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Robust Stability and Performance via Fixed-Order Dynamic Compensation with Guaranteed Cost Bounds*

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Abstract. A feedback control-design problem involving structured plant parameter uncertainties is considered. Two robust control-design issues are addressed. The Robust Stability Problem involves deterministic bounded structured parameter variations, while the Robust Performance Problem includes, in addition, a quadratic performance criterion averaged over stochastic disturbances and maximized over the admissible parameter variations. The optimal projection approach to fixed-order dynamic compensation is merged with the guaranteed cost control approach to robust stability and performance to obtain a theory of full- and reduced-order robust control design. The principle result is a sufficient condition for characterizing dynamic controllers of fixed dimension which are guaranteed to provide both robust stability and performance. The sufficient conditions involve a system of modified Riccati and Lyapunov equations coupled by an oblique projection and the uncertainty bounds. The full-order result involves a system of two modified Riccati equations and two modified Lyapunov equations coupled by the uncertainty bounds. The coupling illustrates the breakdown of the separation principle for LQG control with structured plant parameter variations.

Key words. Robust stability, Robust performance, LQG control, Dynamic compensation, structured uncertainty.

1. Introduction

The direct method of Lyapunov has proven to be an effective approach to robust analysis and design of feedback control laws. References [B1], [B2], [BCL], [BG2], [CL], [CP], [ER], [GB], [H], [KB], [KBH], [L], [PH], [TB], and [VW] comprise a representative collection of the literature in this area. In performing robust synthesis there are two principal issues, namely, stability robustness and performance robustness. Stability robustness addresses the problem of guaranteeing stability of the closed-loop system for plant perturbations within a specified class of uncertainties. In addition to guaranteeing robust stability, it is often desirable to minimize the worst-case performance degradation within a given robust stability

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range. Although both robust stability and performance are of interest in practice, most of the literature involving quadratic Lyapunov functions deals only with the problem of robust stability. A notable exception is the early work of Chang and Peng [CP] which also provides bounds on worst-case quadratic performance within full-state feedback control design.

The contribution of this paper is a methodology for designing controllers which provide both robust stability and robust performance over a prescribed range of structured plant parameter variations. The feedback law is in the form of a fixed-order (i.e., full- or reduced-order) strictly proper dynamic compensator. The overall approach is based upon the merging of two distinct control-design techniques, namely, the guaranteed cost control approach to robust performance [CP] and the optimal projection approach to fixed-order dynamic compensation [BH3], [HB]. The principal motivation for our approach is to permit greater flexibility in the design of robust feedback laws by providing an alternative to full-state feedback and full-order dynamic compensation.

The guaranteed cost control approach [CP] adopted in this paper utilizes a performance bound to provide robust performance in addition to robust stability. Here, robust performance refers to a guaranteed bound on the worst-case value of the expectation of a quadratic cost criterion over a prescribed uncertainty set. This quadratic criterion is precisely the standard cost functional of linear-quadratic-Gaussian (LQG) control theory. By bounding the worst-case value of this criterion over a specified range of plant uncertainties, we effectively bound the variances of specified states and control signals.

To bound the worst-case closed-loop performance, we require a bound on the effect of plant uncertainties on the steady-state closed-loop covariance matrix. The form of the guaranteed cost control bound utilized herein was originally motivated by the effect of multiplicative white noise on the state covariance [B2], [BG2]. Since this bound is differentiable with respect to the covariance matrix and compensator gains, it permits optimal design via first-order necessary conditions. This approach is not possible using the nondifferentiable bound ordinally proposed in [CP]. An alternative differentiable bound proposed in [PH] for full-state feedback has been extended to fixed-order dynamic compensation in [BH1].

In this paper the guaranteed cost technique is used to bound the closed-loop performance and characterize robustly stabilizing controllers. This performance bound is then interpreted as an *auxiliary cost* which is to be minimized by the choice of compensator gains. The *actual* performance for a given realization of the parameter uncertainty is thus guaranteed to lie below this bound. Assuming stabilizability (disturbability), the robust performance bound automatically implies robust stability. The auxiliary cost and the Lyapunov equation constraint together form the Auxiliary Minimization Problem. Since the Auxiliary Minimization Problem is a nonconvex mathematical programming problem with differentiable data, it is amenable to first-order necessary conditions.

One feature of this approach is that since the necessary conditions are obtained for the Auxiliary Minimization Problem rather than the original problem, extremals are guaranteed to provide both robust stability and performance. Note that this is true for every extremal of the Auxiliary Minimization Problem whether it corresponds to a local minimum, local maximum, or otherwise. Of course, the global minimum is most likely to provide the best worst-case performance over the robust stability range. In any case, necessary conditions for the Auxiliary Minimization Problem effectively serve as *sufficient* conditions for robust stability with a guaranteed performance bound.

This paper presents a rigorous development of sufficient conditions for robust stability and performance via fixed-order dynamic compensation. These sufficient conditions are in the form of a coupled system of algebraic matrix equations consisting of two modified Riccati equations and two modified Lyapunov equations. The coupling is due to the optimal projection, which characterizes reduced-order controllers, and the uncertainty bounds, which account for the effect of parameter uncertainties on the performance functional. When the compensator order is constrained to be equal to the dimension of the plant and the uncertainty bounds are absent, the equations specialize to the usual pair of separated Riccati equations of steady-state LQG theory.

We emphasize that our approach is constructive in nature rather than existential. Our sufficient conditions provide explicit formulae for robust, fixed-order feedback gains when the Auxiliary Minimization Problem has a solution, and in this case our constructive conditions are complementary to existential results on robust stabilizability. The existence of a solution to the Auxiliary Minimization Problem and associated design equations depends upon stabilizability via fixed-order controllers and on the sharpness of the quadratic Lyapunov bounds. The stabilizability problem has been studied using independent methods (see, e.g., [BHK]), while the conservatism of the bounds is considered in [BH2]. Here we state a local existence result for solvability of the design equations which assumes only nominal stabilizability.

The contents and scope of this paper are as follows. In Section 2 we state the robust stability and performance problems for fixed-order dynamic compensation with plant parameter uncertainty. In Section 3 a modified Lyapunov equation is introduced whose solution, when it exists, is guaranteed to bound the steady-state closed-loop covariance over the specified range of plant uncertainty. A performance bound is then given in terms of the covariance bound. In Section 4 we view the performance bound as an auxiliary cost and consider the problem of minimizing the auxiliary cost subject to the modified Lyapunov equation and a definiteness condition as side constraints. These side constraints have the property that all admissible elements provide robust stability and performance (Proposition 4.1). In Section 5 the uncertainty set and bound for constructing the modified Lyapunov equation are given concrete forms. Specifically, the uncertainty set has the form of an ellipsoidal region in parameter space while the modified Lyapunov equation includes additional linear terms to bound the uncertainty. A sufficient condition involving Kronecker sums and products implies the existence of a unique, nonnegative-definite solution to the modified Lyapunov equation. Section 6 presents the first-order necessary conditions (Theorem 6.1) for the Auxiliary Minimization Problem under minor additional technical conditions to ensure the applicability of the Lagrange multiplier technique. As discussed above, these necessary conditions are in the form of extended optimal projection equations. A partial converse of the necessary conditions shows that solutions of these algebraic equations provide, by construction, a solution of the original modified Lyapunov equation. This result is combined in Section 7 (Theorem 7.1) with a stabilizability assumption to guarantee robust stability with a robust performance bound. In addition, we state an existence result for local solvability of the design equations by applying a result from [R1] and [R2] (Theorem 7.2). To draw connections with standard LQG. theory, in Section 8 we specialize Theorem 7.1 to the full-order case. In contrast to the pair of separated Riccati equations of standard LQG theory, the full-order result in the presence of plant parameter variations is given by a coupled system of *four* modified Riccati/Lyapunov equations. In Section 9 the theory is illustrated by means of an example due to Doyle [D]. This problem was also considered in [BG1] before the robustness theory developed herein was available. Hence this paper can be viewed as the rigorous mathematical foundation which legitimizes the heretofore *ad hoc* robustness approach of [BG1].

Notation. Note: All matrices have real entries

$\mathbb{E}$	expected value
I_r, $(\ )^{\mathrm{T}}$, $0_{r\times s}$, 0_r	$r \times r$ identity matrix, transpose, $r \times s$ zero matrix, $0_{r\times r}$
$Z_{(i,j)}$	(i, j)-element of matrix Z
$\oplus$, $\otimes$	Kronecker sum, Kronecker product [B3]
$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices
$Z_1 \leq Z_2$, $Z_1 < Z_2$	$Z_2 - Z_1 \in \mathbb{N}^r$, $Z_2 - Z_1 \in \mathbb{P}^r$, $Z_1, Z_2 \in \mathbb{S}^r$
A_α, $A_{c\alpha}$	$A + (\alpha/2)I_n$, $A_c + (\alpha/2)I_{n_c}$
R_1, R_2	state, control weighting matrices; $R_1 \in \mathbb{N}^n$, $R_2 \in \mathbb{P}^m$
R_{12}	$n \times m$ cross-weighting matrix; $R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}} \geq 0$
$\tilde{R}$	$\begin{bmatrix} R_1 & R_{12}C_c \\ C_c^{\mathrm{T}}R_{12}^{\mathrm{T}} & C_c^{\mathrm{T}}R_2C_c \end{bmatrix}$
$w_1(\cdot)$, $w_2(\cdot)$	n, l-dimensional white noise
V_1, V_2	intensity of $w_1(\cdot)$, $w_2(\cdot)$; $V_1 \in \mathbb{N}^n$, $V_2 \in \mathbb{P}^l$
V_{12}	$n \times l$ cross intensity of $w_1(\cdot)$, $w_2(\cdot)$
$\tilde{w}(\cdot)$, $\tilde{V}$	$\begin{bmatrix} w_1(\cdot) \\ B_c w_2(\cdot) \end{bmatrix}$, $\begin{bmatrix} V_1 & V_{12}B_c^{\mathrm{T}} \\ B_c V_{12}^{\mathrm{T}} & B_c V_2 B_c^{\mathrm{T}} \end{bmatrix}$

2. Robust Stability and Robust Performance Problems

In this section we state the Robust Stability Problem and Robust Performance Problem. Both problems involve a set $\mathscr{U} \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{l\times n} \times \mathbb{R}^{l\times m}$ of uncertain perturbations $(\Delta A, \Delta B, \Delta C, \Delta D)$ of the nominal system matrices (A, B, C, D). The goal of the Robust Stability Problem is to determine a fixed-order, strictly proper dynamic compensator (A_c, B_c, C_c) which stabilizes the plant for all variations in $\mathscr{U}$. In this section and the following section no explicit assumptions are required for the set $\mathscr{U}$. In Section 5 the structure of variations in $\mathscr{U}$ will be specified.

Robust Stability Problem. For fixed $n_c \leq n$ determine (A_c, B_c, C_c) such that the closed-loop system consisting of the nth-order controlled plant,

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t), \qquad t \in [0, \infty), \tag{2.1}$$

measurements

$$y(t) = (C + \Delta C)x(t) + (D + \Delta D)u(t), \tag{2.2}$$

and n_cth-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{2.3}$$

$$u(t) = C_c x_c(t), \tag{2.4}$$

is asymptotically stable for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathscr{U}$.

The Robust Performance Problem involves, in addition, white plant disturbances and measurement noise. The goal of this problem is to determine a fixed-order, strictly proper compensator (A_c, B_c, C_c) which minimizes the worst-case value over the uncertainty set $\mathscr{U}$ of a steady-state average quadratic performance criterion.

Robust Performance Problem. For fixed $n_c \leq n$, determine (A_c, B_c, C_c) such that, for the closed-loop system consisting of the nth-order controlled and disturbed plant

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) + w_1(t), \qquad t \in [0, \infty), \tag{2.5}$$

noisy measurements

$$y(t) = (C + \Delta C)x(t) + (D + \Delta D)u(t) + w_2(t), \tag{2.6}$$

and n_cth-order dynamic compensator (2.3), (2.4), the performance criterion

$$J(A_c, B_c, C_c) \triangleq \sup_{(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathscr{U}} \limsup_{t \to \infty} \mathbb{E}[x^T(t)R_1 x(t) + 2x^T(t)R_{12}u(t) + u^T(t)R_2 u(t)] \tag{2.7}$$

is minimized.

Remark 2.1. The cost functional (2.7) is identical to the standard LQG criterion with the exception of the supremum for evaluating worst-case quadratic performance over $\mathscr{U}$. Note that (2.7) can also be written in terms of an averaged integral, i.e.,

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathscr{U}} \limsup_{t \to \infty} \frac{1}{t} \mathbb{E}\left\{\int_0^t [x^T(s)R_1 x(s) + 2x^T(s)R_{12}u(s) + u^T(s)R_2 u(s)]\, ds\right\}.$$

For practical application, the cost (2.7) provides the means for minimizing the variances of selected state variables and control signals. This can be achieved by appropriate selection of the matrices R_1 and R_2 which serve as design weights. For robust performance the goal is to minimize the worst-case variances of selected variables over the plant uncertainty.

For each uncertain variation $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$, the undisturbed closed-loop system (2.1)–(2.4) can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t), \qquad t \in [0, \infty), \tag{2.8}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \qquad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c + B_cDC_c \end{bmatrix}, \qquad \Delta\tilde{A} \triangleq \begin{bmatrix} \Delta A & \Delta BC_c \\ B_c\Delta C & B_c\Delta DC_c \end{bmatrix}.$$

Similarly, the disturbed closed-loop system (2.3)–(2.6) can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{w}(t), \qquad t \in [0, \infty), \tag{2.9}$$

where the closed-loop disturbace $\tilde{w}(t)$ has intensity $\tilde{V} \in \mathbb{N}^{\tilde{n}}$.

3. Sufficient Conditions for Robust Stability and Performance

In practice, steady-state performance is only of interest when the undisturbed closed-loop system (2.8) is robustly stable over $\mathcal{U}$. The following result, which expresses the performance in terms of the steady-state closed-loop second-moment matrix, is immediate.

Lemma 3.1. *Let (A_c, B_c, C_c) be given and assume the system (2.8) is stable for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$. Then*

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}} \operatorname{tr} \tilde{Q}_{\Delta\tilde{A}} \tilde{R}, \tag{3.1}$$

where $\tilde{Q}_{\Delta\tilde{A}} \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)] \in \mathbb{N}^{\tilde{n}}$ is the unique solution to

$$0 = (\tilde{A} + \Delta\tilde{A})\tilde{Q}_{\Delta\tilde{A}} + \tilde{Q}_{\Delta\tilde{A}}(\tilde{A} + \Delta\tilde{A})^{\mathrm{T}} + \tilde{V}. \tag{3.2}$$

Remark 3.1. When $\mathcal{U}$ is compact, "sup" in (3.1) can be replaced by "max."

The key step in guaranteeing robust stability and performance is to replace the uncertain terms in the covariance Lyapunov equation (3.2) by a bounding function Ω. Note that since $\Delta\tilde{A}$ is independent of A_c, the bounding function need only depend upon B_c and C_c.

Theorem 3.1. *Let $\Omega: \mathbb{N}^{\tilde{n}} \times \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c} \to \mathbb{S}^{\tilde{n}}$ be such that*

$$\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}} \leq \Omega(\mathcal{Q}, B_c, C_c),$$
$$(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}, \qquad (\mathcal{Q}, B_c, C_c) \in \mathbb{N}^{\tilde{n}} \times \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c}, \tag{3.3}$$

and, for given (A_c, B_c, C_c), assume there exists $\mathcal{Q} \in \mathbb{N}^{\tilde{n}}$ satisfying

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^{\mathrm{T}} + \Omega(\mathcal{Q}, B_c, C_c) + \tilde{V}. \tag{3.4}$$

Then

$$(\tilde{A} + \Delta\tilde{A}, [\tilde{V} + \Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}})]^{1/2}) \quad \textit{is stabilizable},$$
$$(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}, \tag{3.5}$$

if and only if

$$\tilde{A} + \Delta\tilde{A} \quad \textit{is asymptotically stable}, \qquad (\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}. \tag{3.6}$$

In this case,

$$\tilde{Q}_{\Delta\tilde{A}} \leq \mathcal{Q}, \qquad (\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}, \tag{3.7}$$

where $\tilde{Q}_{\Delta\tilde{A}}$ *is given by* (3.2), *and*

$$J(A_c, B_c, C_c) \leq \operatorname{tr} \mathcal{Q}\tilde{R}. \tag{3.8}$$

Proof. First note for clarity that in (3.3) $\mathcal{Q}$ denotes an arbitrary element of $\mathbb{N}^{\tilde{n}}$ since (3.3) holds for all $\mathcal{Q} \in \mathbb{N}^{\tilde{n}}$, while in (3.4) $\mathcal{Q}$ denotes a specific solution to (3.4). Now for $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$, (3.4) is equivalent to

$$0 = (\tilde{A} + \Delta\tilde{A})\mathcal{Q} + \mathcal{Q}(\tilde{A} + \Delta\tilde{A}^{\mathrm{T}}) + \Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}}) + \tilde{V}. \tag{3.9}$$

Hence, by assumption, (3.9) has a solution $\mathcal{Q} \in \mathbb{N}^{\tilde{n}}$ for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$ and, by (3.3), $\Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}})$ is nonnegative definite. Now if the stabilizability condition (3.5) holds for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$, it follows from Lemma 12.2 of [W] that $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$. Conversely, if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$, then (3.5) holds. Next, subtracting (3.2) from (3.9) yields

$$0 = (\tilde{A} + \Delta\tilde{A})(\mathcal{Q} - \tilde{Q}_{\Delta\tilde{A}}) + (\mathcal{Q} - \tilde{Q}_{\Delta\tilde{A}})(\tilde{A} + \Delta\tilde{A})^{\mathrm{T}} + \Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}}),$$

or, equivalently, since $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$,

$$\mathcal{Q} - \tilde{Q}_{\Delta\tilde{A}} = \int_0^\infty e^{(\tilde{A}+\Delta\tilde{A})t}[\Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}})]e^{(\tilde{A}+\Delta\tilde{A})^{\mathrm{T}}t}\, dt \geq 0,$$

which implies (3.7). The performance bound (3.8) is now an immediate consequence of (3.7). ■

Remark 3.2. In applying Theorem 3.1 it may be convenient to replace condition (3.5) with a stronger condition which is easier to verify in practice. Clearly, (3.5) is satisfied if $\tilde{V} + \Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}})$ is positive definite for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$. This will be the case, for example, if either $\tilde{V}$ is positive definite or strict inequality holds in (3.3). Also, it follows from Theorem 3.6 of [W] that (3.5) is implied by the stronger condition

$$(\tilde{A} + \Delta\tilde{A}, \tilde{V}^{1/2}) \quad \text{is stabilizable}, \qquad (\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}. \tag{3.10}$$

Remark 3.3. The covariance bound (3.7) can also be used to analyze the effect of disturbances on specified state variables. For example, if $E_1 \in \mathbb{R}^{q \times n}$, then (3.7) implies

$$[E_1 \quad 0_{q \times n_c}]\tilde{Q}_{\Delta\tilde{A}}\begin{bmatrix} E_1^{\mathrm{T}} \\ 0_{n_c \times q} \end{bmatrix} \leq [E_1 \quad 0_{q \times n_c}]\mathcal{Q}\begin{bmatrix} E_1^{\mathrm{T}} \\ 0_{n_c \times q} \end{bmatrix} \tag{3.11}$$

so that the right-hand side of (3.11) serves as a bound on selected state variances. For control-design purposes we effectively set $R_1 = E_1^{\mathrm{T}} E_1$. Similar remarks apply to obtaining bounds on the variances of control signals.

4. The Auxiliary Minimization Problem

The key step in our development involves consideration of the performance bound (3.8) in place of the actual worst-case performance $J(A_c, B_c, C_c)$. This leads to the following problem.

Auxiliary Minimization Problem. Determine $(\mathscr{Q}, A_c, B_c, C_c)$ which minimizes

$$\mathscr{J}(\mathscr{Q}, A_c, B_c, C_c) \triangleq \operatorname{tr} \mathscr{Q}\tilde{R} \tag{4.1}$$

subject to (3.4) and

$$\mathscr{Q} \in \mathbb{N}^{\tilde{n}}. \tag{4.2}$$

The relationship between the Auxiliary Minimization Problem and the Robust Stability and Performance Problems is straightforward as shown by the following observation.

Proposition 4.1. *If* $(\mathscr{Q}, A_c, B_c, C_c)$ *satisfies* (3.4) *and* (4.2) *and the stabilizability condition* (3.5) *holds, then* $\tilde{A} + \Delta\tilde{A}$ *is asymptotically stable for all* $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathscr{U}$, *and*

$$J(A_c, B_c, C_c) \leq \mathscr{J}(\mathscr{Q}, A_c, B_c, C_c). \tag{4.3}$$

Proof. Since (3.4) has a solution $\mathscr{Q} \in \mathbb{N}^{\tilde{n}}$ and the stabilizability condition (3.5) holds, the hypotheses of Theorem 3.1 are satisfied so that robust stability with robust performance bound (3.8) is guaranteed; (4.3) is merely a restatement of (3.8). ■

Several comments are in order. Since the *auxiliary cost* (4.1) is an upper bound for the actual cost (2.7), it is clearly desirable to minimize (4.1) over $\mathscr{Q}$ and the controller gains. Note, however, that the Auxiliary Minimization Problem is a nonconvex mathematical programming problem on a noncompact set. Hence existence of solutions and sufficient conditions for global optimality cannot be obtained without imposing additional restrictive assumptions. To develop nonrestrictive results, we proceed in Section 6 by deriving necessary conditions for optimality which require no further assumptions except that Ω be differentiable and that the minimization be performed over an open set. In the next section we construct a bound Ω which possesses the required smoothness.

5. Uncertainty Structure and the Guaranteed Cost Bound

Having established the theoretical basis for our approach, we now assign explicit structure to the set $\mathscr{U}$ and bounding function Ω. Specifically, the uncertainty set $\mathscr{U}$ is assumed to be of the form

$$\mathscr{U} = \left\{(\Delta A, \Delta B, \Delta C, \Delta D)\colon \Delta A = \sum_{i=1}^{p} \sigma_i A_i,\ \Delta B = \sum_{i=1}^{p} \sigma_i B_i,\ \Delta C = \sum_{i=1}^{p} \sigma_i C_i,\right.$$
$$\left.\Delta D = \sum_{i=1}^{p} \sigma_i D_i,\ \sum_{i=1}^{p} \sigma_i^2/\alpha_i^2 \leq 1\right\}, \tag{5.1}$$

where for $i = 1, \ldots, p$: $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $C_i \in \mathbb{R}^{l \times n}$ and $D_i \in \mathbb{R}^{l \times m}$ are fixed matrices denoting the structure of the parametric uncertainty; α_i is a given positive number; and σ_i is an uncertain real parameter. Note that the uncertain parameters σ_i are assumed to lie in a specified ellipsoidal region in $\mathbb{R}^p$. The closed-loop system (2.8) thus has structured uncertainty of the form

$$\Delta \tilde{A} = \sum_{i=1}^{p} \sigma_i \tilde{A}_i, \tag{5.2}$$

where

$$\tilde{A}_i \triangleq \begin{bmatrix} A_i & B_i C_c \\ B_c C_i & B_c D_i C_c \end{bmatrix}, \qquad i = 1, \ldots, p.$$

The uncertainty set $\mathscr{U}$ is general in the sense that no explicit assumptions such as the matching conditions used in [BCL] will be made with regard to the structure of A_i, B_i, C_i, and D_i. We do, however, require (as is evident from (5.1)) that the uncertain parameters σ_i appear linearly in the off-nominal perturbations which is more confining than matching assumptions. Note that the symmetry of the uncertainty set entails no loss of generality by requiring only a redefinition of the nominal plant matrices.

In order to obtain *explicit* gain expressions for (A_c, B_c, C_c) in Section 6, we shall require one additional technical assumption. Specifically, we assume that, for each $i \in \{1, \ldots, p\}$, at most one of the matrices B_i, C_i, and D_i is nonzero. This condition thus implies that a given uncertain parameter σ_i may appear explicitly in both ΔA and ΔB, or both ΔA and ΔC, or both ΔA and ΔD, or only ΔA, but not (say) in both ΔB and ΔD. Thus we can account partially (but not totally) for *correlated* parameter uncertainties in different plant matrices. If a given uncertain parameter does arise in both (say) ΔB and ΔD, then it must be represented by two distinct uncertain parameters. If this assumption is not imposed, then optimality conditions can still be derived, but at the expense of closed-form gain expressions.

For the structure of $\mathscr{U}$ as specified by (5.1), the bound Ω satisfying (3.3) can now be given a concrete form.

Proposition 5.1. *Let α be an arbitrary positive scalar. Then the function*

$$\Omega(\mathscr{Q}, B_c, C_c) = \alpha \mathscr{Q} + \alpha^{-1} \sum_{i=1}^{p} \alpha_i^2 \tilde{A}_i \mathscr{Q} \tilde{A}_i^{\mathrm{T}} \tag{5.3}$$

satisfies (3.3) *with $\mathscr{U}$ given by* (5.1).

Proof. Note that

$$\begin{aligned} 0 &\leq \sum_{i=1}^{p} [(\alpha^{1/2}\sigma_i/\alpha_i) I_{\tilde{n}} - (\alpha_i/\alpha^{1/2})\tilde{A}_i] \mathscr{Q} [(\alpha^{1/2}\sigma_i/\alpha_i) I_{\tilde{n}} - (\alpha_i/\alpha^{1/2})\tilde{A}_i]^{\mathrm{T}} \\ &= \alpha \sum_{i=1}^{p} (\sigma_i^2/\alpha_i^2)\mathscr{Q} + \alpha^{-1} \sum_{i=1}^{p} \alpha_i^2 \tilde{A}_i \mathscr{Q} \tilde{A}_i^{\mathrm{T}} - \sum_{i=1}^{p} \sigma_i (\tilde{A}_i \mathscr{Q} + \mathscr{Q} \tilde{A}_i^{\mathrm{T}}), \end{aligned}$$

which, since $\sum_{i=1}^{p} \sigma_i^2/\alpha_i^2 \leq 1$, implies (3.3). ∎

Remark 5.1. Note that the bound Ω given by (5.3) consists of two distinct terms. The first term $\alpha \mathscr{Q}$ can be thought of as arising from an exponential time weighting of the cost, or, equivalently, from a uniform right shift of the open-loop dynamics [AM]. The second term $\alpha^{-1}\sum_{i=1}^{p}\alpha_i^2\tilde{A}_i\mathscr{Q}\tilde{A}_i^{\mathrm{T}}$ arises naturally from a multiplicative white-noise model [BG1], [BG2], [B]. Such interpretations have no bearing on the results obtained here since only the bound Ω defined by (5.3) is required. Note that the bound (5.3) is valid for all positive α. A similar bound was also considered in [KB].

With Ω defined by (5.3), the modified Lyapunov equation (3.4) becomes

$$0 = \tilde{A}\mathscr{Q} + \mathscr{Q}\tilde{A}^{\mathrm{T}} + \alpha\mathscr{Q} + \alpha^{-1}\sum_{i=1}^{p}\alpha_i^2\tilde{A}_i\mathscr{Q}\tilde{A}_i^{\mathrm{T}} + \tilde{V} \tag{5.4}$$

or, equivalently,

$$0 = \tilde{A}_\alpha\mathscr{Q} + \mathscr{Q}\tilde{A}_\alpha^{\mathrm{T}} + \sum_{i=1}^{p}\gamma_i\tilde{A}_i\mathscr{Q}\tilde{A}_i^{\mathrm{T}} + \tilde{V}, \tag{5.5}$$

where

$$\tilde{A}_\alpha \triangleq \tilde{A} + \frac{\alpha}{2}I_{\tilde{n}} = \begin{bmatrix} A_\alpha & BC_{\mathrm{c}} \\ B_{\mathrm{c}}C & A_{\mathrm{c}\alpha} + B_{\mathrm{c}}DC_{\mathrm{c}} \end{bmatrix} \tag{5.6}$$

and $\gamma_i \triangleq \alpha_i^2/\alpha$. Note that (5.5) is equivalent to

$$0 = \mathscr{A}\ \mathrm{vec}\ \mathscr{Q} + \mathrm{vec}\ \tilde{V}, \tag{5.7}$$

where "vec" is the column-stacking operation defined in [B3] and $\mathscr{A}$ is defined by

$$\mathscr{A} \triangleq \tilde{A}_\alpha \oplus \tilde{A}_\alpha + \sum_{i=1}^{p}\gamma_i\tilde{A}_i \otimes \tilde{A}_i.$$

Next, using the bound Ω given by (5.3) and $\mathscr{U}$ given by (5.1) we present a result which guarantees the existence of a nonnegative-definite solution to (3.4) or, equivalently, (5.5) for a given controller $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}})$. For the converse we view $\tilde{V}$ as an arbitrary element of $\mathbb{N}^{\tilde{n}}$.

Proposition 5.2. *Let $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}})$ be given and let $\alpha > 0$. If $\mathscr{A}$ is asymptotically stable, then there exists a unique $\tilde{n} \times \tilde{n}$ $\mathscr{Q}$ satisfying (5.5) and, furthermore, $\mathscr{Q} \geq 0$. Conversely, if for all $\tilde{V} \in \mathbb{N}^{\tilde{n}}$ there exists $\mathscr{Q} \geq 0$ satisfying (5.5), then $\mathscr{A}$ is asymptotically stable.*

Proof. Since (5.5) is equivalent to

$$\mathscr{Q} = -\mathrm{vec}^{-1}[\mathscr{A}^{-1}\ \mathrm{vec}\ \tilde{V}], \tag{5.8}$$

existence and uniqueness hold. To prove that $\mathscr{Q}$ is nonnegative definite, we rewrite (5.8) as

$$\mathscr{Q} = \int_0^\infty \mathrm{vec}^{-1}[e^{\mathscr{A}t}\ \mathrm{vec}\ \tilde{V}]\,dt \tag{5.9}$$

and show that the integrand is nonnegative definite for all $t \in [0, \infty)$. (Note that the following argument does not require that $\mathscr{A}$ be stable). Using the Lie exponential product formula, the exponential in (5.9) can be written as

$$e^{\mathscr{A}t} = \lim_{k\to\infty}\left\{\exp\left[\frac{1}{k}(\tilde{A}_\alpha \oplus \tilde{A}_\alpha)t\right]\prod_{i=1}^{p}\exp\left[\frac{1}{k}\gamma_i(\tilde{A}_i \otimes \tilde{A}_i)t\right]\right\}^k. \tag{5.10}$$

For convenience, let S and N be $r \times r$ matrices with $N \geq 0$. Since (see [B3])

$$\text{vec}^{-1}[(S \otimes S)\,\text{vec}\,N] = SNS^{\mathrm{T}} \geq 0 \tag{5.11}$$

and

$$(S^k \otimes S^k)(S \otimes S) = S^{k+1} \otimes S^{k+1}, \tag{5.12}$$

it follows that

$$\text{vec}^{-1}[e^{S\otimes S}\,\text{vec}\,N] = \sum_{k=0}^{\infty} (k!)^{-1} S^k N S^{k\mathrm{T}} \geq 0. \tag{5.13}$$

Furthermore,

$$\text{vec}^{-1}[e^{S\oplus S}\,\text{vec}\,N] = \text{vec}^{-1}[(e^S \otimes e^S)\,\text{vec}\,N] = e^S N e^{S^{\mathrm{T}}} \geq 0. \tag{5.14}$$

Applying (5.13) and (5.14) alternately with (5.10) and using induction on k it follows that the integrand of (5.9) is nonnegative definite. To prove the converse, note that it follows from (5.5) that $\mathscr{Q}$ satisfies

$$\mathscr{Q} = \text{vec}^{-1}[e^{\mathscr{A}t}\,\text{vec}\,\mathscr{Q}] + \int_0^t \text{vec}^{-1}[e^{\mathscr{A}s}\,\text{vec}\,\tilde{V}]\,ds, \qquad t \in [0, \infty). \tag{5.15}$$

Since the integral term on the right-hand side of (5.15) is nonnegative definite, is bounded from above by $\mathscr{Q}$, and $\tilde{V} \in \mathbb{N}^{\tilde{n}}$ is arbitrary, it follows that $\mathscr{A}$ is asymptotically stable. ■

Proposition 5.2 shows that a solution of (5.5) exists as long as $\alpha_1, \ldots, \alpha_p$ are sufficiently small so that $\mathscr{A}$ remains stable for some $\alpha > 0$. The following result characterizes values of $\alpha_1, \ldots, \alpha_p$ for which $\mathscr{A}$ is asymptotically stable. Let $\|\cdot\|$ denote an arbitrary vector norm and its induced matrix norm.

Proposition 5.3. *Let (A_c, B_c, C_c) be given, assume $\tilde{A}$ is asymptotically stable, and let $\alpha, \alpha_1, \ldots, \alpha_p > 0$. If*

$$\left\| (\tilde{A} \oplus \tilde{A})^{-1} \left(\alpha I_{\tilde{n}^2} + \sum_{i=1}^{p} \gamma_i \tilde{A}_i \otimes \tilde{A}_i \right) \right\| < 1, \tag{5.16}$$

then there exists $\mathscr{Q} \in \mathbb{N}^{\tilde{n}}$ satisfying (5.5) and $\mathscr{A}$ is asymptotically stable.

Proof. Define $\{\mathscr{Q}_k\}_{k=0}^{\infty}$ where $\mathscr{Q}_0$ satisfies

$$0 = \tilde{A}\mathscr{Q}_0 + \mathscr{Q}_0\tilde{A}^{\mathrm{T}} + \tilde{V},$$

and $\mathscr{Q}_{k+1}$ satisfies

$$0 = \tilde{A}\mathscr{Q}_{k+1} + \mathscr{Q}_{k+1}\tilde{A}^{\mathrm{T}} + \Omega(\mathscr{Q}_k, B_c, C_c) + \tilde{V}.$$

Note that $\mathscr{Q}_k \geq 0$, $k = 1, 2, \ldots$. Hence it follows that

$$\text{vec}\,\mathscr{Q}_{k+1} - \text{vec}\,\mathscr{Q}_k = -(\tilde{A} \oplus \tilde{A})^{-1}[\text{vec}\,\Omega(\mathscr{Q}_k, B_c, C_c) - \text{vec}\,\Omega(\mathscr{Q}_{k-1}, B_c, C_c)]$$

and thus

$$\|\text{vec}\,\mathscr{Q}_{k+1} - \text{vec}\,\mathscr{Q}_k\| \leq \left\| (\tilde{A} \oplus \tilde{A})^{-1} \left(\alpha I_{\tilde{n}^2} + \sum_{i=1}^{p} \gamma_i \tilde{A}_i \otimes \tilde{A}_i \right) \right\| \|\text{vec}\,\mathscr{Q}_k - \text{vec}\,\mathscr{Q}_{k-1}\|.$$

Using (5.16) it follows that $\mathcal{Q} \triangleq \lim_{k\to\infty} \mathcal{Q}_k$ exists. Thus $\mathcal{Q} \geq 0$ satisfies (5.5). Furthermore, since $\tilde{V} \in \mathbb{N}^{\tilde{n}}$ can be considered arbitrary, Proposition 5.2 implies that $\mathscr{A}$ is asymptotically stable. ∎

6. Necessary Conditions for the Auxiliary Minimization Problem

The derivation of the necessary conditions for the Auxiliary Minimization Problem is based upon the Fritz John form of the Lagrange multiplier theorem. Application of this theorem requires that we further restrict $(\mathcal{Q}, A_c, B_c, C_c)$ to the open set

$$\mathscr{S} \triangleq \{(\mathcal{Q}, A_c, B_c, C_c)\colon \mathcal{Q} \in \mathbb{P}^{\tilde{n}}, \mathscr{A} \text{ is asymptotically stable,}$$
$$\text{and } (A_c, B_c, C_c) \text{ is controllable and observable}\}.$$

As will be seen, the constraint $(\mathcal{Q}, A_c, B_c, C_c) \in \mathscr{S}$ is not required for either robust stability or robust performance since Proposition 4.1 shows that only (3.4), (3.5), and (4.2) are needed. Rather, the set $\mathscr{S}$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the Auxiliary Minimization Problem. Specifically, the condition $\mathcal{Q} > 0$ replaces (4.2) by an open set constraint, the asymptotic stability of $\mathscr{A}$ serves as a normality condition which further implies that the dual $\mathscr{P}$ of $\mathcal{Q}$ is nonnegative definite, and (A_c, B_c, C_c) minimal is a nondegeneracy condition which implies that the lower right $n_c \times n_c$ subblocks of $\mathcal{Q}$ and $\mathscr{P}$ are positive definite thus yielding explicit expressions for B_c and C_c. Note that by Proposition 5.2 the condition that $\mathscr{A}$ be asymptotically stable also implies that (5.5) has a unique, nonnegative solution. Finally, we point out that the stabilizability condition (3.5) and stability condition (3.6) play no role in determining solutions of the Auxiliary Minimization Problem.

In order to state the main results we require some additional notation and a lemma concerning pairs of nonnegative-definite matrices. For a real $n \times n$ matrix Z define the set of real diagonalizing matrices

$$\mathscr{D}(Z) \triangleq \{\Psi \in \mathbb{R}^{n\times n}\colon \Psi^{-1}Z\Psi \text{ is diagonal}\},$$

and, for a pair of $n \times n$ symmetric matrices, X, Y define the set of real *contragrediently diagonalizing* matrices

$$\mathscr{C}(X, Y) \triangleq \{\Psi \in \mathscr{D}(XY)\colon \Psi^{-1}X\Psi^{-\mathrm{T}} \text{ and } \Psi^{\mathrm{T}}Y\Psi \text{ are diagonal}\}$$

and the subset of real *balancing* transformations

$$\mathscr{B}(X, Y) \triangleq \{\Psi \in \mathscr{C}(X, Y)\colon \Psi^{-1}X\Psi^{-\mathrm{T}} = \Psi^{\mathrm{T}}Y\Psi\}.$$

Of course, a necessary condition for $\mathscr{B}(X, Y)$ to be nonempty is that X, Y, and XY all have the same rank. Note that in general

$$\mathscr{B}(X, Y) \subset \mathscr{C}(X, Y) \subset \mathscr{D}(XY). \tag{6.1}$$

Obviously, a diagonalizable matrix is either invertible (has no zero eigenvalues) or has semisimple zero eigenvalues. Hence if $\mathscr{D}(Z) \neq \varnothing$, then the group generalized inverse $Z^{\#}$ exists as a special case of the Drazin generalized inverse [CM]. Note that we limit our consideration to diagonalizable matrices with real eigenvalues.

Also, note that there is no assumption here that Z is symmetric. Of course, when Z is symmetric the group, Drazin, and Moore–Penrose generalized inverses coincide.

Lemma 6.1. *Let $\hat{Q}, \hat{P} \in \mathbb{N}^n$ and let $r = \operatorname{rank} \hat{Q}\hat{P}$. Then the following statements hold:*

(i) *$\hat{Q}\hat{P}$ has nonnegative eigenvalues.*

(ii) *$\mathscr{C}(\hat{Q}, \hat{P}) \neq \varnothing$.*

(iii) *$\hat{Q}\hat{P}$ is diagonalizable.*

(iv) *The $n \times n$ matrix*

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} = (\hat{Q}\hat{P})^{\#}\hat{Q}\hat{P} \tag{6.2}$$

is idempotent, i.e., τ is an oblique projection, and

$$\operatorname{rank} \tau = r. \tag{6.3}$$

(v) *There exist $G, \Gamma \in \mathbb{R}^{r \times n}$ and invertible $M \in \mathbb{R}^{r \times r}$ such that*

$$\hat{Q}\hat{P} = G^{\mathrm{T}} M \Gamma, \tag{6.4}$$

$$\Gamma G^{\mathrm{T}} = I_r. \tag{6.5}$$

(vi) *If $G, \Gamma \in \mathbb{R}^{r \times n}$ and $M \in \mathbb{R}^{r \times r}$ satisfy* (6.4) *and* (6.5), *then*

$$\operatorname{rank} G = \operatorname{rank} \Gamma = \operatorname{rank} M = r, \tag{6.6}$$

$$(\hat{Q}\hat{P})^{\#} = G^{\mathrm{T}} M^{-1} \Gamma, \tag{6.7}$$

$$\tau = G^{\mathrm{T}} \Gamma, \tag{6.8}$$

$$\tau G^{\mathrm{T}} = G^{\mathrm{T}}, \qquad \Gamma \tau = \Gamma. \tag{6.9}$$

(vii) *The matrices $G, \Gamma \in \mathbb{R}^{r \times n}$ and $M \in \mathbb{R}^{r \times r}$ satisfying* (6.4) *and* (6.5) *are unique except for a change of basis in $\mathbb{R}^r$. Furthermore, all such M are diagonalizable with positive eigenvalues.*

(viii) *If $\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = r$, then $\mathscr{B}(\hat{Q}, \hat{P}) \neq \varnothing$ and*

$$\hat{Q} = \tau \hat{Q} = \hat{Q} \tau^{\mathrm{T}} = \tau \hat{Q} \tau^{\mathrm{T}}, \tag{6.10}$$

$$\hat{P} = \tau^{\mathrm{T}} \hat{P} = \hat{P} \tau = \tau^{\mathrm{T}} \hat{P} \tau. \tag{6.11}$$

Proof. See Appendix A. ■

A triple (G, M, Γ) satisfying (6.4) and (6.5) with $G, \Gamma \in \mathbb{R}^{r \times n}$, $M \in \mathbb{R}^{r \times r}$, and $r = \operatorname{rank} \hat{Q}\hat{P}$ is called a *projective factorization* of $\hat{Q}\hat{P}$. In particular, we set $r = n_c$. Furthermore, define the complementary projection

$$\tau_{\perp} \triangleq I_n - \tau, \tag{6.12}$$

and, for arbitrary $Q, P, \hat{Q}, \hat{P} \in \mathbb{R}^{n \times n}$, $G, \Gamma \in \mathbb{R}^{n_c \times n}$, $B_c \in \mathbb{R}^{n_c \times l}$, $C_c \in \mathbb{R}^{m \times n_c}$, and $\alpha > 0$, define the following notation:

$$V_{2s} \triangleq V_2 + \sum_{i=1}^{p} \gamma_i [C_i(Q + \hat{Q})C_i^{\mathrm{T}} + D_i C_c \Gamma \hat{Q} \Gamma^{\mathrm{T}} C_c^{\mathrm{T}} D_i^{\mathrm{T}}],$$

$$R_{2s} \triangleq R_2 + \sum_{i=1}^{p} \gamma_i [B_i^{\mathrm{T}}(P + \hat{P})B_i + D_i^{\mathrm{T}} B_c^{\mathrm{T}} G \hat{P} G^{\mathrm{T}} B_c D_i],$$

$$Q_s \triangleq QC^{\mathrm{T}} + V_{12} + \sum_{i=1}^{p} \gamma_i[A_i(Q+\hat{Q})C_i^{\mathrm{T}} + A_i\hat{Q}\Gamma^{\mathrm{T}}C_c^{\mathrm{T}}D_i^{\mathrm{T}}],$$

$$P_s \triangleq B^{\mathrm{T}}P + R_{12}^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i[B_i^{\mathrm{T}}(P+\hat{P})A_i - D_i^{\mathrm{T}}B_c^{\mathrm{T}}G\hat{P}A_i],$$

$$A_Q \triangleq A_\alpha - Q_sV_{2s}^{-1}C, \qquad A_P \triangleq A_\alpha - BR_{2s}^{-1}P_s.$$

The above definitions are for convenience in stating the necessary conditions for the Auxiliary Minimization Problem. This result provides explicit formulae for extremals $(\mathcal{Q}, A_c, B_c, C_c)$ of the Auxiliary Minimization Problem. A partial converse shows that this form of the necessary conditions represents no loss of generality with regard to the constraint equation (5.5).

Theorem 6.1.

(I) *Suppose $(\mathcal{Q}, A_c, B_c, C_c) \in \mathcal{S}$ solves the Auxiliary Minimization Problem with $\mathcal{U}$ given by* (5.1) *and Ω given by* (5.3). *Then there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ such that, for some projective factorization (G, M, Γ) of $\hat{Q}\hat{P}$, $(\mathcal{Q}, A_c, B_c, C_c)$ are given by*

$$\mathcal{Q} = \begin{bmatrix} Q+\hat{Q} & \hat{Q}\Gamma^{\mathrm{T}} \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}} \end{bmatrix}, \tag{6.13}$$

$$A_c = \Gamma(A - BR_{2s}^{-1}P_s - Q_sV_{2s}^{-1}C + Q_sV_{2s}^{-1}DR_{2s}^{-1}P_s)G^{\mathrm{T}}, \tag{6.14}$$

$$B_c = \Gamma Q_sV_{2s}^{-1}, \tag{6.15}$$

$$C_c = -R_{2s}^{-1}P_sG^{\mathrm{T}}, \tag{6.16}$$

and such that Q, P, $\hat{Q}$, and $\hat{P}$ satisfy

$$0 = A_\alpha Q + QA_\alpha^{\mathrm{T}} + V_1 + \sum_{i=1}^{p} \gamma_i[A_iQA_i^{\mathrm{T}} + (A_i - B_iR_{2s}^{-1}P_s)\hat{Q}(A_i - B_iR_{2s}^{-1}P_s)^{\mathrm{T}}]$$
$$- Q_sV_{2s}^{-1}Q_s^{\mathrm{T}} + \tau_\perp Q_sV_{2s}^{-1}Q_s^{\mathrm{T}}\tau_\perp^{\mathrm{T}}, \tag{6.17}$$

$$0 = A_\alpha^{\mathrm{T}}P + PA_\alpha + R_1 + \sum_{i=1}^{p} \gamma_i[A_i^{\mathrm{T}}PA_i + (A_i - Q_sV_{2s}^{-1}C_i)^{\mathrm{T}}\hat{P}(A_i - Q_sV_{2s}^{-1}C_i)]$$
$$- P_s^{\mathrm{T}}R_{2s}^{-1}P_s + \tau_\perp^{\mathrm{T}}P_s^{\mathrm{T}}R_{2s}^{-1}P_s\tau_\perp, \tag{6.18}$$

$$0 = A_P\hat{Q} + \hat{Q}A_P^{\mathrm{T}} + Q_sV_{2s}^{-1}Q_s^{\mathrm{T}} - \tau_\perp Q_sV_{2s}^{-1}Q_s^{\mathrm{T}}\tau_\perp^{\mathrm{T}}, \tag{6.19}$$

$$0 = A_Q^{\mathrm{T}}\hat{P} + \hat{P}A_Q + P_s^{\mathrm{T}}R_{2s}^{-1}P_s - \tau_\perp^{\mathrm{T}}P_s^{\mathrm{T}}R_{2s}^{-1}P_s\tau_\perp, \tag{6.20}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c. \tag{6.21}$$

Furthermore, the auxiliary cost is given by

$$\mathcal{J}(\mathcal{Q}, A_c, B_c, C_c)$$
$$= \operatorname{tr}[(Q+\hat{Q})R_1 - 2R_{12}R_{2s}^{-1}P_s\hat{Q} + P_s^{\mathrm{T}}R_{2s}^{-1}R_2R_{2s}^{-1}P_s\hat{Q}]. \tag{6.22}$$

(II) *Conversely, if there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying* (6.17)–(6.21) *with B_c and C_c given by* (6.15) *and* (6.16), *then $(\mathcal{Q}, A_c, B_c, C_c)$ given by* (6.13)–(6.16) *satisfy* (4.2) *and* (5.5) *with $\mathcal{J}(\mathcal{Q}, A_c, B_c, C_c)$ given by* (6.22).

Proof. See Appendix B. ■

Remark 6.1. Theorem 6.1 presents necessary conditions for the Auxiliary Minimization Problem which explicitly characterize extremal quadruples $(\mathcal{Q}, A_c, B_c, C_c)$. These necessary conditions consist of a system of two modified Riccati equations and two modified Lyapunov equations coupled by both the optimal projection τ and uncertainty bounds. If the uncertainty bounds are deleted, then the resuls of [HB] are recovered.

Remark 6.2. When solving (6.17)–(6.21) numerically, the uncertainty terms can be adjusted to examine tradeoffs between performance and robustness. Specifically, the bounds α_i and the structure matrices A_i, B_i, C_i, and D_i appearing in V_{2s}, R_{2s}, Q_s, and P_s can be varied systematically to determine the region of solvability of (6.17)–(6.21).

Remark 6.3. Although (6.17)–(6.21) appear formidable, they are, in fact, quite numerically tractable. For related problems involving coupled Riccati equations, homotopic continuation methods have been shown to be effective [KLJ], [MB]. Similar algorithms for solving (6.17)–(6.21) have been developed in [GH], [R1], and [R2], while iterative algorithms are discussed in [G2], [GV], and [CY].

Remark 6.4. Because of the presence of B_c and C_c in the definitions of V_{2s}, R_{2s}, Q_s, and P_s, the optimality conditions (6.17)–(6.20) are coupled with the gain expressions (6.15) and (6.16) for B_c and C_c. When the problem is specialized to the case $D_i = 0$, $i = 1, \ldots, p$, this coupling disappears and (6.17)–(6.20) can be solved without reference to the gain expressions (6.15) and (6.16).

7. Sufficient Conditions for Robust Stability and Performance

In this section we combine Theorem 3.1 with Theorem 6.1(II) to obtain our main result guaranteeing robust stability and performance.

Theorem 7.1. *Assume there exist* Q, P, $\hat{Q}$, $\hat{P} \in \mathbb{N}^{\tilde{n}}$ *satisfying* (6.17)–(6.21) *with* B_c *and* C_c *given by* (6.15) *and* (6.16). *Then, with* $(\mathcal{Q}, A_c, B_c, C_c)$ *given by* (6.13)–(6.16), $(\tilde{A} + \Delta\tilde{A}, [\tilde{V} + \Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}})]^{1/2})$ *is stabilizable for all* $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$ *if and only if* $\tilde{A} + \Delta\tilde{A}$ *is asymptotically stable for all* $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$. *In this case, the performance* (2.7) *of the closed-loop system* (2.9) *satisfies the bound*

$$J(A_c, B_c, C_c) \leq \operatorname{tr}[(Q + \hat{Q})R_1 - 2R_{12}R_{2s}^{-1}P_s\hat{Q} + P_s^{\mathrm{T}}R_{2s}^{-1}R_2R_{2s}^{-1}P_s\hat{Q}]. \tag{7.1}$$

Proof. The converse portion to Theorem 6.1 implies that $\mathcal{Q}$ given by (6.13) is nonnegative definite and satisfies (5.5) or, equivalently, (3.4). It now follows from Theorem 3.1 that the stabilizability condition (3.5) is equivalent to the asymptotic stability of $\tilde{A} + \Delta\tilde{A}$ for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$. In this case Proposition 4.1 yields robust stability and performance. The robust performance bound (7.1) is a restatement of (4.3) utilizing (6.22). ■

Note that Theorem 7.1 is constructive in nature rather than existential. Specifically, Theorem 7.1 involves a coupled system of modified Riccati/Lyapunov equations (6.17)–(6.21) whose solutions, when they exist, are used explicitly to construct the dynamic feedback gains (6.14)–(6.16) which are guaranteed to provide both robust stability and performance. The following existence result concerns the solvability of (6.17)–(6.21). Let n_u denote the dimension of the unstable subspace of the plant dynamics matrix A.

Theorem 7.2. *Assume* $n_c \geq n_u$, $R_1 > 0$, $V_1 > 0$, *suppose the nominal plant, i.e.,* (2.1), (2.2) *with* $\alpha_i = 0$, $i = 1, \ldots, p$, *is stabilizable and detectable and, in addition, is stabilizable by means of an* n_c*th-order strictly proper dynamic compensator* (2.3), (2.4). *Then there exist* $\bar{\alpha}_1, \ldots, \bar{\alpha}_p > 0$ *such that if* $\alpha_i \in [0, \bar{\alpha}_i)$, $i = 1, \ldots, p$, *then* (6.17)–(6.21) *have a solution* $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ *for which* (A_c, B_c, C_c) *given by* (6.14)–(6.16) *solves the Robust Stability Problem with robust performance bound* (6.22).

Proof. From Theorem 3.1 of [R1] and [R2] it follows that there exists a solution to (6.17)–(6.21) which stabilizes the nominal plant. By continuity there exists a neighborhood over which robust stability with performance bound (6.22) holds. ■

Theorem 7.2 is an existence result which guarantees solvability of the sufficiency conditions over a range of parameter uncertainties. The actual range of uncertainty which can be bounded and the conservatism of the performance bound are, of course, problem dependent.

8. Specialization to Full-Order Dynamic Compensation

To draw connections with standard full-order LQG theory, we specialize the results of Sections 6 and 7 to the full-order case, i.e., $n_c = n$. As discussed in [HB], in the full-order case $G = \Gamma^{-1}$ and thus $G = \Gamma = \tau = I_n$ and $\tau_\perp = 0$ without loss of generality. To develop further connections with standard LQG theory assume

$$R_{12} = 0, \qquad V_{12} = 0, \qquad D = \Delta D = 0. \tag{8.1}$$

Since $\Delta D = 0$ we write $(\Delta A, \Delta B, \Delta C)$ in place of $(\Delta A, \Delta B, \Delta C, \Delta D)$. Also, for arbitrary $Q, P, \hat{Q}, \hat{P} \in \mathbb{R}^{n \times n}$ and $\alpha > 0$ define the following notation:

$$\hat{V}_{2s} \triangleq V_2 + \sum_{i=1}^{p} \gamma_i C_i (Q + \hat{Q}) C_i^{\mathrm{T}}, \qquad \hat{R}_{2s} \triangleq R_2 + \sum_{i=1}^{p} \gamma_i B_i^{\mathrm{T}} (P + \hat{P}) B_i,$$

$$\hat{Q}_s \triangleq Q C^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i A_i (Q + \hat{Q}) C_i^{\mathrm{T}}, \qquad \hat{P}_s \triangleq B^{\mathrm{T}} P + \sum_{i=1}^{p} \gamma_i B_i^{\mathrm{T}} (P + \hat{P}) A_i,$$

$$\hat{A}_Q \triangleq A_\alpha - \hat{Q}_s \hat{V}_{2s}^{-1} C, \qquad \hat{A}_P \triangleq A_\alpha - B \hat{R}_{2s}^{-1} \hat{P}_s.$$

Theorem 8.1. *Let* $n_c = n$, *assume* (8.1) *is satisfied, and assume there exist* $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ *satisfying*

$$0 = A_\alpha Q + Q A_\alpha^{\mathrm{T}} + V_1 + \sum_{i=1}^{p} \gamma_i [A_i Q A_i^{\mathrm{T}} + (A_i - B_i \hat{R}_{2s}^{-1} \hat{P}_s) \hat{Q} (A_i - B_i \hat{R}_{2s}^{-1} \hat{P}_s)^{\mathrm{T}}]$$
$$- \hat{Q}_s \hat{V}_{2s}^{-1} \hat{Q}_s^{\mathrm{T}}, \tag{8.2}$$

$$0 = A_\alpha^T P + PA_\alpha + R_1 + \sum_{i=1}^{p} \gamma_i [A_i^T P A_i + (A_i - \hat{Q}_s \hat{V}_{2s}^{-1} C_i)^T \hat{P}(A_i - \hat{Q}_s \hat{V}_{2s}^{-1} C_i)]$$
$$- \hat{P}_s^T \hat{R}_{2s}^{-1} \hat{P}_s, \tag{8.3}$$
$$0 = \hat{A}_P \hat{Q} + \hat{Q} \hat{A}_P^T + \hat{Q}_s \hat{V}_{2s}^{-1} \hat{Q}_s^T, \tag{8.4}$$
$$0 = \hat{A}_Q^T \hat{P} + \hat{P} \hat{A}_Q + \hat{P}_s^T \hat{R}_{2s}^{-1} \hat{P}_s, \tag{8.5}$$

and let $(\mathcal{Q}, A_c, B_c, C_c)$ *be given by*

$$\mathcal{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q} \\ \hat{Q} & \hat{Q} \end{bmatrix}, \tag{8.6}$$
$$A_c = A - B\hat{R}_{2s}^{-1}\hat{P}_s - \hat{Q}_s \hat{V}_{2s}^{-1} C, \tag{8.7}$$
$$B_c = \hat{Q}_s \hat{V}_{2s}^{-1}, \tag{8.8}$$
$$C_c = -\hat{R}_{2s}^{-1}\hat{P}_s. \tag{8.9}$$

Then, $(\tilde{A} + \Delta\tilde{A}, [\tilde{V} + \Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^T)^{1/2}])$ *is stabilizable for all* $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$ *if and only if* $\tilde{A} + \Delta\tilde{A}$ *is asymptotically stable for all* $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$. *In this case the performance of the closed-loop system* (2.9) *satisfies the bound*

$$J(A_c, B_c, C_c) \leq \operatorname{tr}[(Q + \hat{Q})R_1 + \hat{P}_s^T \hat{R}_{2s}^{-1} R_2 \hat{R}_{2s}^{-1} \hat{P}_s \hat{Q}]. \tag{8.10}$$

Proof. The proof follows from the reduced-order case given in Appendix B. ■

Remark 8.1. Theorem 8.1 presents sufficient conditions for robust stability and performance for full-order dynamic compensation. These sufficient conditions comprise a system of two modified Riccati equations and two Lyapunov equations coupled by the uncertainty bounds. This coupling illustrates the breakdown of regulator/estimator separation and shows that the certainty equivalence principle is no longer valid for the LQG result with real-valued structured plant parameter variations. If, however, the uncertainty terms A_i, B_i, C_i are set to zero, it can be seen that (8.4) and (8.5) drop out, while (8.2) and (8.3) reduce to the standard separated Riccati equations of LQG theory.

9. Illustrative Numerical Example

To demonstrate the above results we present an illustrative numerical example. The example chosen was originally used by Doyle [D] to illustrate the lack of a guaranteed gain margin for LQG controllers. This example was also considered in [BG1] for a preliminary robustness study. Define

$$n = 2, \qquad m = l = p = 1,$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = [1 \quad 0], \qquad D = 0,$$

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_1 = [0 \quad 0], \qquad D_1 = 0,$$

$$R_1 = V_1 = \begin{bmatrix} 60 & 60 \\ 60 & 60 \end{bmatrix}, \qquad R_{12} = V_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad R_2 = V_2 = 1.$$

Note that the system is open-loop unstable and becomes unstabilizable at $\sigma_1 = -1$. As can easily be seen using root locus, a strictly proper stabilizing controller must be of at least second order. Hence we consider (6.17)–(6.21) with $n_c = n$ and thus $\tau_\perp = 0$. Using algorithms described in [GH] and [R1], controllers were obtained for $(\alpha, \alpha_1) = (0.1, 0.1), (0.4, 0.2)$, and $(1.6, 0.4)$. Figure 1 compares the guaranteed robust stability region to the actual robust stability region. Note that the design approach yields greater stability than is guaranteed *a priori*. This phenomenon is not surprising since even the LQG result may provide arbitrarily high levels of robustness for particular problems while failing to guarantee even minimal robustness for all problems. These results thus demonstrate the ability of the theory to robustify the LQG result. Interestingly, the form of the actual stability region mimics the classical 6 dB downward/infinite dB upward gain margin of full-state-feedback LQR controllers. Finally, Figure 2 compares guaranteed closed-loop performance to actual closed-loop performance over the guaranteed closed-loop robust stability region. Controller gains are given in Table 1.

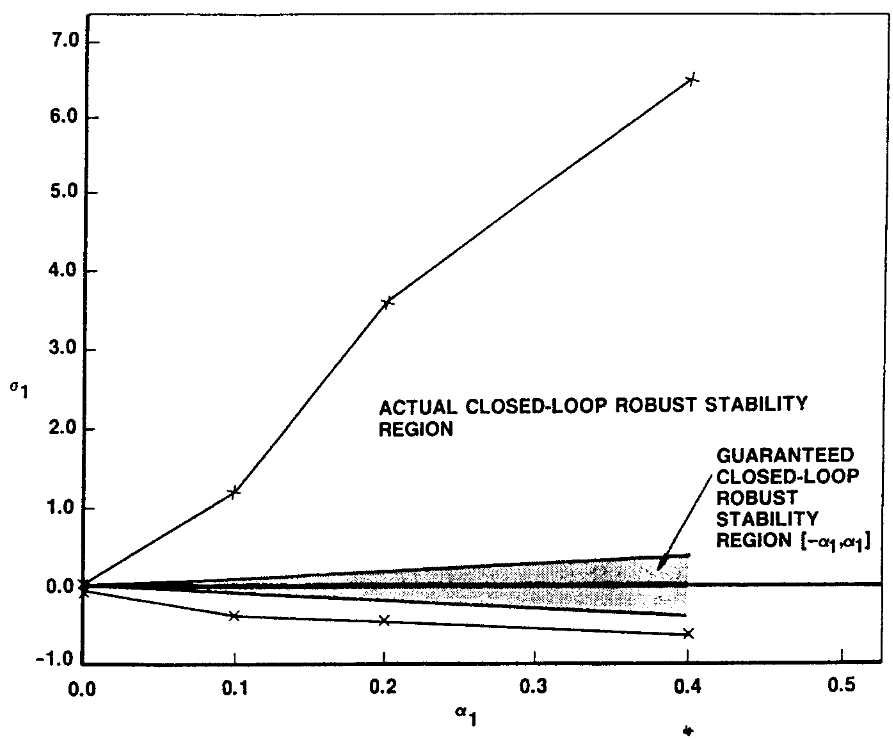

Fig 1

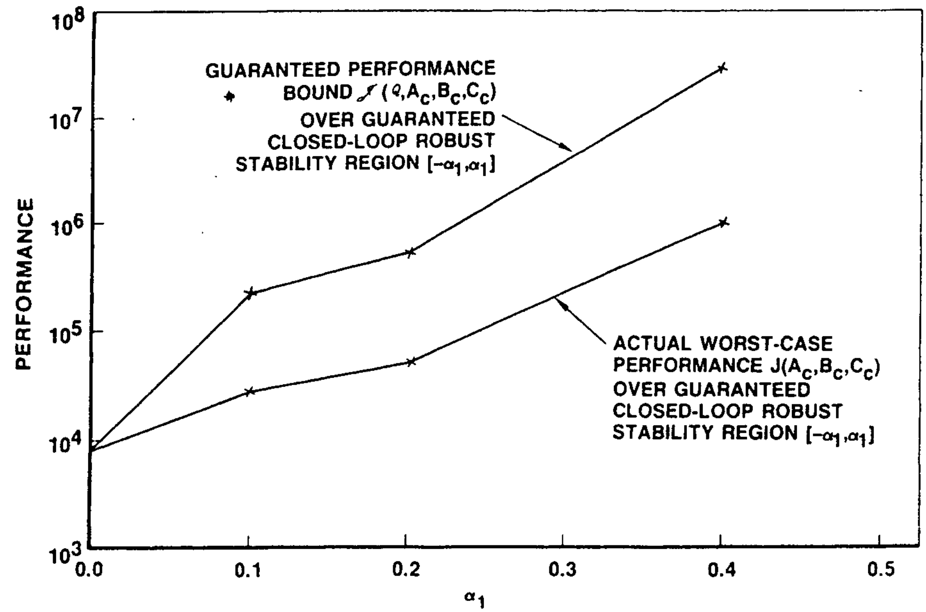

Fig. 2

Table 1

(α, α_1)	A_c	B_c	C_c
(0.1, 0.1)	$\begin{bmatrix} -14.917 & 1.0 \\ -85.177 & 3.9657 \end{bmatrix}$	$\begin{bmatrix} 15.917 \\ 79.959 \end{bmatrix}$	$[-15.2182 \quad -4.9657]$
(0.4, 0.2)	$\begin{bmatrix} -17.963 & 1.0 \\ -133.65 & -4.4614 \end{bmatrix}$	$\begin{bmatrix} 18.963 \\ 127.05 \end{bmatrix}$	$[-6.6011 \quad -5.4614]$
(1.6, 0.4)	$\begin{bmatrix} -47.813 & 1.0 \\ -1087.3 & -6.5463 \end{bmatrix}$	$\begin{bmatrix} 48.813 \\ 1073.5 \end{bmatrix}$	$[-13.766 \quad -7.5463]$

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Appendix A. Proof of Lemma 6.1

(i) Clearly $\hat{Q}\hat{P}$ and $\hat{P}^{1/2}\hat{Q}\hat{P}^{1/2}$ have the same nonzero eigenvalues. Since $\hat{P}^{1/2}\hat{Q}\hat{P}^{1/2}$ is nonnegative definite, $\hat{Q}\hat{P}$ has nonnegative eigenvalues.

(ii) The result follows from Theorem 6.2.5 of [RM], p. 123. See also Theorem 4.3 of [G1].

(iii) This result follows from (ii) and (6.1).

(iv) This result follows from the definition of the group generalized inverse (see [CM]). Alternatively, let $\hat{Q}\hat{P} = \Psi D \Psi^{-1}$, where $\Psi \in \mathscr{D}(\hat{Q}\hat{P})$, $D = \operatorname{diag}(d_1, \ldots, d_n)$. Then $(\hat{Q}\hat{P})^{\#} = \Psi D^{\#} \Psi^{-1}$, where $D^{\#}_{(i,i)} = 1/d_i$ if $d_i \neq 0$, and $D^{\#}_{(i,i)} = 0$, if $d_i = 0$, $i = 1, \ldots, n$. Hence $\hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} = \Psi E \Psi^{-1}$ is idempotent, where E is a diagonal matrix with r ones and $n - r$ zeros on the diagonal. Clearly, (6.3) is valid.

(v) Without loss of generality choose Ψ in the preceding argument so that $D = \text{block-diag}(\hat{D}, 0_{n-r})$, where $\hat{D} = \operatorname{diag}(\hat{d}_1, \ldots, \hat{d}_r)$, $\hat{d}_i > 0$, $i = 1, \ldots, r$. Hence

$$\hat{Q}\hat{P} = \Psi \begin{bmatrix} \hat{D} & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{n-r} \end{bmatrix} \Psi^{-1},$$

and thus (6.5) holds with

$$G = [I_r \quad 0_{r \times (n-r)}]\Psi^{\mathrm{T}}, \qquad M = \hat{D}, \qquad \Gamma = [I_r \quad 0_{r \times (n-r)}]\Psi^{-1}.$$

(vi) Sylvester's inequality and (6.4) imply that

$$r = \operatorname{rank} \hat{Q}\hat{P} \leq \{\operatorname{rank} G, \operatorname{rank} M, \operatorname{rank} \Gamma\} \leq r,$$

which yields (6.6). The expression (6.7) for $(\hat{Q}\hat{P})^{\#}$ follows directly from the definition of the group generalized inverse. Furthermore, (6.2), (6.5), and (6.7) imply (6.8), while (6.5) and (6.8) imply (6.9).

(vii) Let both (G, M, Γ) and an identically dimensioned triple $(\hat{G}, \hat{M}, \hat{\Gamma})$ satisfy (6.4). Then it is easy to verify that $\hat{G} = S^{-1}G$, $\hat{M} = SMS^{-1}$, and $\hat{\Gamma} = S\Gamma$, where $S = \hat{\Gamma}G^{\mathrm{T}}$ and $S^{-1} = \Gamma\hat{G}^{\mathrm{T}}$.

(viii) It follows from (ii) that there exists $\Psi \in \mathscr{C}(\hat{Q}, \hat{P})$ such that

$$\hat{Q} = \Psi \begin{bmatrix} D_{\hat{Q}} & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{n-r} \end{bmatrix} \Psi^{\mathrm{T}}, \qquad \hat{P} = \Psi^{-\mathrm{T}} \begin{bmatrix} D_{\hat{P}} & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{n-r} \end{bmatrix} \Psi^{-1},$$

where $D_{\hat{Q}}$ and $D_{\hat{P}}$ are positive diagonal. Define

$$\hat{\Psi} = \Psi \begin{bmatrix} (D_{\hat{Q}} D_{\hat{P}}^{-1})^{1/4} & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & I_{n-r} \end{bmatrix}$$

so that

$$\hat{\Psi}^{-1}\hat{Q}\hat{\Psi}^{-\mathrm{T}} = \hat{\Psi}^{\mathrm{T}}\hat{P}\hat{\Psi} = \begin{bmatrix} (D_{\hat{Q}} D_{\hat{P}})^{1/2} & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{n-r} \end{bmatrix}$$

and thus $\hat{\Psi} \in \mathscr{B}(\hat{Q}, \hat{P})$. Finally, (6.10) and (6.11) are immediate. ■

Appendix B. Proof of Theorem 6.1

To optimize (4.10) over the open set $\mathscr{S}$ subject to the constraint (5.5), form the Lagrangian

$$\mathscr{L}(\mathscr{Q}, A_c, B_c, C_c, \mathscr{P}, \lambda) \triangleq \operatorname{tr}\left\{\lambda \mathscr{Q}\tilde{R} + [\tilde{A}_a \mathscr{Q} + \mathscr{Q}\tilde{A}_a^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i \tilde{A}_i \mathscr{Q} \tilde{A}_i^{\mathrm{T}} + \tilde{V}]\mathscr{P}\right\}, \quad \text{(B.1)}$$

where the Lagrange multipliers $\lambda \geq 0$ and $\mathscr{P} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ are not both zero. We thus obtain

$$\frac{\partial \mathscr{L}}{\partial \mathscr{Q}} = \tilde{A}_\alpha^{\mathrm{T}}\mathscr{P} + \mathscr{P}\tilde{A}_\alpha + \sum_{i=1}^{p} \gamma_i \tilde{A}_i^{\mathrm{T}}\mathscr{P}\tilde{A}_i + \lambda\tilde{R}. \tag{B.2}$$

Setting $\partial\mathscr{L}/\partial\mathscr{Q} = 0$ yields

$$0 = \tilde{A}_\alpha^{\mathrm{T}}\mathscr{P} + \mathscr{P}\tilde{A}_\alpha + \sum_{i=1}^{p} \gamma_i \tilde{A}_i^{\mathrm{T}}\mathscr{P}\tilde{A}_i + \lambda\tilde{R} \tag{B.3}$$

or, equivalently,

$$\mathscr{A}^{\mathrm{T}} \operatorname{vec} \mathscr{P} = -\lambda \operatorname{vec} \tilde{R}.$$

Since $\mathscr{A}$ is assumed to be stable, $\mathscr{A}^{\mathrm{T}}$ is invertible, and thus $\lambda = 0$ implies $\mathscr{P} = 0$. Hence, it can be assumed without loss of generality that $\lambda = 1$. Furthermore, it follows from Proposition 5.2 with $\mathscr{A}$, $\tilde{V}$ replaced by $\mathscr{A}^{\mathrm{T}}$, $\tilde{R}$ that $\mathscr{P}$ is nonnegative definite.

Now partition $\tilde{n} \times \tilde{n}$ $\mathscr{Q}, \mathscr{P}$ into $n \times n$, $n \times n_{\mathrm{c}}$, and $n_{\mathrm{c}} \times n_{\mathrm{c}}$ subblocks as

$$\mathscr{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2 \end{bmatrix}, \qquad \mathscr{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix},$$

and define the positive-definite matrices

$$V_{2s} \triangleq V_2 + \sum_{i=1}^{p} \gamma_i [C_i Q_1 C_i^{\mathrm{T}} + D_i C_{\mathrm{c}} Q_2 C_{\mathrm{c}}^{\mathrm{T}} D_i^{\mathrm{T}}],$$

$$R_{2s} \triangleq R_2 + \sum_{i=1}^{p} \gamma_i [B_i^{\mathrm{T}} P_1 B_i + D_i^{\mathrm{T}} B_{\mathrm{c}}^{\mathrm{T}} P_2 B_{\mathrm{c}} D_i].$$

Thus, the stationarity conditions for A_{c}, B_{c}, C_{c} are given by

$$\frac{\partial \mathscr{L}}{\partial A_{\mathrm{c}}} = P_{12}^{\mathrm{T}} Q_{12} + P_2 Q_2 = 0, \tag{B.4}$$

$$\frac{\partial \mathscr{L}}{\partial B_{\mathrm{c}}} = P_2 B_{\mathrm{c}} V_{2s} + (P_{12}^{\mathrm{T}} Q_1 + P_2 Q_{12}^{\mathrm{T}}) C^{\mathrm{T}} + P_{12}^{\mathrm{T}}\left[V_{12} + \sum_{i=1}^{p} \gamma_i (A_i Q_1 C_i^{\mathrm{T}} + A_i Q_{12} C_{\mathrm{c}}^{\mathrm{T}} D_i^{\mathrm{T}}) \right] = 0, \tag{B.5}$$

$$\frac{\partial \mathscr{L}}{\partial C_{\mathrm{c}}} = R_{2s} C_{\mathrm{c}} Q_2 + B^{\mathrm{T}}(P_1 Q_{12} + P_{12} Q_2) + \left[R_{12}^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i (B_i^{\mathrm{T}} P_1 A_i + D_i^{\mathrm{T}} B_{\mathrm{c}}^{\mathrm{T}} P_{12}^{\mathrm{T}} A_i) \right] Q_{12} = 0. \tag{B.6}$$

Expanding (5.5) and (B.3) yields

$$0 = A_\alpha Q_1 + Q_1 A_\alpha^{\mathrm{T}} + B C_{\mathrm{c}} Q_{12}^{\mathrm{T}} + Q_{12} C_{\mathrm{c}}^{\mathrm{T}} B^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i [A_i Q_1 A_i^{\mathrm{T}} + B_i C_{\mathrm{c}} Q_{12}^{\mathrm{T}} A_i^{\mathrm{T}} + A_i Q_{12} C_{\mathrm{c}}^{\mathrm{T}} B_i^{\mathrm{T}} + B_i C_{\mathrm{c}} Q_2 C_{\mathrm{c}}^{\mathrm{T}} B_i^{\mathrm{T}}] + V_1, \tag{B.7}$$

$$0 = A_\alpha Q_{12} + Q_{12}A_{c\alpha}^{\mathrm{T}} + Q_{12}C_c^{\mathrm{T}}D^{\mathrm{T}}B_c^{\mathrm{T}} + Q_1C^{\mathrm{T}}B_c^{\mathrm{T}} + BC_cQ_2 + \sum_{i=1}^{p}\gamma_i[A_iQ_1C_i^{\mathrm{T}}B_c^{\mathrm{T}} + A_iQ_{12}C_c^{\mathrm{T}}D_i^{\mathrm{T}}B_c^{\mathrm{T}}] + V_{12}B_c^{\mathrm{T}}, \tag{B.8}$$

$$0 = A_{c\alpha}Q_2 + Q_2A_{c\alpha}^{\mathrm{T}} + B_cCQ_{12} + Q_{12}^{\mathrm{T}}C^{\mathrm{T}}B_c^{\mathrm{T}} + B_cDC_cQ_2 + Q_2C_c^{\mathrm{T}}D^{\mathrm{T}}B_c^{\mathrm{T}} + B_cV_{2s}B_c^{\mathrm{T}}, \tag{B.9}$$

$$0 = A_\alpha^{\mathrm{T}}P_1 + P_1A_\alpha + C^{\mathrm{T}}B_c^{\mathrm{T}}P_{12}^{\mathrm{T}} + P_{12}B_cC + \sum_{i=1}^{p}\gamma_i[A_i^{\mathrm{T}}P_1A_i + C_i^{\mathrm{T}}B_c^{\mathrm{T}}P_{12}^{\mathrm{T}}A_i + A_i^{\mathrm{T}}P_{12}B_cC_i + C_i^{\mathrm{T}}B_c^{\mathrm{T}}P_2B_cC_i] + R_1, \tag{B.10}$$

$$0 = A_\alpha^{\mathrm{T}}P_{12} + P_{12}A_{c\alpha} + P_{12}B_cDC_c + P_1BC_c + C^{\mathrm{T}}B_c^{\mathrm{T}}P_2 + \sum_{i=1}^{p}\gamma_i[A_i^{\mathrm{T}}P_1B_iC_c + A_i^{\mathrm{T}}P_{12}B_cD_iC_c] + R_{12}C_c, \tag{B.11}$$

$$0 = A_{c\alpha}^{\mathrm{T}}P_2 + P_2A_{c\alpha} + C_c^{\mathrm{T}}B^{\mathrm{T}}P_{12} + P_{12}^{\mathrm{T}}BC_c + C_c^{\mathrm{T}}D^{\mathrm{T}}B_c^{\mathrm{T}}P_2 + P_2B_cDC_c + C_c^{\mathrm{T}}R_{2s}C_c. \tag{B.12}$$

Lemma B.1. *Q_2 and P_2 are positive definite.*

Proof. By a minor extension of results from [A], (B.9) can be rewritten as

$$0 = (A_{c\alpha} + B_cDC_c + B_cCQ_{12}Q_2^{+})Q_2 + Q_2(A_{c\alpha} + B_cDC_c + B_cCQ_{12}Q_2^{+})^{\mathrm{T}} + B_cV_{2s}B_c^{\mathrm{T}},$$

where Q_2^{+} is the Moore–Penrose or Drazin generalized inverse of Q_2. Next note that since $(A_{c\alpha}, B_c)$ is controllable then, by Theorem 3.6 of [W], $(A_{c\alpha} + B_cDC_c + B_cCQ_{12}Q_2^{+}, B_cV_{2s}^{1/2})$ is also controllable. Now, since Q_2 and $B_cV_{2s}B_c^{\mathrm{T}}$ are nonnegative definite, it follows from Lemma 12.2 of [W], that Q_2 is positive definite. Using (B.12), similar arguments show that P_2 is positive definite. ■

Since R_{2s}, V_{2s}, Q_2, and P_2 are invertible, (B.4)–(B.6) can be written as

$$-P_2^{-1}P_{12}^{\mathrm{T}}Q_{12}Q_2^{-1} = I_{n_c}, \tag{B.13}$$

$$B_c = -P_2^{-1}\left\{(P_{12}^{\mathrm{T}}Q_1 + P_2Q_{12}^{\mathrm{T}})C^{\mathrm{T}} + P_{12}^{\mathrm{T}}\left[V_{12} + \sum_{i=1}^{p}\gamma_i(A_iQ_1C_i^{\mathrm{T}} + A_iQ_{12}C_c^{\mathrm{T}}D_i^{\mathrm{T}})\right]\right\}V_{2s}^{-1}, \tag{B.14}$$

$$C_c = -R_{2s}^{-1}\left\{B^{\mathrm{T}}(P_1Q_{12} + P_{12}Q_2) + \left[R_{12}^{\mathrm{T}} + \sum_{i=1}^{p}\gamma_i(B_i^{\mathrm{T}}P_1A_i + D_i^{\mathrm{T}}B_c^{\mathrm{T}}P_{12}^{\mathrm{T}}A_i)\right]Q_{12}\right\}Q_2^{-1}. \tag{B.15}$$

Now define the $n \times n$ matrices

$$Q \triangleq Q_1 - Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \qquad P \triangleq P_1 - P_{12}P_2^{-1}P_{12}^{\mathrm{T}},$$

$$\hat{Q} \triangleq Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \qquad \hat{P} \triangleq P_{12}P_2^{-1}P_{12}^{\mathrm{T}},$$

$$\tau \triangleq -Q_{12}Q_2^{-1}P_2^{-1}P_{12}^{\mathrm{T}},$$

and the $n_c \times n$, $n_c \times n_c$, and $n_c \times n$ matrices

$$G \triangleq Q_2^{-1} Q_{12}^{\mathrm{T}}, \qquad M \triangleq Q_2 P_2, \qquad \Gamma \triangleq -P_2^{-1} P_{12}^{\mathrm{T}}.$$

Note that $\tau = G^{\mathrm{T}}\Gamma$.

Clearly, Q, P, $\hat{Q}$, and $\hat{P}$ are symmetric and $\hat{Q}$ and $\hat{P}$ are nonnegative definite. To show that Q and P are also nonnegative definite, note that Q is the upper left-hand block of the nonnegative-definite matrix $\tilde{\mathcal{Q}}\mathcal{Q}\tilde{\mathcal{Q}}^{\mathrm{T}}$, where

$$\tilde{\mathcal{Q}} = \begin{bmatrix} I_n & -Q_{12}Q_2^{-1} \\ 0_{n_c \times n} & I_{n_c} \end{bmatrix}.$$

Similarly, P is nonnegative definite.

Next note that with the above definitions, (B.13) is equivalent to (6.5) and that (6.4) holds. Hence $\tau = G^{\mathrm{T}}\Gamma$ is idempotent, i.e., $\tau^2 = \tau$. Furthermore, it is helpful to note the identities

$$\hat{Q} = Q_{12}G = G^{\mathrm{T}}Q_{12}^{\mathrm{T}} = G^{\mathrm{T}}Q_2G, \qquad \hat{P} = -P_{12}\Gamma = -\Gamma^{\mathrm{T}}P_{12}^{\mathrm{T}} = \Gamma^{\mathrm{T}}P_2\Gamma, \quad \text{(B.16)}$$

$$\tau G^{\mathrm{T}} = G^{\mathrm{T}}, \qquad \Gamma\tau = \Gamma, \quad \text{(B.17)}$$

$$\hat{Q} = \tau\hat{Q}, \qquad \hat{P} = \hat{P}\tau, \quad \text{(B.18)}$$

$$\hat{Q}\hat{P} = -Q_{12}P_{12}^{\mathrm{T}}. \quad \text{(B.19)}$$

Using (B.13) and Sylvester's inequality, it follows that

$$\operatorname{rank} G = \operatorname{rank} \Gamma = \operatorname{rank} Q_{12} = \operatorname{rank} P_{12} = n_c.$$

Now using (B.16) and Sylvester's inequality yields

$$n_c = \operatorname{rank} Q_{12} + \operatorname{rank} G - n_c \le \operatorname{rank} \hat{Q} \le \operatorname{rank} Q_{12} = n_c,$$

which implies that rank $\hat{Q} = n_c$. Similarly, rank $\hat{P} = n_c$, and rank $\hat{Q}\hat{P} = n_c$ follows from (B.19). The components of $\mathcal{Q}$ and $\mathcal{P}$ can be written in terms of Q, P, $\hat{Q}$, $\hat{P}$, G, and Γ as

$$Q_1 = Q + \hat{Q}, \qquad P_1 = P + \hat{P}, \quad \text{(B.20)}$$

$$Q_{12} = \hat{Q}\Gamma^{\mathrm{T}}, \qquad P_{12} = -\hat{P}G^{\mathrm{T}}, \quad \text{(B.21)}$$

$$Q_2 = \Gamma\hat{Q}\Gamma^{\mathrm{T}}, \qquad P_2 = G\hat{P}G^{\mathrm{T}}. \quad \text{(B.22)}$$

The expressions (6.13), (6.15), and (6.16) follow from the definition of $\mathcal{Q}$, (B.14) and (B.15). Substituting (B.20)–(B.22) into (B.7)–(B.12) yields

$$0 = A_\alpha Q + QA_\alpha^{\mathrm{T}} + V_1 + \sum_{i=1}^{p} \gamma_i[A_iQA_i^{\mathrm{T}} + (A_i - B_iR_{2s}^{-1}P_s)\hat{Q}(A_i - B_iR_{2s}^{-1}P_s)^{\mathrm{T}}] + A_P\hat{Q} + \hat{Q}A_P^{\mathrm{T}}, \quad \text{(B.23)}$$

$$0 = [A_P\hat{Q} + \hat{Q}(G^{\mathrm{T}}A_{c\alpha}\Gamma + Q_sV_{2s}^{-1}C)^{\mathrm{T}} + Q_sV_{2s}^{-1}Q_s^{\mathrm{T}}]\Gamma^{\mathrm{T}}, \quad \text{(B.24)}$$

$$0 = \Gamma[(G^{\mathrm{T}}A_{c\alpha}\Gamma + Q_sV_{2s}^{-1}C)\hat{Q} + \hat{Q}(G^{\mathrm{T}}A_{c\alpha}\Gamma + Q_sV_{2s}^{-1}C)^{\mathrm{T}} + Q_sV_{2s}^{-1}Q_s^{\mathrm{T}}]\Gamma^{\mathrm{T}}, \quad \text{(B.25)}$$

$$0 = A_\alpha^{\mathrm{T}}P + PA_\alpha + R_1 + \sum_{i=1}^{p} \gamma_i[A_i^{\mathrm{T}}PA_i + (A_i - Q_sV_{2s}^{-1}C_i)^{\mathrm{T}}\hat{P}(A_i - Q_sV_{2s}^{-1}C_i)] + A_Q^{\mathrm{T}}\hat{P} + \hat{P}A_Q, \quad \text{(B.26)}$$

$$0 = [A_Q^{\mathrm{T}}\hat{P} + \hat{P}(G^{\mathrm{T}}A_{ca}\Gamma + BR_{2s}^{-1}P_s) + P_s^{\mathrm{T}}R_{2s}^{-1}P_s]G^{\mathrm{T}}, \tag{B.27}$$

$$0 = G[(G^{\mathrm{T}}A_{ca}\Gamma + BR_{2s}^{-1}P_s)^{\mathrm{T}}\hat{P} + \hat{P}(G^{\mathrm{T}}A_{ca}\Gamma + BR_{2s}^{-1}P_s) + P_s^{\mathrm{T}}R_{2s}^{-1}P_s]G^{\mathrm{T}}. \tag{B.28}$$

Next, computing either Γ(B.24)–(B.25) or G(B.27)–(B.28) yields (6.14). Substituting this expression for A_c into (B.23), (B.24), (B.27), and (B.28) it follows that (B.25) = Γ(B.24) and (B.28) = G(B.27). Thus, (B.25) and (B.28) are superfluous and can be omitted. Next, using (B.23) + $G^{\mathrm{T}}\Gamma$(B.24)G − (B.24)G − [(B.24)G]$^{\mathrm{T}}$ and $G^{\mathrm{T}}\Gamma$(B.24)G − (B.24)G − [(B.24)G]$^{\mathrm{T}}$ yields (6.17) and (6.19). Using (B.26) + $\Gamma^{\mathrm{T}}G$(B.27)Γ − (B.27)Γ − [(B.27)Γ]$^{\mathrm{T}}$ and $\Gamma^{\mathrm{T}}G$(B.27)Γ − (B.27)Γ − [(B.27)Γ]$^{\mathrm{T}}$ yields (6.18) and (6.20).

Finally, to prove the converse we use (6.13)–(6.21) to obtain (5.5) and (B.3)–(B.6). Let $A_c, B_c, C_c, G, \Gamma, \tau, Q, P, \hat{Q}, \hat{P}, \mathcal{Q}$ be as in the statement of Theorem 6.1 and define $Q_1, Q_{12}, Q_2, P_1, P_{12}, P_2$ by (B.20)–(B.22). Using (6.5), (6.15), and (6.16), it is easy to verify (B.5), (B.6). Finally, substitute the definitions of Q, P, $\hat{Q}$, $\hat{P}$, G, and τ into (6.17)–(6.20), reverse the steps taken earlier in the proof, and use (6.13)–(6.16) along with (6.5) and (6.8)–(6.11) to obtain (5.5) and (B.3). Finally, note that

$$\mathcal{Q} = \begin{bmatrix} Q & 0_{n \times n_c} \\ 0_{n_c \times n} & 0_{n_c} \end{bmatrix} + \begin{bmatrix} I_n \\ \Gamma \end{bmatrix} \hat{Q} [I_n \quad \Gamma^{\mathrm{T}}],$$

which show that $\mathcal{Q} \geq 0$ thus verifying (4.2). ∎

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