}} N^{\mathrm{T}} R_0^{-1} N C_0 B, \\ P_a &\triangleq B^{\mathrm{T}} P + B^{\mathrm{T}} C_0^{\mathrm{T}} N^{\mathrm{T}} R_0^{-1}(\tilde{C} + B_0^{\mathrm{T}} P), \\ \bar{\Sigma} &\triangleq C^{\mathrm{T}} V_2^{-1} C, \quad A_{\hat{P}} \triangleq A_P - Q\bar{\Sigma} + B_0 R_0^{-1} B_0^{\mathrm{T}} P, \\ A_{\hat{Q}} &\triangleq A_P + B_0 R_0^{-1} B_0^{\mathrm{T}} P - (I + B_0 R_0^{-1} N C_0) B R_{2a}^{-1} P_a. \end{aligned}$$

Theorem 6.1: Let $n_c \leq n$, assume $R_0 > 0$, and assume $N \in \mathcal{N}_{\mathrm{nd}}$. Furthermore, suppose there exist $n \times n$ nonnegative-definite matrices $P, Q, \hat{P}, \hat{Q}$ satisfying

$$\begin{aligned} 0 =& A_P^{\mathrm{T}} P + P A_P + R_1 + \tilde{C}^{\mathrm{T}} R_0^{-1}\tilde{C} + P B_0 R_0^{-1} B_0^{\mathrm{T}} P \\ &- P_a^{\mathrm{T}} R_{2a}^{-1} P_a + \tau_\perp^{\mathrm{T}} P_a^{\mathrm{T}} R_{2a}^{-1} P_a \tau_\perp, \end{aligned} \quad (6.12)$$
$$\begin{aligned} 0 =& (A_P + B_0 R_0^{-1} B_0^{\mathrm{T}}[P + \hat{P}])Q \\ &+ Q(A_P + B_0 R_0^{-1} B_0^{\mathrm{T}}[P + \hat{P}])^{\mathrm{T}} \\ &+ V_1 - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}}, \end{aligned} \quad (6.13)$$
$$\begin{aligned} 0 =& A_{\hat{P}}^{\mathrm{T}}\hat{P} + \hat{P}A_{\hat{P}} + \hat{P}B_0 R_0^{-1} B_0^{\mathrm{T}}\hat{P} \\ &+ P_a^{\mathrm{T}} R_{2a}^{-1} P_a - \tau_\perp^{\mathrm{T}} P_a^{\mathrm{T}} R_{2a}^{-1} P_a \tau_\perp, \end{aligned} \quad (6.14)$$
$$0 = A_{\hat{Q}}\hat{Q} + \hat{Q}A_{\hat{Q}}^{\mathrm{T}} + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}}, \quad (6.15)$$
$$\mathrm{rank}\,\hat{Q} = \mathrm{rank}\,\hat{P} = \mathrm{rank}\,\hat{Q}\hat{P} = n_c \quad (6.16)$$

and let A_c, B_c, C_c be given by

$$A_c = \Gamma[A_{\hat{Q}} - Q\bar{\Sigma}]G^{\mathrm{T}}, B_c = \Gamma QC^{\mathrm{T}} V_2^{-1}, C_c = -R_{2a}^{-1} P_a G^{\mathrm{T}}. \quad (6.17)$$

Then $(\tilde{A} + \Delta\tilde{A}, \tilde{E})$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case the performance of the closed-loop system (6.7) satisfies the bound

$$J(A_c, B_c, C_c) \leq \mathrm{tr}[(P + \hat{P})V_1 + \hat{P}Q\bar{\Sigma}Q + C_0^{\mathrm{T}}\mu C_0 V_1]. \quad (6.18)$$

Proof: The proof is constructive in nature and is similar to the proofs given in [14] and [33]. Specifically, first we obtain necessary conditions for the Dynamic Auxiliary Minimization Problem and then show by construction that these conditions serve as sufficient conditions for robust stabilization and provide a worst-case H_2 performance bound. □

Theorem 6.1 provides constructive sufficient conditions that yield dynamic feedback gains A_c, B_c, C_c for robust stability and performance. When solving (6.12)-(6.15) numerically, the matrices M and N and the structure matrices B_0 and C_0 appearing in the design equations can be adjusted to examine tradeoffs between performance and robustness. Finally, to further reduce conservatism, one can view the multiplier matrix N as a free parameter and optimize the worst-case H_2 performance bound $\mathcal{J}$ with respect to N. In particular, setting $\partial\mathcal{J}/\partial N = 0$ yields

$$\begin{aligned}0 =&\frac{1}{2}M\tilde{C}_0\tilde{V}\tilde{C}_0^{\mathrm{T}} + [(M^{-1} - N\tilde{C}_0\tilde{B}_0) + (M^{-1} - N\tilde{C}_0\tilde{B}_0)^{\mathrm{T}}]^{-1} \\ &\times (\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^{\mathrm{T}}\tilde{P})\tilde{Q}\tilde{A}^{\mathrm{T}}\tilde{C}_0^{\mathrm{T}} + [(M^{-1} - N\tilde{C}_0\tilde{B}_0) \\ &+ (M^{-1} - N\tilde{C}_0\tilde{B}_0)^{\mathrm{T}}]^{-1}(\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^{\mathrm{T}}\tilde{P})\tilde{Q}(\tilde{C}_0 \\ &+ N\tilde{C}_0\tilde{A} + \tilde{B}_0^{\mathrm{T}}\tilde{P})^{\mathrm{T}}[(M^{-1} - N\tilde{C}_0\tilde{B}_0) \\ &+ (M^{-1} - N\tilde{C}_0\tilde{B}_0)^{\mathrm{T}}]^{-1}\tilde{B}_0^{\mathrm{T}}\tilde{C}_0^{\mathrm{T}}\end{aligned} \tag{6.19}$$

where $\tilde{Q}$ satisfies

$$\begin{aligned}0 =&(\tilde{A} + \tilde{B}_0R_0^{-1}N\tilde{C}_0\tilde{A} + \tilde{B}_0R_0^{-1}\tilde{C}_0 + \tilde{B}_0R_0^{-1}\tilde{B}_0^{\mathrm{T}}\tilde{P})\tilde{Q} \\ &+ \tilde{Q}(\tilde{A} + \tilde{B}_0R_0^{-1}N\tilde{C}_0\tilde{A} + \tilde{B}_0R_0^{-1}\tilde{C}_0 + \tilde{B}_0R_0^{-1}\tilde{B}_0^{\mathrm{T}}\tilde{P})^{\mathrm{T}} + \tilde{V}.\end{aligned} \tag{6.20}$$

By using (6.19) within a numerical search algorithm, the optimal compensator and multiplier N can be determined simultaneously, thus avoiding the need to iterate between controller design and optimal multiplier evaluation.

Remark 6.1: Several special cases can immediately be discerned from Theorem 6.1. For example, in the full-order case, set $n_c = n$ so that $\tau = G = \Gamma = I_n$ and $\tau_\perp = 0$. In this case the last term in each of (6.12)-(6.15) is zero and (6.15) is superfluous. Alternatively, letting $B_0 = 0, C_0 = 0$ and retaining the reduced-order constraint $n_c \le n$ yields the result of [32].

VII. Numerical Algorithm and Illustrative Results

In this section we describe a numerical algorithm for solving the Riccati equation (6.10) along with the expression (6.19) for the optimal multiplier N. We also present numerical results for controller synthesis via an illustrative example.

To synthesize dynamic compensators, we let $\mu = (M_2 - M_1)N$ in (6.9) and determine (A_c, B_c, C_c, N) to minimize $\mathcal{J}(A_c, B_c, C_c, N)$ subject to (6.10) with $\tilde{P} \in \mathbb{N}^{\tilde{n}}$. To do this we form the Lagrangian

$$\begin{aligned}\mathcal{L}(A_c, B_c, C_c, N, \tilde{P}, \tilde{Q}) =&\mathrm{tr}[(\tilde{P} + \tilde{C}_0^{\mathrm{T}}(M_2 - M_1)N\tilde{C}_0)\tilde{V} \\ &+ \{(\tilde{A} + \tilde{B}_0M_1\tilde{C}_0)^{\mathrm{T}}\tilde{P} \\ &+ \tilde{P}(\tilde{A} + \tilde{B}_0M_1\tilde{C}_0) \\ &+ [\tilde{C}_0 + N\tilde{C}_0(\tilde{A} + \tilde{B}_0M_1\tilde{C}_0) + \tilde{B}_0^{\mathrm{T}}\tilde{P}]^{\mathrm{T}} \\ &\cdot \tilde{R}_0^{-1}[\tilde{C}_0 + N\tilde{C}_0(\tilde{A} + \tilde{B}_0M_1\tilde{C}_0) \\ &+ \tilde{B}_0^{\mathrm{T}}\tilde{P}] + \tilde{R}\}\tilde{Q}]\end{aligned} \tag{7.1}$$

where

$$\tilde{R}_0 \triangleq [(M_2-M_1)^{-1} - N\tilde{C}_0\tilde{B}_0] + [(M_2-M_1)^{-1} - N\tilde{C}_0\tilde{B}_0]^{\mathrm{T}} \tag{7.2}$$

and $\tilde{Q} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ is a Lagrange multiplier. The partial derivatives of $\mathcal{L}$ are then used in the search procedure. Note that the shifted version of (6.10) discussed in Remark 4.2 is used in (7.1) to address uncertainties with upper and lower bounds of the form $M_1 \le F \le M_2$.

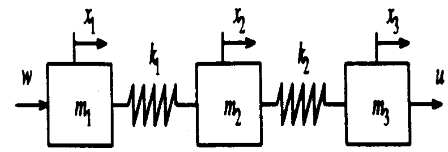

Fig. 1. Three-mass system.

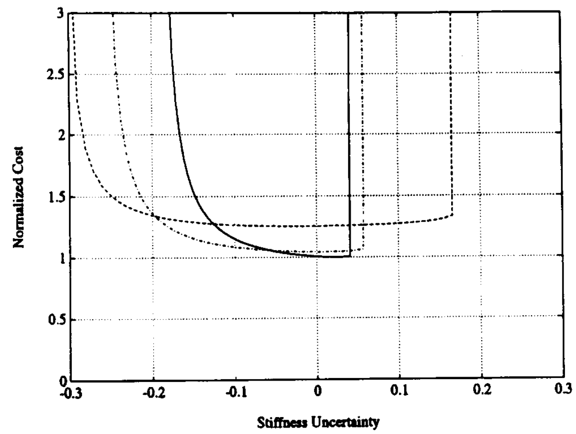

Fig. 2. Performance vs. robustness trade-off for LQG and Popov controllers.

A quasi-Newton search algorithm was initialized with $N = 0$ and the LQG gains. For given values of the robustness bounds M_1 and M_2, the search algorithm was used to find A_c, B_c, C_c and N satisfying the necessary conditions. After each iteration, M_1 and M_2 were increased and the current design was used as the initial step for the next iteration. Since the optimal compensator and multiplier are found simultaneously, there is no need to iterate between controller design and optimal multiplier evaluation.

Consider the three-mass, two-spring system shown in Fig. 1 with $m_1 = m_2 = m_3 = 1$ and an uncertain spring stiffness k_2. A control force acts on mass 3 while the position of mass 1 is measured resulting in a noncolocated control problem. The nominal dynamics, with state variables defined in Fig. 1, are given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k_1 & k_1 & 0 & 0 & 0 & 0 \\ k_1 & -(k_1 + k_{2\mathrm{nom}}) & k_{2\mathrm{nom}} & 0 & 0 & 0 \\ 0 & k_{2\mathrm{nom}} & -k_{2\mathrm{nom}} & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [110000], \quad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$D_2 = [0\ 1]$, and $k_1 = k_{2\mathrm{nom}} = 1$. The actual spring stiffness of the second spring can be written as $k_2 = k_{2\mathrm{nom}} + \Delta k$ so that the actual dynamics are given by $A(\Delta k) = A + B_0\Delta kC_0$, where $B_0 = [0000-11]^{\mathrm{T}}$ and $C_0 = [01-1000]$. Furthermore, let

$$E_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

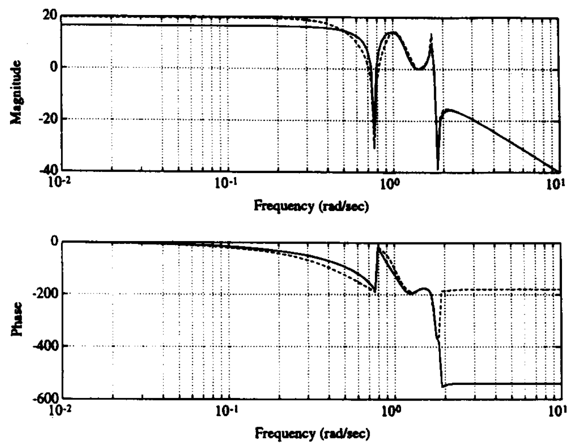

Fig. 3. Popov (dashed) and LQG (solid) controllers.

Two full-order ($n_c = n$) Popov compensators were designed. Fig. 2 compares performance versus robustness trade-offs of the Popov compensators (dashed) with the normalized LQG design (solid). Fig. 3 shows the magnitude and phase of both a Popov design and the LQG design. Note that the Popov design robustified the LQG controller notch by increasing both the width and the depth of the notch.

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