ODEs with Differentiated Inputs

Specifying Preinitial Conditions

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The correspondence between transfer functions and state-space realizations is a bedrock principle in systems and control theory. We routinely extract transfer functions from state-space realizations and, conversely, construct realizations of transfer functions. In doing so, we recognize that a transfer function does not capture initial conditions and thus ignores the free (unforced) response of the system, that is, the response to initial conditions. Aside from this consideration, the equivalence is viewed as exact.

In the time domain, we can, as an alternative to state-space models, express the system as an ordinary differential equation of higher order, that is, with derivatives of order higher than one. For mechanical systems with inertia, models with second derivatives are widely thought to be more natural than state-space models, despite the fact that much control theory is most conveniently developed for the latter representation.

Given the single-input, single-output (SISO) transfer function $G(s) = n(s)/d(s)$, the degree of the denominator $d(s)$ determines the highest-order derivative of the output appearing in the differential equation, while the degree of $n(s)$ determines the highest-order derivative of the input. The presence of differentiated inputs is a distinguishing feature compared to state-space models, in which differentiated inputs do not appear.

The purpose of this note is to stress the critical role of the initial conditions of the input when the differential equation involves differentiated inputs.

A SIMPLE EXAMPLE

We consider the second-order forced ordinary differential equation (ODE) in the scalar variable $q(t)$ given by

$$\ddot{q}(t) + \dot{q}(t) + q(t) = \dot{u}(t) + u(t), \quad t \geq 0, \tag{1}$$

with the initial conditions

$$\dot{q}(0) = 0, \qquad q(0) = 0. \tag{2}$$

The input signal $u(t)$ is taken to be

$$u(t) = e^{-t}, \quad t \geq 0. \tag{3}$$

We wish to determine the solution to (1)–(3).

A difficulty with (1) is immediately evident, namely, how can we specify the values $u(0)$ and $\dot{u}(0)$? Although $\dot{u}(t) + u(t) = 0$ for all $t > 0$, the value of $u(0) + \dot{u}(0)$ is not clear in light of the fact that the initial-value problem is concerned only with time $t \geq 0$. In short, is there a discontinuity at $u(0)$ that causes an impulse in $\dot{u}(t)$?

LAPLACE TRANSFORM SOLUTIONS

Taking the $\mathcal{L}_+$ Laplace transform of (1) yields

$$Q(s) = \frac{U(s) + sU(s) - u(0^+) + \dot{q}(0^+) + sq(0^+) + q(0^+)}{s^2 + s + 1} \tag{4}$$

where $U(s) = \mathcal{L}\{e^{-t}\} = 1/(s+1)$. Thus, $U(s) + sU(s) = 1$.

If we choose to interpret (2) as the *post-initial conditions*

$$\dot{q}(0^+) = 0, \qquad q(0^+) = 0 \tag{5}$$

then, along with $u(0^+) = 1$, it follows that $Q(s) = 0$, and thus the solution to (1) and (5) is $q(t) = 0$ for all $t \geq 0$.

Alternatively, using the $\mathcal{L}_-$ transform, as preferred in many texts [1]–[9],

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt,$$

where the domain of integration fully includes the origin, yields

$$Q(s) = \frac{1 - u(0^-) + \dot{q}(0^-) + sq(0^-) + q(0^-)}{s^2 + s + 1}. \tag{6}$$

If we choose to interpret (2) as the *pre-initial conditions*

$$\dot{q}(0^-) = 0, \qquad q(0^-) = 0, \tag{7}$$

we find

$$Q(s) = \frac{1 - u(0^-)}{s^2 + s + 1}, \tag{8}$$

where $u(0^-)$ is the pre-initial condition on u.

If we choose to define $u(t)$ continuously for $t < 0$, then $u(0^-) = 1$, and the solution to (1) and (7) is

$$q(t) = 0 \text{ for all } t \geq 0, \tag{9}$$

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State-Space Formulation

Transformation to state-space coordinates has the dubious benefit of burying the pre-initial conditions of the input. To circumvent the problem of specifying the value $u(0)+\dot{u}(0)$, we ignore initial conditions and realize the transfer function

$$G(s)=\frac{s+1}{s^2+s+1} \tag{S1}$$

corresponding to (1) in state space. A minimal realization of (S1) is given by

$$\dot{x}=Ax+Bu, \tag{S2}$$
$$q=Cx, \tag{S3}$$

where

$$x \triangleq \begin{bmatrix} -\dot{q}+u \\ q+\dot{q}-u \end{bmatrix}, \quad A \triangleq \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},$$
$$B \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C \triangleq [1 \quad 1].$$

As a check, taking the Laplace transform of (S2) and (S3), yields

$$\hat{q}(s)=\frac{[x_1(0)+x_2(0)]s+x_2(0)}{s^2+s+1}+\frac{s+1}{s^2+s+1}.$$

The free response of (S2) and (S3) thus involves the value $u(0)$, although it is embedded in the initial state $x(0)$. However, it remains to be determined whether the correct value is $u(0^+)$ or $u(0^-)$. In particular, this transformation merely obscures the issue since the initial condition of the state is now suspect. Consequently, the use of transfer functions to solve differential equations such as (1)–(3) ultimately requires that the pre-initial conditions of the input be specified. In every instance the user is required to determine the physical rationale for specifying these values.

as above. On the other hand, if we define $u(t)=0$ for all $t<0$, then $u(0^-)=0$, and we have

$$Q(s)=\frac{1}{s^2+s+1}, \tag{10}$$

which yields

$$q(t)=\frac{2}{\sqrt{3}}e^{-t/2}\sin\left[(\sqrt{3}/2)t\right]. \tag{11}$$

The different solutions we obtain using the $\mathcal{L}_-$ transform arise from the different pre-initial conditions that we assume in each case. In effect, the two different solutions arise from the fact that we are solving two different problems. So the question remains: How do we determine the pre-initial condition $u(0^-)$?

HOW TO SPECIFY PREINITIAL CONDITIONS

When using the $\mathcal{L}_-$ Laplace transform, we obtained two different solutions of the ODE (1) depending on how the pre-initial condition $u(0^-)$ is specified. In particular, $u(0^-)=0$ is assumed in [3, p. 80]. But on what *rational basis* is the value $u(0^-)$ determined in general?

In defining a function $u(t)$ on the interval $[0,\infty)$, the pre-initial values of $u(t)$ and its derivatives are required pieces of information [10], [11]. This information cannot be "left to the reader." Thus, the input function

$$u(t)=e^{-t}, \quad t\geq 0, \quad u(0^-)=1,$$

which is used to find (9), and the input function

$$u(t)=e^{-t}, \quad t\geq 0, \quad u(0^-)=0,$$

which is used to find (11), are different functions. As such, they produce different solutions to (1) and (2). The unilateral Laplace transform doesn't care what $u(t)$ is for $t<0$, but the value of $u(0^-)$ is a special piece of extra information that we require to be able to take derivatives on the interval $[0,\infty)$.

The need for pre-initial information is explained by the brief theory of generalized functions in [11]. If we want to define a function on the interval $[0,\infty)$, and we want to be able to take derivatives, then we need complete information at $t=0^-$. Since these pre-initial values are not implied by the behavior on $t\geq 0$, this extra information must be explicitly included.

CONCLUSIONS

When a differential equation has differentiated inputs, the pre-initial values of the input must be specified in order for the problem to be well posed. These values are not determined by the mathematics but rather by the physical situation. The issue persists in state-space models, even though no derivatives of the input appear explicitly (see "State-Space Formulation"). Consequently, care must be taken to determine the pre-initial conditions to obtain the physically meaningful solution.

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The Final Value Theorem Revisited

Infinite Limits and Irrational Functions

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The final value theorem is an extremely handy result in Laplace transform theory. In many cases, such as in the analysis of proportional-integral-derivative (PID) controllers, it is necessary to determine the asymptotic value of a signal. The final value theorem provides an easy-to-use technique for determining this value without having to first invert the Laplace transform to determine the time signal.

The standard assumptions for the final value theorem [1, p. 34] require that the Laplace transform have all of its poles either in the open-left-half plane (OLHP) or at the origin, with at most a single pole at the origin. In this case, the time function has a finite limit.

Although no limit exists when the Laplace transform has a nonzero pole on the imaginary axis, some textbooks note that the final value theorem can be used when the limit is infinite. For example, in [2, p. 104], (1) given below is used to obtain infinite limits of the closed-loop transfer function for type-0 and type-1 systems with ramp commands as well as for type-1 systems with parabolic commands. Furthermore, [3, p. 96] states that, for poles at the origin, (1) "gives the final value $f(\infty) = \infty$" for a time function $f(t)$. In addition, [4, p. 567], allows poles in the OLHP or at the origin.

The goal of this note is to publicize and prove the "infinite-limit" version of the final value theorem. The version we provide is a slight refinement of the classical literature in that we require that s approach zero through the right-half plane to obtain the correct sign of the infinite limit. We first consider the case of rational Laplace transforms and then state a version that applies to irrational functions.

FINITE-LIMIT CASE

Let $y(t)$ be a signal on $[0, \infty)$, let $\hat{y}(s)$ be its Laplace transform, and define

$$y(\infty) \triangleq \lim_{t\to\infty} y(t)$$

whenever this limit exists. By "exists" we mean that $y(\infty)$ is a real number and $y(t) - y(\infty) \to 0$ as $t \to \infty$. We stress that ∞ and $-\infty$ are not real numbers. For now, we assume that $\hat{y}(s)$ is a proper rational function.

Standard Final Value Theorem

Assume that every pole of $\hat{y}(s)$ is either in the OLHP or at the origin, and assume that $\hat{y}(s)$ has at most a single pole at the origin. Then $y(\infty)$ exists and is given by

$$y(\infty) = \lim_{s\to 0} s\hat{y}(s). \quad (1)$$

Note the "reversal" between t and s in that $t \to \infty$ corresponds to $s \to 0$. Note also the factor of s that precedes $\hat{y}(s)$. A similar reversal occurs in the initial value theorem, which includes a factor of s as well.

As an example, let $\hat{y}(s) = (3s+2)/(s(s+1))$. It thus follows from (1) that $y(\infty) = 2$. Indeed, $y(t) = 2 + e^{-t}$.

To see how this result can fail when its hypotheses are not satisfied, consider $y(t) = \sin \omega_0 t$, where $\omega_0 > 0$, so that $\hat{y}(s) = \omega_0/(s^2 + \omega_0^2)$. Since the poles of $\hat{y}(s)$ are not in the OLHP or at the origin, the final value theorem cannot be applied. Although $\lim_{s\to 0} s\hat{y}(s)$ exists for this example, the limiting value 0 is useless since $y(\infty)$ does not exist.

INFINITE-LIMIT CASE

We wish to extend the applicability of (1) beyond the stated conditions on $\hat{y}(s)$. To do this, suppose that $y(\infty)$ does not exist, but assume that $\lim_{t\to\infty} y(t) = \infty$ or $\lim_{t\to\infty} y(t) = -\infty$. Let $y(\infty)$ denote $\pm\infty$ in these cases. For convenience we say that $y(\infty)$ *does not exist but is infinite*.

Note that this definition does not apply to signals such as $y(t) = e^t \sin t$. Alternatively, consider $y(t) = e^t$, so that $y(\infty) = \infty$. Since $\hat{y}(s) = 1/(s-1)$ it follows that $\lim_{s\to 0} s\hat{y}(s) = 0$, and thus (1) is not satisfied. However, the following result encompasses infinite limits arising from multiple poles at the origin. In the following statement, the notation "$s \downarrow 0$" means that s approaches 0 through the positive numbers.

Note that the limit $s \downarrow 0$ is consistent with the fact that $\hat{y}(s)$ has poles only in the CLHP and is analytic in the ORHP. Hence, the Laplace transform converges in the ORHP and the limit can be taken along the positive real axis, whereas the limit may not exist when taken from the CLHP.