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Robust H_∞ Stabilization via Parameterized Lyapunov Bounds

Wassim M. Haddad, Vikram Kapila, and Dennis S. Bernstein

***Abstract*—The parameterized Lyapunov bounding technique of Haddad and Bernstein is extended to include an H_∞-disturbance attenuation constraint. The results presented in this paper provide a framework for designing fixed-order (i.e., full- and reduced-order) controllers that guarantee robust H_2 and H_∞ performance in the presence of structured constant real parameter variations in the state space model.**

***Index Terms*—H_2/H_∞ design, real parameter uncertainty, parameter-dependent Lyapunov functions.**

Nomenclature

$\Re$, $\Re^{r\times s}$, $\Re^r$	Real numbers, $r\times s$ real matrices, $\Re^{r\times 1}$.
$(\,)^T$, $(\,)^{-1}$, tr$(\,)$, $\mathcal{E}$	Transpose, inverse, trace, expectation.
I_r, 0_r	$r\times r$ identity matrix, $r\times r$ zero matrix.
$\mathcal{S}^r$, $\mathcal{N}^r$, $\mathcal{P}^r$	$r\times r$ symmetric, nonnegative-definite, positive-definite matrices.
$Z_1 \le Z_2$, $Z_1 < Z_2$	$Z_2 - Z_1 \in \mathcal{N}^r$, $Z_2 - Z_1 \in \mathcal{P}^r$; $Z_1, Z_2 \in \mathcal{S}^r$.
$n, l, m, p, p_\infty, q, n_c, \tilde{n}$	Positive integers; $1 \le n_c \le n$; $\tilde{n} = n + n_c$.
$x, u, y, z, x_c, \tilde{x}$	n-, m-, l-, q-, n_c-, $\tilde{n}$-dimensional vectors.
$w(\cdot)$, $w_\infty(\cdot)$	p-, p_∞-dimensional white noise, L_2 signals.
A, B, C	$n\times n$, $n\times m$, $l\times n$ matrices.
A_c, B_c, C_c	$n_c\times n_c$, $n_c\times l$, $m\times n_c$ matrices.
$D_1, D_2, D_{1\infty}, D_{2\infty}$	$n\times p$, $l\times p$, $n\times p_\infty$, $l\times p_\infty$ matrices.
E_1, E_2	$q\times n$, $q\times m$ matrices.

I. Introduction

In a recent series of papers [9]–[12], a refined Lyapunov function technique was developed to overcome some of the current limitations of Lyapunov function theory for the problem of robust stability and performance in the presence of constant real parameter uncertainty. Since, as noted in [9]–[11], conventional Lyapunov bounding techniques guarantee stability with respect to time-varying parameter perturbations, a feedback controller designed for time-varying parameter variations will unnecessarily sacrifice performance when the uncertain real parameters are actually constant. To overcome some of the limitations of conventional Lyapunov bounding techniques, the authors in [9]–[11] developed a general framework for robust controller analysis and synthesis based on *parameter-dependent Lyapunov functions* that is both flexible in addressing a large class of uncertainty structures and restrictive in excluding uncertainties that are not physically meaningful. Specifically, in this framework, the Lyapunov function is allowed to be a function of the uncertain parameters, thus guaranteeing robust stability and performance via a family of Lyapunov functions. As demonstrated in [9]–[11], the form of the parameterized Lyapunov bounding function proves to be critical because the presence of uncertainty within the Lyapunov function curtails arbitrary time-variation of the uncertain parameters, thus yielding a more effective robust analysis and synthesis framework for constant real parameter uncertainty.

In this paper, we extend the results of [9]–[11] to guarantee robust H_2 and H_∞ performance in the presence of constant real-valued parameter uncertainty. Thus, the results presented herein provide a further refinement of the results in [13] which considered the design of H_∞ robust controllers in the presence of arbitrarily time-varying real-valued parameter variations.

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W. M. Haddad is with the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150 USA (e-mail: wm.haddad@aerospace.gatech.edu).

V. Kapila is with the Department of Mechanical Engineering, Polytechnic University, Brooklyn, NY 11201 USA.

D. S. Bernstein is with the Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2118 USA.

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II. Robust Fixed-Order Dynamic Compensation with H_∞-Disturbance Attenuation

In this section, we introduce the robust stability and H_2 performance problem with an H_∞-disturbance attenuation constraint. Specifically, we consider a fixed-order dynamic output-feedback control design problem with constant real parametric uncertainty and constrained H_∞-disturbance attenuation. This problem involves a set $\mathcal{U} \subset \Re^{n\times n}$ of constant uncertain perturbation ΔA of the nominal system matrix A. The goal of the problem is to determine a fixed-order strictly proper dynamic compensator (A_c, B_c, C_c) which i) stabilizes the plant for all variations in $\mathcal{U}$, ii) satisfies an H_∞ constraint on disturbance rejection for all variations in $\mathcal{U}$, and iii) minimizes the worst case value over the uncertainty set $\mathcal{U}$of a steady-state H_2-performance criterion.

In this and the following section, no explicit structure is assumed for the elements of $\mathcal{U}$. In Section IV, a specific structure of variations in $\mathcal{U}$ will be introduced.

A. H_∞-Constrained Robust Dynamic Compensation Problem

Given the nth-order stabilizable and detectable plant with constant real parametric variations

$$\dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + D_1 w(t) + D_{1\infty} w_\infty(t), \qquad t \geq 0 \tag{1}$$

$$y(t) = Cx(t) + D_2 w(t) + D_{2\infty} w_\infty(t) \tag{2}$$

determine an n_cth-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{3}$$

$$u(t) = C_c x_c(t) \tag{4}$$

which satisfies the following design criteria.

i) The closed-loop system (1)–(4) is asymptotically stable for all $\Delta A \in \mathcal{U}$.

ii) The $q \times p_\infty$ closed-loop transfer function from disturbances $w_\infty(t)$ to performance variables $z(t) = E_1 x(t) + E_2 u(t)$ given by

$$\tilde{H}_{\Delta\tilde{A}}(s) \triangleq \tilde{E}[sI_{\tilde{n}} - (\tilde{A} + \Delta\tilde{A})]^{-1}\tilde{D}_\infty \tag{5}$$

where

$$\tilde{D}_\infty \triangleq \begin{bmatrix} D_{1\infty} \\ B_c D_{2\infty} \end{bmatrix}, \quad \tilde{E} \triangleq [E_1 \quad E_2 C_c]$$

$$\tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}$$

$$\Delta\tilde{A} \triangleq \begin{bmatrix} \Delta A & 0 \\ 0 & 0 \end{bmatrix}$$

satisfies the constraint

$$\|\tilde{H}_{\Delta\tilde{A}}(s)\|_\infty \leq \gamma, \qquad \Delta A \in \mathcal{U} \tag{6}$$

where $\gamma > 0$ is a given constant.

iii) The performance functional

$$J(A_c, B_c, C_c) \triangleq \sup_{\Delta A\in\mathcal{U}} \limsup_{t\to\infty} \frac{1}{t}\mathcal{E}\int_0^t z^T(s)z(s)\,ds \tag{7}$$

is minimized.

Note that for each uncertain variation $\Delta A \in \mathcal{U}$, the closed-loop system (1)–(4) can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{D}w(t) + \tilde{D}_\infty w_\infty(t), \qquad t \geq 0 \tag{8}$$

$$z(t) = \tilde{E}\tilde{x}(t) \tag{9}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix}$$

and (7) becomes

$$J(A_c, B_c, C_c) = \sup_{\Delta A\in\mathcal{U}} \limsup_{t\to\infty} \mathcal{E}[\tilde{x}^T(t)\tilde{R}\tilde{x}(t)] \tag{10}$$

where

$$\tilde{R} \triangleq \begin{bmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{bmatrix}, \quad R_1 \triangleq E_1^T E_1, \quad R_2 \triangleq E_2^T E_2 > 0.$$

Note that the problem stated above involves distinct H_2- and H_∞-disturbance weights. As in [1], $w(t)$ is interpreted as white noise for the H_2 design and $w_\infty(t)$ is interpreted as an L_2 signal, each of whose components has norm less than one. In particular, the matrices $V_1 \triangleq D_1 D_1^T$ and $V_2 \triangleq D_2 D_2^T > 0$ are the H_2 disturbance and sensor noise intensities. For the H_∞-disturbance attenuation constraint, the dynamic system given by (1) and (2) involves disturbance and sensor weights $V_{1\infty} \triangleq D_{1\infty} D_{1\infty}^T$ and $V_{2\infty} \triangleq D_{2\infty} D_{2\infty}^T > 0$. Although we do not require that $V_{1\infty}$ and $V_{2\infty}$ be equal to V_1 and V_2, respectively, we shall require for technical reasons that $V_{2\infty} = \beta^2 V_2$, where the nonnegative scalar β is a design variable.

Before continuing it is useful to note that for a given compensator (A_c, B_c, C_c), if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$, then the performance (7) is given by

$$J(A_c, B_c, C_c) = \sup_{\Delta A\in\mathcal{U}} \operatorname{tr} \tilde{P}_{\Delta\tilde{A}}\tilde{V} \tag{11}$$

where

$$\tilde{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix}$$

and $\tilde{P}_{\Delta\tilde{A}} \in \Re^{\tilde{n}\times\tilde{n}}$ is the unique nonnegative definite solution to

$$0 = (\tilde{A} + \Delta\tilde{A})^T \tilde{P}_{\Delta\tilde{A}} + \tilde{P}_{\Delta\tilde{A}}(\tilde{A} + \Delta\tilde{A}) + \tilde{R}. \tag{12}$$

In the present paper, our approach is to obtain robust stability as a consequence of sufficient conditions for robust performance. Such conditions are developed in the following section.

III. Sufficient Conditions for Robust H_∞ Stabilization via Parameter-Dependent Bounding Functions

In this section, we determine an upper bound for $J(A_c, B_c, C_c)$ given by (11). The key step in obtaining robust stability and performance is to bound the uncertain terms in the Lyapunov equation (12), i.e., $\Delta\tilde{A}^T\tilde{P}_{\Delta\tilde{A}} + \tilde{P}_{\Delta\tilde{A}}\Delta\tilde{A}$, by means of a *parameter-dependent* bounding function $\Omega(\mathcal{P}, \Delta\tilde{A})$. As discussed in [9] a key aspect of this approach is the fact that it constrains the class of allowable time-varying uncertainties, thus reducing conservatism in the presence of constant real parameter uncertainty and hence providing sharper H_2-performance bounds. Furthermore, the H_∞-disturbance attenuation constraint (6) is enforced for all $\Delta A \in \mathcal{U}$ by replacing the modified algebraic Lyapunov equation (12) by an algebraic Riccati equation which overbounds the closed-loop Lyapunov equation (12). The following result is fundamental and forms the basis for all later developments.

Theorem 3.1: Let $\Omega_0\colon \mathcal{N}^{\tilde{n}} \to \mathcal{S}^{\tilde{n}}$ and $\mathcal{P}_0\colon \mathcal{U} \to \mathcal{S}^{\tilde{n}}$ be such that

$$\begin{aligned}\Delta\tilde{A}^T\mathcal{P} + \mathcal{P}\Delta\tilde{A} \leq \Omega_0(\mathcal{P}) - [&\{(\tilde{A} + \Delta\tilde{A})^T\mathcal{P}_0(\Delta\tilde{A}) \\ &+ \mathcal{P}_0(\Delta\tilde{A})(\tilde{A} + \Delta\tilde{A})\} \\ &+ \gamma^{-2}\{\mathcal{P}_0(\Delta\tilde{A})\tilde{V}_\infty\mathcal{P}_0(\Delta\tilde{A}) \\ &+ \mathcal{P}\tilde{V}_\infty\mathcal{P}_0(\Delta\tilde{A}) + \mathcal{P}_0(\Delta\tilde{A})\tilde{V}_\infty\mathcal{P}\}] \\ &\Delta A \in \mathcal{U}, \quad \mathcal{P} \in \mathcal{N}^{\tilde{n}}\end{aligned} \tag{13}$$

where $\tilde{V}_\infty \triangleq \tilde{D}_\infty\tilde{D}_\infty^T$, and for a given (A_c, B_c, C_c), suppose there exists $\mathcal{P} \in \mathcal{N}^{\tilde{n}}$ satisfying

$$0 = \tilde{A}^T\mathcal{P} + \mathcal{P}\tilde{A} + \gamma^{-2}\mathcal{P}\tilde{V}_\infty\mathcal{P} + \Omega_0(\mathcal{P}) + \tilde{R} \tag{14}$$

such that $\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})$ is nonnegative definite for all $\Delta A \in \mathcal{U}$. Then

$$(\tilde{A}+\Delta\tilde{A},\, \tilde{E}) \text{ is detectable}, \quad \Delta A \in \mathcal{U} \tag{15}$$

if and only if

$$\tilde{A}+\Delta\tilde{A} \text{ is asymptotically stable}, \qquad \Delta A \in \mathcal{U}. \tag{16}$$

In this case

$$\|\tilde{H}_{\Delta\tilde{A}}(s)\|_\infty \le \gamma, \qquad \Delta A \in \mathcal{U} \tag{17}$$

and

$$\tilde{P}_{\Delta\tilde{A}} \le \mathcal{P}+\mathcal{P}_0(\Delta\tilde{A}), \qquad \Delta A \in \mathcal{U} \tag{18}$$

where $\tilde{P}_{\Delta\tilde{A}}$ is given by (12). Consequently

$$J(A_c,\, B_c,\, C_c) \le \operatorname{tr}\mathcal{P}\tilde{V} + \sup_{\Delta A\in\mathcal{U}} \operatorname{tr}\mathcal{P}_0(\Delta\tilde{A})\tilde{V}. \tag{19}$$

If, in addition, there exists $\overline{\mathcal{P}}_0 \in \mathcal{S}^{\tilde{n}}$ such that

$$\mathcal{P}_0(\Delta\tilde{A}) \le \overline{\mathcal{P}}_0, \qquad \Delta A \in \mathcal{U} \tag{20}$$

then

$$J(A_c,\, B_c,\, C_c) \le \operatorname{tr}[(\mathcal{P}+\overline{\mathcal{P}}_0)\tilde{V}]. \tag{21}$$

Proof: The proof of (15), (16), (18)–(21) is similar to the proof of [11, Th. 3.1]. To prove (17), note that for $\Delta A \in \mathcal{U}$, (14) is equivalent to

$$\begin{aligned} 0 = {} & (\tilde{A}+\Delta\tilde{A})^T[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})] \\ & + [\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})](\tilde{A}+\Delta\tilde{A}) \\ & + \gamma^{-2}[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})]\tilde{V}_\infty[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})] \\ & + \Omega(\mathcal{P},\, \Delta\tilde{A}) + \tilde{R} \end{aligned} \tag{22}$$

where

$$\begin{aligned} \Omega(\mathcal{P},\, \Delta\tilde{A}) \triangleq {} & \Omega_0(\mathcal{P}) - [\{(\tilde{A}+\Delta\tilde{A})^T\mathcal{P}_0(\Delta\tilde{A}) \\ & + \mathcal{P}_0(\Delta\tilde{A})(\tilde{A}+\Delta\tilde{A})\} \\ & + \gamma^{-2}\{\mathcal{P}_0(\Delta\tilde{A})\tilde{V}_\infty\mathcal{P}_0(\Delta\tilde{A}) \\ & + \mathcal{P}\tilde{V}_\infty\mathcal{P}_0(\Delta\tilde{A}) + \mathcal{P}_0(\Delta\tilde{A})\tilde{V}_\infty\mathcal{P}\}]. \end{aligned} \tag{23}$$

Next, replace $\tilde{V}_\infty$ by $\tilde{D}_\infty\tilde{D}_\infty^T$ and $\tilde{R}$ by $\tilde{E}^T\tilde{E}$ so that (22) becomes

$$\begin{aligned} 0 = {} & (\tilde{A}+\Delta\tilde{A})^T[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})] \\ & + [\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})](\tilde{A}+\Delta\tilde{A}) \\ & + \gamma^{-2}[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})]\tilde{D}_\infty\tilde{D}_\infty^T[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})] \\ & + \overline{\Omega}(\mathcal{P},\, \Delta\tilde{A}) + \tilde{E}^T\tilde{E} \end{aligned} \tag{24}$$

where $\overline{\Omega}(\mathcal{P},\, \Delta\tilde{A}) \triangleq \Omega(\mathcal{P},\, \Delta\tilde{A}) - (\Delta\tilde{A}^T\mathcal{P}+\mathcal{P}\Delta\tilde{A}) \ge 0$. Next, add and subtract $\jmath\omega[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})]$ to and from (24) so that (24) becomes

$$\begin{aligned} 0 = {} & (\jmath\omega I_{\tilde{n}}+\tilde{A}+\Delta\tilde{A})^T[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})] \\ & + [\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})](-\jmath\omega I_{\tilde{n}}+\tilde{A}+\Delta\tilde{A}) \\ & + \gamma^{-2}[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})]\tilde{D}_\infty\tilde{D}_\infty^T[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})] \\ & + \overline{\Omega}(\mathcal{P},\, \Delta\tilde{A}) + \tilde{E}^T\tilde{E} \end{aligned} \tag{25}$$

or, equivalently

$$\begin{aligned} \tilde{E}^T\tilde{E} = {} & [-\jmath\omega I_{\tilde{n}}-(\tilde{A}+\Delta\tilde{A})]^T[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})] \\ & + [\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})][\jmath\omega I_{\tilde{n}}-(\tilde{A}+\Delta\tilde{A})] \\ & - \gamma^{-2}[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})] \\ & \cdot \tilde{D}_\infty\tilde{D}_\infty^T[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})] - \overline{\Omega}(\mathcal{P},\, \Delta\tilde{A}). \end{aligned} \tag{26}$$

Now, forming

$$\tilde{D}_\infty^T[-\jmath\omega I_{\tilde{n}}-(\tilde{A}+\Delta\tilde{A})]^{-T}(26)[\jmath\omega I_{\tilde{n}}-(\tilde{A}+\Delta\tilde{A})]^{-1}\tilde{D}_\infty$$

yields

$$\tilde{H}^*_{\Delta\tilde{A}}(\jmath\omega)\tilde{H}_{\Delta\tilde{A}}(\jmath\omega) = U + U^* - \gamma^{-2}U^*U - \hat{\Sigma} \tag{27}$$

where

$$\begin{aligned} U \triangleq {} & \tilde{D}_\infty^T[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})][\jmath\omega I_{\tilde{n}}-(\tilde{A}+\Delta\tilde{A})]^{-1}\tilde{D}_\infty \\ \hat{\Sigma} \triangleq {} & \tilde{D}_\infty^T[-\jmath\omega I_{\tilde{n}}-(\tilde{A}+\Delta\tilde{A})]^{-T} \\ & \cdot \overline{\Omega}(\mathcal{P},\, \Delta\tilde{A})[\jmath\omega I_{\tilde{n}}-(\tilde{A}+\Delta\tilde{A})]^{-1}\tilde{D}_\infty \end{aligned}$$

or, equivalently, multiplying (27) by -1, adding $\gamma^{-2}I_{p_\infty}$ to both sides of (27), and noting that $\hat{\Sigma} \ge 0$ since $\overline{\Omega}(\mathcal{P},\, \Delta\tilde{A}) \ge 0$, yields

$$\begin{aligned} & \gamma^2 I_{p_\infty} - \tilde{H}^*_{\Delta\tilde{A}}(\jmath\omega)\tilde{H}_{\Delta\tilde{A}}(\jmath\omega) \\ & \quad = \gamma^2 I_{p_\infty} - U - U^* + \gamma^{-2}U^*U + \hat{\Sigma} \\ & \quad = (\gamma I_{p_\infty} - \gamma^{-1}U)^*(\gamma I_{p_\infty} - \gamma^{-1}U) + \hat{\Sigma} \\ & \quad \ge 0 \end{aligned}$$

which implies $\tilde{H}^*_{\Delta\tilde{A}}(\jmath\omega)\tilde{H}_{\Delta\tilde{A}}(\jmath\omega) \le \gamma^2 I_{p_\infty}$. This proves (17). □

Note that with $\Omega(\mathcal{P},\, \Delta\tilde{A})$ defined by (23) condition (13) can be written as

$$\Delta\tilde{A}^T\mathcal{P}+\mathcal{P}\Delta\tilde{A} \le \Omega(\mathcal{P},\, \Delta\tilde{A}), \qquad \Delta A \in \mathcal{U}, \qquad \mathcal{P} \in \mathcal{N}^{\tilde{n}}. \tag{28}$$

For convenience we shall say that $\Omega(\cdot,\, \cdot)$ is a parameter-dependent bounding function.

Note that the preceding framework establishing robust stability is equivalent to the existence of a parameter-dependent Lyapunov function of the form $V(\tilde{x}) = \tilde{x}^T[\mathcal{P}+\mathcal{P}_0(\Delta\tilde{A})]\tilde{x}$, which also establishes robust stability [9]–[11].

IV. Uncertainty Structure and a Parameter-Dependent Bounding Function

Having established the theoretical basis for our approach, we now assign explicit structure to the uncertainty set $\mathcal{U}$ and the parameter-dependent bounding function $\Omega(\cdot,\, \cdot)$. Specifically, the uncertainty set $\mathcal{U}$ is defined by

$$\mathcal{U} \triangleq \{\Delta A \in \Re^{n\times n}\colon \Delta A = B_0 F C_0,\, F \in \mathcal{F}\} \tag{29}$$

where $\mathcal{F}$ satisfies

$$\mathcal{F} \subseteq \hat{\mathcal{F}} \triangleq \{F \in \mathcal{S}^{m_0}\colon 0 \le F \le M\} \tag{30}$$

and $B_0 \in \Re^{n\times m_0}$, $C_0 \in \Re^{m_0\times n}$ are fixed matrices denoting the structure of uncertainty, $F \in \mathcal{S}^{m_0}$ is an uncertain symmetric matrix, and $M \in \mathcal{P}^{m_0}$ is a symmetric positive definite matrix. We restrict our attention to symmetric uncertainties for convenience only. More general uncertainty sets as in [9] can also be considered.

The closed-loop system (8) thus has structured uncertainty of the form

$$\Delta\tilde{A} = \tilde{B}_0 F \tilde{C}_0$$

where

$$\tilde{B}_0 \triangleq \begin{bmatrix} B_0 \\ 0 \end{bmatrix}, \quad \tilde{C}_0 \triangleq [C_0 \quad 0].$$

Next, define the set of compatible scaling matrices $\mathcal{N}_{\rm s}$ and $\mathcal{N}_{\rm nd}$ by

$$\mathcal{N}_{\rm s} \triangleq \{N \in \Re^{m_0\times m_0}\colon FN = N^TF,\, F \in \mathcal{F}\} \tag{31}$$

$$\mathcal{N}_{\rm nd} \triangleq \{N \in \Re^{m_0\times m_0}\colon FN = N^TF \ge 0,\, F \in \mathcal{F}\}. \tag{32}$$

Before specifying the parameter-dependent bounding function $\Omega(\cdot, \cdot)$ satisfying (13), we need to define the following additional notation:

$$\hat{B}_0 \triangleq \begin{bmatrix} C_0^T N^T \\ 0_{n_c \times m_0} \end{bmatrix}, \quad \hat{C}_0 \triangleq [C_0 V_{1\infty} \quad 0_{m_0 \times n_c}]$$
$$\tilde{R}_0 \triangleq \begin{bmatrix} \hat{V}_{1\infty} & 0_{n \times n_c} \\ 0_{n_c \times n} & 0_{n_c \times n_c} \end{bmatrix}$$

where $\hat{V}_{1\infty} \in \Re^{n \times n}$.

Proposition 4.1: Let $N \in \mathcal{N}_{\rm s}$, let $\hat{V}_{1\infty} \in \mathcal{N}^n$ be such that $C_0^T F N C_0 V_{1\infty} C_0^T F N C_0 \leq \hat{V}_{1\infty}$ for all $F \in \mathcal{F}$, and assume

$$R_0 \triangleq [M^{-1} - N\tilde{C}_0 \tilde{B}_0] + [M^{-1} - N\tilde{C}_0 \tilde{B}_0]^T > 0. \tag{33}$$

Furthermore, let $\mathcal{U}$ be defined by (29) and define $\Omega_0(\mathcal{P})$ and $\mathcal{P}_0(F)$ by

$$\begin{aligned}\Omega_0(\mathcal{P}) \triangleq\ & (\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P})^T R_0^{-1} \\ & \cdot (\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P}) \\ & + \gamma^{-2}(\hat{B}_0 M^2 \hat{B}_0^T + \mathcal{P}\hat{C}_0^T\hat{C}_0\mathcal{P} + \tilde{R}_0) \end{aligned} \tag{34}$$

$$\mathcal{P}_0(F) \triangleq \tilde{C}_0^T F N \tilde{C}_0. \tag{35}$$

Then (13) is satisfied.

Proof: Note that since by assumption $C_0^T F\ N C_0 V_{1\infty}\ C_0^T\ FN\ C_0 \leq \hat{V}_{1\infty}$ for all $F \in \mathcal{F}$, it follows that $\mathcal{P}_0(F)\tilde{V}_\infty \mathcal{P}_0(F) \leq \tilde{R}_0$ for all $F \in \mathcal{F}$. Next, note that $0 \leq F \leq M$ if and only if $0 \leq (F - FM^{-1}F)$ [9]. Hence, it follows that

$$\begin{aligned}
0 \leq\ & [(\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P}) - R_0 F\tilde{C}_0]^T R_0^{-1} \\
& \cdot [(\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P}) - R_0 F\tilde{C}_0] \\
& + 2\tilde{C}_0^T(F - FM^{-1}F)\tilde{C}_0 \\
& + \gamma^{-2}(\hat{B}_0 F - \mathcal{P}\hat{C}_0^T)(\hat{B}_0 F - \mathcal{P}\hat{C}_0^T)^T \\
& + \gamma^{-2}[\tilde{R}_0 - \mathcal{P}_0(F)\tilde{V}_\infty\mathcal{P}_0(F)] \\
\leq\ & (\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P})^T R_0^{-1}(\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P}) \\
& + \tilde{C}_0^T F R_0 F\tilde{C}_0 - \tilde{C}_0^T F(\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P}) \\
& - (\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P})^T F\tilde{C}_0 \\
& + 2\tilde{C}_0^T(F - FM^{-1}F)\tilde{C}_0 \\
& + \gamma^{-2}[\hat{B}_0 M^2\hat{B}_0^T + \mathcal{P}\hat{C}_0^T\hat{C}_0\mathcal{P} \\
& - \mathcal{P}\hat{C}_0^T F\hat{B}_0^T - \hat{B}_0 F\hat{C}_0\mathcal{P} \\
& + \tilde{R}_0 - \mathcal{P}_0(F)\tilde{V}_\infty\mathcal{P}_0(F)] \\
=\ & \Omega_0(\mathcal{P}) - [\tilde{C}_0^T F\tilde{B}_0^T\mathcal{P} + \mathcal{P}\tilde{B}_0 F\tilde{C}_0 \\
& + \tilde{A}^T\tilde{C}_0^T N^T F\tilde{C}_0 + \tilde{C}_0^T FN\tilde{C}_0\tilde{A} \\
& + \tilde{C}_0^T F\tilde{B}_0^T\tilde{C}_0^T N^T F\tilde{C}_0 + \tilde{C}_0^T FN\tilde{C}_0\tilde{B}_0 F\tilde{C}_0 \\
& + \gamma^{-2}\{\mathcal{P}\hat{C}_0^T F\hat{B}_0^T + \hat{B}_0 F\hat{C}_0\mathcal{P} \\
& + \mathcal{P}_0(F)\tilde{V}_\infty\mathcal{P}_0(F)\}] \\
=\ & \Omega_0(\mathcal{P}) - [\Delta\tilde{A}^T\mathcal{P} + \mathcal{P}\Delta\tilde{A} + (\tilde{A} + \Delta\tilde{A})^T\mathcal{P}_0(F) \\
& + \mathcal{P}_0(F)(\tilde{A} + \Delta\tilde{A}) + \gamma^{-2}\{\mathcal{P}\tilde{V}_\infty\mathcal{P}_0(F) \\
& + \mathcal{P}_0(F)\tilde{V}_\infty\mathcal{P} + \mathcal{P}_0(F)\tilde{V}_\infty\mathcal{P}_0(F)\}]
\end{aligned}$$

which proves (13) with $\mathcal{U}$ given by (29). □

Note that with $N \in \mathcal{N}_{\rm s}$, it follows from (30) that there exists a matrix $\mu \in \mathcal{N}^{m_0}$ such that $FN \leq \mu$ for all $F \in \mathcal{F}$. Next, using Theorem 3.1 and Proposition 4.1 we have the following immediate result.

Theorem 4.1: Let $N \in \mathcal{N}_{\rm nd}$, let $\hat{V}_{1\infty} \in \mathcal{N}^n$ be such that $C_0^T F N C_0 V_{1\infty} C_0^T F N C_0 \leq \hat{V}_{1\infty}$ for all $F \in \mathcal{F}$, and suppose (33) is satisfied. Furthermore, suppose there exists a nonnegative-definite matrix $\mathcal{P}$ satisfying

$$\begin{aligned}0 =\ & \tilde{A}^T\mathcal{P} + \mathcal{P}\tilde{A} + \gamma^{-2}(\mathcal{P}\tilde{V}_\infty\mathcal{P} + \hat{B}_0 M^2\hat{B}_0^T + \mathcal{P}\hat{C}_0^T\hat{C}_0\mathcal{P} + \tilde{R}_0) \\ & + (\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P})^T \cdot R_0^{-1}(\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P}) + \tilde{R}. \end{aligned} \tag{36}$$

Then

$$(\tilde{A} + \Delta\tilde{A},\ \tilde{E}) \text{ is detectable}, \qquad \Delta A \in \mathcal{U} \tag{37}$$

if and only if

$$\tilde{A} + \Delta\tilde{A} \text{ is asymptotically stable}, \qquad \Delta A \in \mathcal{U}. \tag{38}$$

In this case

$$\|\tilde{H}_{\Delta\tilde{A}}(s)\|_\infty \leq \gamma, \qquad \Delta A \in \mathcal{U} \tag{39}$$

and

$$\begin{aligned} J(A_c, B_c, C_c) \leq\ & \mathcal{J}(\mathcal{P}, A_c, B_c, C_c) \\ \triangleq\ & \text{tr}\,[(\mathcal{P} + \tilde{C}_0^T\mu\tilde{C}_0)\tilde{V}]. \end{aligned} \tag{40}$$

Proof: The result is a direct specialization of Theorem 3.1 using Proposition 4.1. We only note that $\mathcal{P}_0(\Delta\tilde{A})$ now has the form $\mathcal{P}_0(F) = \tilde{C}_0^T FN\tilde{C}_0$. Since by assumption $N \in \mathcal{N}_{\rm nd}$ for all $F \in \mathcal{F}$, it follows that $\mathcal{P} + \mathcal{P}_0(F)$ is nonnegative definite for all $F \in \mathcal{F}$ as required by Theorem 3.1. □

Remark 4.1: The condition that $FN = N^T F$, $F \in \mathcal{F}$, represents an intimate relationship between the matrix N and the structure of $\mathcal{F}$. As noted in [9], this condition allows for a generalization of mixed-μ analysis to address nondiagonal real uncertain matrix blocks.

Remark 4.2: Standard loop-shifting techniques [9], [11], and [12] can be used to consider uncertainties with upper and lower bounds of the form $M_1 \leq F \leq M_2$, where $F \in \hat{\mathcal{F}}$ and $M_1,\ M_2 \in \mathcal{S}^{m_0}$. In this case, Proposition 4.1 holds with F, $\tilde{A}$, and M replaced by $F - M_1$, $\tilde{A} + \tilde{B}_0 M_1\tilde{C}_0$, and $M_2 - M_1$, respectively. Similar modifications can be made to Theorem 4.1.

V. The Auxiliary Minimization Problem

As shown in the previous section, the replacement of (12) by (36) enforces the H_∞-disturbance attenuation constraint and yields an upper bound for the worst case H_2-performance criterion. That is, given a compensator (A_c, B_c, C_c) for which there exists a nonnegative-definite solution to (36), the actual worst case H_2 performance $J(A_c, B_c, C_c)$ of the compensator is guaranteed to be no worse than the bound given by $\mathcal{J}(\mathcal{P}, A_c, B_c, C_c)$. Hence $\mathcal{J}(\mathcal{P}, A_c, B_c, C_c)$ can be interpreted as an auxiliary cost which leads to the following minimization problem.

Auxiliary Minimization Problem: Determine $(\mathcal{P}, A_c, B_c, C_c)$ with $\mathcal{P} \in \mathcal{N}^{\tilde{n}}$, which minimizes $\mathcal{J}(\mathcal{P}, A_c, B_c, C_c)$ subject to (36).

It follows from Theorem 4.1 that the satisfaction of (36) for $\mathcal{P} \in \mathcal{N}^{\tilde{n}}$ along with the generic condition (37) leads to i) closed-loop stability for all $\Delta A \in \mathcal{U}$, ii) prespecified H_∞-disturbance attenuation for all $\Delta A \in \mathcal{U}$, and iii) an upper bound for the worst case H_2-performance criterion. Hence, it remains to determine (A_c, B_c, C_c) which minimizes $\mathcal{J}(\mathcal{P}, A_c, B_c, C_c)$ and thus provides an upper bound for the actual worst case H_2 performance $J(A_c, B_c, C_c)$ over all $\Delta A \in \mathcal{U}$. This framework is similar in spirit to the mixed-norm H_2/H_∞ control problem considered in [1], where the H_2 performance bound was shown to correspond to an entropy functional [14].

VI. Robust H_∞ Controller Synthesis via Parameter-Dependent Bounding Functions

In this section, we state constructive sufficient conditions for characterizing fixed-order (full- and reduced-order) robust H_∞ controllers. To state the main result of this section we require some additional notation and a lemma concerning pairs of nonnegative definite matrices.

Lemma 6.1 [2]: Let $\hat{Q}$, $\hat{P}$ be $n \times n$ nonnegative-definite matrices, and suppose that rank $\hat{Q}\hat{P} = n_c$. Then there exist $n_c \times n$ matrices G, Γ and an $n_c \times n_c$ invertible matrix $\hat{M}$, unique except for a change of basis in $\Re^{n_c}$ such that

$$\hat{Q}\hat{P} = G^T \hat{M} \Gamma, \quad \Gamma G^T = I_{n_c}. \tag{41}$$

Furthermore, the $n \times n$ matrices

$$\tau \triangleq G^T \Gamma, \quad \tau_\perp \triangleq I_n - \tau \tag{42}$$

are idempotent and have rank n_c and $n - n_c$, respectively.

For convenience in stating the main result of this section define the notation $S \triangleq (I + \beta^2 \gamma^{-2} Q\hat{P})^{-1}$, for arbitrary $Q, \hat{P} \in \mathcal{N}^n$, and

$$\begin{aligned}
\overline{\Sigma} &\triangleq C^T V_2^{-1} C \\
\hat{V}_\infty &\triangleq V_{1\infty} C_0^T C_0 V_{1\infty} \\
\tilde{C} &\triangleq C_0 + N C_0 A \\
R_{2a} &\triangleq R_2 + B^T C_0^T N^T R_0^{-1} N C_0 B_0 \\
P_a &\triangleq B^T P + B^T C_0^T N^T R_0^{-1} (\tilde{C} + B_0^T P) \\
A_P &\triangleq A + B_0 R_0^{-1} \tilde{C} \\
A_{\hat{P}} &\triangleq A_P - SQ\overline{\Sigma} + [B_0 R_0^{-1} B_0^T + \gamma^{-2}(V_{1\infty} + \hat{V}_\infty)]P \\
A_Q &\triangleq A_P + [B_0 R_0^{-1} B_0^T + \gamma^{-2}(V_{1\infty} + \hat{V}_\infty)]\,(P + \hat{P}) \\
A_{\hat{Q}} &\triangleq A_P + [B_0 R_0^{-1} B_0^T + \gamma^{-2}(V_{1\infty} + \hat{V}_\infty)]P \\
&\quad - (I + B_0 R_0^{-1} N C_0) B R_{2a}^{-1} P_a
\end{aligned}$$

for arbitrary $P, Q, \hat{P} \in \Re^{n \times n}$. Note that since $Q, \hat{P} \in \mathcal{N}^n$, and the eigenvalues of $Q\hat{P}$ coincide with the eigenvalues of the nonnegative-definite matrix $Q^{1/2}\hat{P}Q^{1/2}$, it follows that $Q\hat{P}$ has nonnegative eigenvalues. Thus, the eigenvalues of $I + \beta^2\gamma^{-2}Q\hat{P}$ are all greater than one so that S exists.

Theorem 6.1: Let $n_c \le n$, assume $R_0 > 0$, and $N \in \mathcal{N}_{\rm nd}$, and let $\hat{V}_{1\infty} \in \mathcal{N}^n$ be such that $C_0^T F N C_0 V_{1\infty} C_0^T F N C_0 \le \hat{V}_{1\infty}$ for all $F \in \mathcal{F}$. Furthermore, assume that there exist $n \times n$ nonnegative-definite matrices $P, Q, \hat{P}$, and $\hat{Q}$ satisfying

$$\begin{aligned}
0 &= A_P^T P + P A_P + R_1 + \tilde{C} R_0^{-1} \tilde{C} \\
&\quad + \gamma^{-2}(C_0^T N^T M^2 N C_0 + \hat{V}_{1\infty}) \\
&\quad + P[B_0 R_0^{-1} B_0^T + \gamma^{-2}(V_{1\infty} + \hat{V}_\infty)]P \\
&\quad - P_a^T R_{2a}^{-1} P_a + \tau_\perp^T P_a^T R_{2a}^{-1} P_a \tau_\perp
\end{aligned} \tag{43}$$

$$0 = A_Q Q + Q A_Q^T + V_1 - SQ\overline{\Sigma}QS^T + \tau_\perp SQ\overline{\Sigma}QS^T \tau_\perp^T \tag{44}$$

$$\begin{aligned}
0 &= A_{\hat{P}}^T \hat{P} + \hat{P} A_{\hat{P}} + \hat{P}[B_0 R_0^{-1} B_0^T \\
&\quad + \gamma^{-2}(V_{1\infty} + \hat{V}_\infty) + \beta^2 SQ\overline{\Sigma}QS^T]\hat{P} \\
&\quad + P_a^T R_{2a}^{-1} P_a - \tau_\perp^T P_a^T R_{2a}^{-1} P_a \tau_\perp
\end{aligned} \tag{45}$$

$$0 = A_{\hat{Q}} \hat{Q} + \hat{Q} A_{\hat{Q}}^T + SQ\overline{\Sigma}QS^T - \tau_\perp SQ\overline{\Sigma}QS^T \tau_\perp^T \tag{46}$$

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_c \tag{47}$$

and let A_c, B_c, and C_c be given by

$$A_c = \Gamma[A_{\hat{Q}} - SQ\overline{\Sigma}]G^T \tag{48}$$

$$B_c = \Gamma SQC^T V_2^{-1} \tag{49}$$

$$C_c = -R_{2a}^{-1} P_a G^T. \tag{50}$$

Then $(\tilde{A} + \Delta\tilde{A}, \tilde{E})$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case, the closed-loop transfer function $\tilde{H}_{\Delta\tilde{A}}(s)$ given by (5) satisfies the H_∞-disturbance attenuation constraint (39), and the worst case H_2-performance criterion (11) satisfies the bound

$$\begin{aligned}
\mathcal{J}(\tilde{P}, A_c, B_c, C_c) &= \operatorname{tr}[(P + \hat{P})V_1 + \hat{P}SQ\overline{\Sigma}QS^T \\
&\quad + C_0^T \mu C_0 V_1].
\end{aligned} \tag{51}$$

Proof: The proof is constructive in nature. Specifically, first we obtain necessary conditions for the auxiliary minimization problem and show by construction that these conditions serve as sufficient conditions for closed-loop stability and prespecified disturbance attenuation and provide a worst case H_2-performance bound. For details of a similar proof see [1]. □

Remark 6.1: In the full-order case, set $n_c = n$ so that $G = \Gamma = \tau = I_n$ and $\tau_\perp = 0$. In this case the last term in each of (43)–(46) is zero and (46) is superfluous. If, alternatively, the reduced-order constraint is retained but the H_∞ constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, then the results of [9] are recovered.

Remark 6.2: When solving (43)–(46) numerically, the matrices M and N, the structure matrices B_0 and C_0, and the scalar γ appearing in the design equations can be adjusted to examine the tradeoffs between H_2 performance, H_∞ performance, and robustness. As discussed in [11], to further reduce conservatism, one can view the scaling matrix N as a free parameter and optimize the H_2-performance bound $\mathcal{J}(\cdot)$ with respect to N. In particular, setting $\partial\mathcal{J}/\partial N = 0$ yields

$$\begin{aligned}
0 &= \frac{1}{2}\,\frac{\partial \mathcal{J}}{\partial N} \\
&= \tfrac{1}{2} M\tilde{C}_0 \tilde{V} \tilde{C}_0^T + R_0^{-1}(\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0\mathcal{P})\tilde{Q} \\
&\quad \cdot [\tilde{A} + \tilde{B}_0 R_0^{-1}(\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0\mathcal{P})]^T \tilde{C}_0^T
\end{aligned} \tag{52}$$

where $\tilde{Q}$ satisfies

$$\begin{aligned}
0 &= [\tilde{A} + \tilde{B}_0 R_0^{-1}(\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P}) \\
&\quad + \gamma^{-2}(\tilde{V}_\infty + \hat{C}_0^T\hat{C}_0)\mathcal{P}]\tilde{Q} \\
&\quad + \tilde{Q}[\tilde{A} + \tilde{B}_0 R_0^{-1}(\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P}) \\
&\quad + \gamma^{-2}(\tilde{V}_\infty + \hat{C}_0^T\hat{C}_0)\mathcal{P}]^T + \tilde{V}.
\end{aligned} \tag{53}$$

By using (52) within a numerical search algorithm, the optimal robust reduced-order controller and the scaling matrix N can be determined simultaneously, thus avoiding the need to iterate between robust reduced-order controller design and optimal N-scale evaluation.

Although (43)–(46) appear formidable, they are, in fact, numerically tractable. In particular, for related problems involving coupled Riccati equations, a new class of numerical algorithms has been developed, based on homotopic continuation methods [3] and [4]. These methods operate by first replacing the original problem by a simpler problem with a known solution. The desired solution is then reached by integrating along a path which connects the starting problem to the original problem. Alternatively, the reduced-order robust H_∞ problem (without the H_2-performance bound) can be approached by solving bilinear-matrix-inequalities (BMI's) [5], [6], [15]. However, since BMI's are not convex and as shown in [16] are NP-hard, it is difficult to develop computationally feasible algorithms that guarantee convergence. Finally, it has also been shown that the reduced-order H_∞ problem without parametric uncertainty can be addressed using alternating projection methods [7] and [8]. This class of algorithms is developed by posing the design problem in terms of linear-matrix-inequalities (LMI's) with an associated rank condition. This rank condition makes the problem nonconvex, and once again guaranteed convergence is not assured.

VII. Conclusion

The parameter-dependent Lyapunov function approach of [9]–[11] for robust controller synthesis with constant real parameter uncertainty was extended to account for H_∞-disturbance rejection. Specifically, by merging the results of [1] and [9]–[11], controller synthesis design equations are presented that guarantee robust stability and robust mixed H_2/H_∞ performance over a specified range of constant real parameter uncertainty.

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Robust, Reduced-Order Modeling for State-Space Systems via Parameter-Dependent Bounding Functions

Wassim M. Haddad and Vikram Kapila

***Abstract*— One of the most important problems in dynamic systems theory is to approximate a higher-order system model with a low-order, relatively simpler model. However, the nominal high-order model is never an exact representation of the true physical system. In this paper the problem of approximating an uncertain high-order system with constant real parameter uncertainty by a robust reduced-order model is considered. A *parameter-dependent* quadratic bounding function is developed that bounds the effect of uncertain real parameters on the model-reduction error. An auxiliary minimization problem is formulated that minimizes an upper bound for the model-reduction error. The principal result is a necessary condition for solving the auxiliary minimization problem which effectively provides sufficient conditions for characterizing robust reduced-order models.**

***Index Terms*— Real parameter uncertainty, reduced-order modeling, uncertain systems.**

Nomenclature

$\Re$, $\Re^{r\times s}$, $\Re^r$	Real numbers, $r \times s$ real matrices, $\Re^{r\times 1}$.
$()^T$, $()^{-1}$, $\mathrm{tr}\,()$, $\mathcal{E}$	Transpose, inverse, trace, expectation.
I_r, 0_r	$r \times r$ identity matrix, $r \times r$ zero matrix.
$\mathcal{S}^r$, $\mathcal{N}^r$, $\mathcal{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices.
$Z_1 \le Z_2$, $Z_1 < Z_2$	$Z_2 - Z_1 \in \mathcal{N}^r$, $Z_2 - Z_1 \in \mathcal{P}^r$; $Z_1, Z_2 \in \mathcal{S}^r$.
$n, l, m, n_m, \tilde{n}$	Positive integers; $1 \le n_m \le n$; $\tilde{n} = n + n_m$.
$x, y, x_m, y_m, \tilde{x}$	$n-$, $l-$, n_m-, $l-$, $\tilde{n}-$ dimensional vectors.
A, ΔA; B, C	$n \times n$; $n \times m$, $l \times n$ matrices.
A_m, B_m, C_m	$n_m \times n_m$, $n_m \times m$, $l \times n_m$ matrices.
R	Model-reduction error-weighting matrix, $R \in \mathcal{P}^l$.
$w(\cdot)$, V	m-dimensional white noise, intensity of $w(\cdot)$, $V \in \mathcal{P}^m$.

I. Introduction

One of the most important problems in dynamic systems theory is to approximate a higher-order system model with a low-order, relatively simpler model [8], [9], [11], [12]. However, the nominal high-order model is never an exact representation of the true physical system. This necessitates design tools that allow robust reduced-order modeling with respect to uncertainties in the high-order model. In many physical systems, such as flexible structures with uncertain frequency and damping, these uncertainties are characterized as highly structured, constant real parametric errors. Hence, to guarantee the best performance possible in the presence of these uncertainties it

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W. M. Haddad is with the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150 USA (wm.haddad@aerospace.gatech.edu).

V. Kapila is with the Department of Mechanical Engineering, Polytechnic University, Brooklyn, NY 11201 USA.

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