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A Popov Criterion for Uncertain Linear Multivariable Systems*

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Key Words—Popov criterion; uncertain dynamic systems; robust stability; structured singular value.

Abstract—The Popov absolute stability criterion is traditionally proved using a Lyapunov function and the positive real lemma. In this paper a simplified proof of the multivariable Popov criterion is given for the case of one-sided, sector-bounded real parameter uncertainty. A loop-shifting transformation is then used to extend the Popov criterion to two-sided, sector-bounded uncertain matrices. Specialization of this result to norm-bounded uncertain matrices leads to an upper bound for the structured singular value for block-structured, real parameter uncertainty.

1. *Introduction*

Absolute stability theory has traditionally been used to analyze the stability of systems with unknown, sector-bounded nonlinearities represented as feedback elements (Hsu and Meyer, 1968; Narendra and Taylor, 1973; Khalil, 1992). In particular, the positivity and Popov criteria (Haddad and Bernstein, 1991) provide sufficient conditions for the stability of a linear system in a negative feedback interconnection with a nonlinear element. In the scalar case both of these absolute stability criteria guarantee stability by restricting the Nyquist plot of the nominal linear system to a specified region of the complex plane. In particular, positivity requires that the Nyquist plot lie in a half-plane that depends upon the uncertain sector containing the nonlinearity, while the less conservative Popov criterion utilizes a frequency-dependent multiplier to restrict the Nyquist plot to a frequency-dependent, rotated half-plane.

An alternative view of absolute stability theory has led to its use in robust stability analysis and synthesis (Haddad and Bernstein, 1991; Haddad, *et al.*, 1992; Chiang and Safonov, 1992; How and Hall, 1993). By specializing the nonlinear elements in the feedback path to the linear case, absolute stability theory provides sufficient conditions for robust stability with real parameter uncertainty. The connection between the Popov criterion and the structured singular value as a measure of robustness to constant, real parameter uncertainty was discussed in How and Hall (1992).

In this paper the uncertain system is represented as a nominal transfer function in a negative feedback interconnection with a matrix representing the uncertain parameters. The multivariable Popov criterion is proved for one-sided, sector-bounded, symmetric uncertain matrices. Whereas the Popov criterion is traditionally proved using a Lyapunov function and the positive real lemma, here the proof is based upon frequency-domain arguments. A loop-shifting transformation is used to extend the Popov criterion to two-sided, sector-bounded uncertain matrices. Finally, an upper bound for the structured singular value for robust stability with respect to real parameter uncertainty is formulated by specializing the shifted Popov criterion to block-structured, norm-bounded uncertain matrices.

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2. *The Popov criterion*

In this section we prove the multivariable Popov criterion for the case of real, symmetric, sector-bounded parameter uncertainty. Consider a square nominal transfer function $G(s)$ in a negative feedback interconnection with a real, square, uncertain matrix F as shown in Fig. 1, where F belongs to the set of symmetric, sector-bounded matrices

$$\mathcal{F} \triangleq \{F \in \mathbb{R}^{m\times m}: F = F^{\mathrm{T}}, 0 \le F \le M\},$$

where $M \in \mathbb{R}^{m\times m}$ is positive-definite. This type of sector bound is referred to as one-sided to denote the fact that the lower bound is zero. The following result provides alternative characterizations of $\mathcal{F}$.

Lemma 2.1. Let $F \in \mathbb{R}^{m\times m}$ by symmetric. Then the following statements are equivalent:

(i) $0 \le F \le M$;
(ii) $FM^{-1}F \le F$;
(iii) $\sigma_{\max}(M^{-1/2}FM^{-1/2} - \frac{1}{2}I) \le \frac{1}{2}$.

Proof. (i)$\Rightarrow$(ii): Suppose $0 \le F \le M$, so that $0 \le M^{-1/2}FM^{-1/2} \le I$. Since $M^{-1/2}FM^{-1/2}$ is symmetric, there exist an orthogonal matrix S and a real diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ such that $0 \le S\Lambda S^{\mathrm{T}} = M^{-1/2}FM^{-1/2} \le I$, so that $0 \le \lambda_i \le 1$, $i = 1, \ldots, m$. Hence, since $\Lambda^2 \le \Lambda$, it follows that

$$(M^{-1/2}FM^{-1/2})(M^{-1/2}FM^{-1/2}) = S\Lambda^2 S^{\mathrm{T}} \le S\Lambda S^{\mathrm{T}} \le M^{-1/2}FM^{-1/2},$$

and thus $FM^{-1}F \le F$.

(ii)$\Rightarrow$(iii): Suppose $FM^{-1}F \le F$, so that

$$M^{-1/2}FM^{-1/2}M^{-1/2}FM^{-1/2} \le M^{-1/2}FM^{-1/2}.$$

Adding $\frac{1}{4}I$ to each side and rearranging yields

$$(M^{-1/2}FM^{-1/2} - \tfrac{1}{2}I)(M^{-1/2}FM^{-1/2} - \tfrac{1}{2}I) \le \tfrac{1}{4}I,$$

which is equivalent to $\sigma_{\max}(M^{-1/2}FM^{-1/2} - \frac{1}{2}I) \le \frac{1}{2}$.

(iii)$\Rightarrow$(i): Suppose $\sigma_{\max}(M^{-1/2}FM^{-1/2} - \frac{1}{2}I) \le \frac{1}{2}$. Then

$$(M^{-1/2}FM^{-1/2} - \tfrac{1}{2}I)(M^{-1/2}FM^{-1/2} - \tfrac{1}{2}I) \le \tfrac{1}{4}I.$$

Rearranging yields

$$(M^{-1/2}FM^{-1/2})(M^{-1/2}FM^{-1/2}) \le M^{-1/2}FM^{-1/2},$$

so that $0 \le (M^{-1/2}FM^{-1/2})(M^{-1/2}FM^{-1/2}) \le M^{-1/2}FM^{-1/2}$. Since $M^{-1/2}FM^{-1/2}$ is symmetric, there exist an orthogonal matrix S and a real diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ such that $S\Lambda S^{\mathrm{T}} = M^{-1/2}FM^{-1/2}$. It follows that $0 \le S\Lambda^2 S^{\mathrm{T}} \le S\Lambda S^{\mathrm{T}}$, so that $0 \le \Lambda^2 \le \Lambda$ and thus $0 \le \lambda_i \le 1$, $i = 1, \ldots, m$.

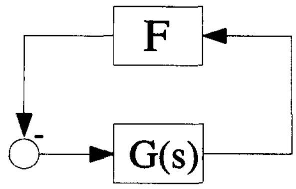

Fig. 1. Interconnection of transfer function $G(s)$ with real uncertain matrix F.

Hence $0 \le S\Lambda S^{T} \le I$, so that $0 \le M^{-1/2}FM^{-1/2} \le I$ and hence $0 \le F \le M$. □

For robust stability, we consider the set of real, symmetric, block-structured matrices $\mathscr{F}_{bs} \subset \mathscr{F}$ defined by

$$\mathscr{F}_{bs} \triangleq \{F \in \mathscr{F}: F = \text{block-diag}(I_{l_1} \otimes F_1, \ldots, I_{l_r} \otimes F_r),\ F_i = F_i^T \in \mathbb{R}^{m_i \times m_i}, i = 1, \ldots, r\},$$

where $A \otimes B$ denotes the Kronecker product of the matrices A and B, and where the dimension $m_i \times m_i$ of each block and the number of repetitions l_i of each block are given.

For the following lemma, define the set

$$\mathscr{N} \triangleq \{N \in \mathbb{R}^{m \times m}: N = \text{block-diag}(N_1 \otimes I_{m_1}, \ldots, N_r \otimes I_{m_r}),\ N_i = N_i^T \in \mathbb{R}^{l_i \times l_i}, i = 1, \ldots, r\}.$$

Note that if $N \in \mathscr{N}$ and $F \in \mathscr{F}_{bs}$ then $FN = NF = \text{block-diag}(N_1 \otimes F_1, \ldots, N_r \otimes F_r)$; that is, every element of $\mathscr{N}$ commutes with every element of $\mathscr{F}_{bs}$. Finally, the Hermitian part of G is denoted by $\text{He}\, G \triangleq \frac{1}{2}(G + G^*)$.

Lemma 2.2. Let $\omega \in \mathbb{R}$. If there exists $N \in \mathscr{N}$ such that

$$\text{He}\,[M^{-1} + (I + j\omega N)G(j\omega)] > 0 \qquad (1)$$

then $\det(I + G(j\omega)F) \neq 0$ for all $F \in \mathscr{F}_{bs}$.

Proof. Suppose that there exists $F \in \mathscr{F}_{bs}$ such that $\det[I + G(j\omega)F] = 0$. Then there exists $x \in \mathbb{C}^m$, $x \neq 0$, such that $[I + FG(j\omega)]x = 0$. Hence $x = -FG(j\omega)x$ and $x^* = -x^*G^*(j\omega)F$. Since, by Lemma 2.1, $FM^{-1}F \le F$, it follows that

$$\begin{aligned} 2x^*M^{-1}x &= 2x^*G^*(j\omega)FM^{-1}FG(j\omega)x \\ &\le x^*G^*(j\omega)FG(j\omega)x + x^*G^*(j\omega)FG(j\omega)x \\ &= -x^*[G^*(j\omega) + G(j\omega)]x, \end{aligned}$$

so that $x^*[2M^{-1} + G^*(j\omega) + G(j\omega)]x \le 0$.

Next note that, since (1) is equivalent to

$$2M^{-1} + G^*(j\omega) + G(j\omega) + j\omega[NG(j\omega) - G^*(j\omega)N] > 0,$$

it follows that

$$\begin{aligned} &x^*[2M^{-1} + G^*(j\omega) + G(j\omega)]x \\ &\quad > -j\omega x^*[NG(j\omega) - G^*(j\omega)N]x, \\ &\quad = -j\omega[x^*NG(j\omega)x - x^*G^*(j\omega)Nx] \\ &\quad = j\omega[x^*G^*(j\omega)FNG(j\omega)x - x^*G^*(j\omega)NFG(j\omega)x] \\ &\quad = j\omega x^*G^*(j\omega)(FN - NF)G(j\omega)x. \end{aligned}$$

Now $FN = NF$ implies that $x^*[2M^{-1} + G^*(j\omega) + G(j\omega)]x > 0$, which is a contradiction. Hence $\det[I + G(j\omega)F] \neq 0$ for all $F \in \mathscr{F}_{bs}$. □

We now prove the multivariable Popov criterion for one-sided, sector-bounded, uncertain matrices.

Theorem 2.1. Let $G(s)$ be an asymptotically stable transfer function. If there exists $N \in \mathscr{N}$ such that

$$\text{He}\,[M^{-1} + (I + sN)G(s)] > 0 \qquad (2)$$

for all $s = j\omega$ than the negative feedback interconnection of $G(s)$ and F is asymptotically stable for all $F \in \mathscr{F}_{bs}$.

Proof. Let $F \in \mathscr{F}_{bs}$ and $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$, where A is Hurwitz, so that the negative feedback interconnection of $G(s)$ and F is

$$[I + G(s)F]^{-1}G(s) \sim \left[\begin{array}{c|c} A - BFC & B \\ \hline C & 0 \end{array}\right].$$

Suppose that there exists $F \in \mathscr{F}_{bs}$ such that $[I + G(s)F]^{-1}G(s)$ is not asymptotically stable. Since A is Hurwitz, there exists $\varepsilon \in (0, 1]$ such that $A - \varepsilon BFC$ has an eigenvalue $j\hat{\omega}$ on the imaginary axis.

Next note that

$$\begin{aligned} \det[I + \varepsilon G(s)F] &= \det[I + \varepsilon C(sI - A)^{-1}BF] \\ &= \det[I + \varepsilon BFC(sI - A)^{-1}] \\ &= \det(sI - A)^{-1}\det[sI - (A - \varepsilon BFC)]. \end{aligned}$$

Hence

$$\det[I + \varepsilon G(j\hat{\omega})F] = \det(j\hat{\omega}I - A)^{-1} \det[j\hat{\omega}I - (A - \varepsilon BFC)] = 0.$$

However, since $\varepsilon F \in \mathscr{F}_{bs}$, Lemma 2.2 implies that $\det[I + \varepsilon G(j\hat{\omega})F] \neq 0$, which is a contradiction. □

Remark 2.1. The set of matrices $\mathscr{N}$ defined here is larger than that used by Haddad and Bernstein (1991), Haddad *et al.* (1992) and How and Hall (1993), who proved the Popov criterion by constructing a suitable Lyapunov function. In that case the matrices N must be nonnegative-definite so that the parameter-dependent Lyapunov candidate $x^TPx + x^TC^TFNCx$ is positive-definite.

By setting $N = 0$, we obtain the multivariable positivity criterion for one-sided, sector-bounded uncertain matrices.

Corollary 2.1. Let $G(s)$ be an asymptotically stable transfer function. If

$$\text{He}\,[M^{-1} + G(s)] > 0, \qquad (3)$$

for all $s = j\omega$ then the negative feedback interconnection of $G(s)$ and F is asymptotically stable for all $F \in \mathscr{F}_{bs}$.

3. *Extension to two-sided uncertainty*

In this section we use a loop-shifting transformation to extend the Popov criterion from the case of one-sided, sector-bounded, block-structured uncertain matrices to two-sided, sector-bounded, block-structured uncertain matrices. Let M_1 and M_2 be symmetric, block-structured matrices such that $M = M_2 - M_1$ is positive-definite. Then consider the set of real, symmetric, block-structured, two-sided, sector-bounded matrices defined by

$$\Delta_{bs} \triangleq \{\Delta \in \mathbb{R}^{m \times m}: \Delta - M_1 \in \mathscr{F}_{bs}\}.$$

Hence $F \in \mathscr{F}_{bs}$ if and only if $\Delta = F + M_1 \in \Delta_{bs}$. Note that if $\Delta \in \Delta_{bs}$ then $0 \le \Delta - M_1 \le M_2 - M_1$, and thus $M_1 \le \Delta \le M_2$, which is a two-sided sector bound.

Next define the shifted transfer function $G_s(s)$ by

$$G_s(s) \triangleq [I + G(s)M_1]^{-1}G(s).$$

Then the asymptotic stability of $G(s)$ in a negative feedback interconnection with the uncertainty $\Delta \in \Delta_{bs}$ is equivalent to the asymptotic stability of $G_s(s)$ in a negative feedback interconnection with the uncertainty $F \in \mathscr{F}_{bs}$. This equivalence can be seen from Fig. 2 (for details see Vidyasagar, 1993, pp. 340–341).

We now state the multivariable Popov criterion for the case of two-sided, sector-bounded uncertain matrices as a corollary to Theorem 2.1.

Corollary 3.1. Suppose $G_s(s)$ is asymptotically stable. If there exists $N \in \mathscr{N}$ such that

$$\text{He}\,[(M_2 - M_1)^{-1} + (I + sN)G_s(s)] > 0 \qquad (4)$$

for all $s = j\omega$ than the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \Delta_{bs}$.

Proof. Since $G_s(s)$ is asymptotically stable and there exists $N \in \mathscr{N}$ such that $\text{He}\,[(M_2 - M_1)^{-1} + (I + sN)G_s(s)] > 0$ for all $s = j\omega$, it follows from Theorem 2.1 that the negative

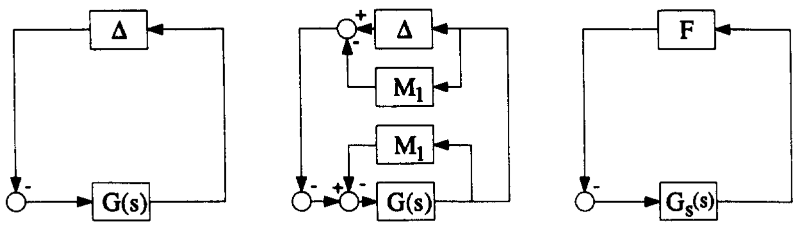

Fig. 2. Loop-shifting transformation.

feedback interconnection of $G_s(s)$ and F is asymptotically stable for all $F \in \mathscr{F}_{bs}$. By writing $F = \Delta - M_1$, it follows from the loop-shifting transformation shown in Fig. 2 that the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_{bs}$. □

4. *Specialization to norm-bounded, block-structured uncertainty*

Next we specialize the Popov criterion for two-sided, sector-bounded, block-structured uncertainty to the case of norm-bounded, block-structured uncertainty in order to obtain a bound on the structured singular value for real parameter uncertainty. Letting $M_1 = -\gamma^{-1}I$ and $M_2 = \gamma^{-1}I$, it follows that $M = 2\gamma^{-1}I$, so that $M^{-1} = \frac{1}{2}\gamma I$. The sets $\mathscr{F}$, $\mathscr{F}_{bs}$ and $\mathbf{\Delta}_{bs}$ thus become

$$\mathscr{F} = \{F \in \mathbb{R}^{m\times m} : 0 \le F \le 2\gamma^{-1}I\},$$

$$\mathscr{F}_{bs} \triangleq \{F \in \mathbb{R}^{m\times m} : F = \text{block-diag}\,(I_{l_1} \otimes F_1, \ldots, I_{l_r} \otimes F_r),\ 0 \le F_i \le 2\gamma^{-1}I_{m_i}, i = 1, \ldots, r\},$$

$$\mathbf{\Delta}_{bs} = \{\Delta \in \mathbb{R}^{m\times m} : \Delta + \gamma^{-1}I \in \mathscr{F}_{bs}\}.$$

Furthermore, since $-\gamma^{-1}I \le \Delta \le \gamma^{-1}I$, it follows that

$$\mathbf{\Delta}_{bs} = \{\Delta \in \mathbb{R}^{m\times m} : \Delta = \text{block-diag}\,(I_{l_1} \otimes \Delta_1, \ldots, I_{l_r} \otimes \Delta_r),\ \Delta_i = \Delta_i^{T} \in \mathbb{R}^{m_i\times m_i}, \sigma_{\max}(\Delta_i) \le \gamma^{-1}, i = 1, \ldots, r\}.$$

In addition, the shifted transfer function $G_s(s)$ becomes

$$G_s(s) = [I - \gamma^{-1}G(s)]^{-1}G(s).$$

We now state the Popov criterion for the case of norm-bounded, block-structured uncertain matrices as a corollary to Corollary 3.1.

Corollary 4.1. Let $\gamma > 0$ and suppose $G_s(s)$ is asymptotically stable. If there exists $N \in \mathscr{N}$ such that

$$\text{He}\,[\tfrac{1}{2}\gamma I + (I + sN)G_s(s)] > 0 \qquad (5)$$

for all $s = j\omega$ then the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_{bs}$.

Proof. This follows by letting $M_1 = -\gamma^{-1}I$ and $M_2 = \gamma^{-1}I$ in Corollary 3.1. □

Remark 4.1. A special case of the sets $\mathbf{\Delta}_{bs}$ and $\mathscr{N}$ is worth noting. Specifically, let $m_i = 1$, $i = 1, \ldots, r$, so that $\mathbf{\Delta}_{bs}$ is the set of diagonal matrices with possibly repeated real scalar elements given by

$$\mathbf{\Delta}_{bs} = \{\Delta \in \mathbb{R}^{m\times m} : \Delta = \text{block-diag}\,(\delta_1 I_{l_1}, \ldots, \delta_r I_{l_r m_r}),\ |\delta_i| \le \gamma^{-1}, i = 1, \ldots, r\}.$$

Then $\mathscr{N}$ is the set of real, symmetric, block-structured matrices given by

$$\mathscr{N} = \{N \in \mathbb{R}^{m\times m} : N = \text{block-diag}\,(N_1, \ldots, N_r),\ N_i = N_i^{T} \in \mathbb{R}^{l_i \times l_i}, i = 1, \ldots, r\}.$$

In this case $N\Delta = \Delta N = \text{block-diag}\,(\delta_1 N_1, \ldots, \delta_r N_r)$.

5. *Real structured singular value upper bound*

We now obtain an upper bound on the structured singular value for real parameter uncertainty. This bound is based upon the Popov criterion specialized to norm-bounded, block-structured uncertain matrices given in the previous section. The structured singular value (Doyle, 1985) of a complex matrix $G(j\omega)$ for real parameter uncertainty is defined by

$$\mu(G(j\omega)) \triangleq \frac{1}{\min\{\sigma_{\max}(\Delta) : \Delta \in \mathbf{\Delta}_{bs,0}\}},$$

where

$$\mathbf{\Delta}_{bs,0} \triangleq \{\Delta \in \mathbf{\Delta}_{bs} : \det\,[I + G(j\omega)\Delta] = 0\}.$$

If $\mathbf{\Delta}_{bs,0}$ is empty then $\mu(G(j\omega)) \triangleq 0$.

Next, define $\mu_{\text{Popov}}(G(j\omega))$ by

$$\mu_{\text{Popov}}(G(j\omega)) \triangleq \inf\{\gamma > 0 : \text{there exists } N \in \mathscr{N} \text{ such that } \text{He}\,[\tfrac{1}{2}\gamma I + (I + j\omega N)G_s(j\omega)] > 0\}.$$

To show that $\mu_{\text{Popov}}(G(j\omega))$ is an upper bound on $\mu(G(j\omega))$, we require the following intermediate result.

Lemma 5.1. Let $\omega \in \mathbb{R}$. If $\mu_{\text{Popov}}(G(j\omega)) < \gamma$ then there exists $N \in \mathscr{N}$ such that

$$\text{He}\,[\tfrac{1}{2}\gamma I + (I + j\omega N)G_s(j\omega)] > 0. \qquad (6)$$

Furthermore, $\det\,[I + G(j\omega)\Delta] \ne 0$ for all $\Delta \in \mathbf{\Delta}_{bs}$.

Proof. Since $\mu_{\text{Popov}}(G(j\omega)) < \gamma$, there exists γ_1 satisfying $\mu_{\text{Popov}}(G(j\omega)) < \gamma_1 < \gamma$ and such that there exists $N \in \mathscr{N}$ such that $\text{He}\,[\frac{1}{2}\gamma_1 I + (I + j\omega N)G_s(j\omega)] > 0$. Then $\text{He}\,[\frac{1}{2}\gamma I + (I + j\omega N)G_s(j\omega)] = \frac{1}{2}(\gamma - \gamma_1)I + \text{He}\,[\frac{1}{2}\gamma_1 I + (I + j\omega N)G_s(j\omega)] > 0$, which proves (6).

Next choose $\hat{\gamma} > \mu_{\text{Popov}}(G(j\omega))$ such that $\hat{\gamma}$ is not an eigenvalue of $G(j\omega)$. Applying Lemma 2.2 with $M = 2\hat{\gamma}^{-1}I$ and $G(j\omega)$ replaced by $\hat{G}_s(j\omega) \triangleq [I - \hat{\gamma}^{-1}G(j\omega)]^{-1}G(j\omega)$, the condition (6) with γ and G_s replaced by $\hat{\gamma}$ and $\hat{G}_s$ implies that $\det\,[I + \hat{G}_s(j\omega)F] \ne 0$ for all $F \in \mathscr{F}_{bs}$. Now let $\Delta \in \mathbf{\Delta}_{bs}$. Since $\Delta = F - \hat{\gamma}^{-1}I$, where $F \in \mathscr{F}_{bs}$, it thus follows that

$$\begin{aligned}
&\det\,[I + G(j\omega)\Delta]\\
&= \det\,[I + G(j\omega)(F - \hat{\gamma}^{-1}I)]\\
&= \det\,[I - \hat{\gamma}^{-1}G(j\omega) + G(j\omega)F]\\
&= \det\,[I - \hat{\gamma}^{-1}G(j\omega)]\det\{I + [I - \hat{\gamma}^{-1}G(j\omega)]^{-1}G(j\omega)F\}\\
&= \det\,[I - \hat{\gamma}^{-1}G(j\omega)]\det\,[I + \hat{G}_s(j\omega)F]\\
&\ne 0.
\end{aligned}$$

Hence, $\det\,[I + G(j\omega)\Delta] \ne 0$ for all $\Delta \in \mathbf{\Delta}_{bs}$. □

Theorem 5.1. Let $\omega \in \mathbb{R}$. Then

$$\mu(G(j\omega)) \le \mu_{\text{Popov}}(G(j\omega)). \qquad (7)$$

Proof. Suppose $\mu_{\text{Popov}}(G(j\omega)) < \mu(G(j\omega))$. By the definition of $\mu(G(j\omega))$, there exists $\Delta_0 \in \mathbf{\Delta}_{bs,0}$ such that $\sigma_{\max}(\Delta_0) = 1/\mu(G(j\omega))$. Therefore $\sigma_{\max}(\Delta_0) < 1/\mu_{\text{Popov}}(G(j\omega))$. Now, let γ satisfy $\sigma_{\max}(\Delta_0) < 1/\gamma < 1/\mu_{\text{Popov}}(G(j\omega))$. Then, since $\mu_{\text{Popov}}(G(j\omega)) < \gamma$ and $\Delta_0 \in \mathbf{\Delta}_{bs}$, Lemma 5.1 implies that $\det\,[I + G(j\omega)\Delta_0] \ne 0$, which contradicts the fact that $\det\,[I + G(j\omega)\Delta_0] = 0$. □

6. *Summary and conclusions*

An alternative, simplified proof has been given of the multivariable Popov absolute stability criterion for symmetric, block-structured, one-sided, sector-bounded uncertain matrices. A loop-shifting transformation is used to transform the Popov criterion to two-sided, sector-bounded uncertain matrices. Specialization to norm-bounded, block-structured uncertain matrices leads to a robust stability test for real parameter uncertainty. Finally, the Popov criterion has been used to formulate an upper bound for the structured singular value for real parameter uncertainty.

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