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H_2 optimal control with an α-shifted pole constraint

Y. WILLIAM WANG† and DENNIS S. BERNSTEIN†

Modified Lyapunov equations for regional pole constraints in H_2 synthesis have been considered in recent papers. This approach involves a constraint equation for enforcing the pole constraints and leads to an auxiliary cost that overbounds the original H_2 performance. The auxiliary cost can then be used for optimization with respect to the modified Lyapunov equations that characterize the constraint region. In this paper, we consider an α-shifted constraint region and show that an augmented system whose order is twice that of the original system can be constructed to eliminate the need for an overbound so that *exact* H_2 cost optimization can be performed. This augmented system is then utilized for closed-loop controller synthesis within a decentralized static output feedback setting. The construction of the augmented system is a refinement of the approach of Gu *et al.* (1992). A numerical algorithm based upon the BFGS quasi-Newton method is used for computing the optimal controller gains. The numerical results are compared to the classical exponential cost weighting technique of Anderson and Moore (1989).

Notation

spec (A)	the set of eigenvalues of A
$\otimes$	Kronecker product
LFT	linear fractional transformation
$\mathcal{R}$, $\mathcal{C}$	real numbers, complex numbers
OLHP	open left half (complex) plane
$\mathcal{E}$	expectation
tr	trace operator
e_i	r-dimensional unit vector having one in the ith component and zeros elsewhere
$e_{ik}^{p\times q}$	$e_i e_k^{\mathrm{T}}$ where $e_i \in \mathcal{R}^p$ and $e_k \in \mathcal{R}^q$
$U_{p\times q}$	the $p \times q$ permutation matrix defined in Brewer (1978)
I_r, $0_{p\times q}$	$r \times r$ identity matrix, $p \times q$-dimensional zero matrix
$\lVert\cdot\rVert$	H_2 norm

1. Introduction

It has been shown in a recent paper (Haddad and Bernstein 1992) that modified Lyapunov equations can be used to enforce regional pole constraints while optimizing an auxiliary cost that overbounds the H_2 performance of the closed-loop system. Our goal in this paper is to eliminate this performance bound by preserving the closed-loop transfer function and hence the original H_2 cost of the closed-loop system while constraining the closed-loop poles to lie in a prescribed region. The basis for our approach is a refinement of the results given in Gu *et al.* (1992) in which an augmented system is constructed to

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constrain the closed-loop poles while preserving the H_2 cost. For the α-shifted open left half plane $\mathscr{C}_\alpha \triangleq \{\lambda \in \mathscr{C}\colon \operatorname{Re}\lambda < -\alpha\}$, where $\alpha > 0$, the augmented system we construct has order $2n$, for an original n-dimensional system. Since this approach leads to a decentralized controller representation involving repeated controller gains, we therefore formulate a general problem that includes repeated decentralized controller gains as in Bernstein *et al.* (1989), Haddad *et al.* (1989), Seinfeld (1991) and Seinfeld *et al.* (1991).

We will start in § 2 with a general control problem involving r vector inputs and r vector outputs with decentralized output feedback controller gains. We then turn to the main results in § 3 where the exact H_2 norm is preserved while the closed-loop poles are constrained to lie in $\mathscr{C}_\alpha$. This problem is related to the classical technique of Anderson and Moore (1989) in which the quadratic cost is weighted by the exponential function $\exp(2\alpha t)$. Since this performance measure is not an H_2 cost functional, this approach may yield conservative H_2 designs. The H_2 conservatism of Anderson and Moore's technique for the α-shifted left half plane is illustrated in § 4 where we compare the numerical results from Anderson and Moore's approach with the results obtained from the augmentation approach. In fact, we use the controller gains obtained from Anderson and Moore's approach as initial conditions for computing the controller gains via the augmentation approach. Finally, future research directions are discussed in § 5.

2. H_2-optimal decentralized static output feedback with possibly repeated gains

Consider the r-vector-input, r-vector-output decentralized plant $G_d(s)$ with realization given by

$$G_d(s) \sim \left[\begin{array}{c|ccccc} \mathscr{A} & \mathscr{D}_1 & \mathscr{B}_1 & \mathscr{B}_2 & \cdots & \mathscr{B}_r \\ \hline \mathscr{E}_1 & 0 & \mathscr{E}_{21} & \mathscr{E}_{22} & \cdots & \mathscr{E}_{2r} \\ \mathscr{C}_1 & \mathscr{D}_{21} & \mathscr{F}_{11} & \mathscr{F}_{12} & \cdots & \mathscr{F}_{1r} \\ \mathscr{C}_2 & \mathscr{D}_{22} & \mathscr{F}_{21} & \mathscr{F}_{22} & \cdots & \mathscr{F}_{2r} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \mathscr{C}_r & \mathscr{D}_{2r} & \mathscr{F}_{r1} & \mathscr{F}_{r2} & \cdots & \mathscr{F}_{rr} \end{array}\right] = \left[\begin{array}{c|cc} \mathscr{A} & \mathscr{D}_1 & \mathscr{B} \\ \hline \mathscr{E}_1 & 0 & \mathscr{E}_2 \\ \mathscr{C} & \mathscr{D}_2 & \mathscr{F} \end{array}\right] \tag{1}$$

The corresponding state space equations are

$$\dot{x}(t) = \mathscr{A}x(t) + \sum_{i=1}^{r} \mathscr{B}_i u_i(t) + \mathscr{C}_1 w(t) \tag{2}$$

$$y_i(t) = \mathscr{C}_i x(t) + \sum_{j=1}^{r} \mathscr{F}_{ij} u_j(t) + \mathscr{D}_{2i} w(t), \quad i = 1, \ldots, r \tag{3}$$

with the performance variable specified by

$$z(t) = \mathscr{E}_1 x(t) + \sum_{i=1}^{r} \mathscr{E}_{2i} u_i(t) \tag{4}$$

Our goal is to consider a decentralized static output feedback control law

$$u_i(t) = \mathscr{K}_i y_i(t), \quad i = 1, \ldots, r \tag{5}$$

that stabilizes the closed-loop system and minimizes the performance measure

$$J(\mathcal{K}_1, \ldots, \mathcal{K}_r) \triangleq \lim_{t\to\infty} \mathcal{E}[z^{\mathrm{T}}(t)z(t)] \tag{6}$$

where $w(t)$ represents a standard white noise. In this problem formulation the decentralized output feedback gains $\mathcal{K}_i$ need not be distinct, that is, one may require *a priori* that specific gains be identical as in Haddad *et al*. (1989). The relevance of this problem to the α-shifted pole placement problem as well as to the standard LQG problem will be described in later sections.

It is well known that the performance measure (6) is equivalent to the H_2 norm

$$J(\mathcal{K}_1, \ldots, \mathcal{K}_r) = \|G_{zw}(s)\|_2 = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\,[G_{zw}^{\mathrm{T}}(-j\omega)G_{zw}(j\omega)]\,\mathrm{d}\omega\right]^{1/2}$$

where $G_{zw}(s) = \mathrm{LFT}(G_d(s), \tilde{\mathcal{K}})$ is the closed-loop transfer function from z to w of system (2)–(5) with $\tilde{\mathcal{K}} \triangleq \text{block-diag}(\mathcal{K}_1, \ldots, \mathcal{K}_r)$. Assuming $(I - \tilde{\mathcal{K}}\mathcal{F})^{-1}$ exists and combining (2), (4) and (5) yields the closed-loop system

$$\dot{x}(t) = \tilde{\mathcal{A}}x(t) + \tilde{\mathcal{D}}w(t) \tag{7}$$

$$z(t) = \tilde{\mathcal{E}}x(t) + \mathcal{E}_2(I - \tilde{\mathcal{K}}\mathcal{F})^{-1}\tilde{\mathcal{K}}\mathcal{D}_2 w(t) \tag{8}$$

where

$$\tilde{\mathcal{A}} \triangleq \mathcal{A} + \mathcal{B}(I - \tilde{\mathcal{K}}\mathcal{F})^{-1}\tilde{\mathcal{K}}\mathcal{C},\ \tilde{\mathcal{D}} \triangleq \mathcal{D}_1 + \mathcal{B}(I - \tilde{\mathcal{K}}\mathcal{F})^{-1}\tilde{\mathcal{K}}\mathcal{D}_2$$

$$\tilde{\mathcal{E}} \triangleq \mathcal{E}_1 + \mathcal{E}_2(I - \tilde{\mathcal{K}}\mathcal{F})^{-1}\tilde{\mathcal{K}}\mathcal{C}$$

Thus $G_{zw}(s)$ has the realization

$$G_{zw}(s) \sim \left[\begin{array}{c|c} \tilde{\mathcal{A}} & \tilde{\mathcal{D}} \\ \hline \tilde{\mathcal{E}} & \mathcal{E}_2(\mathrm{I} - \tilde{\mathcal{K}}\mathcal{F})^{-1}\tilde{\mathcal{K}}\mathcal{D}_2 \end{array}\right] \tag{9}$$

To guarantee that the H_2 cost is finite, we require that $\mathcal{E}_2(I - \tilde{\mathcal{K}}\mathcal{F})^{-1}\tilde{\mathcal{K}}\mathcal{D}_2 = 0$. The following result provides sufficient conditions for this to hold.

Lemma 2.1: *The following statements hold.*

(*a*) *Suppose* $\mathcal{F} = 0$. *If* $\mathcal{E}_{2i} = 0$ *or* $\mathcal{D}_{2i} = 0$, $i = 1, \ldots, r$, *then* $\mathcal{E}_2(I - \tilde{\mathcal{K}}\mathcal{F})^{-1}\tilde{\mathcal{K}}\mathcal{D}_2 = 0$.

(*b*) *Suppose there exists* $i \in \{1, \ldots, r-1\}$, *such that* $\mathcal{F}_{kj} = 0$, $k \neq i$, $j = 1, \ldots, r$, *and* $\mathcal{F}_{ij} = 0$, $j \leq i$. *Then* $(I - \tilde{\mathcal{K}}\mathcal{F})^{-1} = I + \tilde{\mathcal{K}}\mathcal{F}$. *Furthermore, if* $\mathcal{E}_{2i} \neq 0$, $\mathcal{D}_{2j} = 0$, $j = i, \ldots, r$, *and either* $\mathcal{E}_{2k} = 0$ *or* $\mathcal{D}_{2k} = 0$, $k = 1, \ldots, i-1$, *then* $\mathcal{E}_2(I - \tilde{\mathcal{K}}\mathcal{F})^{-1}\tilde{\mathcal{K}}\mathcal{D}_2 = 0$. *Finally, if* $\mathcal{E}_{2i} = 0$, *and either* $\mathcal{E}_{2j} = 0$ *or* $\mathcal{D}_{2j} = 0$, $j = 1, \ldots, r$, *then* $\mathcal{E}_2(I - \tilde{\mathcal{K}}\mathcal{F})^{-1}\tilde{\mathcal{K}}\mathcal{D}_2 = 0$.

Proof: The results follow from algebraic manipulation. □

Using (8), it is seen that (6) is given by

$$J(\mathcal{K}_1, \ldots, \mathcal{K}_r) = \mathrm{tr}\,\tilde{\mathcal{Q}}\tilde{\mathcal{R}} \tag{10}$$

where $\tilde{\mathcal{R}} \triangleq \tilde{\mathcal{E}}^{\mathrm{T}}\tilde{\mathcal{E}}$, $\tilde{\mathcal{V}} \triangleq \tilde{\mathcal{D}}\tilde{\mathcal{D}}^{\mathrm{T}}$ and $\tilde{\mathcal{Q}}$ is given by

$$\tilde{\mathcal{Q}} = \int_0^{\infty} \exp(\tilde{\mathcal{A}}\tau)\tilde{\mathcal{V}}\exp(\mathcal{A}^{\mathrm{T}}\tau)\,\mathrm{d}\tau$$

which satisfies

$$0 = \tilde{\mathcal{A}}\tilde{\mathfrak{Q}} + \tilde{\mathfrak{Q}}\tilde{\mathcal{A}}^{\mathrm{T}} + \tilde{\mathcal{V}} \tag{11}$$

Hence, the original optimization problem is equivalent to the problem of minimizing (10) subject to the constraint equation (11). This problem can be treated by means of Lagrange multipliers. Defining the Lagrangian by

$$L(\mathcal{K}_1, \ldots, \mathcal{K}_r, \tilde{\mathfrak{Q}}) = \mathrm{tr}\,[\tilde{\mathfrak{Q}}\tilde{\mathcal{R}} + \tilde{\mathcal{P}}(\tilde{\mathcal{A}}\tilde{\mathfrak{Q}} + \tilde{\mathfrak{Q}}\tilde{\mathcal{A}}^{\mathrm{T}} + \tilde{\mathcal{V}})]$$

and letting $\partial L/\partial\tilde{\mathfrak{Q}} = 0$, we obtain

$$0 = \tilde{\mathcal{A}}^{\mathrm{T}}\tilde{\mathcal{P}} + \tilde{\mathcal{P}}\tilde{\mathcal{A}} + \tilde{\mathcal{R}} \tag{12}$$

Since repeated gains are allowed in the following development, we let $\mathcal{I}_i$ denote the index set of all k such that $\mathcal{K}_k = \mathcal{K}_i$. Next note that $\tilde{\mathcal{K}}^{\mathrm{T}}$ can be written as

$$\tilde{\mathcal{K}}^{\mathrm{T}} = \sum_{j=1}^{r}(e_j \otimes I_{q_j})\mathcal{K}_j^{\mathrm{T}}(e_j^{\mathrm{T}} \otimes I_{p_j}) \tag{13}$$

where $\mathcal{K}_i \in \mathcal{R}^{p_i \times q_i}$. For the ith gain, the gradient of L with respect to $\mathcal{K}_i$ is given by

$$\begin{aligned}\frac{1}{2}\frac{\partial L}{\partial \mathcal{K}_i} = \sum_{j\in\mathcal{I}_i} &(e_j^{\mathrm{T}} \otimes I_{p_j})(I - \tilde{\mathcal{K}}\mathcal{F})^{-\mathrm{T}}(\mathscr{C}_2^{\mathrm{T}}\mathscr{E}\tilde{\mathfrak{Q}}\mathscr{C}^{\mathrm{T}} \\ &+ \mathcal{B}^{\mathrm{T}}\tilde{\mathcal{P}}\tilde{\mathfrak{Q}}\mathscr{C}^{\mathrm{T}} + \mathcal{B}^{\mathrm{T}}\tilde{\mathcal{P}}\tilde{\mathfrak{D}}\mathfrak{D}_2^{\mathrm{T}})(I - \tilde{\mathcal{K}}\mathcal{F})^{-\mathrm{T}}(e_j \otimes I_{q_j})\end{aligned} \tag{14}$$

The expression (14) can be simplified in the case in which $\mathcal{F}$ has the form

$$\mathcal{F} = \begin{bmatrix} 0 & \mathcal{F}_{12} & \mathcal{F}_{13} & \cdots & \mathcal{F}_{1r} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

If we also assume that $\mathcal{K}_1, \ldots, \mathcal{K}_r$ are distinct, then, for $i = 2, \ldots, r$, (14) becomes

$$\frac{1}{2}\frac{\partial \mathrm{L}}{\partial \mathcal{K}_i} = \mathscr{C}_{2i}^{\mathrm{T}}\mathscr{E}\tilde{\mathfrak{Q}}\mathscr{C}_i^{\mathrm{T}} + \mathcal{B}_i^{\mathrm{T}}\tilde{\mathcal{P}}\tilde{\mathfrak{Q}}\mathscr{C}_i^{\mathrm{T}} + \mathcal{B}_i^{\mathrm{T}}\tilde{\mathcal{P}}\tilde{\mathfrak{D}}\mathfrak{D}_{2i}^{\mathrm{T}}, \tag{15}$$

while for the gain $\mathcal{K}_1$, the gradient is given by

$$\begin{aligned}\frac{1}{2}\frac{\partial \mathrm{L}}{\partial \mathcal{K}_1} = {}&\mathscr{C}_{21}^{\mathrm{T}}\mathscr{E}\tilde{\mathfrak{Q}}\mathscr{C}_1^{\mathrm{T}} + \sum_{j=2}^{r}\mathscr{C}_{21}^{\mathrm{T}}\mathscr{E}\tilde{\mathfrak{Q}}(\mathcal{F}_{1j}\mathcal{K}_j\mathscr{C}_j)^{\mathrm{T}} + \mathcal{B}_1^{\mathrm{T}}\tilde{\mathcal{P}}\tilde{\mathfrak{Q}}\mathscr{C}_1^{\mathrm{T}} \\ &+ \sum_{j=2}^{r}\mathcal{B}_1^{\mathrm{T}}\tilde{\mathcal{P}}\tilde{\mathfrak{Q}}(\mathcal{F}_{1j}\mathcal{K}_j\mathscr{C}_j)^{\mathrm{T}} + \mathcal{B}_1^{\mathrm{T}}\tilde{\mathcal{P}}\tilde{\mathfrak{D}}\mathfrak{D}_{21}^{\mathrm{T}} + \sum_{j=2}^{r}\mathcal{B}_1^{\mathrm{T}}\tilde{\mathcal{P}}\tilde{\mathfrak{D}}(\mathcal{F}_{1j}\mathcal{K}_j\mathfrak{D}_{2j})^{\mathrm{T}}\end{aligned} \tag{16}$$

The following result given in Seinfeld (1991) shows that the gradient of L with respect to $\mathcal{K}_i$ is equal to the Frechet derivative of J with respect to $\mathcal{K}_i$. For this result, we define $\mathcal{S}$ as

$$\mathcal{S} \triangleq \{\tilde{\mathcal{K}}\colon \tilde{\mathcal{A}} \text{ is asymptotically stable}\}$$

Proposition 2.1: *Let $J(\tilde{\mathfrak{Q}}, \tilde{\mathcal{K}})$ be defined as in* (10), *define* $g(\tilde{\mathfrak{Q}}, \tilde{\mathcal{K}}) \triangleq \tilde{\mathcal{A}}\tilde{\mathfrak{Q}} + \tilde{\mathfrak{Q}}\tilde{\mathcal{A}}^{\mathrm{T}} + \tilde{\mathcal{V}}$, *where* $\tilde{\mathcal{K}} \in \mathcal{S}$, *and let* $L(\tilde{\mathfrak{Q}}, \tilde{\mathcal{K}}) \triangleq J(\tilde{\mathfrak{Q}}, \tilde{\mathcal{K}}) + \operatorname{tr}\tilde{\mathcal{P}}g(\tilde{\mathfrak{Q}}, \tilde{\mathcal{K}})$, *where* $\tilde{\mathcal{P}} \in \mathcal{R}^{n\times n}$. *If* $g(\tilde{\mathfrak{Q}}_0, \tilde{\mathcal{K}}_0) = 0$ *and* $(\partial L/\partial\tilde{\mathfrak{Q}})_{\tilde{\mathfrak{Q}}_0,\tilde{\mathcal{K}}_0} = 0$, *then*

$$\left(\frac{\mathrm{d}J}{\mathrm{d}\tilde{\mathcal{K}}}\right)_{\tilde{\mathfrak{Q}}_0,\tilde{\mathcal{K}}_0} = \left(\frac{\partial L}{\partial\tilde{\mathcal{K}}}\right)_{\tilde{\mathfrak{Q}}_0,\tilde{\mathcal{K}}_0} \tag{17}$$

Proof: Since $\tilde{\mathcal{A}}$ is asymptotically stable, it follows that $g(\tilde{\mathfrak{Q}}, \tilde{\mathcal{K}}) = 0$ if and only if $\tilde{\mathfrak{Q}} = \tilde{\mathfrak{Q}}(\tilde{\mathcal{K}}) \triangleq -\mathrm{vec}^{-1}[(\tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}})^{-1}\mathrm{vec}\,\tilde{\mathcal{V}}]$. Therefore, $g(\tilde{\mathfrak{Q}}(\tilde{\mathcal{K}}), \tilde{\mathcal{K}}) = 0$, for all $\tilde{\mathcal{K}} \in \mathcal{S}$. Hence, it follows that

$$0 = \frac{\partial g}{\partial\tilde{\mathfrak{Q}}}\frac{\mathrm{d}\tilde{\mathfrak{Q}}}{\mathrm{d}\tilde{\mathcal{K}}} + \frac{\partial g}{\partial\tilde{\mathcal{K}}}, \quad \tilde{\mathcal{K}} \in \mathcal{S}$$

Thus,

$$\left(\frac{\mathrm{d}\tilde{\mathfrak{Q}}}{\mathrm{d}\tilde{\mathcal{K}}}\right)_{\tilde{\mathfrak{Q}}_0,\tilde{\mathcal{K}}_0} = -\left(\left(\frac{\partial g}{\partial\tilde{\mathfrak{Q}}}\right)^{-1}\frac{\partial g}{\partial\tilde{\mathcal{K}}}\right)_{\tilde{\mathfrak{Q}}_0,\tilde{\mathcal{K}}_0}$$

The Frechet derivative of $J(\tilde{\mathfrak{Q}}(\tilde{\mathcal{K}}), \tilde{\mathcal{K}})$ with respect to $\tilde{\mathcal{K}}$ is given by

$$\frac{\mathrm{d}J}{\mathrm{d}\tilde{\mathcal{K}}} = \frac{\partial J}{\partial\tilde{\mathcal{K}}} + \frac{\partial J}{\partial\tilde{\mathfrak{Q}}}\frac{\mathrm{d}\tilde{\mathfrak{Q}}}{\mathrm{d}\tilde{\mathcal{K}}}$$

Thus,

$$\left(\frac{\mathrm{d}J}{\mathrm{d}\tilde{\mathcal{K}}}\right)_{\tilde{\mathfrak{Q}}_0,\tilde{\mathcal{K}}_0} = \left(\frac{\partial J}{\partial\tilde{\mathcal{K}}} - \frac{\partial J}{\partial\tilde{\mathfrak{Q}}}\left(\frac{\partial g}{\partial\tilde{\mathfrak{Q}}}\right)^{-1}\frac{\partial g}{\partial\tilde{\mathcal{K}}}\right)_{\tilde{\mathfrak{Q}}_0,\tilde{\mathcal{K}}_0}$$

The gradient of $L(\tilde{\mathfrak{Q}}, \tilde{\mathcal{K}})$ with respect to $\tilde{\mathfrak{Q}}$ is given by

$$\frac{\partial L}{\partial\tilde{\mathfrak{Q}}} = \frac{\partial J}{\partial\tilde{\mathfrak{Q}}} + \tilde{\mathcal{P}}\frac{\partial g}{\partial\tilde{\mathfrak{Q}}}$$

Since, by the assumption, $(\partial L/\partial\tilde{\mathfrak{Q}})_{\tilde{\mathfrak{Q}}_0,\tilde{\mathcal{K}}_0} = 0$, $\tilde{\mathcal{P}}$ is given by

$$\tilde{\mathcal{P}} = -\left(\frac{\partial J}{\partial\tilde{\mathfrak{Q}}}\left(\frac{\partial g}{\partial\tilde{\mathfrak{Q}}}\right)^{-1}\right)_{\tilde{\mathfrak{Q}}_0,\tilde{\mathcal{K}}_0}$$

The gradient of $L(\tilde{\mathfrak{Q}}, \tilde{\mathcal{K}})$ with respect to $\tilde{\mathcal{K}}$ is given by

$$\frac{\partial L}{\partial\tilde{\mathcal{K}}} = \frac{\partial J}{\partial\tilde{\mathcal{K}}} + \tilde{\mathcal{P}}\frac{\partial g}{\partial\tilde{\mathcal{K}}}$$

Thus,

$$\left(\frac{\partial L}{\partial\tilde{\mathcal{K}}}\right)_{\tilde{\mathfrak{Q}}_0,\tilde{\mathcal{K}}_0} = \left(\frac{\partial J}{\partial\tilde{\mathcal{K}}} - \frac{\partial J}{\partial\tilde{\mathfrak{Q}}}\left(\frac{\partial g}{\partial\tilde{\mathfrak{Q}}}\right)^{-1}\frac{\partial g}{\partial\tilde{\mathcal{K}}}\right)_{\tilde{\mathfrak{Q}}_0,\tilde{\mathcal{K}}_0}$$

Consequently, we obtain

$$\left(\frac{\mathrm{d}J}{\mathrm{d}\tilde{\mathcal{K}}}\right)_{\tilde{\mathfrak{Q}}_0,\tilde{\mathcal{K}}_0} = \left(\frac{\partial L}{\partial\tilde{\mathcal{K}}}\right)_{\tilde{\mathfrak{Q}}_0,\tilde{\mathcal{K}}_0}$$

This completes the proof. □

Now we show that the standard centralized LQG control problem is a special case of the above decentralized static output feedback problem formulation.

Consider a two-input, two-output plant

$$G(s) \sim \left[\begin{array}{c|cc} A & D_1 & B \\ \hline E_1 & 0 & E_2 \\ C & D_2 & F \end{array}\right] \tag{18}$$

with state space equations

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t) \tag{19}$$

$$y(t) = Cx(t) + Fu(t) + D_2 w(t) \tag{20}$$

and performance variables

$$z(t) = E_1 x(t) + E_2 u(t) \tag{21}$$

Further consider a dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{22}$$

$$u(t) = C_c x_c(t) \tag{23}$$

such that the closed-loop system is asymptotically stable. Combining (19), (20), (22) and (23) yields

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} A & BC_c \\ B_c C & A_c + B_c F C_c \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix} w(t)$$

Here A_c, B_c and C_c can be treated as decentralized gains so that

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u_2(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u_3(t) + \begin{bmatrix} D_1 \\ 0 \end{bmatrix} w(t)$$

$$y_1(t) = [0 \quad I] \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}$$

$$y_2(t) = [C \quad 0] \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + Fu_3(t) + D_2 w(t)$$

$$y_3(t) = [0 \quad I] \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}$$

$$z(t) = [E_1 \quad 0] \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + [0 \quad 0 \quad E_2] \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$$

$$u_1(t) = A_c y_1(t), \quad u_2(t) = B_c y_2(t), \quad u_3(t) = C_c y_3(t)$$

Using the notation defined in (2)–(5), it can be seen that

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_1 = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{B}_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{B}_3 = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\mathcal{C}_1 = [0 \quad I], \quad \mathcal{C}_2 = [C \quad 0], \quad \mathcal{C}_3 = [0 \quad I]$$

$$\mathcal{D}_1 = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}, \quad \mathcal{D}_{22} = D_2, \quad \mathcal{F}_{23} = F, \quad \mathcal{E}_1 = [E_1 \quad 0], \quad \mathcal{E}_2 = [0 \quad 0 \quad E_2]$$

$$\mathcal{K}_1 = A_c, \quad \mathcal{K}_2 = B_c, \quad \mathcal{K}_3 = C_c$$

Hence the centralized controller gains can be treated as decentralized gains in the feedback design. Moreover, due to the special structure of $\mathcal{F}$, Lemma 2.1, statement (*b*), can be applied to guarantee that

$$(I - \tilde{\mathcal{K}}\mathcal{F})^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & B_cF \\ 0 & 0 & I \end{bmatrix}$$

and that

$$\mathcal{E}_2(I - \tilde{\mathcal{K}}\mathcal{F})^{-1}\tilde{\mathcal{K}}\mathcal{D}_2 = [0 \quad 0 \quad E_2]\begin{bmatrix} I & 0 & 0 \\ 0 & I & B_cF \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} 0 \\ B_cD_2 \\ 0 \end{bmatrix} = 0$$

For further details, see Bernstein *et al.* (1989).

3. Characterization of the augmented plant for the α-shifted open left half plane

In this section we show our main result on the construction of an augmented plant that preserves the closed-loop transfer function while the closed-loop poles are constrained to lie in the shifted half plane $\mathcal{C}_\alpha$. For the plant $G(s)$ given in (18) and the static output feedback control law $u = Ky$, the closed-loop transfer function $G_{zw}(s) = \mathrm{LFT}(G(s), K)$, is given by

$$G_{zw}(s) \sim \left[\begin{array}{c|c} A + B(I - KF)^{-1}KC & D_1 + B(I - KF)^{-1}KD_2 \\ \hline E_1 + E_2(I - KF)^{-1}KC & E_2(I - KF)^{-1}KD_2 \end{array}\right] \tag{24}$$

It is seen that the poles of the closed-loop system are the roots of

$$\det[\lambda I - (A + B(I - KF)^{-1}KC)] = 0 \tag{25}$$

Now suppose that the roots of (25) are required to lie in $\mathcal{C}_\alpha \triangleq \{\lambda \in \mathcal{C}: \mathrm{Re}\,\lambda < -\alpha\}$. To enforce this constraint, we construct a new plant and controller with the same closed-loop transfer function $G_{zw}(s)$ and having the property that if its realization is asymptotically stable, then the roots of (25) lie in $\mathcal{C}_\alpha$. We now show that these requirements are met by the augmented TITO plant $G_a(s)$ given by

$$G_a(s) \sim \left[\begin{array}{c|cc} A_a & D_{1a} & B_a \\ \hline E_{1a} & 0 & E_{2a} \\ C_a & D_{2a} & F_a \end{array}\right] \tag{26}$$

which the augmented feedback gain K_a given by

$$K_a \triangleq \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} = I_2 \otimes K$$

where

$$A_a \triangleq \begin{bmatrix} A + \alpha I & -\alpha I \\ 0 & A \end{bmatrix}$$

$$D_{1a} \triangleq \begin{bmatrix} D_1 \\ D_1 \end{bmatrix}, \quad D_{2a} \triangleq \begin{bmatrix} D_2 \\ D_2 \end{bmatrix}, \quad B_a \triangleq \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

$$E_{1a} \triangleq [E_1 \quad 0], \quad E_{2a} \triangleq [E_2 \quad 0], \quad C_a \triangleq \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}, \quad F_a \triangleq \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}$$

The following result is a refinement of results given in Gu *et al*. (1992).

Theorem 3.1: *Let the realization of the system* $G(s)$ *and the realization of the system* $G_a(s)$ *be given by* (18) *and* (26). *Then*

(*a*) LFT $(G(s), K) = \mathrm{LFT}(G_a(s), K_a)$; *and*

(*b*) $\mathrm{spec}(A + B(I - KF)^{-1}KC) \subset \mathscr{C}_\alpha$ *if and only if* $\mathrm{spec}(A_a + B_a(I - K_aF_a)^{-1}K_aC_a) \subset \mathrm{OLHP}$.

Proof: Note that the $\mathrm{LFT}(G_a(s), K_a)$ has the realization

$\mathrm{LFT}(G_a(s), K_a) \sim$

$$\left[\begin{array}{c|c} A_a + B_a(I - K_aF_a)^{-1}K_aC_a & D_{1a} + B_a(I - K_aF_a)^{-1}K_aD_{2a} \\ \hline E_{1a} + E_{2a}(I - K_aF_a)^{-1}K_aC_a & E_{2a}(I - K_aF_a)^{-1}K_aD_{2a} \end{array}\right]$$

Equivalently, this realization can be written as

$$\mathrm{LFT}(G_a(s), K_a) = [E_1 + E_2(I - KF)^{-1}KC \quad 0]$$

$$\cdot \begin{bmatrix} sI - (A + \alpha I + B(I - KF)^{-1}KC) & \alpha I \\ 0 & sI - (A + B(I - KF)^{-1}KC) \end{bmatrix}^{-1}$$

$$\cdot \begin{bmatrix} D_1 + B(I - KF)^{-1}KD_2 \\ D_1 + B(I - KF)^{-1}KD_2 \end{bmatrix} + E_2(I - KF)^{-1}KD_2$$

Letting $\Lambda = (I - KF)^{-1}$, it follows that

$$\begin{aligned} \mathrm{LFT}(G_a(s), K_a) &= (E_1 + E_2\Lambda KC)[sI - (A + \alpha I + B\Lambda KC)]^{-1}(D_1 + B\Lambda KD_2) \\ &\quad -\alpha(E_1 + E_2\Lambda KC)[sI - (A + \alpha I + B\Lambda KC)]^{-1} \\ &\quad \times [sI - (A + B\Lambda KC)]^{-1}(D_1 + B\Lambda KD_2) + E_2\Lambda KD_2 \\ &= (E_1 + E_2\Lambda KC)[sI - (A + B\Lambda KC)]^{-1}(D_1 + B\Lambda KD_2) + E_2\Lambda KD_2 \end{aligned}$$

This is equivalent to the realization $G_{zw}(s)$ given by (24). Hence, (*a*) is proved. To show (*b*), it is seen that $\mathrm{spec}(A_a + B_a(I - K_aF_a)^{-1}K_aC_a) =$

$\mathrm{spec}(A + B(I - KF)^{-1}KC) \cup [\alpha + \mathrm{spec}(A + B(I - KF)^{-1}KC)]$. Thus, if $A_a + B_a(I - K_aF_a)^{-1}K_aC_a$ is asymptotically stable then $\mathrm{spec}(A + B(I - KF)^{-1}KC)$ must be a subset of $\mathscr{C}_\alpha$. Conversely, if $\mathrm{spec}(A + B(I - KF)^{-1}KC)$ is in $\mathscr{C}_\alpha$ then it follows that $A_a + B_a(I - K_aF_a)^{-1}K_aC_a$ is asymptotically stable, which proves (b). □

Note that the closed-loop system formed from the augmented plant and controller, namely, $\mathrm{LFT}(G_a(s), K_a)$, is not minimal. In fact, it can be shown that $(A_a + B_a(I - K_aF_a)^{-1}K_aC_a,\ D_{1a} + B_a(I - K_aF_a)^{-1}K_aD_{2a})$ is not controllable. However, it can be shown that (A, B) is stabilizable if and only if (A_a, B_a) is stabilizable, while (A, C) is detectable if and only if (A_a, C_a) is detectable. Similarly, (A, D_1) is stabilizable if and only if (A_a, D_{1a}) is stabilizable, while (A, E_1) is detectable if and only if (A_a, E_{1a}) is detectable.

Now for the standard LQG control system (19)–(23), we first write (18) as decentralized static output feedback gains and then apply (26) in which $K = \text{block-diag}[A_c, B_c, C_c]$, so that the corresponding augmented system for constraining poles in $\mathscr{C}_\alpha$ is given by

$$\mathscr{A}_a = \begin{bmatrix} A+\alpha I & 0 & -\alpha I & 0 \\ 0 & \alpha I & 0 & -\alpha I \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathscr{B}_a = \begin{bmatrix} 0 & 0 & B & 0 & 0 & 0 \\ I & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B \\ 0 & 0 & 0 & I & I & 0 \end{bmatrix}, \quad \mathscr{C}_a = \begin{bmatrix} 0 & I & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

$$\mathscr{D}_{1a} = \begin{bmatrix} D_1 \\ 0 \\ D_1 \\ 0 \end{bmatrix}, \quad \mathscr{D}_{2a} = \begin{bmatrix} 0 \\ D_2 \\ 0 \\ 0 \\ D_2 \\ 0 \end{bmatrix}$$

$$\mathscr{E}_{1a} = [E_1 \quad 0 \quad 0 \quad 0], \quad \mathscr{E}_{2a} = [0 \quad 0 \quad E_2 \quad 0 \quad 0 \quad 0]$$

$$\mathscr{F}_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & F \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathscr{K}_a = \text{block-diag}(A_c, B_c, C_c, A_c, B_c, C_c)$$

Thus we can summarize the augmented system with an α-shifted constraint region for LQG controller design as follows. The cost is given by

$$J(A_c, B_c, C_c) = \operatorname{tr} \tilde{\mathfrak{Q}}\tilde{\mathfrak{R}} \tag{27}$$

where

$$\tilde{\mathfrak{R}} \triangleq (\mathscr{C}_{1a} + \mathscr{C}_{2a}(I + \mathscr{K}_a\mathscr{F}_a)\mathscr{K}_a\mathscr{C}_a)^{\mathrm{T}}(\mathscr{C}_{1a} + \mathscr{C}_{2a}(I + \mathscr{K}_a\mathscr{F}_a)\mathscr{K}_a\mathscr{C}_a)$$

and $\tilde{\mathfrak{Q}}$ satisfies

$$0 = \tilde{\mathscr{A}}\tilde{\mathfrak{Q}} + \tilde{\mathfrak{Q}}\tilde{\mathscr{A}}^{\mathrm{T}} + \tilde{\mathscr{V}}$$

where

$$\tilde{\mathscr{A}} \triangleq \mathscr{A}_a + \mathscr{B}_a(I + \mathscr{K}_a\mathscr{F}_a)\mathscr{K}_a\mathscr{C}_a$$

$$\tilde{\mathscr{V}} \triangleq (\mathscr{D}_{1a} + \mathscr{B}_a(I + \mathscr{K}_a\mathscr{F}_a)\mathscr{K}_a\mathscr{D}_{2a})(\mathscr{D}_{1a} + \mathscr{B}_a(I + \mathscr{K}_a\mathscr{F}_a)\mathscr{K}_a\mathscr{D}_{2a})^{\mathrm{T}}$$

Remark 3.1: The closed-loop dynamics $\tilde{\mathscr{A}}$ has order $2(n + n_c)$, where n_c is the order of the compensator. Hence it is possible to use our development given so far for calculating reduced-order decentralized controller gains. □

4. Numerical algorithm and illustrative results

The construction of the augmented plant in § 3 involves the repetition of a constant output feedback gain. Recall that the order of the augmented system is twice that of the original system. In this section we provide the basis for a computational algorithm for decentralized static output feedback with repeated gains. This computational technique, which has been considered in Seinfeld *et al.* (1991), involves the gradients of the Lagrangian with respect to the gains. As in Seinfeld *et al.*, we have chosen the BFGS algorithm to perform the numerical computation in which (14) provides the cost gradient and (10) is the cost. In our application the closed-loop system must always be stable while the algorithm is searching for the optimal decentralized gains. To guarantee stability, a sub-routine is used to decrease the step length so that the new search direction will not cause system instability. The optimization toolbox in the MATLAB software package has been utilized for implementing this algorithm.

In this section, we consider two examples. Both of the examples focus on H_2 cost minimization with closed-loop poles constrained to lie in $\mathscr{C}_\alpha$. In these examples, E_1 and E_2 are chosen so that $E_1^{\mathrm{T}}E_2 = 0$. Thus the state cost is given by $\operatorname{tr} E_1\tilde{Q}E_1^{\mathrm{T}}$, while the control cost is given by $\operatorname{tr}(\sum_{i=1}^{r} E_{2i}K_iC_i)\tilde{Q}(\sum_{i=1}^{r} E_{2i}K_iC_i)^{\mathrm{T}}$.

Example 1: Consider the mass-spring-damper system

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -0{\cdot}02 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

Our goal is to locate the closed-loop poles in $\mathscr{C}_\alpha$, where $\alpha = 1{\cdot}2$. We began the numerical calculation by carrying out Anderson and Moore's technique for pole shifting. The resulting control gains then serve as initial conditions for the augmentation approach developed in § 3. The results, which are given in Table 1, show improvement in both the state and control costs compared with

Table 1.

Methods	Results		
	State cost	Control cost	Eigenvalues
Open loop	100·1	–	$-0{\cdot}01 \pm j0{\cdot}998$
LQR	14·07	10·54	$-0{\cdot}0714 \pm j0{\cdot}998$
A & M's approach	1·804	3453	$-2{\cdot}391 \pm j0{\cdot}99$
Aug. approach	0·989	272·96	$-1{\cdot}201 \pm j0{\cdot}834$

Anderson and Moore's approach. We also observe that in the augmentation approach the closed-loop eigenvalues are placed near the boundary of the desired region $\mathscr{C}_\alpha$ whereas Anderson and Moore's approach pushes the closed-loop eigenvalues substantially into the desired region. This excessive pole shifting appears to account for the larger state control costs required by Anderson and Moore's approach. A plot of the state and control costs is shown in Fig. 1, in which we began with the cost corresponding to Anderson and Moore's approach. □

Example 2: In this example, taken with modification from Haddad and Bernstein (1992), we consider a simply supported uniform beam with force actuator and position sensor. The beam deflection $w(x, t)$ is governed by

$$m \frac{\partial^2 w(x, t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI \frac{\partial^2 w(x, t)}{\partial x^2}\right] = f(x, t)$$

with boundary conditions

$$w(x, t)\Big|_{x=0, L} = 0, \quad EI \frac{\partial^2 w(x, t)}{\partial x^2}\Big|_{x=0, L} = 0$$

where m is the beam mass and EI is the flexural rigidity. By standard modal decomposition, we have

$$w(x, t) = \sum_{r=1}^{\infty} W_r(x) q_r(t), \quad r = 1, 2, \ldots$$

where

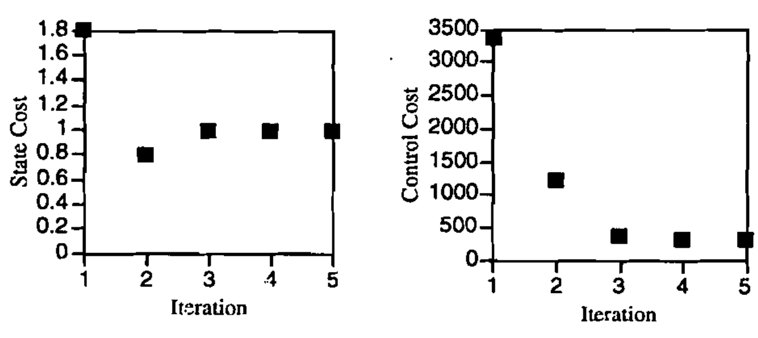

Figure 1.

$$\int_0^L mW_r^2(x)\,dx = 1, \quad W_r(x) = \left(\frac{2}{mL}\right)^{1/2} \sin\frac{r\pi x}{L}$$

Hence, in modal coordinates, it follows that

$$\ddot{q}_r(t) + 2\zeta\omega_r\dot{q}_r(t) + \omega_r^2 q_r(t) = \int_0^L f(x, t)W_r(x)\,dx, \quad r = 1, 2, \ldots$$

In this example, we place the sensor and actuator at $x = 0{\cdot}45L$ and $x = 0{\cdot}65L$, respectively. The disturbance is located at $x = 0{\cdot}7L$, while the performance variable corresponds to the transverse beam velocity at $0{\cdot}53L$. For simplicity, we truncate the model so that five low-frequency modes are considered. In this case, we set $\omega_i = i$, $L = \pi$, $m = EI = 2/\pi$ and $\zeta = 0{\cdot}01$ so that the state space model is given by

$$A = \underset{i=1,\ldots,5}{\text{block-diag}}\begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix}$$

$$B = [0 \quad 0{\cdot}891 \quad 0 \quad -0{\cdot}809 \quad 0 \quad -0{\cdot}156 \quad 0 \quad 0{\cdot}951 \quad 0 \quad -0{\cdot}7071]^{\mathrm{T}}$$

$$D_1 = \begin{bmatrix} 0 & 0{\cdot}809 & 0 & -0{\cdot}951 & 0 & 0{\cdot}309 & 0 & 0{\cdot}5878 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

$$D_2 = [0 \quad 1],$$

$$C = [0{\cdot}9877 \quad 0 \quad -0{\cdot}309 \quad 0 \quad -0{\cdot}891 \quad 0 \quad 0{\cdot}5878 \quad 0 \quad 0{\cdot}7071 \quad 0]$$

$$E_1 = \begin{bmatrix} 0 & 0{\cdot}9956 & 0 & -0{\cdot}1873 & 0 & -0{\cdot}9603 & 0 & 0{\cdot}3681 & 0 & 0{\cdot}891 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

Now, we wish to design an LQG compensator for this system such that the closed-loop poles are constrained to lie in $\mathscr{C}_\alpha$, where $\alpha = 0{\cdot}1$. As in the previous example, we use Anderson and Moore's approach to provide a starting point for the numerical optimization algorithm. Since LQG design involves three gains, the corresponding augmented system has three distinct gains with each gain repeated twice for a total of six gains. Using the algorithm developed in § 3, we obtain the results shown in Table 2. The state and control costs are shown in Fig. 2, while the closed-loop poles for both Anderson and Moore's approach and the transformation approach are plotted in Fig. 3. Note that for this problem the augmented system has order 20.

	Results	
Methods	State cost	Control cost
Open loop	26·83	–
LQG	18·52	2·23
A & M's approach	7·26	380·74
Aug. approach	6·73	228·94

Table 2.

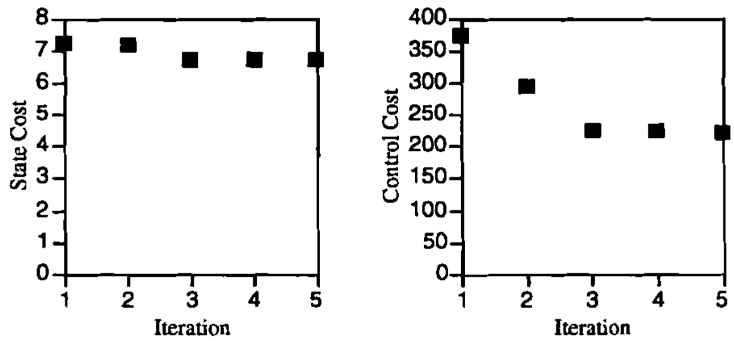

Figure 2.

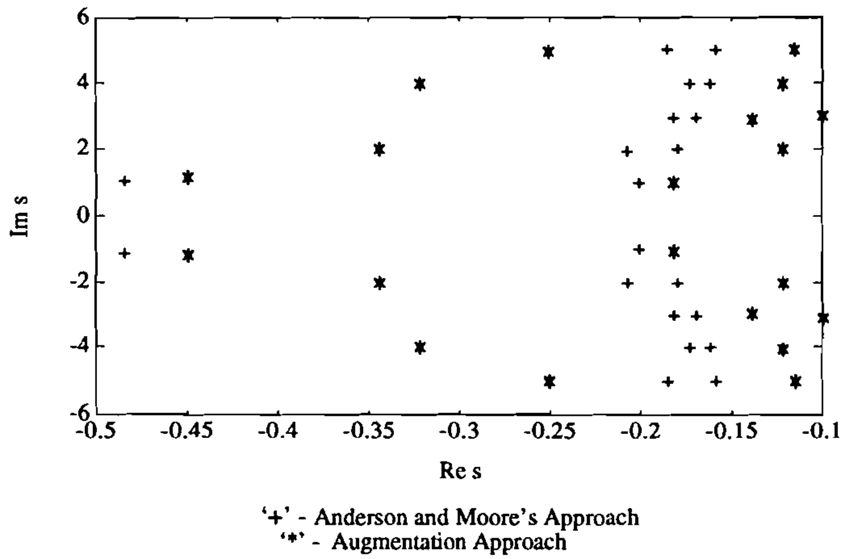

Figure 3. +, Anderson and Moore's approach; $*$, augmentation approach.

5. Conclusion

We have shown that it is possible to construct an α-shifted augmented system which is twice that of the original system such that the exact H_2 norm is preserved. We then developed an algorithm that calculates the decentralized repeated gains. Based on numerical results, it is then concluded that the new approach yields lower costs than Anderson and Moore's approach in the sense that both the state and control costs are smaller. This was to be expected, however, since Anderson and Moore's approach does not claim to solve an H_2 cost optimization problem. In principle, the new approach is applicable to more general constraint regions, for example, circular region, parabolic region and so on. Future investigation will focus on developing techniques for generating augmented realizations that characterize such constraint regions as considered in Haddad and Bernstein (1992) so that the corresponding augmented system can be used for controller synthesis.

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References

Anderson, B. D. O., and Moore, J. B., 1989, *Optimal Control Linear Quadratic Methods* (Englewood Cliffs, NJ: Prentice Hall).

Bernstein, D. S., Haddad, W. M., and Nett, C. N., 1989, Minimal complexity control law synthesis. Part 2: problem solution via H_2/H_∞ optimal static output feedback. *Proceedings of the American Control Conference*, pp. 2506–2512.

Brewer, J. W., 1978, Kronecker products and matrix calculus in system theory. *IEEE Transactions on Circuits and Systems*, **25**, 772–781.

Gu, G., Nett, C. N., and Xiong, D., 1992, Optimal constrained structure control law design subject to regional closed-loop pole constraints.

Haddad, W. M., and Bernstein, D. S., 1992, Controller design with regional pole constraints. *IEEE Transactions on Automatic Control*, **37**, 54–69.

Haddad, W. M., Bernstein, D. S., and Nett, C. N., 1989, Decentralized H_2/H_∞ controller design: the discrete-time case. *Proceedings of the IEEE Conference on Decision and Control*, pp. 932–933.

Nett, C. N., Bernstein, D. S., and Haddad, W. M., 1989, Minimal complexity control law synthesis. Part 1: problem solution via H_2/H_∞ optimal static output feedback. *Proceedings of the American Control Conference*, pp. 2056–2063.

Seinfeld, D. R., 1991, H_2/H_∞ Optimal controller synthesis via quasi-Newton methods. M. S. thesis, Florida Institute of Technology.

Seinfeld, D. R., Haddad, W. M., Bernstein, D. S., and Nett, C. N., 1991, H_2/H_∞ controller synthesis: illustrative numerical results via quasi-Newton methods. *Proceedings of the American Control Conference*, pp. 1155–1156.