

Robust, reduced-order, nonstrictly proper state estimation via the optimal projection equations with Petersen–Hollot bounds

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Abstract: A state-estimation design problem involving parametric plant uncertainties is considered. An error bound suggested by recent work of Petersen and Hollot is utilized for guaranteeing robust estimation. Necessary conditions which generalize the optimal projection equations for reduced-order state estimation are used to characterize the estimator which minimizes the error bound. The design equations thus effectively serve as sufficient conditions for synthesizing robust estimators. An additional feature is the presence of a static estimation gain in conjunction with the dynamic (Kalman) estimator, i.e., a nonstrictly proper estimator.

Keywords: Robust Kalman filter, Error bounds, Reduced-order state estimation.

1. Introduction

As is well known [2,5–8,11,14,15] optimal filters based upon nominal parameter values may be severely degraded in the presence of parameter deviations. Thus, it is desirable to obtain robust state estimators which provide acceptable performance over the range of parametric uncertainty. The approach of the present paper is related to the guaranteed cost approach developed for control in [4,16] and applied to estimation in [11]. Specifically, the main idea is to bound the effect of the uncertain parameters on the estimation error over the uncertainty range and then choose estimator gains to minimize the estimation bound. Thus the actual estimation error is guaranteed to lie below the prescribed upper bound.

The technique used to determine minimizing estimator gains is based upon a generalization of the optimal projection equations for reduced-order state estimation [1]. Thus the results of the present paper effectively extend the results of [1] to the case of system uncertainties. It should be noted that the optimal projection equations, which are necessary conditions for optimality, now serve as sufficient conditions for robust estimation by virtue of the fact that a bound on the estimation error is being minimized rather than the estimation error itself. The bound utilized in the present paper is an extension of the approach developed in [12,13] for constructing Lyapunov functions for full-state feedback and utilized in [10] to characterize the structured stability radius.

An additional feature of the present paper is the inclusion of a static feedback gain in conjunction with the dynamic estimator. Thus the results of the present paper represent a generalization of standard results to the case of nonstrictly proper estimation.

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2. Notation and definitions

Note: All matrices have real entries.

$\mathbb{R}, \mathbb{R}^{r \times s}, \mathbb{R}^r, \mathbb{E}$	real numbers, $r \times s$ real matrices, $\mathbb{R}^{r \times 1}$, expected value.
$I_r, (\cdot)^T$	$r \times r$ identity matrix, transpose
$\mathbb{S}^r, \mathbb{N}^r, \mathbb{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices.
$Z_1 \leq Z_2, Z_1 < Z_2$	$Z_2 - Z_1 \in \mathbb{N}^r, Z_2 - Z_1 \in \mathbb{P}^r, Z_1, Z_2 \in \mathbb{S}^r$.
$n, l, \hat{l}, n_e, p, q; \tilde{n}$	positive integers; $n + n_e$.
$x, y, \hat{y}, y_e, x_e, \tilde{x}$	$n, l, \hat{l}, q, n_e, \tilde{n}$ -dimensional vectors.
$A, \Delta A; C, \Delta C$	$n \times n$ matrices; $l \times n$ matrices.
\hat{C}, L, R	$\hat{l} \times n$ matrix, $q \times n$ matrix, $q \times q$ matrix, $R \in \mathbb{P}^q$.
A_e, B_e, C_e, D_e	$n_e \times n_e, n_e \times l, q \times n_e, q \times l$ matrices.
$\tilde{A}, \Delta \tilde{A}$	$\begin{bmatrix} A & 0 \\ B_e C & A_e \end{bmatrix}, \begin{bmatrix} \Delta A & 0 \\ B_e \Delta C & 0 \end{bmatrix}$.
\tilde{R}	$\begin{bmatrix} L^T R L - L^T R D_e \hat{C} - \hat{C}^T D_e^T R L + \hat{C}^T D_e^T R D_e \hat{C} & -L^T R C_e + \hat{C}^T D_e^T R C_e \\ -C_e^T R L + C_e^T R D_e \hat{C} & C_e^T R C_e \end{bmatrix}$.
$w_0(\cdot), w_1(\cdot)$	n, l -dimensional white noise.
V_0, V_1	intensity of $w_0(\cdot), w_1(\cdot)$; $V_0 \in \mathbb{N}^n, V_1 \in \mathbb{P}^l$.
V_{01}	$n \times l$ cross intensity of $w_0(\cdot), w_1(\cdot)$.
$\tilde{w}(\cdot), \tilde{V}$	$\begin{bmatrix} w_0(\cdot) \\ B_e w_1(\cdot) \end{bmatrix}, \begin{bmatrix} V_0 & V_{01} B_e^T \\ B_e V_{01}^T & B_e V_1 B_e^T \end{bmatrix}$.

3. Robust estimation problem

Let $\mathcal{Q} \subset \mathbb{R}^{n \times n} \times \mathbb{R}^{l \times n}$ denote the set of uncertain perturbations $(\Delta A, \Delta C)$ of the nominal plant matrices A and C .

Robust Estimation Problem. For fixed $n_e \leq n$, determine (A_e, B_e, C_e, D_e) such that, for the system consisting of the n -th-order disturbed plant

$$\dot{x}(t) = (A + \Delta A)x(t) + w_0(t), \quad t \in [0, \infty), \quad (3.1)$$

noisy and nonnoisy measurements

$$y(t) = (C + \Delta C)x(t) + w_1(t), \quad (3.2)$$

$$\hat{y}(t) = \hat{C}x(t), \quad (3.3)$$

and n_e -th-order nonstrictly proper state estimator

$$\dot{x}_e(t) = A_e x_e(t) + B_e y(t), \quad (3.4)$$

$$y_e(t) = C_e x_e(t) + D_e \hat{y}(t), \quad (3.5)$$

the state-estimation error criterion

$$J(A_e, B_e, C_e, D_e) \triangleq \sup_{(\Delta A, \Delta C) \in \mathcal{Q}} \limsup_{t \rightarrow \infty} \mathbb{E} [Lx(t) - y_e(t)]^T R [Lx(t) - y_e(t)] \quad (3.6)$$

is minimized.

Note that the augmented system (3.1)–(3.5) can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta \tilde{A})\tilde{x}(t) + \tilde{w}(t), \quad t \in [0, \infty), \quad (3.7)$$

where $\tilde{x}(t) \triangleq [x^T(t), x_e^T(t)]^T$. The cost can be expressed in terms of the augmented second-moment matrix.

Proposition 3.1. For given (A_e, B_e, C_e, D_e) and $(\Delta A, \Delta C) \in \mathcal{U}$ the second-moment matrix

$$\tilde{Q}_{\Delta \tilde{A}}(t) \triangleq \mathbb{E}[\tilde{x}(t)\tilde{x}^T(t)], \quad t \in [0, \infty), \quad (3.8)$$

satisfies

$$\dot{\tilde{Q}}_{\Delta \tilde{A}}(t) = (\tilde{A} + \Delta \tilde{A})\tilde{Q}_{\Delta \tilde{A}}(t) + \tilde{Q}_{\Delta \tilde{A}}(t)(\tilde{A} + \Delta \tilde{A})^T + \tilde{V}, \quad t \in [0, \infty). \quad (3.9)$$

Furthermore,

$$J(A_e, B_e, C_e, D_e) = \sup_{(\Delta A, \Delta C) \in \mathcal{U}} \limsup_{t \rightarrow \infty} \text{tr } \tilde{Q}_{\Delta \tilde{A}}(t)\tilde{R}. \quad (3.10)$$

4. Sufficient conditions for robust performance

Lemma 4.1. Suppose the system (3.7) is stable for all $(\Delta A, \Delta C) \in \mathcal{U}$. Then

$$J(A_e, B_e, C_e, D_e) = \sup_{(\Delta A, \Delta C) \in \mathcal{U}} \text{tr } \tilde{Q}_{\Delta \tilde{A}}\tilde{R}, \quad (4.1)$$

where $\tilde{Q}_{\Delta \tilde{A}} \in \mathbb{N}^{\tilde{n}}$ is the unique solution to

$$0 = (\tilde{A} + \Delta \tilde{A})\tilde{Q}_{\Delta \tilde{A}} + \tilde{Q}_{\Delta \tilde{A}}(\tilde{A} + \Delta \tilde{A})^T + \tilde{V}. \quad (4.2)$$

We now seek upper bounds for $J(A_e, B_e, C_e, D_e)$.

Theorem 4.1. Let $\Omega: \mathbb{N}^{\tilde{n}} \times \mathbb{R}^{n_e \times l} \rightarrow \mathbb{S}^{\tilde{n}}$ be such that

$$\Delta \tilde{A}\mathcal{Q} + \mathcal{Q}\Delta \tilde{A}^T \leq \Omega(\mathcal{Q}, B_e), \quad (\Delta A, \Delta C) \in \mathcal{U}, \quad (\mathcal{Q}, B_e) \in \mathbb{N}^{\tilde{n}} \times \mathbb{R}^{n_e \times l}, \quad (4.3)$$

and, for given (A_e, B_e, C_e, D_e) , suppose there exists $\mathcal{Q} \in \mathbb{N}^{\tilde{n}}$ satisfying

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \Omega(\mathcal{Q}, B_e) + \tilde{V}, \quad (4.4)$$

and suppose the pair $(\tilde{V}^{1/2}, \tilde{A} + \Delta \tilde{A})$ is detectable for all $(\Delta A, \Delta C) \in \mathcal{U}$. Then A_e is asymptotically stable, $A + \Delta A$ is asymptotically stable for all $(\Delta A, \Delta C) \in \mathcal{U}$,

$$\tilde{Q}_{\Delta \tilde{A}} \leq \mathcal{Q}, \quad (\Delta A, \Delta C) \in \mathcal{U}, \quad (4.5)$$

where $\tilde{Q}_{\Delta \tilde{A}}$ satisfies (4.2), and

$$J(A_e, B_e, C_e, D_e) \leq \text{tr } \mathcal{Q}\tilde{R}. \quad (4.6)$$

Proof. For all $(\Delta A, \Delta C) \in \mathcal{U}$, (4.4) is equivalent to

$$0 = (\tilde{A} + \Delta \tilde{A})\mathcal{Q} + \mathcal{Q}(\tilde{A} + \Delta \tilde{A})^T + \Psi(\mathcal{Q}, B_e, \Delta \tilde{A}) + \tilde{V}, \quad (4.7)$$

where

$$\Psi(\mathcal{Q}, B_e, \Delta \tilde{A}) \triangleq \Omega(\mathcal{Q}, B_e) - (\tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T).$$

Note that by (4.3), $\Psi(\mathcal{Q}, B_e, \Delta \tilde{A}) \geq 0$ for all $(\Delta A, \Delta C) \in \mathcal{U}$. Since $(\tilde{V}^{1/2}, \tilde{A} + \Delta \tilde{A})$ is detectable for all $(\Delta A, \Delta C) \in \mathcal{U}$, it follows from Theorem 3.6 of [17] that $((\tilde{V} + \Psi(\mathcal{Q}, B_e, \Delta \tilde{A}))^{1/2}, \tilde{A} + \Delta \tilde{A})$ is detectable for all $(\Delta A, \Delta C) \in \mathcal{U}$. Hence Lemma 12.2 of [17] implies $\tilde{A} + \Delta \tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta C) \in \mathcal{U}$. Since $\tilde{A} + \Delta \tilde{A}$ is lower block triangular, A_e is asymptotically stable and $A + \Delta A$ is asymptotically stable for all $(\Delta A, \Delta C) \in \mathcal{U}$. Next, subtracting (4.2) from (4.7) yields

$$0 = (\tilde{A} + \Delta \tilde{A})(\mathcal{Q} - \tilde{Q}_{\Delta \tilde{A}}) + (\mathcal{Q} - \tilde{Q}_{\Delta \tilde{A}})(\tilde{A} + \Delta \tilde{A})^T + \Psi(\mathcal{Q}, B_e, \Delta \tilde{A}),$$

or, equivalently, (since $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable)

$$\mathcal{Q} - \tilde{Q}_{\Delta\tilde{A}} = \int_0^{\infty} e^{(\tilde{A} + \Delta\tilde{A})t} \Psi(\mathcal{Q}, B_e, \Delta\tilde{A}) e^{(\tilde{A} + \Delta\tilde{A})^T t} dt \geq 0,$$

which implies (4.5). Finally, (4.5) and (4.1) yield (4.6). \square

5. Uncertainty structure

The uncertainty set \mathcal{U} is assumed to be of the form

$$\mathcal{U} = \left\{ (\Delta A, \Delta C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{l \times n}: \Delta A = \sum_{i=1}^p D_i M_i N_i E_i, \right. \\ \left. \Delta C = \sum_{i=1}^p F_i M_i N_i E_i, M_i M_i^T \leq \bar{M}_i, N_i^T N_i \leq \bar{N}_i, i = 1, \dots, p \right\}, \quad (5.1)$$

where, for $i = 1, \dots, p$, $D_i \in \mathbb{R}^{n \times r_i}$, $E_i \in \mathbb{R}^{l_i \times n}$ and $F_i \in \mathbb{R}^{l_i \times r_i}$ are fixed matrices denoting the structure of the uncertainty; $\bar{M}_i \in \mathbb{N}^{r_i}$ and $\bar{N}_i \in \mathbb{N}^{l_i}$ are given uncertainty bounds; and $M_i \in \mathbb{R}^{r_i \times s_i}$, $N_i \in \mathbb{R}^{s_i \times l_i}$ are uncertain matrices. The closed-loop system thus has structured uncertainty of the form

$$\Delta\tilde{A} = \sum_{i=1}^p \tilde{D}_i M_i N_i \tilde{E}_i, \quad (5.2)$$

where

$$\tilde{D}_i \triangleq \begin{bmatrix} D_i \\ B_e F_i \end{bmatrix}, \quad \tilde{E}_i \triangleq [E_i \ 0]. \quad (5.3)$$

The special case $\bar{M}_i = \mu_i^2 I_{r_i}$, $\bar{N}_i = \nu_i^2 I_{l_i}$ is worth noting.

Proposition 5.1. *Let $\mu_i, \nu_i \geq 0$, $i = 1, \dots, p$. Then $M_i M_i^T \leq \mu_i^2 I_{r_i}$ and $N_i^T N_i \leq \nu_i^2 I_{l_i}$ if and only if $\sigma_{\max}(M_i) \leq \mu_i$ and $\sigma_{\max}(N_i) \leq \nu_i$.*

Remark 5.1. \mathcal{U} given by (5.1) is directly related to the structured stability radius introduced in [10]. Setting $p = 1$, $\bar{M}_1 = \mu_1 I_{r_1}$, $r_1 = s_1$, $\bar{N}_1 = I_{r_1}$ and $N_1 \equiv I_{r_1}$ yields the setting of [10]. For a similar formulation, see [13].

6. The Petersen–Hollot bound

Given \mathcal{U} as defined in (5.1), we now specify Ω satisfying (4.3).

Proposition 6.1. *The bound Ω given by*

$$\Omega(\mathcal{Q}, B_e) \triangleq \sum_{i=1}^p \tilde{D}_i \bar{M}_i \tilde{D}_i^T + \mathcal{Q} \tilde{E}_i^T \bar{N}_i \tilde{E}_i \mathcal{Q} \quad (6.1)$$

satisfies (4.3) with \mathcal{U} given by (5.1).

Proof. For $i = 1, \dots, p$,

$$\begin{aligned} 0 &\leq [\tilde{D}_i M_i - \mathcal{Q} \tilde{E}_i^T N_i^T] [\tilde{D}_i M_i - \mathcal{Q} \tilde{E}_i^T N_i^T]^T \\ &= \tilde{D}_i M_i M_i^T \tilde{D}_i^T + \mathcal{Q} \tilde{E}_i^T N_i^T N_i \tilde{E}_i \mathcal{Q} - (\tilde{D}_i M_i N_i \tilde{E}_i \mathcal{Q} + \mathcal{Q} \tilde{E}_i^T N_i^T M_i^T \tilde{D}_i^T) \\ &\leq \tilde{D}_i \bar{M}_i \tilde{D}_i^T + \mathcal{Q} \tilde{E}_i^T \bar{N}_i \tilde{E}_i \mathcal{Q} - (\tilde{D}_i M_i N_i \tilde{E}_i \mathcal{Q} + \mathcal{Q} \tilde{E}_i^T N_i^T M_i^T \tilde{D}_i^T). \end{aligned}$$

Summing over i yields (4.3). \square

Remark 6.1. The bound (6.1) is used in [12] for unit-rank perturbations while a more general treatment appears in [13].

7. The auxiliary minimization problem

Our goal is to minimize the error bound (4.6).

Auxiliary Minimization Problem. Determine $(\mathcal{Q}, A_e, B_e, C_e, D_e)$ with $\mathcal{Q} \in \mathbb{N}^{\bar{n}}$ which minimizes

$$\mathcal{J}(\mathcal{Q}, A_e, B_e, C_e, D_e) \triangleq \text{tr } \mathcal{Q} \tilde{R} \quad (7.1)$$

subject to

$$0 = \tilde{A} \mathcal{Q} + \mathcal{Q} \tilde{A}^T + \sum_{i=1}^p [\tilde{D}_i \bar{M}_i \tilde{D}_i^T + \mathcal{Q} \tilde{E}_i^T \bar{N}_i \tilde{E}_i \mathcal{Q}] + \tilde{V}, \quad (7.2)$$

and

$$(\tilde{V}^{1/2}, \tilde{A} + \Delta \tilde{A}) \text{ is detectable, } (\Delta A, \Delta C) \in \mathcal{U}. \quad (7.3)$$

Proposition 7.1. If $(\mathcal{Q}, A_e, B_e, C_e, D_e)$ satisfies (7.2) and (7.3) with $\mathcal{Q} \geq 0$, then $\tilde{A} + \Delta \tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta C) \in \mathcal{U}$, and

$$J(A_e, B_e, C_e, D_e) \leq \mathcal{J}(\mathcal{Q}, A_e, B_e, C_e, D_e). \quad (7.4)$$

Proof. With Ω given by (6.1), (7.2) is equivalent to (4.4). Hence, with (7.3), the hypotheses of Theorem 4.1 are satisfied so that the augmented system is stable over \mathcal{U} with estimation bound (4.6). Note that with (7.1), (7.4) is merely a restatement of (4.6). \square

8. Necessary conditions for the auxiliary minimization problem

Rigorous application of the Lagrange multiplier technique requires additional technical assumptions. Specifically, we further restrict $(\mathcal{Q}, A_e, B_e, C_e, D_e)$ to the set

$$\begin{aligned} \mathcal{S} \triangleq \{ &(\mathcal{Q}, A_e, B_e, C_e, D_e) : \mathcal{Q} \in \mathbb{P}^{\bar{n}}, \tilde{\mathcal{A}} \text{ is asymptotically stable, } (A_e, B_e, C_e) \text{ is controllable} \\ &\text{and observable, and } \hat{C}(Q_1 - Q_{12} Q_2^{-1} Q_{12}^T) \hat{C}^T > 0 \}, \end{aligned}$$

where (\oplus) denotes Kronecker sum [3])

$$\tilde{\mathcal{A}} \triangleq \left(\tilde{A} + \sum_{i=1}^p \tilde{E}_i^T \bar{N}_i \tilde{E}_i \mathcal{Q} \right) \oplus \left(\tilde{A} + \sum_{i=1}^p \tilde{E}_i^T \bar{N}_i \tilde{E}_i \mathcal{Q} \right)$$

and \mathcal{Q} is partitioned as in Appendix A. As shown in Appendix A, Q_2 is invertible since (A_e, B_e) is controllable. The positive definiteness condition holds when \hat{C} has full row rank and \mathcal{Q} is positive definite.

As can be seen from the proof of Theorem 8.1 in Appendix A, this condition implies the existence of the projection τ_1 defined below. Note that \mathcal{S} is open.

Remark 8.1. The constraint $(\mathcal{Q}, A_e, B_e, C_e, D_e) \in \mathcal{S}$ is not required for robust estimation. As will be seen from the proof of Theorem 8.1, the set \mathcal{S} constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the Auxiliary Minimization Problem. Specifically, asymptotic stability of \mathcal{A} serves as a normality condition which further implies that the dual \mathcal{P} of \mathcal{Q} satisfying (A.2) is nonnegative definite. Furthermore, (A_e, B_e, C_e) minimal is a nondegeneracy condition which implies that the lower right $n_e \times n_e$ subblocks of \mathcal{Q} and \mathcal{P} are positive definite. It is extremely important to emphasize that Proposition 7.1 shows that it is *not* necessary for guaranteed robust estimation that an admissible quadruple obtained by solving the necessary conditions actually be shown to be an element of \mathcal{S} .

For arbitrary $Q \in \mathbb{R}^{n \times n}$ define the following notation:

$$V_{1a} \triangleq V_1 + \sum_{i=1}^p F_i \bar{M}_i F_i^T, \quad Q_a \triangleq V_{01} + Q C^T + \sum_{i=1}^p D_i \bar{M}_i F_i^T,$$

$$D \triangleq \sum_{i=1}^p D_i \bar{M}_i D_i^T, \quad E \triangleq \sum_{i=1}^p E_i^T \bar{N}_i E_i, \quad A_Q \triangleq A - Q_a V_{1a}^{-1} C.$$

The following factorization lemma is needed for the statement of the main result. See [1] for details.

Lemma 8.1. *If $\hat{Q}, \hat{P} \in \mathbb{N}^n$ and $\text{rank } \hat{Q}\hat{P} = n_e$, then there exist $n_e \times n$ G , Γ and $n_e \times n_e$ invertible M such that*

$$\hat{Q}\hat{P} = G^T M \Gamma, \quad \Gamma G^T = I_{n_e}. \quad (8.1), (8.2)$$

Furthermore, G , M and Γ are unique except for a change of basis in \mathbb{R}^{n_e} .

Since $\hat{Q}\hat{P}$ is diagonalizable it has a group generalized inverse $(\hat{Q}\hat{P})^\# = G^T M^{-1} \Gamma$ and

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^\# = G^T \Gamma \quad (8.3)$$

is an oblique projection. Define the complementary projection $\tau_\perp \triangleq I_n - \tau$ and call (G, M, Γ) satisfying (8.1) and (8.2) a *projective factorization* of $\hat{Q}\hat{P}$.

Theorem 8.1. *$(\mathcal{Q}, A_e, B_e, C_e, D_e) \in \mathcal{S}$ is an extremal of the Auxiliary Minimization problem with \mathcal{U} given by (5.1) if and only if there exist $Q, \hat{Q}, \hat{P} \in \mathbb{N}^n$ such that $\mathcal{Q}, A_e, B_e, C_e, D_e$ are given by*

$$\mathcal{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^T \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^T \end{bmatrix}, \quad (8.4)$$

$$A_e = \Gamma(A - Q_a V_{1a}^{-1} C + Q E) G^T, \quad (8.5)$$

$$B_e = \Gamma Q_a V_{1a}^{-1}, \quad (8.6)$$

$$C_e = L \tau_{1\perp} G^T, \quad (8.7)$$

$$D_e = L Q \hat{C}^T (\hat{C} Q \hat{C}^T)^{-1}, \quad (8.8)$$

for some projective factorization (G, M, Γ) of $\hat{Q}\hat{P}$, and such that Q, \hat{Q}, \hat{P} satisfy

$$0 = A Q + Q A^T + V_0 + D + Q E Q - Q_a V_{1a}^{-1} Q_a^T + \tau_\perp Q_a V_{1a}^{-1} Q_a^T \tau_\perp^T, \quad (8.9)$$

$$0 = (A + Q E) \hat{Q} + \hat{Q} (A + Q E)^T + \hat{Q} E \hat{Q} + Q_a V_{1a}^{-1} Q_a^T - \tau_\perp Q_a V_{1a}^{-1} Q_a^T \tau_\perp^T, \quad (8.10)$$

$$0 = (A_Q + Q E)^T \hat{P} + \hat{P} (A_Q + Q E) + \tau_{1\perp}^T L^T R L \tau_{1\perp} - \tau_\perp^T \tau_{1\perp}^T L^T R L \tau_{1\perp} \tau_\perp, \quad (8.11)$$

$$\text{rank } \hat{Q} = \text{rank } \hat{P} = \text{rank } \hat{Q}\hat{P} = n_e, \quad (8.12)$$

where

$$\tau_1 \triangleq Q\hat{C}^T(\hat{C}Q\hat{C}^T)^{-1}\hat{C}, \quad \tau_{1\perp} \triangleq I_n - \tau_1. \quad (8.13)$$

Theorem 8.1 (proved in Appendix A) presents necessary conditions for the Auxiliary Minimization Problem which explicitly characterize extremals $(\mathcal{Q}, A_e, B_e, C_e, D_e)$. These necessary conditions consist of a system of two modified Lyapunov equations and one modified Riccati equation coupled by two oblique projections τ and τ_1 and uncertainty terms. The projections τ and τ_1 correspond to reduced estimator order and singular observation noise, respectively.

Several special cases can immediately be discerned. For example, in the full-order estimator case $n_e = n$, set $\tau = I_n$ so that $\tau_{1\perp} = 0$. Now the last term in each of (8.9)–(8.11) can be deleted and G and Γ in (8.4)–(8.7) can be taken to be the identity. Furthermore, since \hat{Q} and \hat{P} now play no role in determining the optimal estimator, equations (8.10) and (8.11) are superfluous. If, furthermore, D_i, E_i and F_i are zero, then (8.9) reduces to the standard observer Riccati equation of steady-state Kalman filter theory. Alternatively, the case in which the static estimator gain D_e is absent can be handled by ignoring (8.8) and setting $\tau_1 = 0$. If, furthermore, the uncertainty terms are deleted then the results of [1] are recovered.

9. Sufficient conditions for robust, reduced-order estimation

The main result guaranteeing robust estimation can now be stated.

Theorem 9.1. *Suppose there exist $Q, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying (8.9)–(8.12), let A_e, B_e, C_e, D_e be given by (8.5)–(8.8), and suppose that $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is detectable for all $(\Delta A, \Delta C) \in \mathcal{U}$ with \mathcal{U} given by (5.1). Then A_e is asymptotically stable, $A + \Delta A$ is asymptotically stable for all $(\Delta A, \Delta C) \in \mathcal{U}$, and the estimation error satisfies the bound*

$$J(A_e, B_e, C_e, D_e) \leq \text{tr } Q\tau_{1\perp}^T L^T R L \tau_{1\perp}. \quad (9.1)$$

Proof. Theorem 8.1 implies \mathcal{Q} given by (8.4) satisfies (7.2). With the detectability assumption the result follows from Proposition 7.1. \square

Remark 9.1. Note that if $\hat{C} = L$ then $C_e = 0$ and the estimation bound (9.1) is zero since $\hat{C}\tau_{1\perp} = 0$. This is, of course, to be expected since perfect estimation is achievable in this case.

Remark 9.2. The problem of designing reduced-order, robust estimators for unstable systems remains an area for future research.

Appendix A: Proof of Theorem 8.1

Partition $\tilde{n} \times \tilde{n}$ \mathcal{Q}, \mathcal{P} into $n \times n, n \times n_e,$ and $n_e \times n_e$ subblocks as

$$\mathcal{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix},$$

and define the $n \times n$ nonnegative-definite matrices

$$Q \triangleq Q_1 - Q_{12}Q_2^{-1}Q_{12}^T, \quad P \triangleq P_1 - P_{12}P_2^{-1}P_{12}^T, \\ \hat{Q} \triangleq Q_{12}Q_2^{-1}Q_{12}^T, \quad \hat{P} \triangleq P_{12}P_2^{-1}P_{12}^T,$$

and the $n_e \times n, n_e \times n_e, n_e \times n$ matrices

$$G \triangleq Q_2^{-1}Q_{12}^T, \quad M \triangleq Q_2P_2, \quad \Gamma \triangleq -P_2^{-1}P_{12}^T.$$

The existence of Q_2^{-1} and P_2^{-1} is shown below.

To optimize (7.1) over the open set \mathcal{S}' , where $\mathcal{S}' \triangleq \{(\mathcal{Q}, A_e, B_e, C_e, D_e) \in \mathcal{S} : (7.3) \text{ is satisfied}\}$, subject to the constraint (7.2), form the Lagrangian

$$\mathcal{L}(\mathcal{Q}, A_e, B_e, C_e, D_e) \triangleq \text{tr} \left[\lambda \tilde{R} + \left(\tilde{A} \mathcal{Q} + \mathcal{Q} \tilde{A}^T + \sum_{i=1}^p \tilde{D}_i \bar{M}_i \tilde{D}_i^T + \mathcal{Q} \tilde{E}_i^T \bar{N}_i \tilde{E}_i \mathcal{Q} + \tilde{V} \right) \mathcal{P} \right],$$

where the multipliers $\lambda \geq 0$ and $\mathcal{P} \in \mathbb{R}^{n \times n}$ are not both zero. We thus obtain

$$\frac{\partial \mathcal{L}}{\partial \mathcal{Q}} = \tilde{A}^T \mathcal{P} + \mathcal{P} \tilde{A} + \sum_{i=1}^p \tilde{E}_i^T \bar{N}_i \tilde{E}_i \mathcal{Q} \mathcal{P} + \mathcal{P} \mathcal{Q} \tilde{E}_i^T \bar{N}_i \tilde{E}_i + \lambda \tilde{R}.$$

Setting $\partial \mathcal{L} / \partial \mathcal{Q} = 0$ yields ('vec' is defined in [3])

$$\tilde{\mathcal{A}}^T \text{vec } \mathcal{P} = -\lambda \text{vec } \tilde{R}.$$

Since $\tilde{\mathcal{A}}$ is assumed to be invertible, $\lambda = 0$ implies $\mathcal{P} = 0$. Hence, without loss of generality, set $\lambda = 1$. Since, furthermore, $\tilde{\mathcal{A}}$ is assumed to be asymptotically stable, \mathcal{P} is nonnegative definite. The stationarity conditions are given by

$$\frac{\partial \mathcal{L}}{\partial \mathcal{P}} = \tilde{A} \mathcal{Q} + \mathcal{Q} \tilde{A}^T + \sum_{i=1}^p \tilde{D}_i \bar{M}_i \tilde{D}_i^T + \mathcal{Q} \tilde{E}_i^T \bar{N}_i \tilde{E}_i \mathcal{Q} + \tilde{V} = 0, \quad (\text{A.1})$$

$$\frac{\partial \mathcal{L}}{\partial \mathcal{Q}} = \tilde{A}^T \mathcal{P} + \mathcal{P} \tilde{A} + \sum_{i=1}^p \tilde{E}_i^T \bar{N}_i \tilde{E}_i \mathcal{Q} \mathcal{P} + \mathcal{P} \mathcal{Q} \tilde{E}_i^T \bar{N}_i \tilde{E}_i + \tilde{R} = 0, \quad (\text{A.2})$$

$$\frac{\partial \mathcal{L}}{\partial A_e} = P_{12}^T Q_{12} + P_2 Q_2 = 0, \quad (\text{A.3})$$

$$\frac{\partial \mathcal{L}}{\partial B_e} = P_{12}^T V_{01} + (P_{12}^T Q_{12} + P_2 Q_{12}^T) C^T + P_2 B_e V_{1a} = 0, \quad (\text{A.4})$$

$$\frac{\partial \mathcal{L}}{\partial C_e} = -R L Q_{12} + R D_e C Q_{12} + R C_e Q_2 = 0, \quad (\text{A.5})$$

$$\frac{\partial \mathcal{L}}{\partial D_e} = -R L Q_1 \hat{C}^T + R D_e \hat{C} Q_1 \hat{C}^T + R C_e Q_{12}^T \hat{C}^T = 0. \quad (\text{A.6})$$

Expanding (A.1) and (A.2) yields

$$0 = A Q_1 + Q_1 A^T + D + Q_1 E Q_1 + V_0, \quad (\text{A.7})$$

$$0 = A Q_{12} + Q_1 C^T B_e^T + Q_{12} A^T + Q_1 E Q_{12} + V_{01} B_e^T, \quad (\text{A.8})$$

$$0 = B_e C Q_{12} + A_e Q_2 + Q_{12}^T C^T B_e^T + Q_2 A^T + B_e V_{1a} B_e^T + Q_{12}^T E Q_{12}, \quad (\text{A.9})$$

$$0 = P_{12} A_e + A^T P_{12} + C^T B_e^T P_2 + E (P_{12}^T Q_{12} + P_2 Q_{12}^T)^T - L^T R C_e + \hat{C}^T D_e^T R C_e, \quad (\text{A.10})$$

$$0 = P_2 A_e + A_e^T P_2 + C_e^T R C_e. \quad (\text{A.11})$$

Note that the (1,1) subblock of equation (A.2) characterizing P_1 has been omitted from the above equations since the estimator gains are independent of P_1 . Writing (A.9) as (see [1,9])

$$0 = (A_e + B_e C Q_{12} Q_2^+) Q_2 + Q_2 (A_e + B_e C Q_{12} Q_2^+)^T + Q_2 (Q_{12} Q_2^+)^T E Q_{12} Q_2^+ Q_2 + B_e V_{1a} B_e^T$$

where Q_2^+ is the Moore–Penrose or Drazin generalized inverse of Q_2 , it follows from [17], Lemmas 2.1 and 12.2, that Q_2 is positive definite. Similarly, (A.11) implies that P_2 is positive definite.

Next (8.4), (8.6)–(8.8) follow from the definition of \mathcal{Q} , (A.4)–(A.6) by using the identities

$$Q_1 = Q + \hat{Q}, \quad P_1 = P + \hat{P}, \quad Q_{12} = \hat{Q} \Gamma^T, \quad P_{12} = -\hat{P} G^T, \quad Q_2 = \Gamma \hat{Q} \Gamma^T, \quad P_2 = G \hat{P} G^T.$$

Computing either $\Gamma(A.8)$ –(A.9) or $G(A.10) + (A.11)$ yields (8.5). Inserting (8.5)–(8.8) into (A.7)–(A.11) and using $(A.7) + G^T \Gamma(A.8) - (A.8)G - (A.8G)^T$ and $G^T \Gamma(A.8) - (A.8)G - (A.8G)^T$ yields (8.9) and (8.10). Similarly, $\Gamma^T G(A.10) \Gamma - (A.10) \Gamma - (A.10 \Gamma)^T$ yields (8.11).

Finally, the proof can be reversed so that (8.5)–(8.12) yield (A.1)–(A.6) and (7.2). See [9] for details. \square

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