Are All Full-Order Dynamic Compensators Observer Based?

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Linear-quadratic-Gaussian (LQG) control theory states that the optimal compensator for a linear plant with white, Gaussian process and sensor noise and with suitable stabilizability and detectability assumptions is given by an observer-based compensator. The observer-based structure of the compensator reflects the *separation principle*, wherein an optimal state estimate is fed back by an optimal static full-state-feedback control law and where the observer and regulator gains are determined independently [1]–[3]. This result implies that every full-order dynamic compensator that is not observer based must be suboptimal in the sense of LQG control.

To clarify the distinction between an observer-based compensator and a compensator of arbitrary structure, consider the single-input, single-output plant

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$
$$y(t) = Cx(t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, and $C \in \mathbb{R}^{1\times n}$. Assume that (A, B, C) is controllable and observable, and write a compensator in the form

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{3}$$
$$u(t) = C_c x_c(t), \tag{4}$$

where the dimension n_c of x_c may be the same or different from the dimension n of the state of the plant (1), (2). To illustrate an observer-based compensator, let $F \in \mathbb{R}^{n\times 1}$ and consider the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + F[y(t) - \hat{y}(t)], \tag{5}$$
$$\hat{y}(t) = C\hat{x}(t), \tag{6}$$

where $\hat{x}(t) \in \mathbb{R}^n$. Note that (1) and (5) can be written as

$$\dot{x}(t) = (A - FC)x(t) + Bu(t) + Fy(t), \tag{7}$$
$$\dot{\hat{x}}(t) = (A - FC)\hat{x}(t) + Bu(t) + Fy(t). \tag{8}$$

Defining the error state $e(t) \triangleq x(t) - \hat{x}(t)$ and subtracting (8) from (7) yields

$$\dot{e}(t) = (A - FC)e(t). \tag{9}$$

If $A - FC$ is asymptotically stable, then e converges to zero for all $x(0)$ and $\hat{x}(0)$. Note that, since $x(t)$ is not measured, $e(t)$ is unknown and thus (9) is used only for analysis.

Next, let $K \in \mathbb{R}^n$ and consider the observer-based feedback control law $u(t) = K\hat{x}(t)$ in (8). The observer-based compensator is

$$\dot{\hat{x}}(t) = (A + BK - FC)\hat{x}(t) + Fy(t), \tag{10}$$
$$u(t) = K\hat{x}(t). \tag{11}$$

Notice that (10), (11) is a full-order dynamic compensator of the form (3), (4) with

$$n_c = n,\ A_c = A + BK - FC,\ B_c = F,\ C_c = K,\ x_c(t) = \hat{x}(t). \tag{12}$$

The only distinction between (3), (4) with $n_c = n$ and (10), (11) is the fact that the dynamics matrix A_c in (3) has the observer-based form $A + BK - FC$ in (10). The structure of A_c suggests that observer-based compensators of the form (10), (11) comprise a subset of full-order compensators relative to the arbitrary structure (3), (4). LQG theory chooses an optimal compensator from this subset.

Comparing (3), (4) to (10), (11) motivates the question in the title of this article, namely, are all full-order compensators observer based? It is shown in the following that the answer to this question is "no." In particular, note that the closed-loop system (1)–(4) is given by

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t), \tag{13}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix},\quad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}.$$

For the observer-based compensator, $\tilde{A}$ has the form

$$\tilde{A} = \begin{bmatrix} A & BK \\ FC & A + BK - FC \end{bmatrix}. \tag{14}$$

Using (11), (1) can be written as

$$\dot{x}(t) = (A + BK)x(t) - BKe(t). \tag{15}$$

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Combining (9) and (15) yields

$$\dot{\tilde{x}}'(t) = \tilde{A}'\tilde{x}'(t), \tag{16}$$

where

$$\tilde{x}'(t) \triangleq \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}, \quad \tilde{A}' \triangleq \tilde{S}\tilde{A}\tilde{S}^{-1} = \begin{bmatrix} A+BK & -BK \\ 0 & A-FC \end{bmatrix},$$
$$\tilde{S} \triangleq \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix}, \tag{17}$$

and $\tilde{A}$ is given by (14). Since $\tilde{A}$ and $\tilde{A}'$ are similar, they have the same eigenvalues. In addition, the eigenvalues of $\tilde{A}'$ consist of the eigenvalues of $A+BK$ and the eigenvalues of $A-FC$. Since $A+BK$ is a real matrix, it follows that, if n is odd, then $A+BK$ has at least one real eigenvalue. The same statement can be made for $A-FC$. Consequently, if n is odd, then $\tilde{A}'$ and thus $\tilde{A}$ must have at least two real eigenvalues. This observation is made in [4, p. 43] to stress the distinction between observer-based controllers and dynamic compensators for pole placement that are not intended to estimate inaccessible states.

Consider the closed-loop system (13) consisting of (1)–(4), where (3), (4) is a full-order compensator. If n is odd and $\tilde{A}$ has no real eigenvalues, then the above discussion shows that (3), (4) cannot be an observer-based compensator. This result leads to the following fundamental question: Is this the *only* situation where the full-order compensator is not observer based in the sense that there does not exist a basis such that (3), (4) can be written in the form of (11), (12)? The main contribution of this article is to show that this is indeed the case.

To set the stage for the subsequent development, it is useful to recall that pole-placement techniques can be used to assign the eigenvalues of $A+BK$ and $A-FC$. Therefore, if either

$$n \text{ is even} \tag{18}$$

or

$$n \text{ is odd, and } \tilde{A} \text{ has at least two real eigenvalues}, \tag{19}$$

then, for each full-order compensator (3), (4), there exists an observer-based compensator that replicates the closed-loop spectrum. However, this does not prove that (3), (4) is observer based because it is not known whether (or not) the pole-placement compensator that replicates the closed-loop spectrum arising from (3), (4) is the *unique* full-order compensator with this property. The goal of this article is thus to demonstrate uniqueness.

It is important to stress that the focus in this article is on uniqueness rather than existence. The existence of dynamic pole-placement controllers is extensively addressed in the literature. For example, sufficient conditions are given in [4] for the existence of a dynamic compensator of specified order that is able to place an arbitrary conjugate-symmetric set of closed-loop poles.

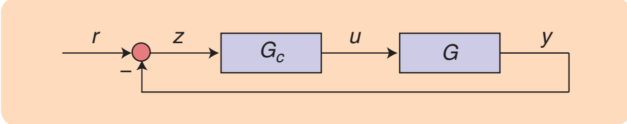

FIGURE 1 The servo problem.

ANALYSIS OF THE SENSITIVITY FUNCTION

To clarify the required uniqueness property, consider the servo problem in Figure 1, where $G(s) \triangleq C(sI-A)^{-1}B = N(s)/D(s)$,

$$z(t) \triangleq r(t) - y(t), \tag{20}$$

$y(t) \in \mathbb{R}$ is the measurement, and $r(t) \in \mathbb{R}$ is the command. Note that D is monic and, since (A, B, C) is controllable and observable, D and N are coprime. The closed-loop transfer function from r to z is given by the sensitivity function

$$S \triangleq \frac{1}{1+GG_c} = \frac{DD_c}{\tilde{D}}, \tag{21}$$

where $G_c = N_c/D_c$ is a proper compensator of order $n_c \geq 1$, and D_c is monic. The closed-loop characteristic polynomial is defined by

$$\tilde{D} \triangleq DD_c + NN_c. \tag{22}$$

It follows from (21) that G_c is given by

$$G_c = \frac{1-S}{SG}, \tag{23}$$

which shows that G_c is uniquely specified by S. Therefore, if two compensators G_{c1} and G_{c2} of arbitrary order give rise to the same sensitivity function S, then $G_{c1} = G_{c2}$. However, this does not show that if two compensators G_{c1} and G_{c2} give rise to the same characteristic polynomial, then $G_{c1} = G_{c2}$. It is shown in the following section that, if $\deg(N_{c1}) \leq \min\{n_c, n-1\}$ and $\deg(N_{c2}) \leq \min\{n_c, n-1\}$, then the two compensators $G_{c1} = N_{c1}/D_{c1}$ and $G_{c2} = N_{c2}/D_{c2}$ that give rise to the same characteristic polynomial are equal. So, if either (18) or (19) is satisfied, then every full-order compensator is observer based. Before demonstrating this fact, the next section reviews pole placement using observer-based compensation.

POLE PLACEMENT USING OBSERVER-BASED COMPENSATION

This section reviews pole placement using observer-based compensation. The regulator and observer are discussed separately, and then several examples are presented. It is shown that the observer-based compensator is independent of how the closed-loop poles are allocated to the regulator and the observer.

Designing the Regulator

Let the characteristic polynomial of A be given by $p(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$. As shown in [5, pp. 309–311], since (A,B) is controllable, there exists a change of basis matrix $S_C \in \mathbb{R}^{n\times n}$ such that

$$A + BK = S_C(A_C + B_C K_C)S_C^{-1}, \tag{24}$$

where $K_C \triangleq KS_C^{-1} = [K_{C,1} \cdots K_{C,n}]$,

$$A_C \triangleq S_C A S_C^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}, \; B_C \triangleq S_C B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{25}$$

and thus

$$A_C + B_C K_C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 + K_{C,1} & -\alpha_1 + K_{C,2} & -\alpha_2 + K_{C,3} & \cdots & -\alpha_{n-1} + K_{C,n} \end{bmatrix}. \tag{26}$$

Note that $A + BK$ and $A_C + B_C K_C$ have the same eigenvalues and that the eigenvalues of $A_C + B_C K_C$ can be placed arbitrarily by the choice of K_C. The regulator gain K can then be determined using $K = K_C S_C$. Finally, since $\mathcal{C}(A_C, B_C) = S_C \mathcal{C}(A,B)$, it follows that $S_C = \mathcal{C}(A_C,B_C)\,\mathcal{C}(A,B)^{-1}$, where $\mathcal{C}(A,B)$ is the controllability matrix of the pair (A,B).

Designing the Observer

Let the characteristic polynomial of A be given by $p(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$. Since (A,C) is observable, there exists a change of basis matrix $S_O \in \mathbb{R}^{n\times n}$ such that

$$A - FC = S_O^{-1}(A_O - F_O C_O)S_O, \tag{27}$$

where $F_O \triangleq S_O F = [F_{O,1} \;\; \cdots \;\; F_{O,n}]^T$,

$$A_O = S_O A S_O^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & \cdots & 0 & -\alpha_2 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix},$$
$$C_O = CS_O^{-1} = [0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1], \tag{28}$$

and thus

$$A_O - F_O C_O = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 - F_{O,1} \\ 1 & 0 & \cdots & 0 & -\alpha_1 - F_{O,2} \\ 0 & 1 & \cdots & 0 & -\alpha_2 - F_{O,3} \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} - F_{O,n} \end{bmatrix}. \tag{29}$$

Note that $A - FC$ and $A_O - F_O C_O$ have the same eigenvalues and that the eigenvalues of $A_O - F_O C_O$ can be placed arbitrarily by the choice of F_O. The observer gain F can then be determined using $F = S_O^{-1} F_O$. Finally, since $\mathcal{O}(A_O, C_O) = \mathcal{O}(A,C)S_O^{-1}$, it follows that $S_O = \mathcal{O}(A_O,C_O)^{-1}\mathcal{O}(A,C)$, where $\mathcal{O}(A,C)$ is the observability matrix of the pair (A, C).

Example 1

Let

$$A = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \;\; 0],$$

and assign the eigenvalues –5 and –4 to $A + BK$ and the eigenvalues –2 and –6 to $A - FC$. Solving (26) and (29) for K and F yields $K = [-3 \;\; -5]$ and $F = [4 \;\; -0.75]^T$. The transfer function representation of the observer-based compensator is

$$G_c(s) = \frac{8.26s + 23.25}{s^2 + 13s + 49}. \tag{30}$$

As a check, use the realization

$$A_c = \begin{bmatrix} -5 & 4 \\ -2.25 & -8 \end{bmatrix}, \; B_c = \begin{bmatrix} 4 \\ -0.75 \end{bmatrix}, \; C_c = [-3 \;\; -5]$$

of $G_c(s)$ to find $\tilde{A}$. The closed-loop eigenvalues are given by the eigenvalues of

$$\tilde{A} = \begin{bmatrix} -1 & 4 & 0 & 0 \\ 0 & -3 & -3 & -5 \\ 4 & 0 & -5 & 4 \\ -0.75 & 0 & -2.25 & -8 \end{bmatrix},$$

which are the desired values $-2, -4, -5, -6$. ■

Proposition 1

Let $\tilde{D}$ be a monic polynomial of degree $2n$ with real coefficients, and assume that either n is even or both n is odd and $\tilde{D}$ has at least two real roots. Furthermore, let (10), (11) be an observer-based compensator such that the eigenvalues of $\tilde{A}$ are given by the roots of $\tilde{D}$. Then, G_c corresponding to (10), (11) is independent of how the poles are allocated to the regulator and observer.

The proof of Proposition 1 is presented after Proposition 2. Proposition 1 shows that the same observer-based compensator is obtained regardless of how the desired closed-loop poles are allocated between the regulator and observer dynamics, as long as a pair of complex poles is not separated between the observer and the regulator. This result is illustrated by revisiting Example 1.

Example 2

Reconsider Example 1, but this time assign the eigenvalues –5 and –6 to $A + BK$ and the eigenvalues –2 and –4 to $A - FC$. Solving (26) and (29) for K and F yields $K = [-5 \;\; -7]$ and $F = [2 \;\; -0.25]^T$. The transfer function representation of the resulting observer-based compensator is again (30), as guaranteed by Proposition 1. ■

UNIQUENESS OF THE COMPENSATOR BASED ON ONLY THE CLOSED-LOOP POLES

Given $\tilde{D}$, the objective of pole-placement design is to find a proper dynamic compensator G_c that assigns the $n+n_c$ closed-loop poles. This section presents sufficient conditions for uniqueness of G_c based on only $\tilde{D}$. Pole placement using dynamic compensators is considered in [4].

Lemma 1

Let a, b, p, and q be polynomials such that $\deg(a) \leq \deg(b)$, $\deg(p) < \deg(q)$, q is monic, $p \neq 0$, and p and q are coprime. Then $bq + ap = 0$ if and only if $a = b = 0$.

Proof

Sufficiency is immediate. To prove necessity, suppose that $bq + ap = 0$, $a \neq 0$, and $b = 0$. Then $ap = 0$. However, since $p \neq 0$, it follows that $a = 0$, which is a contradiction. Next, suppose that $bq + ap = 0$, $a = 0$, and $b \neq 0$. Then $bq = 0$, However, since q is monic, it follows that $b = 0$, which is a contradiction. Finally, suppose that $bq + ap = 0$, $a \neq 0$, and $b \neq 0$. Then $\deg(ap) = \deg(bq)$. However, since $\deg(a) \leq \deg(b)$ and $\deg(p) < \deg(q)$, it follows that $\deg(ap) < \deg(bq)$, which is a contradiction. ■

Proposition 2

Let $\tilde{D}$ be given by (22), where $\deg(\tilde{D}) = n + n_c$, and let $\deg(N_c) \leq \min\{n_c, n-1\}$. Then, N_c and D_c are uniquely determined.

Proof

Let $G_{c1} = N_{c1}/D_{c1}$ and $G_{c2} = N_{c2}/D_{c2}$ be such that $\deg(D_{c1}) = \deg(D_{c2}) = n_c$, $m_{c1} \triangleq \deg(N_{c1}) \leq \min\{n_c, n-1\}$, $m_{c2} \triangleq \deg(N_{c2}) \leq \min\{n_c, n-1\}$, and $\tilde{D} = DD_{c1} + NN_{c1} = DD_{c2} + NN_{c2}$. Define $a \triangleq N_{c1} - N_{c2}$, $b \triangleq D_{c1} - D_{c2}$, $p \triangleq N$, and $q \triangleq D$. Then $\deg(a) \leq \max\{m_{c1}, m_{c2}\}$, $\deg(b) \leq n_c$, and $\deg(p) < \deg(q)$. Suppose that $\deg(a) > \deg(b)$. Then $N_{c1} - N_{c2} \neq 0$ and $N/D = -b/a = -(D_{c1} - D_{c2})/(N_{c1} - N_{c2})$. Since $\deg(D) = n$, it follows that $\deg(a) = \deg(N_{c1} - N_{c2}) \geq n$. Hence, $n \leq \deg(a) \leq \max\{m_{c1}, m_{c2}\} \leq \min\{n_c, n-1\} \leq n-1$, which is a contradiction. Therefore, $\deg(a) \leq \deg(b)$. Lemma 1 thus implies that $N_{c1} - N_{c2} = D_{c1} - D_{c2} = 0$. Therefore, $G_{c1} = G_{c2}$, and thus N_c and D_c are uniquely determined. ■

Proposition 2 can now be used to prove Proposition 1.

Proof of Proposition 1

For the observer-based compensator (10), (11), $\deg(N_c) = n-1$ and $n_c = n$. Since $\tilde{D}$ is monic, Proposition 2 implies that there exist unique polynomials N_c and D_c satisfying (22). Hence, G_c is uniquely determined and is independent of how the poles are allocated to the regulator and observer. ■

Although Proposition 2 provides only sufficient conditions for uniqueness, the following examples show that uniqueness can fail if these conditions are not satisfied.

Example 3

Let $G(s) = 1/(s+2)$, and consider $G_{c1}(s) = (-s-1)/(s+1)$ and $G_{c2}(s) = (-2s-3)/(s+2)$. Note that $n = 1$, $n_c = 1$, and $\deg(N_c) = 1$, and thus the assumption $\deg(N_c) \leq \min\{n_c, n-1\}$ of Proposition 2 is not satisfied. For both compensators, $\tilde{D}(s) = s^2 + 2s + 1$. ■

Example 4

Let $G(s) = 1/(s+2)$, and consider $G_{c1}(s) = (-3s-1)/(s^2+2s+1)$ and $G_{c2}(s) = (-4s-3)/(s^2+2s+2)$. Note that $n = 1$, $n_c = 2$, and $\deg(N_c) = 1$, and thus the assumption $\deg(N_c) \leq \min\{n_c, n-1\}$ of Proposition 2 is not satisfied. For both compensators, $\tilde{D}(s) = s^3 + 4s^2 + 2s + 1$. ■

The main result, which answers the question posed in the title of this article, can now be stated.

Theorem 1

Let $G_c = N_c/D_c$ be a full-order strictly proper compensator. Then, G_c is observer based if and only if either 1) n is even or 2) n is odd and $\tilde{D} = DD_c + NN_c$ has at least two real roots.

Proof

To prove necessity, note that, since $\deg(D_c) = n$ and $\deg(N_c) \leq n-1$, Proposition 2 implies that N_c and D_c are the only polynomials that satisfy (22). Therefore, G_c is the unique observer-based compensator such that $\tilde{D} \triangleq DD_c + NN_c$. Since G_c is an observer-based compensator, it follows that n of the $2n$ closed-loop eigenvalues are eigenvalues of the observer dynamics, while the remaining n closed-loop eigenvalues are eigenvalues of the regulator dynamics. Hence, in the case where n is odd, it follows that $\tilde{D}$ has at least two real roots. Conversely, since either n is even or both n is odd and $\tilde{D}$ has at least two real roots, Proposition 2 implies that there exists a unique compensator $G_{c,\text{obc}}$ with closed-loop poles given by the roots of $\tilde{D}$ and, in addition, $G_{c,\text{obc}}$ is observer based. ■

POLE PLACEMENT WITHOUT OBSERVER-BASED COMPENSATION

Theorem 1 shows that an observer-based compensator cannot be used in all cases to assign the closed-loop poles. For example, if n is odd, $n_c = n$, and $\tilde{D}$ has no real roots, then the closed-loop eigenvalues cannot be allocated to an observer and a regulator, and thus no observer-based compensator that assigns the desired poles exists. However, by using a dynamic compensator, it is nevertheless possible to assign the desired closed-loop spectrum, albeit with a compensator that is not observer based. This section thus concerns the existence and uniqueness of a pole-placement dynamic compensator in cases where an observer-based compensator does not exist. An algorithm based on the Sylvester

Sylvester Resultant

The Sylvester resultant provides a necessary and sufficient condition for the coprimeness of two polynomials [S1, pp. 459–461], [S2, pp. 234–236], and [S3, pp. 140–142].

THEOREM S1

Let k and l be nonnegative integers such that $k+l \geq 1$, let $a(s) = a_k s^k + a_{k-1} s^{k-1} + \cdots + a_1 s + a_0$ and $b(s) = b_l s^l + b_{l-1} s^{l-1} + \cdots + b_1 s + b_0$, and assume that $a_k \neq 0$ and $l \leq k$. Furthermore, define

$$\mathcal{M}_1(a,b) \triangleq \begin{bmatrix} a_k & 0 & \cdots & 0 & b_l & 0 & \cdots & 0 \\ a_{k-1} & a_k & \ddots & \vdots & b_{l-1} & b_l & \ddots & \vdots \\ \vdots & a_{k-1} & \ddots & 0 & \vdots & b_{l-1} & \ddots & \vdots \\ a_1 & \vdots & \ddots & a_k & b_1 & \vdots & \ddots & b_l \\ a_0 & a_1 & \ddots & a_{k-1} & b_0 & b_1 & \ddots & b_{l-1} \\ 0 & a_0 & \ddots & \vdots & 0 & b_0 & \ddots & \vdots \\ \vdots & 0 & \ddots & a_1 & \vdots & \vdots & \ddots & b_1 \\ 0 & \cdots & \cdots & a_0 & 0 & \cdots & \cdots & b_0 \end{bmatrix} \in \mathbb{R}^{(k+l)\times(k+l)}. \quad \text{(S1)}$$

Then the number of common roots of a and b is $k+l-\text{rank}(\mathcal{M}_1(a,b))$. Furthermore, a and b are coprime if and only if $\mathcal{M}_1(a,b)$ is nonsingular.

The Sylvester resultant $\mathcal{M}_1(a,b)$ is constructed by listing the coefficients of a from a_k to a_0 starting at the top of the first column. In the second column, the coefficients of a are shifted downward by one row; l columns are constructed this way. Next, list the coefficients of b from b_l to b_0 starting at the top of column $l+1$. In the next column, the coefficients of b are shifted downward by one row; k columns are constructed this way. The final matrix has $k+l$ rows and $k+l$ columns.

Note that in Theorem S1 b_l may be zero, and thus $\deg(b) \leq l \leq k = \deg(a)$. Therefore, since $l \leq k$, without loss of generality, $b(s)$ can be written with $k+l-1$ additional leading zeros of the form $b(s) = 0s^k + \cdots + 0s^{l+1} + b_l s^l + \cdots + b_0$, and Theorem S1 can be rewritten as follows.

THEOREM S2

Let k be a nonnegative integer, let $a(s) = a_k s^k + a_{k-1} s^{k-1} + \cdots + a_1 s + a_0$, where $a_k \neq 0$, and $b(s) = b_k s^k + b_{k-1} s^{k-1} + \cdots + b_1 s + b_0$. Furthermore, define

$$\mathcal{M}_2(a,b) \triangleq \begin{bmatrix} a_k & 0 & \cdots & 0 & b_k & 0 & \cdots & 0 \\ a_{k-1} & a_k & \ddots & \vdots & b_{k-1} & b_k & \ddots & \vdots \\ \vdots & a_{k-1} & \ddots & 0 & \vdots & b_{k-1} & \ddots & \vdots \\ a_1 & \vdots & \ddots & a_k & b_1 & \vdots & \ddots & b_k \\ a_0 & a_1 & \ddots & a_{k-1} & b_0 & b_1 & \ddots & b_{k-1} \\ 0 & a_0 & \ddots & \vdots & 0 & b_0 & \ddots & \vdots \\ \vdots & 0 & \ddots & a_1 & \vdots & \vdots & \ddots & b_1 \\ 0 & \cdots & \cdots & a_0 & 0 & \cdots & \cdots & b_0 \end{bmatrix} \in \mathbb{R}^{2k\times 2k}. \quad \text{(S2)}$$

Then the number of common roots of a and b is $2k-\text{rank}(\mathcal{M}_1(a,b))$. Furthermore, a and b are coprime if and only if $\mathcal{M}_1(a,b)$ is nonsingular. Note that in Theorem S2 b_k may be zero, and thus $\deg(b) \leq k$. $\mathcal{M}_1(a,b)$ appears in [S3], whereas $\mathcal{M}2(a,b)$ appears in [S1] and [S2].

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resultant (see "Sylvester Resultant") for designing pole-placement dynamic compensators is given in [6]. Let

$$N(s) = N_m s^m + \cdots + N_1 s + N_0, \quad (31)$$

$$D(s) = s^n + D_{n-1} s^{n-1} + \cdots + D_1 s + D_0, \quad (32)$$

$$N_c(s) = N_{c,\hat{m}_c} s^{\hat{m}_c} + \cdots + N_{c,1} s + N_{c,0}, \quad (33)$$

$$D_c(s) = s^{n_c} + D_{c,n_c-1} s^{n_c-1} + \cdots + D_{c,1} s + D_{c,0}, \quad (34)$$

$$\tilde{D}(s) = s^{n+n_c} + \tilde{D}_{n+n_c-1} s^{n+n_c-1} + \cdots + \tilde{D}_1 s + \tilde{D}_0, \quad (35)$$

where $\hat{m}_c \leq n_c$. Note that $N_{c,\hat{m}_c}$ may or may not be zero, and thus (33) implies that $m_c \triangleq \deg(N_c) \leq \hat{m}_c$. Substituting (31)–(35) into (22) and matching like powers of s yields the linear system of equations

$$M \begin{bmatrix} D_{c,n_c-1} \\ \vdots \\ D_{c,0} \\ N_{c,\hat{m}_c} \\ \vdots \\ N_{c,0} \end{bmatrix} = \begin{bmatrix} \tilde{D}_{n+n_c-1} - D_{n-1} \\ \vdots \\ \tilde{D}_{n_c} - D_0 \\ \tilde{D}_{n_c-1} \\ \vdots \\ \tilde{D}_0 \end{bmatrix}, \quad (36)$$

where $M \in \mathbb{R}^{(n+n_c)\times(\hat{m}_c+n_c+1)}$ is defined by

$$M \triangleq \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\ D_{n-1} & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & D_{n-1} & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & N_m & 0 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & N_{m-1} & N_m & \ddots & \vdots \\ D_1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & N_{m-1} & \ddots & 0 \\ D_0 & D_1 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \vdots & \ddots & N_m \\ 0 & D_0 & \ddots & \ddots & \ddots & \ddots & D_{n-1} & \vdots & \vdots & \ddots & N_{m-1} \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & N_1 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & N_0 & N_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & N_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & D_1 & \vdots & \vdots & \ddots & N_1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & D_0 & 0 & \cdots & \cdots & N_0 \end{bmatrix}. \quad (37)$$

The matrix M is constructed by listing the coefficients of D from one to D_0 starting at the top of the first column. In the second column, the coefficients of D are shifted downward by one row; n_c columns are constructed this way. Next, in column n_c+1, the coefficients of N from N_m to N_0 are

EXAMPLE S1

Let $a(s) = s+1$ and $b(s) = s+2$, which are coprime. Then, using (S1),

$$\mathcal{M}_1(a,b) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

which is nonsingular. ■

EXAMPLE S2

Let $a(s) = s+1$ and $b(s) = (s+1)(s+2) = s^2+3s+2$, which are not coprime. Then, using (S1),

$$\mathcal{M}_1(a,b) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix},$$

which has rank 2 and thus is singular. The number of common roots is $k+l-\operatorname{rank}(\mathcal{M}_1(a,b)) = 1+2-2 = 1$. ■

EXAMPLE S3

Let $a(s) = s^3+2s^2+4s+1$ and $b(s) = 6$. Since b is a constant and nonzero, it has no roots, and thus a and b are coprime. Furthermore, using (S1),

$$\mathcal{M}_1(a,b) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix},$$

which is nonsingular. ■

EXAMPLE S4

This example uses (S2) instead of (S1), and leading zeros are added to b in Example S3 by writing $b(s) = 0s^3+0s^2+0s+6$. Thus, using (S2),

$$\mathcal{M}_2(a,b) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 2 & 6 & 0 & 0 \\ 0 & 1 & 4 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 & 6 \end{bmatrix},$$

which is nonsingular. In fact, $\det(\mathcal{M}_2(a,b)) = 216$, as in Example S3. ■

EXAMPLE S5

Let $a(s) = s^3+2s^2+4s+1$ and $b(s) = 0s^3+0s^2+0s+0$ so that b is the zero polynomial. Then, using (S2),

$$\mathcal{M}_2(a,b) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since $\operatorname{rank}(\mathcal{M}_2(a,b)) = 3$, Theorem S1 implies that the number of common roots of a and b is $2k-\operatorname{rank}(\mathcal{M}_2(a,b)) = 6-3 = 3$, which shows that every root of a is also a root of b. ■

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listed starting after $n+n_c-m-\hat{m}_c-1$ zeros. In the next column, the coefficients of N are shifted downward by one row; $\hat{m}_c+1$ columns are constructed this way. The matrix M thus has $n+n_c$ rows and $\hat{m}_c+n_c+1$ columns. Hence, M is square if and only if $\hat{m}_c = n-1$. Therefore, M can be made square by choosing $\hat{m}_c = n-1$ if and only if $m_c \le n-1$. Since $\hat{m}_c \le n_c$, if M is square, then $n_c \ge n-1$. Consequently, M can be made square if and only if $m_c \le n-1 \le n_c$.

The following result relates M defined by (37) with the Sylvester resultants $\mathcal{M}_1$ and $\mathcal{M}_2$ defined by (S1) and (S2), respectively. Note that both Sylvester resultants are square.

Proposition 3

$M = \mathcal{M}_1(D,N)$ if and only if $\hat{m}_c = n-1$ and $n_c \le n$.

Proof

Suppose that $M = \mathcal{M}_1(D,N)$. Then M is square, and thus $\hat{m}_c = n-1$. Hence, $\mathcal{M}_1(D,N) = M \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$. Since $n = \deg(D)$, it follows from the construction of $\mathcal{M}_1(D,N)$ in Theorem S1 (note that $l \le k$ in Theorem S1) that $n_c \le n$. Conversely, suppose that $\hat{m}_c = n-1$ and $n_c \le n$. Then M is square, $M \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$, and $m \le n-1 = \hat{m}_c \le n_c$. Therefore, $M = \mathcal{M}_1(D,N)$. ■

Proposition 4

$M = \mathcal{M}_2(D,N)$ if and only if $\hat{m}_c = n-1$ and $n_c = n$.

Proof

Suppose that $M = \mathcal{M}_2(D,N)$. Then M is square, and thus $\hat{m}_c = n-1$. Hence, $M \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$. Furthermore,

TABLE 1 Cases $n_c = n-1$ and $n_c = n$. Note that, in the case $n_c = n-1$, G_c may be exactly proper or strictly proper; whereas, in the case $n_c = n$, G_c must be strictly proper.

	$n_c = n-1$	$n_c = n$
$\deg(N_c)$	$\le n-1$	$\le n-1$
$\deg(D_c)$	$n-1$	n
$\deg(\tilde{D})$	$2n-1$	$2n$

$\mathcal{M}_2(D,N) \in \mathbb{R}^{2n\times 2n}$. Therefore, $n_c = n$. Conversely, suppose that $\hat{m}_c = n-1$ and $n_c = n$. Then M is square, $M \in \mathbb{R}^{2n\times 2n}$, and $m < n$. Therefore, $M = \mathcal{M}_2(D,N)$. ■

Since $\hat{m}_c \le n-1$, the above discussion shows that M is a Sylvester resultant if and only if $\hat{m}_c = n-1$ and $n-1 \le n_c \le n$. The case where $m_c = n-1$ and $n_c = n-1$ is discussed in [7]. These cases are summarized in Table 1.

Proposition 5

Assume that either $M = \mathcal{M}_1(D,N)$ or $M = \mathcal{M}_2(D,N)$. Then there exist unique N_c and D_c satisfying (36).

Proof

Since N and D are coprime, Theorems S1 and S2 state that $\mathcal{M}_1(D,N)$ and $\mathcal{M}_2(D,N)$ are nonsingular. It thus follows that there exist unique N_c and D_c satisfying (36). ■

In [6, p. 182], necessary and sufficient conditions for the existence and uniqueness of N_c and D_c are given for the case where $n_c = n-1$, and, in [6, p. 182], sufficient conditions for existence are given in the case where $n_c > n-1$. The results in this article complement the results given in [6] by providing sufficient conditions for uniqueness in the case where $n_c = n$ and $\deg(N_c) \le n-1$.

Now revisit Example 1, which considers pole placement using an observer-based compensator, but instead of designing an observer-based compensator with specified observer and regulator poles, apply Proposition 5 by solving (36) to determine a compensator that places all $2n$ poles directly. Theorem 1 implies that there exists a unique nth-order compensator that places the closed-loop poles, and thus it is expected that the same compensator found in Example 1 will be obtained.

Example 5

Revisit Example 1, and place the poles at –6, –5, –4, and –2 without observer-based compensation. Solving (36) for N_c and D_c yields $N_c(s) = 8.26s + 23.25$ and $D_c(s) = s^2 + 13s + 49$. Note that G_c is precisely the observer-based compensator obtained in Example 1. ■

Now consider an example where pole placement using an observer-based compensator is impossible. The example takes advantage of Proposition 5.

Example 6

Consider the state-space equations for a dc motor, where

$$x \triangleq \begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix},\quad A = \begin{bmatrix}0 & 1 & 0\\ 0 & -2 & 0.85\\ 0 & -3 & -1\end{bmatrix},\quad B = \begin{bmatrix}0\\0\\1\end{bmatrix},\quad C = [1\ \ 0\ \ 0].$$

Assign the closed-loop poles to $-0.5 \pm 0.1j$, $-1 \pm 0.5j$, $-2 \pm j$. Solving (36) for N_c and D_c yields $N_c(s) = 5.585s^2 + 8.034s + 1.912$ and $D_c(s) = s^3 + 4s^2 + 3.96s + 0.73$. However, since three pairs of complex poles cannot be allocated separately to the regulator and observer dynamics, it is impossible to design an observer-based compensator that yields the desired closed-loop poles. Hence, $G_c(s) = (5.585s^2 + 8.034s + 1.912)/(s^3 + 4s^2 + 3.96s + 0.73)$ is not observer based and therefore must be suboptimal in the sense of LQG control. ■

CONCLUSIONS

A full-order dynamic compensator is observer based if, in some basis, it has the structure of an observer followed by state-estimate feedback. This article shows that almost all full-order compensators are, in fact, observer based. The essential idea of the proof is that the observer-based compensator that achieves the desired spectrum is unique. An exception to this fact, however, is the case where the plant order is odd and the closed-loop spectrum has no real eigenvalues. In this case, the closed-loop spectra cannot be partitioned into conjugate-symmetric regulator and observer spectra, and therefore the compensator is not observer based. All such compensators are, of course, suboptimal in the sense of LQG control.

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