Infinite-Horizon Linear-Quadratic Control by Forward Propagation of the Differential Riccati Equation

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One of the foundational principles of optimal control theory is that optimal control laws are propagated backward in time. For linear-quadratic control, this means that the solution of the Riccati equation must be obtained from backward integration from a final-time condition. These features are a direct consequence of the transversality conditions of optimal control, which imply that a free final state corresponds to a fixed final adjoint state [1], [2]. In addition, the principle of dynamic programming and the associated Hamilton–Jacobi–Bellman equation is an inherently backward-propagating methodology [3].

The need for backward propagation means that, in practice, the control law must be computed in advance, stored, and then implemented forward in time. The control law may be either open loop or closed loop (as in the linear-quadratic case) but, in both cases, must be computed in advance. Fortunately, the dual case of optimal observers, such as the Kalman filter, is based on forward propagation of the error covariance and thus is more amenable to practical implementation.

For linear time-invariant (LTI) plants, a practical suboptimal solution is to implement the asymptotic control law based on the algebraic Riccati equation (ARE). For plants with linear time-varying (LTV) dynamics, perhaps arising from the linearization of a nonlinear plant about a specified trajectory, the main drawback of backward propagation is the fact that the future dynamics of the plant must be known. To circumvent this requirement, at least partially, various forward-propagating control laws have been developed, such as receding-horizon control and model predictive control [4]–[7]. Although these techniques require that the future dynamics of the plant be known, the control law is determined over a limited horizon, and thus the user can tailor the control law based on the available modeling information. Of course, all such control laws are suboptimal over the entire horizon.

An alternative approach to linear-quadratic control is to modify the sign of the Riccati equation and integrate forward, in analogy with the Kalman filter. This approach, which is described in [8] and [9], requires knowledge of the dynamics at only the present time. As shown in [9], stability is guaranteed for plants with symmetric closed-loop dynamics as well as for plants with sufficiently fast dynamics. However, a proof of stability for larger classes of plants remains open. Finally, the reinforcement learning approach of [10] is also based on forward integration, as is the "cost-to-come" technique in [11].

In view of the need for forward-integration techniques for control that depend only on the present dynamics, this article revisits linear-quadratic control for LTI plants. The first step is to review the basic features of the backward-propagating Riccati equation (BPRE), including the convergence of the Riccati solution to the ARE solution as the final time approaches infinity. These results are based on [12]–[15]. Stronger assumptions on the plant and cost weightings are adopted in this article to simplify the analysis. In particular, it is assumed that (A,B) is controllable and (A,C) is observable, whereas in [12]–[14], [16] the weaker assumptions that (A,B) is stabilizable and (A,C) is detectable are invoked.

The forward-propagating Riccati equation (FPRE) is introduced next, which is analogous to the BPRE but different due to the absence of the minus sign along with an initial condition rather than a final condition. We then show that the results for the BPRE have a dual form for the case of the FPRE. To emphasize the similarities and differences relative to the case of the BPRE, this section is presented in a parallel fashion.

Although the BPRE and FPRE can be viewed as dual equations, a crucial difference is the fact that the BPRE is meaningful over only a finite horizon, whereas the FPRE can be extended to infinity. This fact raises the question as to whether the FPRE control law is stabilizing. Since the solution of the FPRE converges exponentially to the solution of the ARE, it seems reasonable to conjecture that this is true. The main contribution of this article is to prove this fact. Since the control laws and Lyapunov function are both time varying, Lyapunov methods for time-varying systems are required. The required results can be found in [17].

Various examples are given to illustrate properties of the BPRE and FPRE. Of special interest is the case of unstable plants for which the closed-loop dynamics are unstable during the latter part of the time interval for the BPRE and the early part for the FPRE. State and control Pareto plots

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are presented to illustrate the suboptimality of the FPRE as well as the dependence on the initial condition of the FPRE. The article closes with numerical examples for LTV plants as motivation for future research in this direction.

The audience for this article includes two distinct groups of readers. For students learning optimal control, the analysis of the BPRE is largely self-contained and is not available in this compact form in optimal control textbooks. For experts in optimal control, the analysis of the FPRE shows that, for LTI plants, this approach yields closed-loop stability and motivates future research on the FPRE for LTV systems.

BPRE CONTROL

For $t \in [0, t_f]$, consider the LTI plant

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and (A, B) is stabilizable, with the finite-horizon quadratic cost function

$$J(u) = x^{\mathrm{T}}(t_f) P_f x(t_f) + \int_0^{t_f} [x^{\mathrm{T}}(t) R_1 x(t) + u^{\mathrm{T}}(t) R_2 u(t)]\,\mathrm{d}t, \tag{2}$$

where $R_1, P_f \in \mathbb{R}^{n\times n}$ are positive semidefinite and $R_2 \in \mathbb{R}^{m\times m}$ is positive definite. If (A, R_1) is observable, then the control $u:[0, t_f] \to \mathbb{R}^m$ that minimizes (2) is [2]

$$u(t) = K(t)x(t), \tag{3}$$

where

$$K(t) \triangleq -R_2^{-1} B^{\mathrm{T}} P(t) \tag{4}$$

and $P:[0, t_f] \to \mathbb{R}^{n\times n}$ satisfies the backward-in-time differential Riccati equation

$$-\dot{P}(t) = A^{\mathrm{T}} P(t) + P(t)A - P(t)SP(t) + R_1, \qquad P(t_f) = P_f, \tag{5}$$

where $S \triangleq BR_2^{-1}B^{\mathrm{T}}$. For $t \in [0, t_f]$, the closed-loop dynamics are

$$\dot{x}(t) = A_{\mathrm{cl}}(t)x(t), \tag{6}$$

where $A_{\mathrm{cl}}(t) \triangleq A + BK(t) = A - SP(t)$. Note that (5) can be written as

$$\begin{aligned} -\dot{P}(t) &= A_{\mathrm{cl}}^{\mathrm{T}}(t)P(t) + P(t)A_{\mathrm{cl}}(t) + P(t)SP(t) + R_1, \\ P(t_f) &= P_f. \end{aligned} \tag{7}$$

For $t_f = \infty$, the infinite horizon cost is

$$J(u) = \int_0^{\infty} [x^{\mathrm{T}}(t) R_1 x(t) + u^{\mathrm{T}}(t) R_2 u(t)]\,\mathrm{d}t, \tag{8}$$

and the optimal feedback law $u:[0, \infty) \to \mathbb{R}^m$ is

$$u(t) = \bar{K}x(t), \tag{9}$$

where $\bar{K} \triangleq -R_2^{-1}B^{\mathrm{T}}\bar{P}$ and, assuming that (A, R_1) has no unobservable eigenvalues on the imaginary axis, $\bar{P}$ is the unique positive-semidefinite stabilizing solution of the ARE

$$A^{\mathrm{T}}\bar{P} + \bar{P}A - \bar{P}S\bar{P} + R_1 = 0. \tag{10}$$

For $t \in [0, \infty)$, the asymptotically stable closed-loop dynamics are

$$\dot{x}(t) = \bar{A}x(t), \tag{11}$$

where $\bar{A} \triangleq A + B\bar{K} = A - S\bar{P}$. Note that (10) can be written as

$$\bar{A}^{\mathrm{T}}\bar{P} + \bar{P}\bar{A} + \bar{P}S\bar{P} + R_1 = 0. \tag{12}$$

Under stronger assumptions on A, B, and R_1 the following result holds.

Proposition 1

Assume that (A, B) is controllable and (A, R_1) is observable. Then $\bar{P}$ is positive definite and, for all $t > 0$,

$$W(t) \triangleq \int_0^t e^{\bar{A}s} S\, e^{\bar{A}^{\mathrm{T}}s}\mathrm{d}s \tag{13}$$

is positive definite. Furthermore, for all $t_2 > t_1 > 0$,

$$\bar{P} \le \bar{W}^{-1} < W^{-1}(t_2) < W^{-1}(t_1), \tag{14}$$

where

$$\bar{W} \triangleq \lim_{t\to\infty} W(t) = \int_0^{\infty} e^{\bar{A}s} S\, e^{\bar{A}^{\mathrm{T}}s}\mathrm{d}s \tag{15}$$

is positive definite and satisfies

$$\bar{A}\bar{W} + \bar{W}\bar{A}^{\mathrm{T}} + S = 0. \tag{16}$$

Proof

Corollary 12.19.2 of [18] implies that $\bar{P}$ is positive definite. Multiplying (12) on both sides by $\bar{P}^{-1}$ yields

$$\bar{A}\bar{P}^{-1} + \bar{P}^{-1}\bar{A}^{\mathrm{T}} + \bar{P}^{-1}R_1\bar{P}^{-1} + S = 0. \tag{17}$$

Subtracting (16) from (17) yields

$$\bar{A}(\bar{P}^{-1} - \bar{W}) + (\bar{P}^{-1} - \bar{W})\bar{A}^{\mathrm{T}} + \bar{P}^{-1}R_1\bar{P}^{-1} = 0.$$

Since $\bar{A}$ is asymptotically stable, it follows that

$$\bar{P}^{-1} - \bar{W} = \int_0^{\infty} e^{\bar{A}s}\bar{P}^{-1}R_1\bar{P}^{-1}e^{\bar{A}^{\mathrm{T}}s}\,\mathrm{d}s \ge 0.$$

Hence,

$$\bar{W} \le \bar{P}^{-1}. \tag{18}$$

Now, let $t_2 > t_1 > 0$. Then

$$\begin{aligned}W(t_2)-W(t_1) &= \int_0^{t_2} e^{\bar{A}s} S\, e^{\bar{A}^{\mathrm{T}}s} \mathrm{d}s - \int_0^{t_1} e^{\bar{A}s} S\, e^{\bar{A}^{\mathrm{T}}s} \mathrm{d}s \\ &= \int_{t_1}^{t_2} e^{\bar{A}s} S\, e^{\bar{A}^{\mathrm{T}}s} \mathrm{d}s \\ &= e^{\bar{A}t_1} \int_{t_1}^{t_2} e^{\bar{A}(s-t_1)} S\, e^{\bar{A}^{\mathrm{T}}(s-t_1)} \mathrm{d}s\, e^{\bar{A}^{\mathrm{T}}t_1} \\ &= e^{\bar{A}t_1} \int_0^{t_2-t_1} e^{\bar{A}s} S\, e^{\bar{A}^{\mathrm{T}}s} \mathrm{d}s\, e^{\bar{A}^{\mathrm{T}}t_1} \\ &= e^{\bar{A}t_1}\, W(t_2-t_1)\, e^{\bar{A}^{\mathrm{T}}t_1}. \end{aligned} \tag{19}$$

Since (A,B) is controllable, it follows that $(\bar{A},B)$ is controllable, and thus $W(t) > 0$ for all $t > 0$. Therefore, $W(t_2-t_1) > 0$, and thus (19) implies that $W(t_1) < W(t_2)$. Hence, $W^{-1}(t_2) < W^{-1}(t_1)$. Furthermore, $W(t_2) < \bar{W}$, and thus $\bar{W}$ is positive definite. Hence, (18) implies that $\bar{P} \le \bar{W}^{-1} < W^{-1}(t_2)$. ■

Theorem 1

Assume that (A,B) is controllable and (A,R_1) is observable. Then, for all $t \in [0,t_{\mathrm{f}}]$, the solution $P(t)$ of (5) is

$$P(t) = \bar{P} + e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)}(P_{\mathrm{f}}-\bar{P})[I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})]^{-1} e^{\bar{A}(t_{\mathrm{f}}-t)}, \tag{20}$$

where $W(t_{\mathrm{f}}-t)$ is given by (13). Furthermore, for all $t \in [0,t_{\mathrm{f}}]$, $P(t)$ is positive semidefinite, and, for all $t \ge 0$,

$$\lim_{t_{\mathrm{f}}\to\infty} P(t) = \bar{P}. \tag{21}$$

Now assume that $P_{\mathrm{f}}-\bar{P}$ is nonsingular. Then, for all $t \in [0,t_{\mathrm{f}}]$,

$$P(t) = \bar{P} + e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)}[(P_{\mathrm{f}}-\bar{P})^{-1} + W(t_{\mathrm{f}}-t)]^{-1} e^{\bar{A}(t_{\mathrm{f}}-t)}, \tag{22}$$

and

$$P(t) = \bar{P} + Z^{-1}(t). \tag{23}$$

$Z\colon [0,t_{\mathrm{f}}] \to \mathbb{R}^{n\times n}$ defined by

$$Z(t) \triangleq e^{\bar{A}(t-t_{\mathrm{f}})}[(P_{\mathrm{f}}-\bar{P})^{-1} + \bar{W}]e^{\bar{A}^{\mathrm{T}}(t-t_{\mathrm{f}})} - \bar{W} \tag{24}$$

is nonsingular and satisfies

$$\dot{Z}(t) = \bar{A}Z(t) + Z(t)\bar{A}^{\mathrm{T}} - S. \tag{25}$$

Proof

For all $0 \le t < t_{\mathrm{f}}$, Proposition 1 implies that $\bar{P} < W^{-1}(t_{\mathrm{f}}-t)$. Therefore, for all $t \in [0,t_{\mathrm{f}}]$,

$$\begin{aligned}&\det[I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})] \\ &\qquad = (\det W(t_{\mathrm{f}}-t)) \times \det[W^{-1}(t_{\mathrm{f}}-t) - \bar{P} + P_{\mathrm{f}}] > 0,\end{aligned}$$

and thus $I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})$ is nonsingular. In fact, $I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})$ is nonsingular for all $t \in [0,t_{\mathrm{f}}]$.

To show that (20) is symmetric, note that, for all $t \in [0,t_{\mathrm{f}}]$, $[I+(P_{\mathrm{f}}-\bar{P})W(t_{\mathrm{f}}-t)](P_{\mathrm{f}}-\bar{P}) = (P_{\mathrm{f}}-\bar{P})[I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})]$, and thus

$$\begin{aligned}&(P_{\mathrm{f}}-\bar{P})[I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})]^{-1} \\ &\qquad = [I+(P_{\mathrm{f}}-\bar{P})W(t_{\mathrm{f}}-t)]^{-1}(P_{\mathrm{f}}-\bar{P}) \\ &\qquad = [I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})]^{-\mathrm{T}}(P_{\mathrm{f}}-\bar{P}) \\ &\qquad = [(P_{\mathrm{f}}-\bar{P})[I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})]^{-1}]^{\mathrm{T}}.\end{aligned}$$

To show that, for all $t \in [0,t_{\mathrm{f}}]$, $P(t)$ is positive semidefinite, rewrite (5) as

$$\dot{P}(t) = -A_{\mathrm{cl}}^{\mathrm{T}}(t)P(t) - P(t)A_{\mathrm{cl}}(t) - P(t)SP(t) - R_1. \tag{26}$$

Then, for all $t \in [0,t_{\mathrm{f}}]$, $P(t)$ satisfies

$$P(t) = \Phi(t,t_{\mathrm{f}})P_{\mathrm{f}}\Phi^{\mathrm{T}}(t,t_{\mathrm{f}}) + \int_t^{t_{\mathrm{f}}} \Phi(t,s)[P(s)SP(s)+R_1]\Phi^{\mathrm{T}}(t,s)\,\mathrm{d}s, \tag{27}$$

which is positive semidefinite, where, for all $t,s \in [0,t_{\mathrm{f}}]$, the state transition matrix $\Phi(t,s)$ of the dual closed-loop system satisfies $(\partial/\partial t)\Phi(t,s) = -A_{\mathrm{cl}}^{\mathrm{T}}(\mathrm{t})\Phi(t,s)$, $\Phi(t,t) = I$. To show that (27) is the solution of (26), note that, by Leibniz's rule,

$$\begin{aligned}\dot{P}(t) &= \frac{\partial}{\partial t}\Phi(t,t_{\mathrm{f}})P_{\mathrm{f}}\Phi^{\mathrm{T}}(t,t_{\mathrm{f}}) + \Phi(t,t_{\mathrm{f}})P_{\mathrm{f}}\frac{\partial}{\partial t}\Phi^{\mathrm{T}}(t,t_{\mathrm{f}}) \\ &\quad + \int_t^{t_{\mathrm{f}}} \frac{\partial}{\partial t}\Phi(t,s)(P(s)SP(s)+R_1)\Phi^{\mathrm{T}}(t,s)\,\mathrm{d}s \\ &\quad + \int_t^{t_{\mathrm{f}}} \Phi(t,s)(P(s)SP(s)+R_1)\frac{\partial}{\partial t}\Phi^{\mathrm{T}}(t,s)\,\mathrm{d}s \\ &\quad - \Phi(t,t)(P(t)SP(t)+R_1)\Phi^{\mathrm{T}}(\mathrm{t},\mathrm{t}) \\ &= -A_{\mathrm{cl}}^{\mathrm{T}}(t)\Phi(t,t_{\mathrm{f}})P_{\mathrm{f}}\Phi^{\mathrm{T}}(t,t_{\mathrm{f}}) - \Phi(t,t_{\mathrm{f}})P_{\mathrm{f}}\Phi^{\mathrm{T}}(t,t_{\mathrm{f}})A_{\mathrm{cl}}(t) \\ &\quad - \int_t^{t_{\mathrm{f}}} A_{\mathrm{cl}}^{\mathrm{T}}(t)\Phi(t,s)(P(s)SP(s)+R_1)\Phi^{\mathrm{T}}(t,s)\,\mathrm{d}s \\ &\quad - \int_t^{t_{\mathrm{f}}} \Phi(t,s)(P(s)SP(s)+R_1)\Phi^{\mathrm{T}}(t,s)A_{\mathrm{cl}}(t)\,\mathrm{d}s \\ &\quad - P(t)SP(t) - R_1 \\ &= -A_{\mathrm{cl}}^{\mathrm{T}}(t)\big(\Phi t,t_f)P_f\Phi^T(t,t_f) \\ &\quad + \int_t^{t_{\mathrm{f}}} \Phi(t,s)(P(s)SP(s)+R_1)\Phi^T(t,s)\,\mathrm{d}s\big) \\ &\quad - \big(\Phi(t,t_f)P_f\Phi^T(t,t_f) \\ &\quad + \int_t^{t_{\mathrm{f}}} \Phi(t,s)(P(s)SP(s)+R_1)\Phi^T(t,s)\,\mathrm{d}s\big)A_{\mathrm{cl}}(t) \\ &\quad - P(t)SP(t) - R_1 \\ &= -A_{\mathrm{cl}}^{\mathrm{T}}(t)P(t) - P(t)A_{\mathrm{cl}}(t) - P(t)SP(t) - R_1.\end{aligned}$$

To show that (20) satisfies (5), note that $(\mathrm{d}/\mathrm{d}t)W(t_{\mathrm{f}}-t) = -e^{\bar{A}(t_{\mathrm{f}}-t)}Se^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)}$, and thus

$$\begin{aligned}\dot{P}(t) &= -\bar{A}^{\mathrm{T}}e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)}(P_{\mathrm{f}}-\bar{P})[I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})]^{-1}e^{\bar{A}(t_{\mathrm{f}}-t)} \\ &\quad - e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)}(P_{\mathrm{f}}-\bar{P})[I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})]^{-1}e^{\bar{A}(t_{\mathrm{f}}-t)}\bar{A} \\ &\quad - e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)}(P_{\mathrm{f}}-\bar{P})[I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})]^{-1} \\ &\quad \times\Big[\frac{\mathrm{d}}{\mathrm{d}t}W(t_{\mathrm{f}}-t)\Big](P_{\mathrm{f}}-\bar{P})[I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})]^{-1}e^{\bar{A}(t_{\mathrm{f}}-t)} \\ &= -\bar{A}^{\mathrm{T}}(P(t)-\bar{P}) - (P(t)-\bar{P})\bar{A} \\ &\quad + e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)}(P_{\mathrm{f}}-\bar{P})[I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})]^{-1}e^{\bar{A}(t_{\mathrm{f}}-t)}S \\ &\quad \times e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)}(P_{\mathrm{f}}-\bar{P})[I+W(t_{\mathrm{f}}-t)(P_{\mathrm{f}}-\bar{P})]^{-1}e^{\bar{A}(t_{\mathrm{f}}-t)} \\ &= -\bar{A}^{\mathrm{T}}(P(t)-\bar{P}) - (P(t)-\bar{P})\bar{A} + (P(t)-\bar{P})S(P(t)-\bar{P}) \\ &= -\bar{A}^{\mathrm{T}}P(t) - P(t)\bar{A} + P(t)SP(t) \\ &\quad - P(t)S\bar{P} - \bar{P}SP(t) - R_1 + \bar{A}^{\mathrm{T}}\bar{P} + \bar{P}\bar{A} + \bar{P}S\bar{P} + R_1\end{aligned}$$

$$= -\bar{A}^{\mathrm{T}} P(t) - P(t)\bar{A} + P(t)SP(t) - P(t)S\bar{P} - \bar{P}SP(t) - R_1$$
$$= -A^{\mathrm{T}} P(t) - P(t)A + P(t)SP(t) - R_1.$$

Since, for all $t \geq 0$, $e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)} \to 0$ as $t_{\mathrm{f}} \to \infty$, it follows from (20) that, for all $t \geq 0$,

$$\begin{aligned}
&\lim_{t_{\mathrm{f}}\to\infty} P(t) \\
&= \bar{P} + \lim_{t_{\mathrm{f}}\to\infty} [e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)} (P_{\mathrm{f}} - \bar{P})[I + W(t_{\mathrm{f}}-t)(P_{\mathrm{f}} - \bar{P})]^{-1} e^{\bar{A}(t_{\mathrm{f}}-t)}] \\
&= \bar{P} + \left(\lim_{t_{\mathrm{f}}\to\infty} e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)}\right)\left(\lim_{t_{\mathrm{f}}\to\infty} [(P_{\mathrm{f}} - \bar{P})[I + W(t_{\mathrm{f}}-t)(P_{\mathrm{f}} - \bar{P})]^{-1}]\right) \\
&\quad \times \lim_{t_{\mathrm{f}}\to\infty} e^{\bar{A}(t_{\mathrm{f}}-t)} \\
&= \bar{P}.
\end{aligned}$$

Now assume that $P_{\mathrm{f}} - \bar{P}$ is nonsingular. Then (20) implies (22). To show that (23) is equivalent to (22), note that

$$\begin{aligned}
&[e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)} [(P_{\mathrm{f}} - \bar{P})^{-1} + W(t_{\mathrm{f}}-t)]^{-1} e^{\bar{A}(t_{\mathrm{f}}-t)}]^{-1} \\
&= e^{-\bar{A}(t_{\mathrm{f}}-t)} \left[(P_{\mathrm{f}} - \bar{P})^{-1} + \int_0^{t_{\mathrm{f}}-t} e^{\bar{A}s} S e^{\bar{A}^{\mathrm{T}}s} \mathrm{d}s\right] e^{-\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)} \\
&= e^{\bar{A}(t-t_{\mathrm{f}})} \left[(P_{\mathrm{f}} - \bar{P})^{-1} + \int_0^{\infty} e^{\bar{A}s} S e^{\bar{A}^{\mathrm{T}}s} \mathrm{d}s\right] e^{\bar{A}^{\mathrm{T}}(t-t_{\mathrm{f}})} \\
&\quad - e^{-\bar{A}(t_{\mathrm{f}}-t)} \int_{t_{\mathrm{f}}-t}^{\infty} e^{\bar{A}s} S e^{\bar{A}^{\mathrm{T}}s} \mathrm{d}s \, e^{-\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)} \\
&= e^{\bar{A}(t-t_{\mathrm{f}})} [(P_{\mathrm{f}} - \bar{P})^{-1} + \bar{W}] e^{\bar{A}^{\mathrm{T}}(t-t_{\mathrm{f}})} \\
&\quad - \int_{t_{\mathrm{f}}-t}^{\infty} e^{\bar{A}(s-t_{\mathrm{f}}+t)} S e^{\bar{A}^{\mathrm{T}}(s-t_{\mathrm{f}}+t)} \mathrm{d}s \\
&= e^{\bar{A}(t-t_{\mathrm{f}})} [(P_{\mathrm{f}} - \bar{P})^{-1} + \bar{W}] e^{\bar{A}^{\mathrm{T}}(t-t_{\mathrm{f}})} - \bar{W} \\
&= Z(t).
\end{aligned}$$

Therefore, $Z(t)$ is nonsingular, and

$$Z^{-1}(t) = e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)} [(P_{\mathrm{f}} - \bar{P})^{-1} + W(t_{\mathrm{f}}-t)]^{-1} e^{\bar{A}(t_{\mathrm{f}}-t)},$$

which shows that (22) and (23) are equivalent.

To show that (24) satisfies (25), note that (16) implies that

$$\begin{aligned}
\dot{Z}(t) &= \bar{A} e^{\bar{A}(t-t_{\mathrm{f}})} [(P_{\mathrm{f}} - \bar{P})^{-1} + \bar{W}] e^{\bar{A}^{\mathrm{T}}(t-t_{\mathrm{f}})} \\
&\quad + e^{\bar{A}(t-t_{\mathrm{f}})} [(P_{\mathrm{f}} - \bar{P})^{-1} + \bar{W}] e^{\bar{A}^{\mathrm{T}}(t-t_{\mathrm{f}})} \bar{A}^{\mathrm{T}} \\
&= \bar{A}(Z(t) + \bar{W}) + (Z(t) + \bar{W})\bar{A}^{\mathrm{T}} \\
&= \bar{A}Z(t) + Z(t)\bar{A}^{\mathrm{T}} + \bar{A}\bar{W} + \bar{W}\bar{A}^{\mathrm{T}} \\
&= \bar{A}Z(t) + Z(t)\bar{A}^{\mathrm{T}} - S.
\end{aligned}$$

□

Note that (20) implies that

$$P(0) = \bar{P} + e^{\bar{A}^{\mathrm{T}} t_{\mathrm{f}}} (P_{\mathrm{f}} - \bar{P})[I + W(t_{\mathrm{f}})(P_{\mathrm{f}} - \bar{P})]^{-1} e^{\bar{A} t_{\mathrm{f}}} \tag{28}$$

and

$$\begin{aligned}
P(t_{\mathrm{f}}) &= \bar{P} + e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t_{\mathrm{f}})} (P_{\mathrm{f}} - \bar{P})[I + W(t_{\mathrm{f}}-t_{\mathrm{f}})(P_{\mathrm{f}} - \bar{P})]^{-1} e^{\bar{A}(t_{\mathrm{f}}-t_{\mathrm{f}})} \\
&= \bar{P} + P_{\mathrm{f}} - \bar{P} = P_{\mathrm{f}}.
\end{aligned} \tag{29}$$

The expression for $P(t)$ given by (23)–(25) is based on [19, pp. 418–419]. The expression for $P(t)$ given by (20) can be viewed as a superposition formula. For details, see [20].

Example 1

Consider the asymptotically stable plant

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & -1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{30}$$

and the unstable plant

$$A = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{31}$$

with $R_1 = I$ and $R_2 = 1$ for both plants. For (30), $\bar{P}$ is

$$\bar{P} = \begin{bmatrix} 1.6 & 0.6 \\ 0.6 & 0.6 \end{bmatrix}, \tag{32}$$

and for (31), $\bar{P}$ is

$$\bar{P} = \begin{bmatrix} 97.1 & 0.06 \\ 0.06 & 12.1 \end{bmatrix}. \tag{33}$$

For both plants, set

$$P_{\mathrm{f}} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}. \tag{34}$$

Figure 1 shows that, for each fixed t, $P(t)$ converges to $\bar{P}$ as t_{f} approaches infinity. ■

Proposition 2

Assume that $P_{\mathrm{f}} < \bar{P}$. Then, for all $t \in [0, t_{\mathrm{f}})$, $(P_{\mathrm{f}} - \bar{P})^{-1} + W(t_{\mathrm{f}} - t) < 0$ and $P(t) < \bar{P}$. If, in addition, P_{f} satisfies

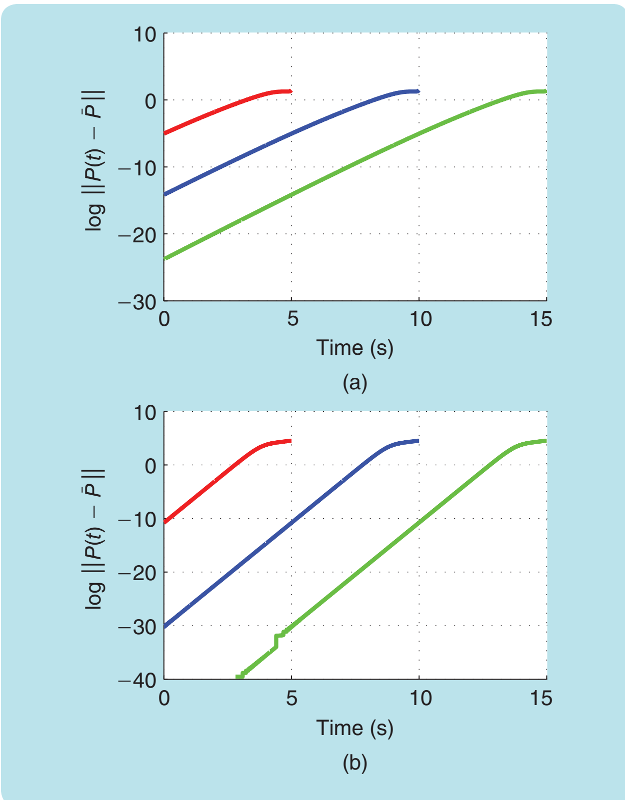


FIGURE 1 These plots illustrate Theorem 1 for the plants (30), (31), with P_{f} given by (34) and t_{f} equal to 5, 10, and 15 s. The norm denotes the largest singular value. For the asymptotically stable plant (30), (a) shows the convergence of $P(t)$ to $\bar{P}$ for each fixed t as t_{f} approaches infinity, whereas (b) shows the convergence of $P(t)$ to $\bar{P}$ for the unstable plant (31) for each fixed t as t_{f} approaches infinity.

$$A^{\mathrm{T}}P_{\mathrm{f}}+P_{\mathrm{f}}A-P_{\mathrm{f}}SP_{\mathrm{f}}+R_1\geq 0, \tag{35}$$

then, for all $t\in[0,t_{\mathrm{f}})$, $P_{\mathrm{f}}\leq P(t)$.

Proof

It follows from (14) and the fact that $P_{\mathrm{f}}\geq 0$ that, for all $t\in[0,t_{\mathrm{f}})$,

$$\bar{P}-P_{\mathrm{f}}\leq\bar{P}<W^{-1}(t_{\mathrm{f}}-t).$$

Therefore, since $\bar{P}-P_{\mathrm{f}}>0$, it follows that, for all $t\in[0,t_{\mathrm{f}})$,

$$W(t_{\mathrm{f}}-t)<(\bar{P}-P_{\mathrm{f}})^{-1}=-(P_{\mathrm{f}}-\bar{P})^{-1}.$$

Therefore, for all $t\in[0,t_{\mathrm{f}})$, $(P_{\mathrm{f}}-\bar{P})^{-1}+W(t_{\mathrm{f}}-t)$ is negative definite, and thus (22) implies that, for all $t\in[0,t_{\mathrm{f}})$, $P(t)<\bar{P}$.

Next, note that (35) can be written as

$$\begin{aligned}(\bar{A}+S\bar{P})^{\mathrm{T}}P_{\mathrm{f}}&+P_{\mathrm{f}}(\bar{A}+S\bar{P})-P_{\mathrm{f}}SP_{\mathrm{f}}+R_1\\&=-\bar{A}^{\mathrm{T}}(\bar{P}-P_{\mathrm{f}})-(\bar{P}-P_{\mathrm{f}})\bar{A}-(\bar{P}-P_{\mathrm{f}})S(\bar{P}-P_{\mathrm{f}})\geq 0,\end{aligned}$$

which implies

$$(\bar{P}-P_{\mathrm{f}})S(\bar{P}-P_{\mathrm{f}})\leq-\bar{A}^{\mathrm{T}}(\bar{P}-P_{\mathrm{f}})-(\bar{P}-P_{\mathrm{f}})\bar{A}. \tag{36}$$

Multiplying (36) on the left and right by $e^{\bar{A}s}(\bar{P}-P_{\mathrm{f}})^{-1}$ and $(\bar{P}-P_{\mathrm{f}})^{-1}e^{\bar{A}^{\mathrm{T}}s}$, respectively, yields

$$\begin{aligned}e^{\bar{A}s}Se^{\bar{A}^{\mathrm{T}}s}&\leq-e^{\bar{A}s}\bar{A}(\bar{P}-P_{\mathrm{f}})^{-1}e^{\bar{A}^{\mathrm{T}}s}-e^{\bar{A}s}(\bar{P}-P_{\mathrm{f}})^{-1}\bar{A}^{\mathrm{T}}e^{\bar{A}^{\mathrm{T}}s},\\&=-\frac{\mathrm{d}}{\mathrm{d}s}e^{\bar{A}s}(\bar{P}-P_{\mathrm{f}})^{-1}e^{\bar{A}^{\mathrm{T}}s}.\end{aligned}$$

Hence,

$$\begin{aligned}W(t_{\mathrm{f}}-t)&=\int_0^{t_{\mathrm{f}}-t}e^{\bar{A}s}Se^{\bar{A}^{\mathrm{T}}s}\mathrm{d}s\\&\leq-\int_0^{t_{\mathrm{f}}-t}\frac{\mathrm{d}}{\mathrm{d}s}e^{\bar{A}s}(\bar{P}-P_{\mathrm{f}})^{-1}e^{\bar{A}^{\mathrm{T}}s}\mathrm{d}s\\&=(\bar{P}-P_{\mathrm{f}})^{-1}-e^{\bar{A}(t_{\mathrm{f}}-t)}(\bar{P}-P_{\mathrm{f}})^{-1}e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)}.\end{aligned}$$

Thus,

$$(\bar{P}-P_{\mathrm{f}})^{-1}\leq e^{-\bar{A}(t_{\mathrm{f}}-t)}[(\bar{P}-P_{\mathrm{f}})^{-1}-W(t_{\mathrm{f}}-t)]e^{-\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)}. \tag{37}$$

Since, for all $t\in[0,t_{\mathrm{f}})$, $(\bar{P}-P_{\mathrm{f}})^{-1}-W(t_{\mathrm{f}}-t)$ is positive definite, (37) implies

$$e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)}[(\bar{P}-P_{\mathrm{f}})^{-1}-W(t_{\mathrm{f}}-t)]^{-1}e^{\bar{A}(t_{\mathrm{f}}-t)}\leq\bar{P}-P_{\mathrm{f}},$$

which is equivalent to

$$P_{\mathrm{f}}\leq\bar{P}+e^{\bar{A}^{\mathrm{T}}(t_{\mathrm{f}}-t)}[(P_{\mathrm{f}}-\bar{P})^{-1}+W(t_{\mathrm{f}}-t)]^{-1}e^{\bar{A}(t_{\mathrm{f}}-t)}=P(t). \quad \square$$

For $n=1$, we now show that $P_{\mathrm{f}}<\bar{P}$ implies that (35) holds. Therefore, in the scalar case, (35) need not be invoked as an assumption in Proposition 2. Define $a\triangleq A$, $b\triangleq B$, $p_{\mathrm{f}}\triangleq P_{\mathrm{f}}$, $\bar{p}\triangleq\bar{P}$, $r_1\triangleq R_1$, and $s\triangleq S$, and assume $R_2=1$. Then $s=b^2$, and the left-hand side of (35) can be written as

$$\begin{aligned}-b^2p_{\mathrm{f}}^2+2ap_{\mathrm{f}}+r_1&=-b^2p_{\mathrm{f}}^2+2ap_{\mathrm{f}}+r_1-(-b^2\bar{p}^2+2a\bar{p}+r_1)\\&=b^2(\bar{p}^2-p_{\mathrm{f}}^2)-2a(\bar{p}-p_{\mathrm{f}})\\&=b^2(\bar{p}-p_{\mathrm{f}})(\bar{p}+p_{\mathrm{f}})-2a(\bar{p}-p_{\mathrm{f}}).\end{aligned} \tag{38}$$

Furthermore, the solution $\bar{p}$ of (10) is

$$\bar{p}=\frac{1}{b^2}(a+\sqrt{a^2+b^2r_1}). \tag{39}$$

Since $\bar{p}-p_{\mathrm{f}}>0$, dividing (38) by $\bar{p}-p_{\mathrm{f}}$ and using (39) yields

$$\begin{aligned}b^2(\bar{p}+p_{\mathrm{f}})-2a&=b^2p_{\mathrm{f}}+a+\sqrt{a^2+b^2r_1}-2a\\&=b^2p_{\mathrm{f}}+\sqrt{a^2+b^2r_1}-a\\&>0,\end{aligned} \tag{40}$$

which implies (35). The following example shows that, for $n\geq 2$, $P_{\mathrm{f}}<\bar{P}$ does not imply (35).

Example 2

Consider the unstable plant

$$A=\begin{bmatrix}0&1\\-0.34&1.2\end{bmatrix},\quad B=\begin{bmatrix}0\\1\end{bmatrix}, \tag{41}$$

with $R_1=I$ and $R_2=1$. For this plant, $\bar{P}$ is

$$\bar{P}=\begin{bmatrix}2.48&0.71\\0.71&3.17\end{bmatrix}. \tag{42}$$

We consider two choices of P_{f}, namely,

$$P_{\mathrm{f}}=\begin{bmatrix}0.8&0.3\\0.3&1.2\end{bmatrix} \tag{43}$$

and

$$P_{\mathrm{f}}=\begin{bmatrix}2&0.1\\0.1&0.5\end{bmatrix}. \tag{44}$$

For P_{f} given by (43), condition (35) is satisfied, whereas, for P_{f} given by (44), condition (35) is not satisfied. Figure 2 shows that, with (43), for all $t\in[0,5]$ s, $P_{\mathrm{f}}\leq P(t)<\bar{P}$, whereas, with (44), for all $t\in[0,5]$ s, $P(t)<\bar{P}$, but $P_{\mathrm{f}}\leq P(t)$ does not hold for all $t\in[0,5]$ s. ■

Numerical examples suggest that (35) is a necessary condition for $P_{\mathrm{f}}\leq P(t)$ on $[0,t_{\mathrm{f}})$. Proof of this conjecture is open.

The following result complements Proposition 2.

Proposition 3

Assume that $\bar{P}<P_{\mathrm{f}}$. Then, for all $t\in[0,t_{\mathrm{f}})$, $(P_{\mathrm{f}}-\bar{P})^{-1}+W(t_{\mathrm{f}}-t)>0$ and $\bar{P}<P(t)$. If, in addition, P_{f} satisfies

$$A^{\mathrm{T}}P_{\mathrm{f}}+P_{\mathrm{f}}A-P_{\mathrm{f}}SP_{\mathrm{f}}+R_1\leq 0, \tag{45}$$

then, for all $t\in[0,t_{\mathrm{f}})$, $P(t)\leq P_{\mathrm{f}}$.

If $P_{\mathrm{f}}>\bar{P}$, then (22) implies that, for all $t\in[0,t_{\mathrm{f}}]$, $P(t)$ is positive semidefinite, and thus (27) is not needed. In fact, if $P_{\mathrm{f}}>\bar{P}$, then it follows from (22) that, for all $t\in[0,t_{\mathrm{f}})$, $P(t)$ is positive definite. Unfortunately, it does not seem to be possible to avoid using (27) for arbitrary positive-semidefinite P_{f}.

FPRE CONTROL

The FPRE control law replaces (5) with the forward-in-time differential Riccati equation [9]

$$\dot{P}(t) = A^{\mathrm{T}}P(t) + P(t)A - P(t)SP(t) + R_1, \ \ P(0) = P_0, \qquad (46)$$

where P_0 is positive semidefinite. Note that (46) can be written as

$$\dot{P}(t) = A_{\mathrm{cl}}^{\mathrm{T}}(t)P(t) + P(t)A_{\mathrm{cl}}(t) + P(t)SP(t) + R_1, \ P(0) = P_0, \ (47)$$

which differs from (5) due to the minus sign and the initial condition. Otherwise, (3) and (6) remain unchanged. However, the FPRE control law is not guaranteed to minimize (2).

Theorem 2

Assume that (A, B) is controllable and (A, R_1) is observable. Then, for all $t \geqslant 0$, $I + W(t)(P_0 - \bar{P})$ is nonsingular, and the positive-semidefinite solution $P(t)$ of (46) is

$$P(t) = \bar{P} + e^{\bar{A}^{\mathrm{T}}t}(P_0 - \bar{P})[I + W(t)(P_0 - \bar{P})]^{-1}e^{\bar{A}t}, \qquad (48)$$

where $W(t)$ is given by (13) and, for all $t \geq 0$,

$$\lim_{t \to \infty} P(t) = \bar{P}. \qquad (49)$$

If $P_0 - \bar{P}$ is nonsingular, then, for all $t \geq 0$,

$$P(t) = \bar{P} + e^{\bar{A}^{\mathrm{T}}t}[(P_0 - \bar{P})^{-1} + W(t)]^{-1}e^{\bar{A}t} \qquad (50)$$

and

$$P(t) = \bar{P} + Z^{-1}(t). \qquad (51)$$

$Z: [0, \infty) \to \mathbb{R}^{n \times n}$, defined by

$$Z(t) \triangleq e^{-\bar{A}t}[(P_0 - \bar{P})^{-1} + \bar{W}]e^{-\bar{A}^{\mathrm{T}}t} - \bar{W}, \qquad (52)$$

is nonsingular and satisfies

$$\dot{Z}(t) = -\bar{A}Z(t) - Z(t)\bar{A}^{\mathrm{T}} + S. \qquad (53)$$

Proof

To show that, for all $t > 0$, $I + W(t)(P_0 - \bar{P})$ is nonsingular, note that, from Proposition 1, $\bar{P} < W^{-1}(t)$. Therefore, for all $t > 0$,

$$\det[I + W(t)(P_0 - \bar{P})] = (\det W(t))\det[W^{-1}(t) - \bar{P} + P_0] > 0.$$

Thus, for all $t > 0$, $I + W(t)(P_0 - \bar{P})$ is nonsingular. In fact, for all $t \geq 0$, $I + W(t)(P_0 - \bar{P})$ is nonsingular.

To show that (48) is symmetric, note that, for all $t \geq 0$,

$$[I + (P_0 - \bar{P})W(t)](P_0 - \bar{P}) = (P_0 - \bar{P})[I + W(t)(P_0 - \bar{P})].$$

Thus

$$\begin{aligned}(P_0 - \bar{P})[I + W(t)(P_0 - \bar{P})]^{-1} &= [I + (P_0 - \bar{P})W(t)]^{-1}(P_0 - \bar{P}) \\ &= [I + W(t)(P_0 - \bar{P})]^{-T}(P_0 - \bar{P}) \\ &= [(P_0 - \bar{P})[I + W(t)(P_0 - \bar{P})]^{-1}]^{\mathrm{T}}.\end{aligned}$$

To show that, for all $t \geq 0$, $P(t)$ is positive semidefinite, note that it follows from (47) that

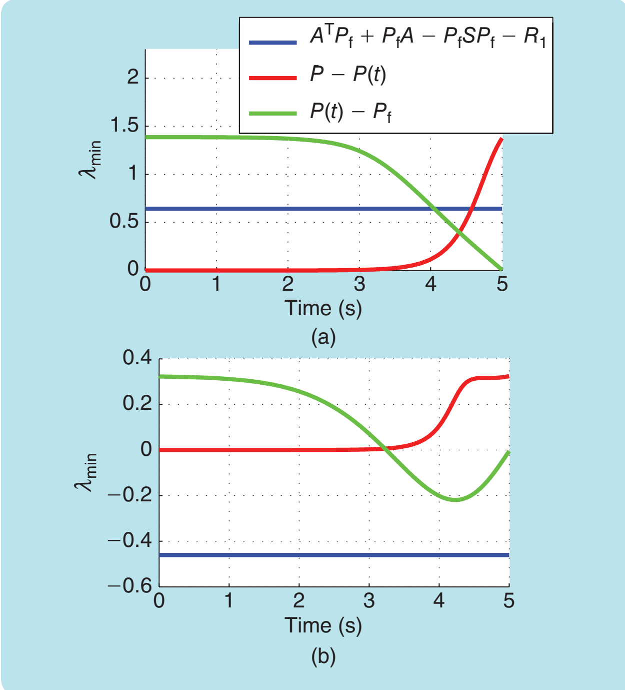


FIGURE 2 These plots illustrate Proposition 2 for the unstable plant (41) with P_f given by (43) and (44), and $t_f = 5$ s. $\lambda_{\min}$ denotes the minimum eigenvalue. For P_f given by (43), (a) shows that (35) is satisfied, and, for all $t \in [0,5]$ s, $P_f \leq P(t) < \bar{P}$. On the other hand, for P_f given by (44), (b) shows that (35) is not satisfied, for all $t \in [0,5]$ s, $P(t) < \bar{P}$, and, for all $t \in [3.2, 5]$ s, $P_f \leq P(t)$ does not hold.

$$P(t) = \Phi(t,0)P_0\Phi^{\mathrm{T}}(t,0) + \int_0^t \Phi(t,s)[P(s)SP(s) + R_1]\Phi^{\mathrm{T}}(t,s)\,\mathrm{d}s \qquad (54)$$

is positive semidefinite, where, for all $t, s \in [0, \infty)$, the state transition matrix $\Phi(t,s)$ of the closed-loop system satisfies $\partial/\partial t\,\Phi(t,s) = A_{\mathrm{cl}}^{T}(t)\Phi(t,s)$, $\Phi(t,t) = I$. To show that (54) satisfies (47), note that, by Leibniz's rule

$$\begin{aligned}\dot{P}(t) &= \frac{\partial}{\partial t}\Phi(t,0)P_0\Phi^{\mathrm{T}}(t,0) + \Phi(t,0)P_0\frac{\partial}{\partial t}\Phi^{\mathrm{T}}(t,0) \\ &\quad + \int_0^t \frac{\partial}{\partial t}\Phi(t,s)[P(s)SP(s) + R_1]\Phi^{\mathrm{T}}(t,s)\,\mathrm{d}s \\ &\quad + \int_0^t \Phi(t,s)[P(s)SP(s) + R_1]\frac{\partial}{\partial t}\Phi^{\mathrm{T}}(t,s)\,\mathrm{d}s \\ &\quad + P(t)SP(t) + R_1 \\ &= A_{\mathrm{cl}}^{\mathrm{T}}(t)\,\Phi(t,0)P_0\Phi^{\mathrm{T}}(t,0) + \Phi(t,0)P_0\Phi^{\mathrm{T}}(t,0)A_{\mathrm{cl}}(t) \\ &\quad + \int_0^t A_{\mathrm{cl}}^{\mathrm{T}}(t)\,\Phi(t,s)[P(s)SP(s) + R_1]\Phi^{\mathrm{T}}(t,s)\,\mathrm{d}s \\ &\quad + \int_0^t \Phi(t,s)[P(s)SP(s) + R_1]\Phi^{\mathrm{T}}(t,s)A_{\mathrm{cl}}(t)\,\mathrm{d}s \\ &\quad + P(t)SP(t) + R_1 \\ &= A_{\mathrm{cl}}^{\mathrm{T}}(t)\Big[\Phi(t,0)P_0\Phi^{T}(t,0) \\ &\quad + \int_0^t \Phi(t,s)[P(s)SP(s) + R_1]\Phi^{\mathrm{T}}(t,s)\,\mathrm{d}s\Big] \\ &\quad + \Big[\Phi(t,0)P_0\Phi^{\mathrm{T}}(t,0) \\ &\quad + \int_0^t \Phi(t,s)[P(s)SP(s) + R_1]\Phi^{\mathrm{T}}(t,s)\,\mathrm{d}s\Big]A_{\mathrm{cl}}(t) \\ &\quad + P(t)SP(t) + R_1 \\ &= A_{\mathrm{cl}}^{\mathrm{T}}(t)P(t) + P(t)A_{\mathrm{cl}}(t) + P(t)SP(t) + R_1. \end{aligned} \qquad (55)$$

Next, we show that (48) satisfies (46). Note that $\mathrm{d}/\mathrm{d}t\, W(t) = e^{\bar{A}t} S e^{\bar{A}^{\mathrm{T}}t}$, and thus

$$\begin{aligned}
\dot{P}(t) &= \bar{A}^{\mathrm{T}} e^{\bar{A}^{\mathrm{T}}t}(P_0-\bar{P})[I+W(t)(P_0-\bar{P})]^{-1} e^{\bar{A}t} \\
&\quad + e^{\bar{A}^{\mathrm{T}}t}(P_0-\bar{P})[I+W(t)(P_0-\bar{P})]^{-1} e^{\bar{A}t}\bar{A} \\
&\quad - e^{\bar{A}^{\mathrm{T}}t}(P_0-\bar{P})[I+W(t)(P_0-\bar{P})]^{-1} \\
&\quad \frac{\mathrm{d}}{\mathrm{d}t} W(t)(P_0-\bar{P})[I+W(t)(P_0-\bar{P})]^{-1} e^{\bar{A}t} \\
&= \bar{A}^{\mathrm{T}}(P(t)-\bar{P}) + (P(t)-\bar{P})\bar{A} - (P(t)-\bar{P})S(P(t)-\bar{P}) \\
&= \bar{A}^{\mathrm{T}}P(t) + P(t)\bar{A} - P(t)SP(t) + P(t)S\bar{P} + \bar{P}SP(t) + R_1 \\
&\quad - (\bar{A}^{\mathrm{T}}\bar{P} + \bar{P}\bar{A} + \bar{P}S\bar{P} + R_1) \\
&= \bar{A}^{\mathrm{T}}P(t) + P(t)\bar{A} - P(t)SP(t) + P(t)S\bar{P} + \bar{P}SP(t) + R_1 \\
&= A^{\mathrm{T}}P(t) + P(t)A - P(t)SP(t) + R_1.
\end{aligned}$$

Since $e^{\bar{A}^{\mathrm{T}}t} \to 0$ as $t \to \infty$, it follows from (48) that

$$\begin{aligned}
\lim_{t\to\infty} P(t) &= \bar{P} + \lim_{t\to\infty}[e^{\bar{A}^{\mathrm{T}}t}(P_0-\bar{P})[I+W(t)\,(P_0-\bar{P})]^{-1} e^{\bar{A}t}] \\
&= \bar{P} + \left(\lim_{t\to\infty} e^{\bar{A}^{\mathrm{T}}t}\right) \\
&\quad \times \left(\lim_{t\to\infty}[(P_0-\bar{P})[I+W(t)\,(P_0-\bar{P})]^{-1}\right)\lim_{t\to\infty} e^{\bar{A}t} \\
&= \bar{P}.
\end{aligned}$$

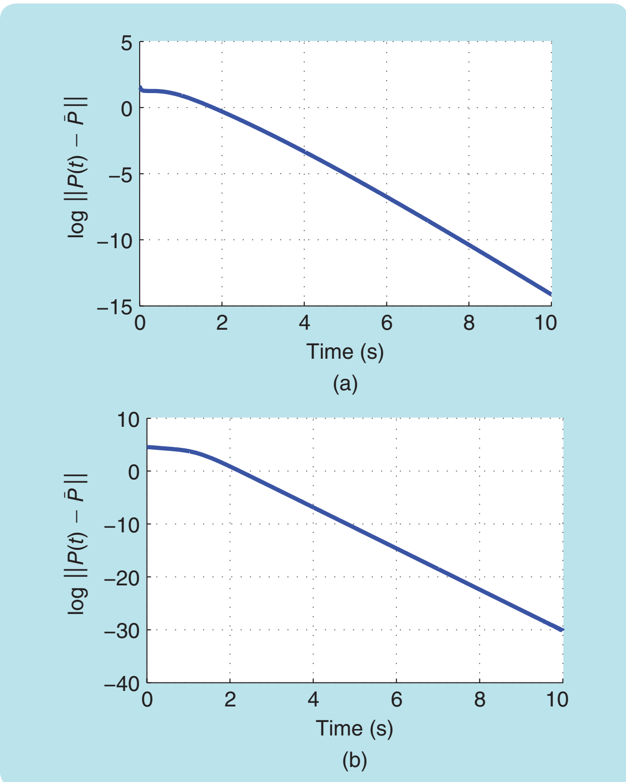


FIGURE 3 These plots illustrate Theorem 2 for the plants (30) and (31) with P_0 equal to P_{f} given by (34). The norm is the largest singular value. The solutions are valid on $[0,\infty)$. For the asymptotically stable plant (30), (a) shows the convergence of $P(t)$ to $\bar{P}$ as $t \to \infty$, whereas (b) shows the convergence of $P(t)$ to $\bar{P}$ for the unstable plant (31) as $t \to \infty$.

Now, assume that $P_0 - \bar{P}$ is nonsingular. Then (50) follows from (48). To show that (51) is equivalent to (50), note that

$$\begin{aligned}
&[e^{\bar{A}^{\mathrm{T}}t}[(P_0-\bar{P})^{-1} + W(t)]^{-1} e^{\bar{A}t}]^{-1} \\
&\quad = e^{-\bar{A}t}\left[(P_0-\bar{P})^{-1} + \int_0^t e^{\bar{A}s} S\, e^{\bar{A}^{\mathrm{T}}s}\mathrm{d}s\right] e^{-\bar{A}^{\mathrm{T}}t} \\
&\quad = e^{-\bar{A}t}\left[(P_0-\bar{P})^{-1} + \int_0^\infty e^{\bar{A}s} S\, e^{\bar{A}^{\mathrm{T}}s}\mathrm{d}s\right] e^{-\bar{A}^{\mathrm{T}}t} \\
&\qquad - e^{-\bar{A}t}\int_t^\infty e^{\bar{A}s} S\, e^{\bar{A}^{\mathrm{T}}s}\mathrm{d}s\, e^{-\bar{A}^{\mathrm{T}}t} \\
&\quad = e^{-\bar{A}t}[(P_0-\bar{P})^{-1} + \bar{W}]\, e^{-\bar{A}^{\mathrm{T}}t} \\
&\qquad - \int_t^\infty e^{\bar{A}(s-t)} S\, e^{\bar{A}^{\mathrm{T}}(s-t)}\mathrm{d}s \\
&\quad = e^{-\bar{A}t}[(P_0-\bar{P})^{-1} + \bar{W}]\, e^{-\bar{A}^{\mathrm{T}}t} - \bar{W} \\
&\quad = Z(t).
\end{aligned}$$

Therefore, for all $t > 0$, $Z(t)$ is nonsingular, and

$$Z^{-1}(t) = e^{\bar{A}^{\mathrm{T}}t}[(P_0-\bar{P})^{-1} + W(t)]^{-1} e^{\bar{A}t},$$

which implies that (50) and (51) are equivalent.

To show that (52) satisfies (53), note that (16) implies that

$$\begin{aligned}
\dot{Z}(t) &= -\bar{A}e^{-\bar{A}t}[(P_0-\bar{P})^{-1} + \bar{W}]e^{-\bar{A}^{\mathrm{T}}t} \\
&\quad - e^{-\bar{A}t}[(P_0-\bar{P})^{-1} + \bar{W}]e^{-\bar{A}^{\mathrm{T}}t}\bar{A}^{\mathrm{T}} \\
&= -\bar{A}(Z(t)+\bar{W}) - (Z(t)+\bar{W})\bar{A}^{\mathrm{T}} \\
&= -\bar{A}Z(t) - Z(t)\bar{A}^{\mathrm{T}} - \bar{A}\bar{W} - \bar{W}\bar{A}^{\mathrm{T}} \\
&= -\bar{A}Z(t) - Z(t)\bar{A}^{\mathrm{T}} + S.
\end{aligned}$$ □

Note that (48) implies that

$$\begin{aligned}
P(0) &= \bar{P} + e^{\bar{A}^{\mathrm{T}}0}(P_0-\bar{P})[I+W(0)\,(P_0-\bar{P})]^{-1} e^{\bar{A}0} \\
&= \bar{P} + P_0 - \bar{P} = P_0
\end{aligned} \tag{56}$$

and

$$P(t_{\mathrm{f}}) = \bar{P} + e^{\bar{A}^{\mathrm{T}}t_{\mathrm{f}}}(P_0-\bar{P})[I+W(t_{\mathrm{f}})\,(P_0-\bar{P})]^{-1} e^{\bar{A}t_{\mathrm{f}}}. \tag{57}$$

Example 3

Consider the plants given by (30) and (31), with $R_1 = I$ and $R_2 = 1$, where $\bar{P}$ is given by (32) and (33), respectively, and P_0 is equal to P_{f} given by (34). Figure 3 shows that $P(t)$ converges to $\bar{P}$. Note that the FPRE solutions $P(t)$ given in Figure 3(a) and (b) on the interval [0, 10] s are the mirror image of the corresponding BPRE solutions $P(t)$ in Figure 1(a) and (b) on the same interval. However, unlike the BPRE solution, the FPRE solution can be extended to $[0,\infty)$. ■

The following result is the FPRE version of Proposition 2.

Proposition 4

Assume that $P_0 < \bar{P}$. Then, for all $t \in [0,\infty)$, $(P_0-\bar{P})^{-1} + W(t) < 0$ and $P(t) < \bar{P}$. If, in addition, P_0 satisfies

$$A^{\mathrm{T}}P_0 + P_0A - P_0SP_0 + R_1 \geq 0, \tag{58}$$

then, for all $t \in [0,\infty)$, $P_0 \leq P(t)$.

Proof

From (14) and the fact that $P_0 \geq 0$, it follows that, for all $t \in [0, \infty)$,

$$\bar{P} - P_0 \leq \bar{P} < W^{-1}(t).$$

Therefore, since $\bar{P} - P_0$ is positive definite, it follows that, for all $t \in [0, \infty)$,

$$W(t) < (\bar{P} - P_0)^{-1} = -(P_0 - \bar{P})^{-1}.$$

Therefore, for all $t \in [0, \infty)$, $(P_0 - \bar{P})^{-1} + W(t)$ is negative definite, and thus (50) implies that, for all $t \in [0, \infty)$, $P(t) < \bar{P}$.

Next, note that (58) can be written as

$$\begin{aligned}(\bar{A} + S\bar{P})^{\mathrm{T}} P_0 + P_0(\bar{A} + S\bar{P}) - P_0 S P_0 + R_1 \\ = -\bar{A}^{\mathrm{T}}(\bar{P} - P_0) - (\bar{P} - P_0)\bar{A} - (\bar{P} - P_0) S (\bar{P} - P_0) \geq 0,\end{aligned}$$

which implies

$$(\bar{P} - P_0) S (\bar{P} - P_0) \leq -\bar{A}^{\mathrm{T}}(\bar{P} - P_0) - (\bar{P} - P_0)\bar{A}. \tag{59}$$

Multiplying (59) on the left and right by $e^{\bar{A}s}(\bar{P} - P_0)^{-1}$ and $(\bar{P} - P_0)^{-1} e^{\bar{A}^{\mathrm{T}}s}$, respectively, yields

$$\begin{aligned}e^{\bar{A}s} S e^{\bar{A}^{\mathrm{T}}s} &\leq -e^{\bar{A}s}\bar{A}(\bar{P} - P_0)^{-1} e^{\bar{A}^{\mathrm{T}}s} - e^{\bar{A}s}(\bar{P} - P_0)^{-1}\bar{A}^{\mathrm{T}} e^{\bar{A}^{\mathrm{T}}s}, \\ &= -\frac{\mathrm{d}}{\mathrm{d}s} e^{\bar{A}s}(\bar{P} - P_0)^{-1} e^{\bar{A}^{\mathrm{T}}s}.\end{aligned}$$

Hence,

$$\begin{aligned}W(t) &= \int_0^t e^{\bar{A}s} S e^{\bar{A}^{\mathrm{T}}s} \mathrm{d}s \\ &\leq -\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} e^{\bar{A}s}(\bar{P} - P_0)^{-1} e^{\bar{A}^{\mathrm{T}}s} \mathrm{d}s, \\ &= (\bar{P} - P_0)^{-1} - e^{\bar{A}t}(\bar{P} - P_0)^{-1} e^{\bar{A}^{\mathrm{T}}t}.\end{aligned}$$

Thus,

$$(\bar{P} - P_0)^{-1} \leq e^{-\bar{A}t}[(\bar{P} - P_0)^{-1} - W(t)] e^{-\bar{A}^{\mathrm{T}}t}. \tag{60}$$

Since, for all $t \in [0, \infty)$, $(\bar{P} - P_0)^{-1} - W(t)$ is positive definite, (60) can be written as

$$e^{\bar{A}^{\mathrm{T}}t}[(\bar{P} - P_0)^{-1} - W(t)]^{-1} e^{\bar{A}t} \leq \bar{P} - P_0,$$

which is equivalent to

$$P_0 \leq \bar{P} + e^{\bar{A}^{\mathrm{T}}t}[(P_0 - \bar{P})^{-1} + W(t)]^{-1} e^{\bar{A}t} = P(t). \qquad \square$$

For $n = 1$, we now show that $P_0 < \bar{P}$ implies that (58) holds. Therefore, in the scalar case, (58) need not be invoked as an assumption in Proposition 4. Define $a \triangleq A$, $b \triangleq B$, $p_0 \triangleq P_0$, $\bar{p} \triangleq \bar{P}$, $r_1 \triangleq R_1$, and $s \triangleq S$, and assume $R_2 = 1$. Then $s = b^2$, and the left-hand side of (58) can be written as

$$\begin{aligned}-b^2 p_0^2 + 2ap_0 + r_1 &= -b^2 p_0^2 + 2ap_0 + r_1 - (-b^2\bar{p}^2 + 2a\bar{p} + r_1) \\ &= b^2(\bar{p}^2 - p_0^2) - 2a(\bar{p} - p_0) \\ &= b^2(\bar{p} - p_0)(\bar{p} + p_0) - 2a(\bar{p} - p_0).\end{aligned} \tag{61}$$

Since $\bar{p} - p_0 > 0$, dividing (61) by $\bar{p} - p_0$ and using (39) yields

$$\begin{aligned}b^2(\bar{p} + p_0) - 2a &= b^2 p_0 + a + \sqrt{a^2 + b^2 r_1} - 2a \\ &= b^2 p_0 + \sqrt{a^2 + b^2 r_1} - a \\ &> 0,\end{aligned} \tag{62}$$

which implies (58).

The following example shows that, for $n \geq 2$, $P_0 < \bar{P}$ does not imply (58).

Example 4

Consider the unstable plant

$$A = \begin{bmatrix} 0 & 1 \\ -1.5 & 2.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{63}$$

with $R_1 = I$ and $R_2 = 1$, where

$$\bar{P} = \begin{bmatrix} 8.8 & 0.3 \\ 0.3 & 5.3 \end{bmatrix}. \tag{64}$$

Two choices of P_0 are considered, namely,

$$P_0 = \begin{bmatrix} 0.6 & -0.5 \\ -0.5 & 1.4 \end{bmatrix} \tag{65}$$

and

$$P_0 = \begin{bmatrix} 2 & 1.8 \\ 1.8 & 4 \end{bmatrix}. \tag{66}$$

For P_0 given by (65), condition (58) is satisfied, whereas, for P_0 given by (66), condition (58) is not satisfied. Figure 4 shows that, with (65), for all $t \in [0, \infty)$, $P_0 \leq P(t) < \bar{P}$, whereas, with (66), for all $t \in [0, \infty)$, $P(t) < \bar{P}$, but $P_0 \leq P(t)$ does not hold for all $t \in [0, \infty)$. ■

Numerical examples suggest that (58) is a necessary condition for $P_0 \leq P(t)$ on $[0, \infty)$. Proof of this conjecture is open.

The following result, which complements Proposition 4, is the FPRE version of Proposition 3.

Proposition 5

Assume that $\bar{P} < P_0$. Then, for all $t \in [0, \infty)$, $(P_0 - \bar{P})^{-1} + W(t) > 0$ and $\bar{P} < P(t)$. If, in addition, P_0 satisfies

$$A^{\mathrm{T}} P_0 + P_0 A - P_0 S P_0 + R_1 \leq 0, \tag{67}$$

then, for all $t \in [0, \infty)$, $P(t) \leq P_0$.

If $P_0 > \bar{P}$, then (50) implies that $P(t)$ is positive semidefinite for all $t \in [0, \infty)$, and thus (54) is not needed. In fact, if $P_0 > \bar{P}$, then it follows from (50) that, for all $t \in [0, \infty)$, $P(t)$ is positive definite. Unfortunately, it does not seem to be possible to avoid using (54) for arbitrary positive-semidefinite P_0.

Lyapunov Analysis of the FPRE

We now state several definitions and a result from [17] that are used below.

Definition 1

Let $a > 0$ and $\gamma : [0, a) \to [0, \infty)$. Then γ is of class $\mathcal{K}$ if $\gamma(0) = 0$ and γ is continuous and strictly increasing.

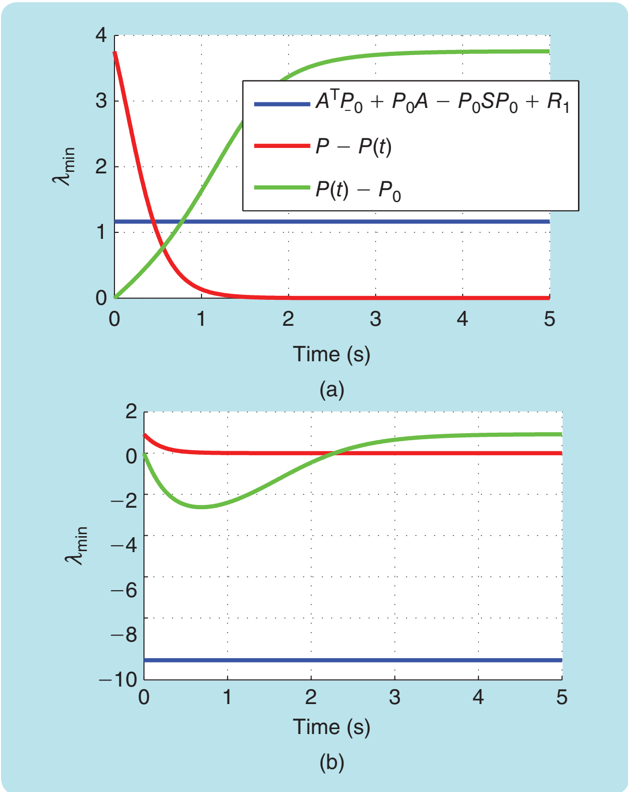


FIGURE 4 These plots illustrate Proposition 4 for the plant (63) with P_0 given by (65) and (66). The solutions are shown for the time interval $[0,5]$ s and are valid on $[0,\infty)$. $\lambda_{\min}$ denotes the minimum eigenvalue. For P_0 given by (65), (a) shows that (58) is satisfied, and, for all $t\in[0,5]$ s, $P_0 \le P(t) < \bar{P}$, whereas, for P_0 given by (66), (b) shows that (58) is not satisfied, and, for all $t\in[0,5]$ s, $P(t) < \bar{P}$, and, for all $t\in[0,2.4]$ s, $P_0 \le P(t)$ does not hold.

Definition 2

Let $f:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$. The solution $x(t)\equiv 0$ of the system $\dot{x}(t)=f(t,x(t))$ is Lyapunov stable if, for all $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that, if $\|x(0)\|<\delta$, then, for all $t\ge 0, \|x(t)\|<\varepsilon$.

Definition 3

Let $f:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$. The solution $x(t)\equiv 0$ of the system $\dot{x}(t)=f(t,x(t))$ is asymptotically stable if it is Lyapunov stable and there exists $\delta>0$ such that, if $\|x(0)\|<\delta$, then $\lim_{t\to\infty}x(t)=0$.

Theorem 3

Let $f:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$ and assume that, for all $t\ge 0$ and $x_0\in\mathbb{R}^n$,

$$\dot{x}(t)=f(t,x(t)), \quad x(0)=x_0, \tag{68}$$

has a unique continuously differentiable solution. Furthermore, assume that there exist continuously differentiable functions $V:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ and $W:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$, and class $\mathcal{K}$ functions α, β, and γ such that $\dot{W}(t,x)$ is bounded from above, and

$$V(t,0)=0, \quad t\in[0,\infty), \tag{69}$$

$$\alpha(\|x\|)\le V(t,x), \quad t\in[0,\infty)\times\mathbb{R}^n, \tag{70}$$

$$W(t,0)=0, \quad t\in[0,\infty), \tag{71}$$

$$\beta(\|x\|)\le W(t,x), \quad t\in[0,\infty)\times\mathbb{R}^n, \tag{72}$$

$$\dot{V}(t,x)\le -\gamma(W(t,x)), \quad t\in[0,\infty)\times\mathbb{R}^n, \tag{73}$$

where

$$\dot{V}(t,x)\triangleq\frac{\partial V}{\partial t}(t,x)+\frac{\partial V}{\partial x}(t,x)f(t,x) \tag{74}$$

and

$$\dot{W}(t,x)\triangleq\frac{\partial W}{\partial t}(t,x)+\frac{\partial W}{\partial x}(t,x)f(t,x). \tag{75}$$

Then the zero solution $x(t)\equiv 0$ to (68) is asymptotically stable.

Theorem 4

Assume that (A,B) is controllable, R_1 is positive definite, and consider plant (1) with the control law (3) and feedback gain (4), where, for all $t\in[0,\infty)$, $P(t)$ is the positive-semidefinite solution of the FPRE (46). Then $\lim_{t\to\infty}x(t)=0$.

Proof

Consider the Lyapunov function candidate

$$V(t,x)\triangleq x^{\mathrm{T}}P(t)x, \tag{76}$$

which satisfies (69). Since $P(t)$ converges to $\bar{P}$ as $t\to\infty$ and $\bar{P}$ is positive definite, there exist $T_1>0$ and $\alpha_1>0$ such that, for all $t>T_1$, $\lambda_{\min}(P(t))>\alpha_1$. Therefore, for all $t>T_1$ and $x\in\mathbb{R}^n$,

$$V(t,x)\ \ge\alpha(\|x\|), \tag{77}$$

where $\alpha:[0,\infty)\to[0,\infty)$ defined by

$$\alpha(z)\triangleq\alpha_1 z^2 \tag{78}$$

is of class $\mathcal{K}$. Hence, $V(t,x)$ satisfies (70) with $[0,\infty)$. replaced by $[T_1,\infty)$.

Define

$$E(t)\triangleq P(t)-\bar{P}=e^{\bar{A}^{\mathrm{T}}t}(P_0-\bar{P})[I+W(t)(P_0-\bar{P})]^{-1}e^{\bar{A}t} \tag{79}$$

and

$$f(t,x)\triangleq(A-SP(t))x. \tag{80}$$

Then, for all $t\ge 0$ and $x\in\mathbb{R}^n$, (74) implies that

$$\begin{aligned}\dot{V}(t,x)&=x^{\mathrm{T}}\dot{P}(t)x+x^{\mathrm{T}}(A-SP(t))^{\mathrm{T}}P(t)x+x^{\mathrm{T}}P(t)(A-SP(t))x\\&=x^{\mathrm{T}}[2A^{\mathrm{T}}P(t)+2P(t)A-3P(t)SP(t)+R_1]x\\&=x^{\mathrm{T}}[2(A^{\mathrm{T}}\bar{P}+\bar{P}A-\bar{P}S\bar{P}+R_1)-\bar{P}S\bar{P}-R_1+2A^{\mathrm{T}}E(t)\\&\quad+2E(t)A-3\bar{P}SE(t)-3E(t)S\bar{P}-3E(t)SE(t)]x\\&=-x^{\mathrm{T}}[\bar{P}S\bar{P}+R_1-Q(t)]x\\&\le -x^{\mathrm{T}}[R_1-Q(t)]x,\end{aligned} \tag{81}$$

where

$$Q(t) \triangleq 2[A^{\mathrm{T}}E(t)+E(t)A]-3[\bar{P}SE(t)+E(t)S\bar{P}+E(t)SE(t))] \tag{82}$$

is symmetric. Since $E(t)\to 0$ as $t\to\infty$, it follows that $Q(t)\to 0$ as $t\to\infty$. Since R_1 is positive definite, there exist $T_2 > T_1$ and $\alpha_2 > 0$ such that, for all $t > T_2$, $R_1 - Q(t) > \alpha_2 I$. Therefore, for all $t > T_2$ and $x\in\mathbb{R}^n$,

$$\dot{V}(t,x)\le -W(t,x), \tag{83}$$

where

$$W(t,x)\triangleq \alpha_2 x^{\mathrm{T}}\bar{P}x, \tag{84}$$

which satisfies (71). Furthermore, for all $t > T_2$ and $x\in R^n$,

$$W(t,x)\ge \beta(\|x\|), \tag{85}$$

where $\beta:[0,\infty)\to[0,\infty)$ defined by

$$\beta(z)\triangleq \alpha_2\lambda_{\min}(\bar{P})z^2 \tag{86}$$

is of class $\mathcal{K}$. Hence, $W(t,x)$ satisfies (72) with $[0,\infty)$ replaced by $[T_2,\infty)$.

To show that, for all $t\in[0,\infty)$ and $x\in\mathbb{R}^n$, $\dot{W}(t,x)$ is bounded from above, note that (75) implies that

$$\begin{aligned}\dot{W}(t,x) &= \alpha_2[x^{\mathrm{T}}(A-SP(t))^{\mathrm{T}}\bar{P}x + x^{\mathrm{T}}\bar{P}(A-SP(t))x]\\ &= \alpha_2 x^{\mathrm{T}}(A_{\mathrm{cl}}^{\mathrm{T}}(t)\bar{P}+\bar{P}A_{\mathrm{cl}}(t))x.\end{aligned} \tag{87}$$

Since $P(t)$ converges to $\bar{P}$ as $t\to\infty$, it follows that

$$\lim_{t\to\infty}(A_{\mathrm{cl}}^{\mathrm{T}}(t)\bar{P}+\bar{P}A_{\mathrm{cl}}(t)) = \bar{A}^{\mathrm{T}}\bar{P}+\bar{P}\bar{A}. \tag{88}$$

Note that, from (12),

$$\bar{A}^{\mathrm{T}}\bar{P}+\bar{P}\bar{A} = -(\bar{P}S\bar{P}+R_1)<0. \tag{89}$$

Hence, there exists $T_4 > T_3$ such that, for all $t > T_4$, $A_{\mathrm{cl}}(t)\bar{P}+\bar{P}A_{\mathrm{cl}}(t)<0$, and thus (87) implies that, for all $t > T_4$ and $x\in\mathbb{R}^n$, $\dot{W}(t,x)\le 0$. Therefore, for all $t > T_4$ and $x\in\mathbb{R}^n$, $\dot{W}(t,x)$ is bounded from above.

To show that $V(t,x)$ satisfies (73), note that (83) implies that

$$\dot{V}(t,x)\le -\gamma(W(t,x)), \tag{90}$$

where $\gamma:[0,\infty)\to[0,\infty)$ defined by

$$\gamma(z)\triangleq z \tag{91}$$

is of class $\mathcal{K}$.

As a result, (69) – (73) hold with $[0,\infty)$ replaced by $[T_4,\infty)$. Then, Theorem 3 implies that, for all $t\in[T_4,\infty)$, $x(t)\to 0$ as $t\to 0$. □

Note that Theorem 4 does not provide Lyapunov stability since the conditions for Lyapunov stability of Theorem 3 are stated for $[0,\infty)$, whereas the proof of Theorem 4 shows that (69)–(73) hold with $[0,\infty)$ replaced by $[T_4,\infty)$.

HOW SUBOPTIMAL IS THE FPRE?

Consider a linearized model of an inverted pendulum mounted on a moving cart. The objective is to bring the pendulum to the upward vertical position. Control is performed by applying a force to the cart. For this system, the state vector is defined as $x=[x\ \dot{x}\ \theta\ \dot{\theta}]^{\mathrm{T}}$, where $x,\dot{x}$ are the horizontal position and velocity of the cart, respectively, and $\theta,\dot{\theta}$ are the angular position and angular velocity of the pendulum, respectively. The upward vertical position of the pendulum corresponds to $\theta=0$ rad. The linearized dynamics of this plant are

$$A=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl}{I(M+m)+Mml^2} & 0\end{bmatrix},$$

$$B=\begin{bmatrix}0\\ \frac{I+ml^2}{I(M+m)+Mml^2}\\ 0\\ \frac{ml}{I(M+m)+Mml^2}\end{bmatrix}. \tag{92}$$

Values of the parameters are given in Table 1 (I is mass moment of inertia of the pendulum, and l is the length to pendulum center of mass).

Initial conditions for the state vector are $x(0)=0.1$ m, $\dot{x}(0)=0$ m/s, $\theta(0)=0.2618$ rad, and $\dot{\theta}(0)=0$ rad/s. Thus, the initial state vector is $x(0)=[0.1\ \ 0\ \ 0.2618\ \ 0]^{\mathrm{T}}$. For the BPRE, the final state weightings are $P_{\mathrm{f}}=0$ and $P_{\mathrm{f}}=I$. For the FPRE, two initial conditions are used, namely, $P_0=\bar{P}+I$ and $P_0=I$, where the solution $\bar{P}$ of ARE (10) is

$$\bar{P}=\begin{bmatrix}1.94 & 1.37 & -3.93 & -0.77\\ 1.37 & 2.20 & -6.85 & -1.33\\ -3.93 & -6.85 & 40.04 & 7.22\\ -0.77 & -1.33 & 7.22 & 1.39\end{bmatrix}. \tag{93}$$

Let $R_1=I$, and $R_2=1$. Figures 5 and 6 show the state trajectories for the BPRE and the FPRE for the plant (92) for the given values of P_{f} for the BPRE, and P_0 for the FPRE. The convergence of $P(t)$ to $\bar{P}$ for the BPRE and the FPRE is shown in terms of $\|P(t)-\bar{P}\|$ in Figure 7.

To compare the performance of the FPRE and the BPRE, Pareto performance tradeoff curves are used to illustrate the efficiency of each control technique in terms of the state and control costs

$$J_{\mathrm{s}}(x_0,u)\triangleq\int_0^{t_{\mathrm{f}}}x^{\mathrm{T}}(t)R_1x(t)\,\mathrm{d}t + x^{\mathrm{T}}P_{\mathrm{f}}\,x \tag{94}$$

and

$$J_{\mathrm{c}}(x_0,u)\triangleq\int_0^{t_{\mathrm{f}}}u^{\mathrm{T}}(t)u(t)\,\mathrm{d}t. \tag{95}$$

For the Pareto plot, let R_2 range from 0.1 to 10. For comparison, we also illustrate the Pareto performance curve of

TABLE 1 Model parameters.

Parameter	Value	Units
Mass of the cart (M)	0.5	kg
Mass of the pendulum (m)	0.2	kg
Friction coefficient of the cart (b)	0.1	N/(m-s)
Mass moment of inertia of the pendulum (I)	0.006	kg-m^2
Length to pendulum center of mass (l)	0.3	m
Gravitational constant (g)	9.8	m/s^2

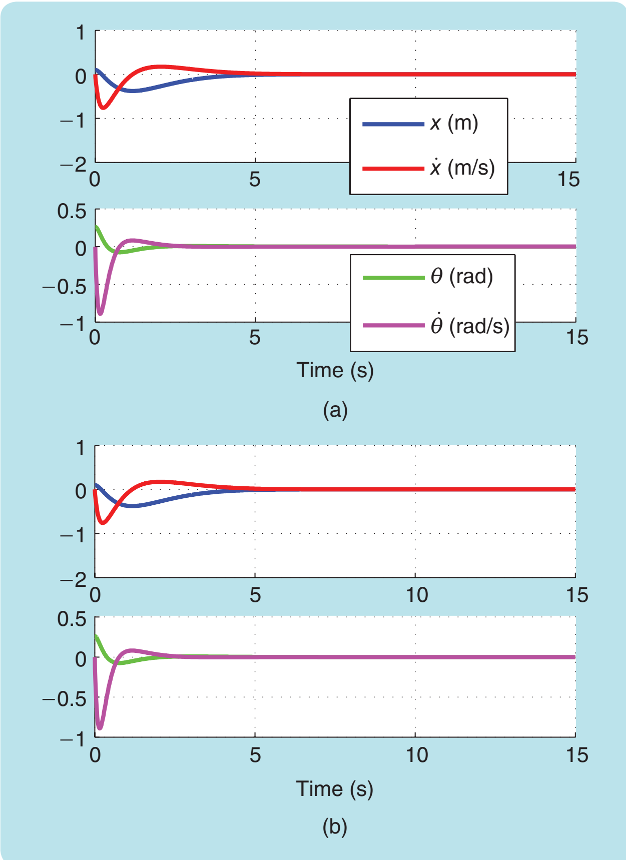


FIGURE 5 State trajectories for the backward-propagating Riccati equation (BPRE) for the inverted pendulum on a cart. This figure shows the state responses for the BPRE controller with $t_f = 15$ s and the final state weightings (a) $P_f = I$ and (b) $P_f = 0$.

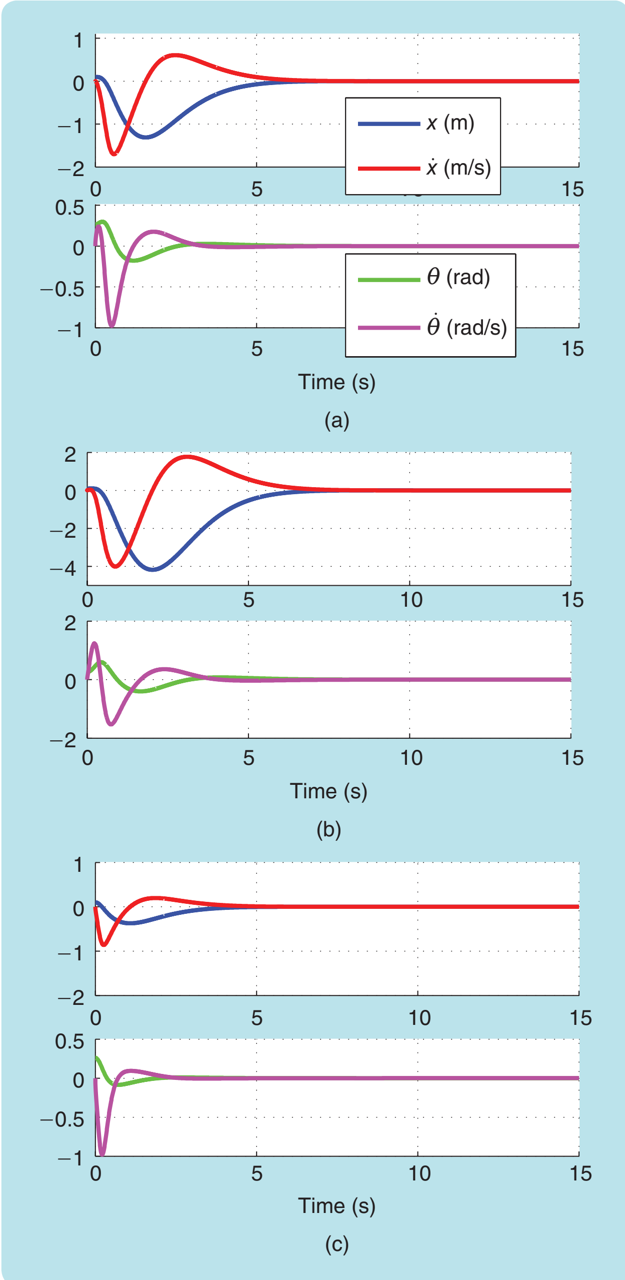


FIGURE 6 State trajectories for the forward-propagating Riccati equation (FPRE) for the inverted pendulum on a cart. (a), (b), (c) show the state responses for the FPRE controller for (92) with the initial conditions $P_0 = I$, $P_0 = 0$, and $P_0 = \bar{P} + I$, respectively. Results are valid for $t \in [0, \infty)$. Note that $P_0 = \bar{P} + I$ provides a better response than $P_0 = I$ and $P_0 = 0$.

the linear-quadratic controller (LQ). Figure 8 shows the Pareto performance curves for the LQ, BPRE, and FPRE for the plant (92).

Next, consider the Lyapunov function candidate and its derivative given by (76) and (81), respectively. Figure 9 shows $V(t, x)$ and $\dot{V}(t, x)$ for the FPRE.

Transient Responses of the BPRE and FPRE

In this section, we show the effect of the final time weighting P_f and the initial condition P_0 on the transient response of the closed-loop system for the BPRE and the FPRE, respectively. In particular, we focus on the case of unstable plants for which the closed-loop dynamics are unstable during the latter part of the time interval for the BPRE and the early part for the FPRE.

Example 5

Consider the unstable plant

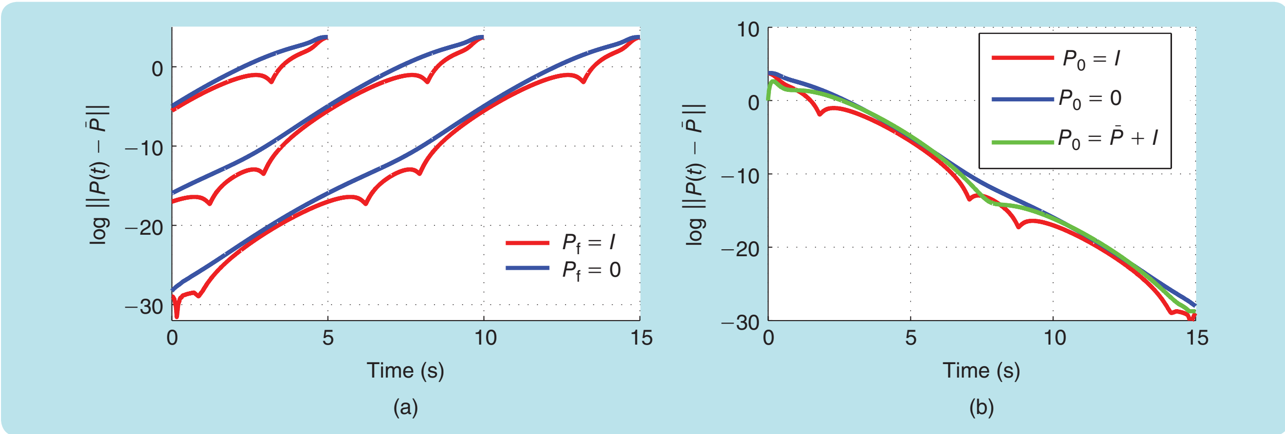
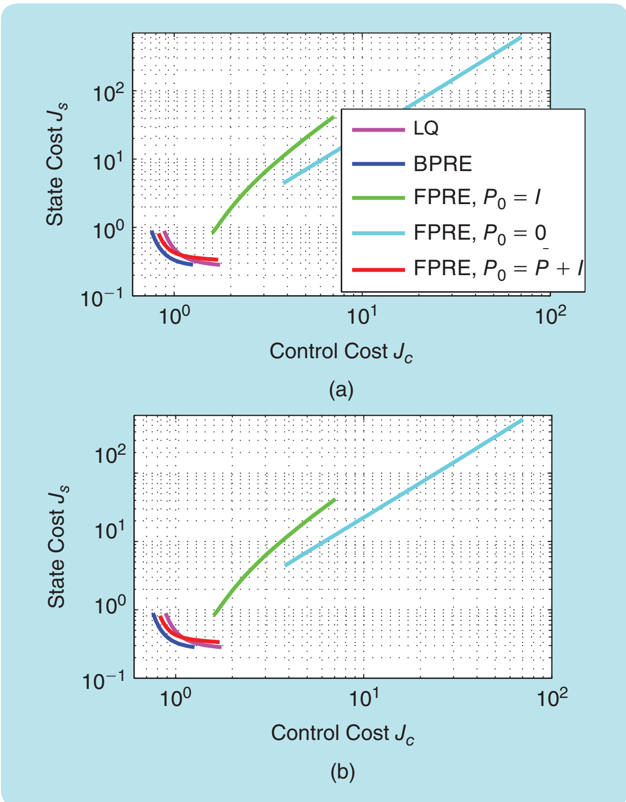


FIGURE 7 Norm of $P(t) - \bar{P}$ for the backward-propagating Riccati equation (BPRE) and the forward-propagating Riccati equation (FPRE) for the inverted pendulum on the cart. Norm denotes the maximum singular value. (a) shows the convergence of $P(t)$ to $\bar{P}$ for the BPRE for each fixed t as t_f approaches infinity with $P_f = I$ and $P_f = 0$, whereas (b) shows that for results for the FPRE with $P_0 = I, P_0 = 0$, and $P_0 = \bar{P} + I$. For the BPRE, the solutions are shown for t_f equal to 5, 10, and 15 s, whereas for the FPRE the solutions are valid on $[0, \infty)$.



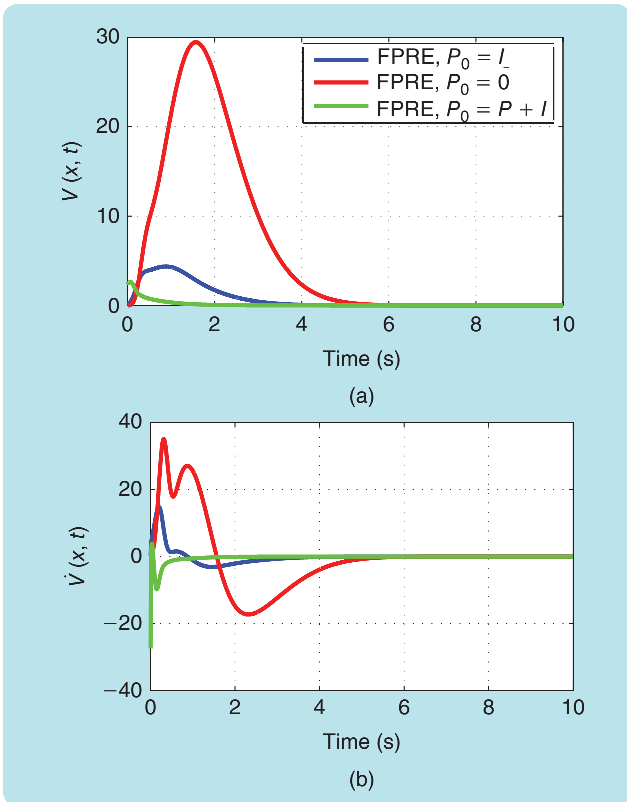


FIGURE 8 Pareto performance tradeoff curves. (a) and (b) show the Pareto curves for the linear-quadratic controller (LQ), the backward-propagating Riccati equation (BPRE), and the forward-propagating Riccati equation (FPRE) with final state weightings $P_f = I$ and $P_f = 0$, respectively. The initial conditions for the FPRE are $P_0 = I, P_0 = 0$, and $P_0 = \bar{P} + I$. The BPRE provides more efficient performance tradeoff curves than the FPRE. For the FPRE, the initial condition $P_0 = \bar{P} + I$ yields a better Pareto curve than $P_0 = I$ and $P_0 = 0$.

FIGURE 9 A Lyapunov-function candidate and its derivative. (a) and (b) show the Lyapunov function candidate $V(t,x)$ and its derivative $\dot{V}(t,x)$ for the forward-propagating Riccati equation (FPRE) for (92) with initial time conditions $P_0 = I, P_0 = 0$, and $P_0 = \bar{P} + I$. (b) shows that $\dot{V}(x,t)$ is positive in an initial time interval, which illustrates the fact that the FPRE does not guarantee Lyapunov stability.

$$A = \begin{bmatrix} 0 & 1 \\ -10 & 7 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (96)$$

with $R_1 = I, R_2 = 1$, and $x(0) = [3 \ \ 2]^{\mathrm{T}}$. For the BPRE, let $P_f = 0$, which implies that $A_{cl}(t_f) = A$ is unstable. Consider $t_f = 3$ s and $t_f = 10$ s. For the FPRE, let $P_0 = 0$, which implies that $A_{cl}(0) = A$ is unstable. The state trajectories, the norm of $P(t) - \bar{P}$, and the spectral abscissa of $A_{cl}(t)$ for the BPRE and the FPRE are shown in Figures 10 and 11, respectively.

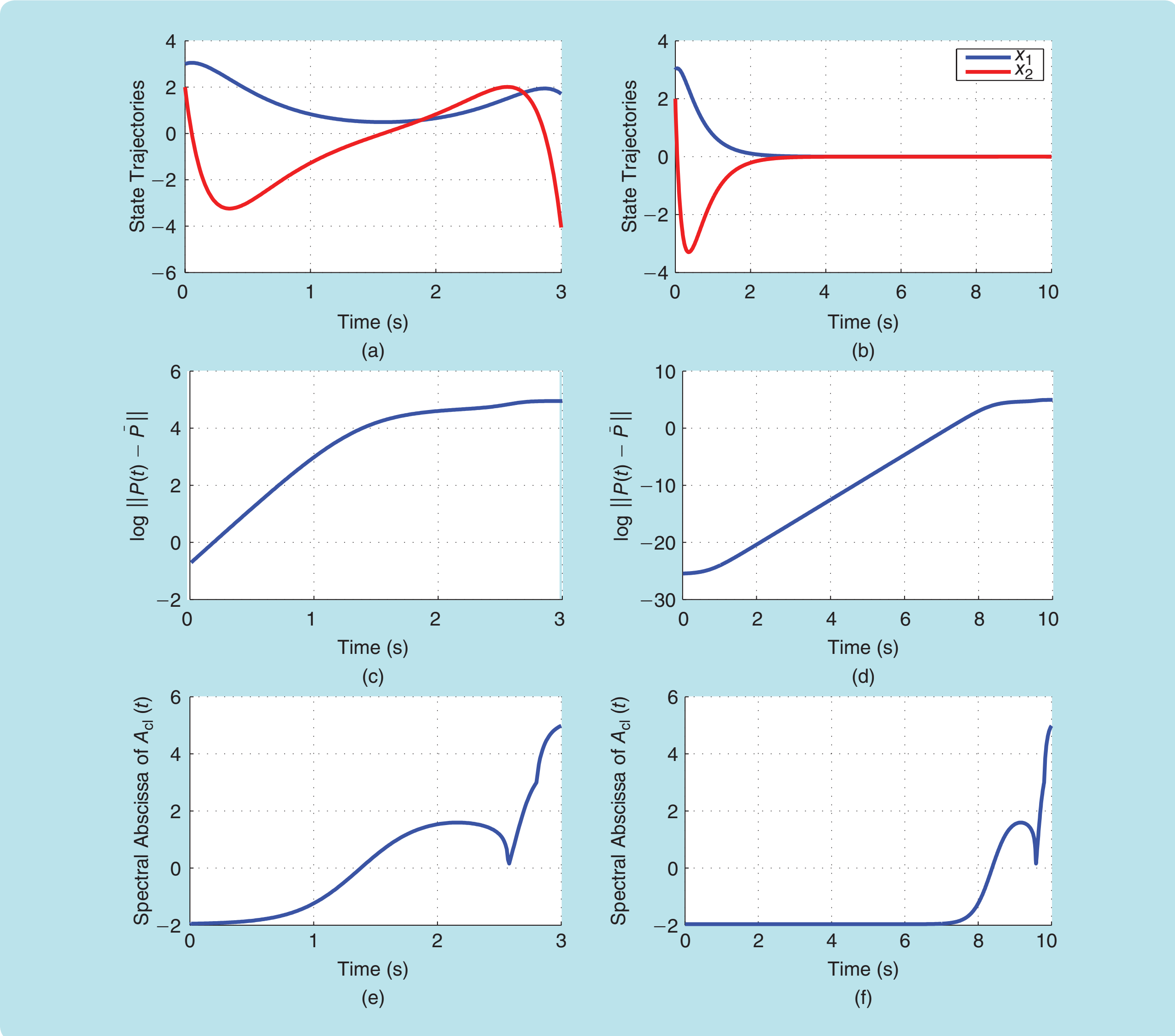


FIGURE 10 State trajectories, maximum singular value of $P(t)-\bar{P}$, and spectral abscissa of $A_{cl}(t)$ for the backward-propagating Riccati equation for the unstable plant (96) with $P_f = 0$. (a) shows that, for $t_f = 3$ s, the state trajectories diverge at t_f due to the fact that the closed-loop dynamics are unstable for $t \in [1.4, 3]$ s, as shown by the spectral abscissa of the closed-loop system in (e). For $t_f = 10$ s, (b) shows that the state trajectories asymptotically approach the origin due to the initially stable dynamics for $t \in [0, 8]$ s, as shown by the spectral abscissa of the closed-loop system in (f). (c) shows that, for $t_f = 3$ s, $P(t)$ is not close to $\bar{P}$, whereas (d) shows that, for $t_f = 10$ s, $P(t)$ is close to $\bar{P}$.

APPLICATION TO LTV AND NONLINEAR SYSTEMS

This section applies the FPRE to an LTV system and a nonlinear system to investigate the facility of the technique beyond LTI plants. For these examples there is no guarantee of stability.

The first example is the Mathieu equation, which has periodic LTV dynamics. The nonlinear Van der Pol oscillator, which is formulated in state-dependent coefficient form, is considered next.

Mathieu Equation

The Mathieu equation [21], [22] is

$$\ddot{q}(t) + (\alpha + \beta\cos(\omega t))q(t) = bu(t), \tag{97}$$

where α, β, ω, and b are real numbers. Defining the state vector $x(t) \triangleq [q(t) \;\; \dot{q}(t)]^{\mathrm{T}}$ yields the LTV dynamics

$$\dot{x}(t) = A(t)x(t) + Bu(t), \tag{98}$$

where

$$A(t) = \begin{bmatrix} 0 & 1 \\ -(\alpha+\beta\cos(\omega t)) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix}. \tag{99}$$

Let $\alpha = 1, \beta = 1, b = 1, \omega = 2$, and $x(0) = [3 \;\; 2]^{\mathrm{T}}$. The phase portrait of the uncontrolled system is shown in Figure 12.

Let $R_1 = I, R_2 = 1$ and consider three choices of P_0, namely, $P_0 = 0$, $P_0 = 10I$, and $P_0 = 100I$. Figure 13 shows the

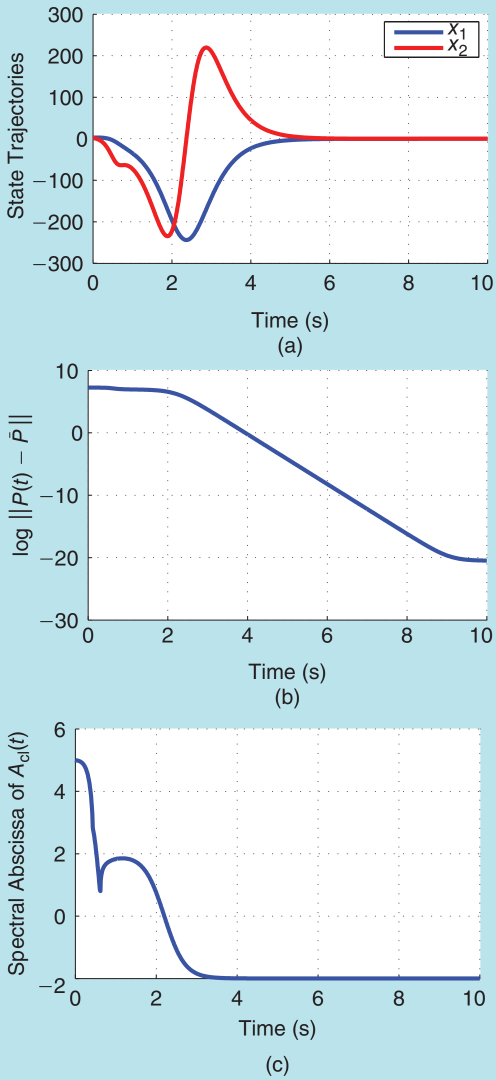


FIGURE 11 State trajectories, maximum singular value of $P(t) - \bar{P}$, and spectral abscissa of $A_{cl}(t)$ for the forward-propagating Riccati equation for the unstable plant (96) with $P_0 = 0$. Results are given for $t \in [0, 10]$ s and are valid for $t \to \infty$. (a) shows that the state trajectory has a large transient due to the initially unstable closed-loop dynamics for $t \in [0, 2]$ s, as shown by the spectral abscissa of the closed-loop system in (c). (b) shows that $P(t)$ approaches $\bar{P}$ as $t \to \infty$.

phase portrait of the closed-loop responses and norm of $P(t)$. Pareto performance curves obtained for R_2 ranging from 0.1 to 10 are given in Figure 14.

Van der Pol Oscillator

Consider the FPRE stabilization of the Van der Pol oscillator

$$\ddot{q}(t) - \mu(1 - q^2(t))\dot{q}(t) + q(t) = bu(t), \tag{100}$$

where $\mu > 0$ and b are real numbers. Defining the state vector $x(t) \triangleq [q(t) \;\; \dot{q}(t)]^{\mathrm{T}}$, (100) can be written in state-dependent coefficient form with

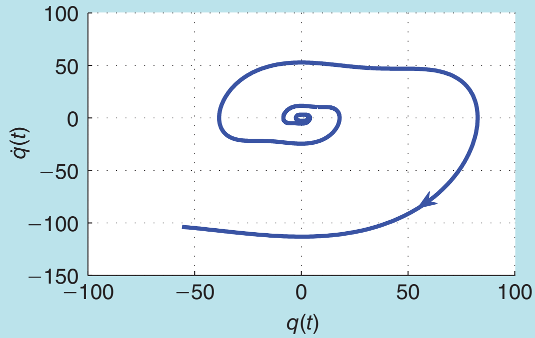


FIGURE 12 A phase portrait. This figure shows the phase portrait of the uncontrolled Mathieu equation for $x(0) = [3 \;\; 2]^{\mathrm{T}}$ and simulation time $t = 20$ s. The state trajectory diverges as $t \to \infty$.

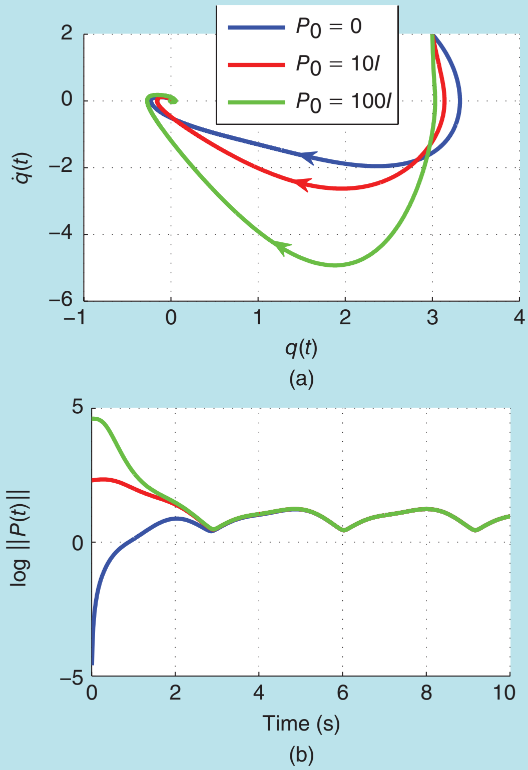


FIGURE 13 A closed-loop phase portrait and norm of $P(t)$. (a) and (b) show the forward-propagating Riccati equation closed-loop phase portraits and maximum singular value of $P(t)$ for the Mathieu equation with $P_0 = 0$, $P_0 = 10I$, and $P_0 = 100I$.

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 & \mu(1 - q^2(t)) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix}. \tag{101}$$

Let $\mu = 0.25$, $b = 1$, and $x(0) = [5 \;\; 3]^{\mathrm{T}}$. The phase portrait for the uncontrolled system is given in Figure 15.

Let $R_1 = I$, and $R_2 = 1$ and consider three choices of P_0, namely, $P_0 = 0$, $P_0 = 10$, and $P_0 = 100$. Figure 16 shows the phase portrait of the closed-loop responses and norm of

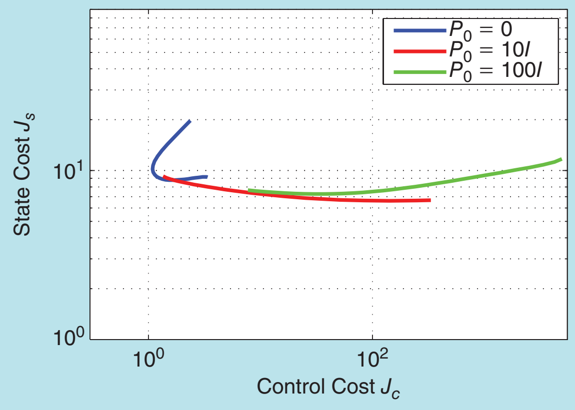


FIGURE 14 Pareto performance tradeoff curves. This plot shows the Pareto performance curves for the forward-propagating Riccati equation for the Mathieu equation with $P_0 = 0, P_0 = 10I$, and $P_0 = 100I$ for R_2 ranging from 0.1 to 10.

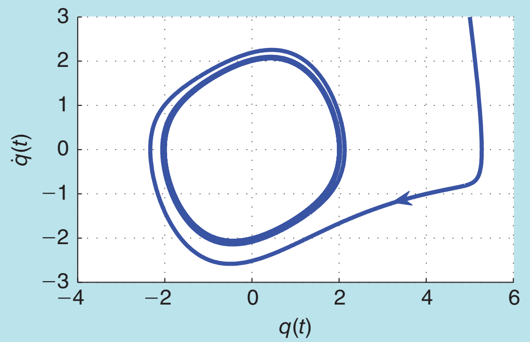


FIGURE 15 A phase portrait. This figure shows the phase portrait of the uncontrolled Van der Pol oscillator for $x(0) = [5 \;\; 3]^{\mathrm{T}}$ and simulation time $t = 50$ s. The state trajectory converges to a limit cycle.

$P(t)$. For the Pareto performance curves shown in Figure 17, R_2 ranges from 0.1 to 100.

CONCLUSION

This article presented analytical expressions for the solutions of the BPRE and the FPRE. With a focus on LTI systems, we proved convergence of the BPRE and FPRE solutions depending on the choice of the final weighting for the BPRE and the initial condition for the FPRE. Lyapunov analysis for LTI systems was used to prove that the FPRE controller provides asymptotic stability. However, this analysis showed that the FPRE controller does not guarantee Lyapunov stability. Numerical examples demonstrated the suboptimality of the FPRE relative to the BPRE. Numerical results were given to motivate future research on the application of the FPRE to LTV and nonlinear systems.

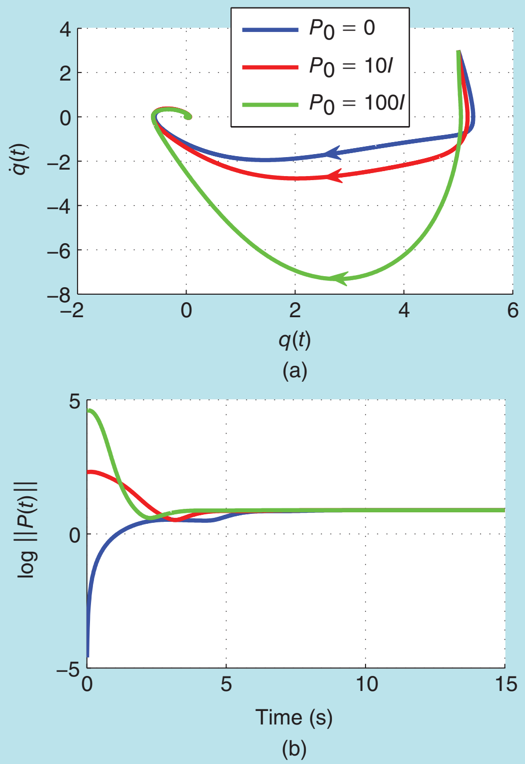


FIGURE 16 A closed-loop phase portrait and norm of $P(t)$. (a) and (b) show the phase portraits of the closed-loop responses and maximum singular value of $P(t)$ for the Van der Pol oscillator with $P_0 = 0$, $P_0 = 10I$, and $P_0 = 100I$.

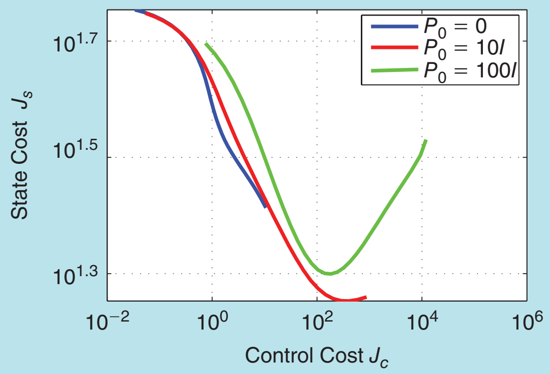


FIGURE 17 Pareto performance tradeoff curves. This plot shows the Pareto performance curves for the forward-propagating Riccati equation for the Van der Pol oscillator with $P_0 = 0, P_0 = 10I$, and $P_0 = 100I$ for R_2 ranging from 0.1 to 100.

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REFERENCES

[1] J. L. Speyer and D. H. Jacobson, *Primer on Optimal Control Theory*. Philadelphia, PA: SIAM, 2010.

[2] D. S. Naidu, *Optimal Control Systems*. Boca Raton, FL: CRC Press, 2002.

[3] R. Bellman, *Dynamic Programming*. Princeton, NJ: Princeton Univ. Press, 1957.

[4] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, no. 6, pp. 789–814, 2000.

[5] J. B. Rawlings and K. R. Muske, "The stability of constrained receding horizon control," *IEEE Trans. Autom. Contr.*, vol. 38, no. 10, pp. 1512–1516, 1993.

[6] D. Q. Mayne and H. Michalska, "Receding horizon control of nonlinear systems," *IEEE Trans. Autom. Contr.*, vol. 35, no. 7, pp. 814–824, 1990.

[7] S. S. Kerthi and E. G. Gilbert, "Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: stability and moving-horizon approximations," *J. Optim. Theory Applicat.*, vol. 57, no. 2, pp. 265–293, 1988.

[8] M.-S. Chen and C.-Y. Kao, "Control of linear time-varying systems using forward Riccati equation," *J. Dyn. Syst. Meas. Control*, vol. 119, no. 3, pp. 536–540, 1997.

[9] A. Weiss, I. Kolmanovsky, and D. S. Bernstein, "Forward-integration Riccati-based output-feedback control of linear time-varying systems," in *Proc. American Control Conf.*, Montreal, QC, Canada, June 2012, pp. 6708–6714.

[10] F. L. Lewis, D. Vrabie, and K. G. Vamvoudakis, "Rreinforcement learning and feedback control: Using natural decision methods to design optimal adaptive controllers," *IEEE Control Syst. Mag.*, vol. 32, no. 6, pp. 76–105, 2012.

[11] J. van den Berg, "Iterated LQR smoothing for locally-optimal feedback control of systems with non-linear dynamics and non-quadratic cost," in *Proc. American Control Conf.*, Portland, OR, June 2014, pp. 1912–1918.

[12] F. M. Callier and J. L. Willems, "Criterion for the convergence of the solution of the Riccati differential equation," *IEEE Trans. Autom. Contr.*, vol. 26, no. 6, pp. 1232–1242, 1981.

[13] F. M. Callier, J. Winkin, and J. L. Willems, "On the exponential convergence of the time-invariant matrix Riccati differential equation," in *Proc. Conf. Decision Control*, Tucson, AZ, Dec. 1992, pp. 1536–1537.

[14] F. M. Callier, J. Winkin, and J. L. Willems, "Convergence of the time-invariant Riccati differential equation and LQ-problem: Mechanisms of attraction," *Int. J. Control*, vol. 59, no. 4, pp. 983–1000, 1994.

[15] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. New York Wiley, 1972.

[16] V. Kucera, "A review of the matrix Riccati equation," *Kybernetika*, vol. 9, no. 1, pp. 42–61, 1973.

[17] W. M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. Princeton, NJ: Princeton Univ. Press, 2011.

[18] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas*. Princeton, NJ: Princeton Univ. Press, 2009.

[19] J. L. Junkins and J. D. Turner, *Optimal Spacecraft Rotational Maneuvers*. Amsterdam, The Netherlands: Elsevier, 1986.

[20] M. Sorine and P. Winternitz, "Superposition laws for solutions of differential matrix Riccati equations arising in control theory," *IEEE Trans. Autom. Contr.*, vol. 30, no. 3, pp. 266–272, 1985.

[21] J. A. Richards, *Analysis of Periodically Time-Varying Systems*. Berlin Heidelberg, Germany: Springer-Verlag, 1983.

[22] A. Prach, O. Tekinalp, and D. S. Bernstein, "A numerical comparison of frozen-time and forward-propagating Riccati equations for stabilization of periodically time-varying Systems," in *Proc. American Control Conf.*, Portland, OR, June 2014, pp. 5633–5638.

» PUBLICATION ACTIVITIES *(continued from page 27)*

Following the suggestion given by the Long-Range Planning Committee (LRPC) and discussed at the LRPC meeting held on June 6, 2014 at Portland during the American Control Conference 2014, a pdf version of the e-Letter with navigation links was released starting with the July 2014 issue. This is an addition to the standard txt version of the e-Letter, which is still distributed via e-mail and now includes a link to its pdf version for possible download.

CSS Web Site

This reporting period has been particularly dense with activities related to the CSS Web site update and restructuring. More specifically, the following three main activities have been undertaken:

1) an update of the IEEE CSS Video Clip Contest Web site
2) restructuring and an update of the technical activities section, with a specific focus on the technical committee Web sites
3) restructuring and an update of the conference section.

Prepared by
Maria Prandini

arXiv ACTIVITIES REPORT

The CSS arXiv moderator team (Marco Lovera, Roberto Tempo, and Yuan Wang) has continued its activities through 2014 along the same lines as in the previous years. The team is jointly moderating the cs.SY (computer science, systems and control) and math.OC (mathematics, optimization and control) categories. The category math.OC was established in 1999; the cs.SY category is relatively young since it was created in 2010.

An analysis was done of the interest for arXiv within the research community in systems and control. The statistics show that the number of submissions to math.OC has been steadily growing at a remarkable rate over the last few years, and that since its inception the cs.SY has been following a similar trend. The data match the moderators' impression about the increasing awareness of the existence and usefulness of the arXiv within our community.

Prepared by
Roberto Tempo