TABLE VI
STATISTICS FOR 800 RANDOM PROBLEM

Estimate computed by	Upper triangular random problems (500) $0 < \Phi_l \cdot \sigma_{min}(Z) \leq 1$ $0.7 \cdot 10^{-11} \leq \sigma_{min}(Z) \leq 0.2 \cdot 10^{-1}$ min	mean	max	Upper quasi-triangular random problems (300) $0 < \Phi_l \cdot \sigma_{min}(Z) \leq 1$ $0.2 \cdot 10^{-7} \leq \sigma_{min}(Z) \leq 0.3 \cdot 10^{-1}$ min	mean	max
Bsolve	0.012	0.190	0.600	0.018	0.197	0.571
Bsolvt	0.008	0.191	0.543	0.015	0.214	0.535
Bsolvc	0.012	0.225	0.515	0.027	0.328	0.708
Bsolvd	0.012	0.225	0.515	0.024	0.325	0.696

show that there are examples where each one of the four estimators wins the contest. That our dif^{-1}-estimators do not always compute a good estimate is illustrated with the following example [2]:

$$A = \begin{bmatrix} 2 & 0 & k & -k \\ 0 & 2 & -k & k \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad B = D = E = I_4.$$

In this example the coefficient matrices of the subsystems (2.5) are all well-conditioned, but Z in (3.3) is not ($\|\Phi^{-1}\| = \sigma_{min}^{-1}(Z)$ is of order $O(k)$ for k large). The ill-conditioning of Z is due to large off-diagonal elements of A, and is only mirrored by our estimators if f in (5.5) is correspondingly large. Here f is $O(1)$ for all subsystems, so our estimators do not signal the ill-conditioning of Z. The global look-ahead algorithm (see [9]) would probably resolve this counterexample. Notice that if we let all nonzero off-diagonal elements of A be $+k$, then $\sigma_{min}^{-1}(Z)$ is still of order $O(k)$ for k large, and now the four dif^{-1}-estimators signal the ill-conditioning of this problem within a factor 10.

VIII. Conclusions

We have presented stable algorithms for solving the generalized Sylvester equation. They are based on orthogonal equivalence transformations of the original problem. During this work we became aware of the paper by Chu [3], where he proposes a method similar to our generalized Schur method. However, he only outlines the main steps of an algorithm without any perturbation of rounding error analysis. We have also presented dif^{-1}-estimators [lower bounds for ($\|\Phi^{-1}\| = \sigma_{min}^{-1}(Z)$] that are incorporated into the generalized Schur algorithm, and which when substituted into the error bound (4.2) produce reliable accuracy estimates of a computed solution. Our heuristic methods to compute lower bounds for dif^{-1} need $O(m^2n + mn^2)$ flops. This should be compared to $O(10m^3n^3)$ flops for computing the exact value of $dif^{-1}(=\sigma_{min}^{-1}(Z))$.

The problem to compute a stable deflating subspace of a matrix pencil has many applications in systems and control theory (e.g., [9:1], [9:4], [11], [15]–[16]). Given an upper block-triangular regular matrix pencil $M - \lambda N$ (1.2) the algorithms and Fortran routines in [10] can be used to compute a lower bound for dif^{-1}, the quantity that determines the sensitivity of the deflating subspace of $M - \lambda N$ (1.2) spanned by the first m columns of the identity matrix I_{m+n}.

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Robust Stability and Performance Analysis for Linear Dynamic Systems

DENNIS S. BERNSTEIN AND WASSIM M. HADDAD

***Abstract*—In a recent paper Zhou and Khargonekar obtained sufficient conditions for robust stability over specified sets of matrix perturbations. In the present note these results are extended to include, in addition, performance bounds. Here performance is defined as the worst-case expected value of a quadratic functional involving the state variables when the system is subjected to white noise disturbances. The results are**

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D. S. Bernstein is with Harris Corporation, Government Aerospace Systems Division, Melbourne, FL 32902.
W. M. Haddad is with the Department of Mechanical and Aerospace Engineering, Florida Institute of Technology, Melbourne, FL 32901.
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illustrated by considering the gain margin of both an LQG controller and a robustified design obtained by Bernstein and Greeley for Doyle's example.

I. Introduction

It is well known that unavoidable discrepancies between mathematical models and real-world systems can result in the degradation of control-system performance. Ideally, feedback control systems should be designed to be *robust* with respect to uncertainties in the plant characteristics. Thus, robustness *analysis* must play a key role in control-system design. That is, given an existing or proposed control system, determine the performance degradation due to variations in the plant. The most fundamental concern in this regard is clearly that of stability. For linear state-space systems with which the present note is concerned, this problem has received increasing attention over the past several years (see, e.g., [1]-[2]).

One of the principal techniques used to assess robust stability is based upon quadratic Lyapunov functions (see [1]-[4], [10]). Quadratic Lyapunov functions have also been used extensively for robust control-system *synthesis*; see [13] for relevant references. The problem of robust synthesis is, however, beyond the scope of the present note.

In addition to assessing robust stability, it is often desirable to quantify performance by considering the degradation of a cost functional as the plant parameters deviate from their nominal values. Although any robustly stable system over a compact set of parameters possesses a worst-case performance, it is desirable in practice to actually determine a bound for the worst-case performance. The concern for *both* robust stability and performance goes back to the early work of Michael and Merriam [14], while more recent references include the work of Chang and Peng [15], Noldus [16], and Petersen [17]. The results of [15]-[17] can be shown to depend upon a modified Lyapunov equation of the form

$$0=AQ+QA^T+\hat{\Omega}(Q)+V \tag{1.1}$$

where the operator $\hat{\Omega}(Q)$ is chosen to bound terms of the form $\Delta AQ + Q\Delta A^T$, where ΔA is an uncertain perturbation of the dynamics matrix A. Since robust performance per se was not discussed in [16], [17], the work most closely related to the present note is that of Chang and Peng [15]. They essentially show that consideration of (1.1) leads to a bound on worst-case performance. Although the development in [15] was carried out for full-state feedback, specialization of their approach to robust performance analysis is straightforward. A systematic, in-depth treatment of robust performance analysis involving the approach of [15] as well as other bounds is given in [18].

The starting point for the present note is the recent paper by Zhou and Khargonekar [10]. By analyzing the Lyapunov equation they obtain a series of stability robustness tests which improve significantly upon earlier work [2]-[4]. In the present note we extend the results of [10] to obtain, in addition, a bound on worst-case performance. As in (1.1) we consider a Lyapunov equation of the form

$$0=AQ+QA^T+\Omega+V \tag{1.2}$$

where Ω bounds uncertainty terms of the form $\Delta AQ + Q\Delta A^T$. The principal difference between (1.1) and (1.2) is that Ω in (1.2) is a constant matrix independent of the solution Q. The case considered in [15] in which Ω is a function of Q is discussed in [18].

The cost functional used in the present note to quantify robust performance is the trace of the output covariance of a system subjected to white noise disturbances. This measure of performance is identical in form to the standard performance criterion of LQG theory. Since we also obtain a bound for the state covariance matrix, our results yield bounds on the variances (mean square response levels) of system states. Although the results of [15] were obtained within a deterministic setting, it is easy to see that the performance criterion of [15] is also of this form.

The contents of the note are as follows. After introducing notation at the end of this section we consider the robust stability and performance problems in Section II. In Section III we present the main result (Theorem 3.1) which provides sufficient conditions for robust stability over a set of parameter variations along with a performance bound. In Section IV we present a dual result (Theorem 4.1) in terms of the dual matrix P. This result serves two purposes. First, it clarifies connections with the previous literature where results are presented in terms of the quadratic Lyapunov function $V(x) = x^TPx$. And, second, we show that the dual performance bound may be much better than the primal bound (and vice versa) for particular problems. The results of Theorems 3.1 and 4.1 are given in terms of a robustness set $\mathfrak{U}$ which is a subset of a maximal set $\tilde{\mathfrak{U}}$. Since $\tilde{\mathfrak{U}}$ is defined implicitly, we provide explicit characterizations of subsets $\mathfrak{U}$ in Section V. Here we restate the principal results of [2]-[4], [10] which, in our context, correspond to particular characterizations of subsets of $\tilde{\mathfrak{U}}$. We also introduce an additional subset of $\tilde{\mathfrak{U}}$ which provides a new robust stability result. Finally, in Section VI we consider a pair of illustrative examples. The first example, which was previously considered in [10], involves two uncertain parameters. It is shown that the new guaranteed stability region is considerably larger for certain parameter values than the regions given in [10] (see Fig. 1). Furthermore, we obtain a robust performance bound, **a result which has no counterpart in [10]**. The second example involves controllers for a second-order open-loop unstable plant originally considered in [19] to demonstrate the lack of a guaranteed stability margin for LQG controllers. We apply Theorems 3.1 and 4.1 to analyze both the LQG design and a robustified design obtained in [20]. We show that the new robust stability test is effective in the sense that the *guaranteed* gain margin for the robustified controller is a factor of 5 larger than the *actual* gain margin of the LQG design.

Notation

Note: All matrices have real entries

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$, $\mathbb{E}$	Real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expectation
I_r	$r \times r$ identity matrix
Asymptotically stable matrix	Matrix with eigenvalues in the open left-half plane
$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices
$Z_1 \geq Z_2$, $Z_1 > Z_2$	$Z_1 - Z_2 \in \mathbb{N}^r$, $Z_1 - Z_2 \in \mathbb{P}^r$, Z_1, $Z_2 \in \mathbb{S}^r$
tr Z, Z^T, co	Trace of Z, transpose of Z, convex hull
$\lambda_{\min}(Z)$, $\lambda_{\max}(Z)$	Smallest and largest eigenvalues of $Z \in \mathbb{S}^r$
$\Vert Z\Vert_s$	Spectral norm
$Z_{(i,j)}$	(i, j) element of matrix Z
$Z \geq\geq 0$	$Z_{(i,j)} \geq 0$, $i, j = 1, \cdots, r$, $Z \in \mathbb{R}^{r\times r}$
$Z \gg 0$	$Z_{(i,j)} > 0$, $i, j = 1, \cdots, r$, $Z \in \mathbb{R}^{r\times r}$
$\vert Z\vert_m$	$\{\vert Z_{(i,j)}\vert\}_{i,j=1}^r$, $Z \in \mathbb{R}^{r\times r}$ (matrix modulus).

II. Robust Stability and Performance Problems

Let $\mathfrak{U} \subset \mathbb{R}^{n\times n}$ denote a set of perturbations ΔA of the nominal dynamics matrix A. Throughout the note it is assumed that A is asymptotically stable. We begin by considering the question of whether or not $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathfrak{U}$.

Robust Stability Problem: Determine whether the linear system

$$\dot{x}(t)=(A+\Delta A)x(t), \qquad t \in [0, \infty) \tag{2.1}$$

is asymptotically stable for all $\Delta A \in \mathfrak{U}$.

The problem of robust performance involves a quadratic form $x^T(t)Rx(t)$, where $R \in \mathbb{N}^n$, when the system is subjected to a white noise disturbance $w(t)$ with nonnegative-definite intensity V. The matrix R can be viewed as a means for selecting output variables of interest while the matrix V can be used to specify disturbance levels.

Robust Performance Problem: For the disturbed linear system

$$\dot{x}(t)=(A+\Delta A)x(t)+w(t), \qquad t \in [0, \infty) \tag{2.2}$$

determine a performance bound β satisfying

$$J(\mathfrak{U}) \triangleq \sup_{\Delta A\in\mathfrak{U}} \limsup_{t\to\infty} \mathbb{E}[x^T(t)Rx(t)] \leq \beta. \tag{2.3}$$

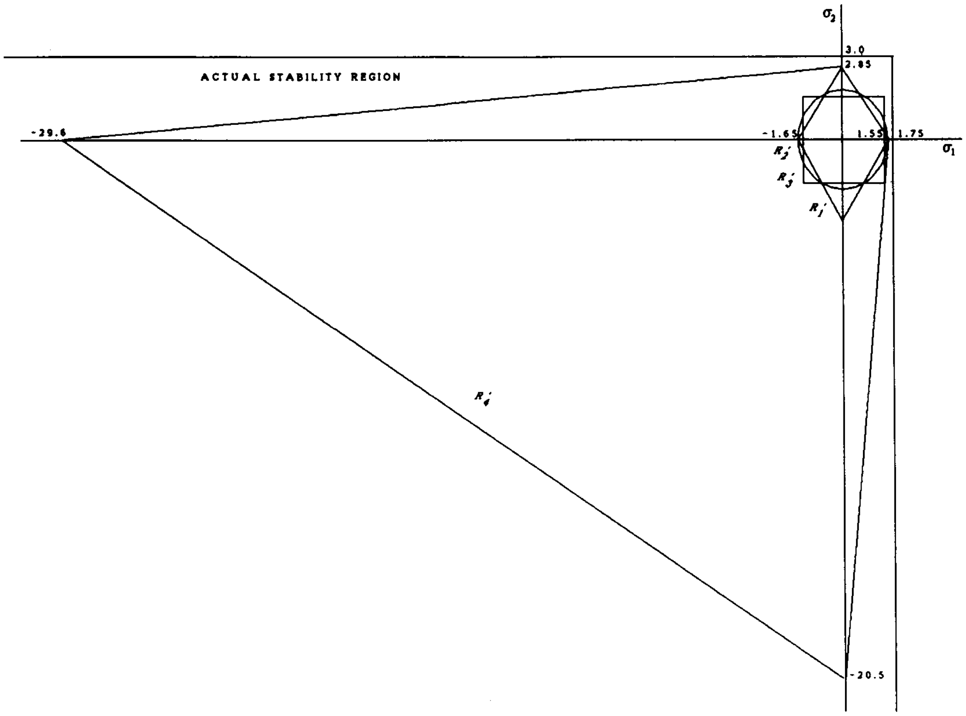

Fig. 1.

The system (2.2) may, for example, denote a control system in closed-loop configuration subjected to external white noise disturbances (see Section VI). Such specializations are not required for this development, however. Note that $J(\mathcal{U})$ represents the worst case (over $\mathcal{U}$) of the average (over the white noise statistics) of quadratically weighted steady-state deviations of the state from the origin. Thus, β represents an upper bound on selected output variances.

Of course, since R and V are only assumed to be nonnegative definite, there may be cases in which a finite performance bound β satisfying (2.3) exists while (2.1) is not asymptotically stable over $\mathcal{U}$. In practice, however, robust performance is mainly of interest when (2.1) is robustly stable. In this case the performance $J(\mathcal{U})$ is given in terms of the steady-state second moment of the state. The following result from linear system theory will be useful.

Lemma 2.1: Suppose (2.1) is asymptotically stable for all $\Delta A \in \mathcal{U}$. Then

$$J(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \operatorname{tr} Q_{\Delta A} R \tag{2.4}$$

where $n \times n$ $Q_{\Delta A} \triangleq \lim_{t \to \infty} \mathbb{E}[x(t)x^T(t)]$ is the unique, nonnegative-definite solution to

$$0 = (A + \Delta A) Q_{\Delta A} + Q_{\Delta A}(A + \Delta A)^T + V. \tag{2.5}$$

In the present note our approach is to obtain sufficient conditions for robust stability as a *consequence* of sufficient conditions for robust performance. Such conditions are developed in the following sections.

III. Sufficient Conditions for Robust Stability and Performance

The key step in obtaining robust stability and performance is to replace the uncertain terms in the Lyapunov equation (2.5) by a bounding matrix Ω. The nonnegative-definite solution Q of this bounding Lyapunov equation is then guaranteed to be an upper bound for $Q_{\Delta A}$. The uncertainty set $\mathcal{U}$ over which robustness is guaranteed then depends upon Q. The following easily proved result is fundamental and forms the basis for all later developments. The hypotheses of this result are of a general nature and are not intended to be directly verifiable. Suitably verifiable specializations of the hypotheses are discussed in Section V.

Theorem 3.1: Let $\Omega \in \mathbb{N}^n$, let $Q \in \mathbb{N}^n$ be the unique solution to

$$0 = AQ + QA^T + \Omega + V \tag{3.1}$$

and let $\mathcal{U}$ be a subset of

$$\hat{\mathcal{U}} \triangleq \{\Delta A \in \mathbb{R}^{n \times n} : \Delta A Q + Q \Delta A^T \leq \Omega\}. \tag{3.2}$$

Then

$$(A + \Delta A, [V + \Omega - (\Delta A Q + Q \Delta A^T)]^{1/2}) \text{ is stabilizable,} \quad \Delta A \in \mathcal{U} \tag{3.3}$$

if and only if

$$A + \Delta A \text{ is asymptotically stable,} \quad \Delta A \in \mathcal{U}. \tag{3.4}$$

In this case,

$$Q_{\Delta A} \leq Q, \quad \Delta A \in \mathcal{U} \tag{3.5}$$

where $Q_{\Delta A} \in \mathbb{N}^n$ is given by (2.5), and

$$J(\mathcal{U}) \leq \operatorname{tr} QR. \tag{3.6}$$

If, in addition, there exists $\Delta A \in \hat{\mathcal{U}}$ such that $(A + \Delta A, [V + \Omega - (\Delta A Q + Q \Delta A^T)]^{1/2})$ is controllable, then Q is positive definite.

Proof: This result is a minor variation of [21, Theorem 3.1] and hence the proof is omitted. □

To apply Theorem 3.1, one first chooses a nonnegative-definite matrix Ω and then solves (3.1) for Q. Next, as shown in Section IV, one examines $\hat{\mathcal{U}}$ to determine subsets $\mathcal{U}$ of perturbations ΔA over which robustness is guaranteed. Note that if $\mathcal{U}_1$ and $\mathcal{U}_2$ are subsets of $\hat{\mathcal{U}}$, then so is the convex hull of their union. (To see this note that $\hat{\mathcal{U}}$ is convex.) The set $\hat{\mathcal{U}}$ is the largest set over which robustness can be guaranteed by Theorem 3.1 for the particular choice of Ω. One may also select several matrices Ω and determine subsets of each resulting $\hat{\mathcal{U}}$ as a constructive approach to determining larger robustness sets. In the next section we examine subsets $\mathfrak{U}$ of $\tilde{\mathfrak{U}}$ of specified structure. Before doing so, we have the following observations.

In applying Theorem 3.1 it may be convenient to replace condition (3.3) with stronger conditions which are easier to verify in practice. The following result is immediate.

Proposition 3.1: Consider the conditions

$$V>0 \tag{3.7}$$

$$(A+\Delta A,\ V^{1/2}) \text{ is stabilizable}, \qquad \Delta A \in \mathfrak{U}, \tag{3.8}$$

$$\Delta A Q + Q \Delta A^T < \Omega, \qquad \Delta A \in \mathfrak{U}, \tag{3.9}$$

$$\Delta A Q + Q \Delta A^T < \Omega + V, \qquad \Delta A \in \mathfrak{U}. \tag{3.10}$$

Then (3.7) ⇒ (3.8) ⇒ (3.3), (3.7) ⇒ (3.10) ⇒ (3.3), and (3.9) ⇒ (3.10) ⇒ (3.3).

If only robust stability is of interest, then the noise intensity V need not have physical significance. In this case one may either set $V = \epsilon I_n$, where $\epsilon > 0$ is small to satisfy (3.7), or set $V = 0$ and confine $\mathfrak{U}$ to perturbations ΔA for which (3.9) holds. This is the case in [3], [4], [10] where $V = 0$, $\Omega = 2I_n$, and the parametric robustness sets are characterized by strict inequality.

Remark 3.1: Since A is asymptotically stable, Q is given by

$$Q=\int_0^\infty e^{At}(\Omega+V)e^{A^Tt}\,dt=\int_0^\infty e^{At}\Omega e^{A^Tt}\,dt+Q_0 \tag{3.11}$$

where $Q_0 \in \mathbb{N}^n$ is given by

$$0=AQ_0+Q_0A^T+V. \tag{3.12}$$

Note that $Q_0 \leq Q$ and that the nominal performance is given by tr Q_0R.

Remark 3.2: Using (3.11) it is also useful to note that the bound for $J(\mathfrak{U})$ given by (3.6) can be written as

$$\text{tr } QR=\text{tr}\int_0^\infty e^{At}(\Omega+V)e^{A^Tt}\,dtR=\text{tr } P_0(\Omega+V) \tag{3.13}$$

where $P_0 \in \mathbb{N}^n$ is given by

$$0=A^TP_0+P_0A+R. \tag{3.14}$$

The bound tr $P_0(\Omega + V)$ can be viewed as a dual formulation of the bound tr QR since the roles of A and A^T are reversed. Dual bounds are developed in the following section. Note that tr Q_0R = tr P_0V.

IV. Dual Sufficient Conditions for Robust Stability and Performance

As noted in Remark 3.2, the performance bound tr QR given by (3.6) can be expressed equivalently in terms of a dual variable P_0 for which the roles of A and A^T are reversed. Using a similar technique, additional conditions for robust stability and performance can be obtained by developing a dual version of Theorem 3.1. A prime motivation for developing such dual bounds is to draw direct connections with previous results in the literature relating to robust stability. Traditionally, the use of the quadratic Lyapunov function $V(x) = x^TPx$ for robust stability leads naturally to the dual formulation. In addition, the dual bounds may, for certain problems, be much sharper than the bounds introduced in the previous section. This point is illustrated at the end of this section by examining an extreme case and in Section VI by means of numerical examples. We note, in addition, that robust performance bounds are more difficult to motivate within the dual formulation without first developing the primal results. The following result is immediate.

Lemma 4.1: Suppose (2.1) is asymptotically stable for all $\Delta A \in \mathfrak{U}$. Then

$$J(\mathfrak{U})=\sup_{\Delta A\in\mathfrak{U}} \text{tr } P_{\Delta A}V \tag{4.1}$$

where $n \times n$ $P_{\Delta A}$ is the unique, nonnegative-definite solution to

$$0=(A+\Delta A)^TP_{\Delta A}+P_{\Delta A}(A+\Delta A)+R. \tag{4.2}$$

The dual of Theorem 3.1 can now be stated.

Theorem 4.1: Let $\Lambda \in \mathbb{N}^n$, let $P \in \mathbb{N}^n$ be the unique solution to

$$0=A^TP+PA+\Lambda+R \tag{4.3}$$

and let $\mathfrak{U}$ be a subset of

$$\tilde{\mathfrak{U}}' \triangleq \{\Delta A \in \mathbb{R}^{n\times n} : \Delta A^TP+P\Delta A \leq \Lambda\}. \tag{4.4}$$

Then

$$([R+\Lambda-(\Delta A^TP+P\Delta A)]^{1/2},\ A+\Delta A) \text{ is detectable}, \qquad \Delta A \in \mathfrak{U}, \tag{4.5}$$

if and only if

$$A+\Delta A \text{ is asymptotically stable}, \qquad \Delta A \in \mathfrak{U}. \tag{4.6}$$

In this case,

$$P_{\Delta A}\leq P, \qquad \Delta A \in \mathfrak{U} \tag{4.7}$$

where $P_{\Delta A} \in \mathbb{N}^n$ is given by (4.2), and

$$J(\mathfrak{U})\leq \text{tr } PV. \tag{4.8}$$

If, in addition, there exists $\Delta A \in \tilde{\mathfrak{U}}'$ such that $([R + \Lambda - (\Delta A^TP + P\Delta A)]^{1/2}, A + \Delta A)$ is observable, then P is positive definite.

The usefulness of Theorem 4.1 resides in the fact that it provides stability and performance bounds which are generally different from those given by Theorem 3.1. Hence, depending upon Ω and Λ either bound (3.6) or bound (4.8) may be better for a particular problem. To illustrate how dual bounds can improve estimates of robust performance, consider the case in which $V = 0$, i.e., plant disturbances are absent. In this case $Q_{\Delta A} = 0$ satisfies (2.5) and thus $J(\mathfrak{U}) = 0$ as long as $A + \Delta A$ is stable for all $\Delta A \in \mathfrak{U}$. The performance bound tr QR given by (3.6) may, however, be arbitrarily large depending upon R since Q may be nonzero due to Ω. Hence, this performance bound may be arbitrarily conservative. The dual bound (4.8), on the other hand, is zero in this case, which completely eliminates the conservatism.

V. Characterization of Subsets of $\tilde{\mathfrak{U}}$ and $\tilde{\mathfrak{U}}'$

To apply Theorems 3.1 and 4.1 it is necessary to explicitly characterize subsets $\mathfrak{U}$ of $\tilde{\mathfrak{U}}$ and $\tilde{\mathfrak{U}}'$ over which robustness is guaranteed. In this section we provide several such characterizations by collecting together and extending known results from the literature.

For the following result let $\Omega = \omega I_n$, where $\omega > 0$, let $W \in \mathbb{R}^{n\times n}$, $W \geqslant 0$, and let $A_1, \cdots, A_p \in \mathbb{R}^{n\times n}$ be arbitrary. Furthermore, for $Q \in \mathbb{P}^n$ satisfying (3.1) define for $i = 1, \cdots, p$:

$$\alpha_i \triangleq \lambda_{\min}(A_iQ+QA_i^T), \qquad \beta_i \triangleq \lambda_{\max}(A_iQ+QA_i^T),$$

$$\begin{aligned}
\mathcal{I}_i &\triangleq (-\infty,\ \infty) && \alpha_i=\beta_i=0,\\
&\triangleq (-\infty,\ \omega/\beta_i), && \alpha_i\geq 0,\ \beta_i>0,\\
&\triangleq (\omega/\alpha_i,\ \infty), && \alpha_i<0,\ \beta_i\leq 0,\\
&\triangleq (\omega/\alpha_i,\ \omega/\beta_i), && \alpha_i<0<\beta_i.
\end{aligned}$$

Finally, let $e_i^{(p)}$ denote the ith column of the $p \times p$ identity matrix.

Proposition 5.1: Let $Q \in \mathbb{P}^n$ satisfy (3.1) with $\Omega = \omega I_n$, where $\omega > 0$. Then the following sets are subsets of $\tilde{\mathfrak{U}}$ which also satisfy (3.9):

$$\mathfrak{U}_1 \triangleq \left\{\Delta A \in \mathbb{R}^{n\times n} : \|\Delta A\|_s<\frac{\omega}{2}\|Q\|_s^{-1}\right\},$$

$$\mathfrak{U}_2 \triangleq \{\Delta A \in \mathbb{R}^{n\times n} : |\Delta A|_m \ll \omega\| W|Q|_m+|Q|_mW^T\|_s^{-1}W\},$$

$$\mathfrak{U}_3 \triangleq \left\{\Delta A \in \mathbb{R}^{n\times n} : \Delta A=\sum_{i=1}^p \sigma_iA_i,\ (\sigma_1,\ \cdots,\ \sigma_p)^T \in \mathcal{R}\right\}$$

where $\mathcal{R}$ is one of the following regions in $\mathbb{R}^p$:

$$\mathcal{R}_1 \triangleq \left\{(\sigma_1, \cdots, \sigma_p) : \sum_{i=1}^{p} |\sigma_i| \, \|A_iQ+QA_i^T\|_s < \omega\right\},$$

$$\mathcal{R}_2 \triangleq \left\{(\sigma_1, \cdots, \sigma_p) : \sum_{i=1}^{p} \sigma_i^2 < \omega^2 \left\| \sum_{i=1}^{p} (A_iQ+QA_i^T)^2 \right\|_s^{-1}\right\},$$

$$\mathcal{R}_3 \triangleq \left\{(\sigma_1, \cdots, \sigma_p) : |\sigma_i| < \omega \left\| \sum_{i=1}^{p} |A_iQ+QA_i^T|_m \right\|_s^{-1}, \quad i=1, \cdots, p\right\},$$

$$\mathcal{R}_4 \triangleq \text{co } \{\sigma_i e_i^{(p)} : \sigma_i \in \mathcal{I}_i, \; i=1, \cdots, p\}.$$

For the dual case we set $\Lambda = \lambda I_n$, where $\lambda > 0$, and define the dual sets $\tilde{\mathcal{U}}_1'$, $\tilde{\mathcal{U}}_2'$, $\tilde{\mathcal{U}}_3'$, $\mathcal{I}_i'$, $\mathcal{R}_1'$, $\mathcal{R}_2'$, $\mathcal{R}_3'$, and $\mathcal{R}_4'$ in an analogous fashion.

Remark 5.1: The proof of Proposition 5.1 is omitted since the results are either known or are immediate. Specifically, $\mathcal{U}_1'$ can be found in [2] while $\mathcal{U}_2'$ appears in [3], [4]. The sets $\mathcal{R}_1'$, $\mathcal{R}_2'$, and $\mathcal{R}_3'$ are given in [10]. The set $\mathcal{R}_4'$ has not appeared previously in the literature although the result is immediate. It is only necessary to diagonalize $A_i^TP + PA_i$ by means of an orthogonal transformation and compare diagonal elements to obtain $\mathcal{I}_i'$. Taking the convex hull over the intervals $\mathcal{I}_i'$ thus yields $\mathcal{R}_4'$. Of course, the required eigenproblem entails additional computation.

Remark 5.2: Although most of the dual of Proposition 5.1 has appeared previously, the primal result Proposition 5.1 has not been discussed in the literature. For robust stability this result can be obtained by considering the stability of A^T in place of A. As will be shown in Section VI, the primal and dual results lead in general to different robust stability regions and performance bounds. It should also be stressed that although most of the dual of Proposition 5.1 has appeared previously, the present note extends its applicability to the problem of robust performance in addition to robust stability.

Remark 5.3: As mentioned previously, the convex hull of the union of any collection of subsets of $\tilde{\mathcal{U}}$ is also a subset of $\tilde{\mathcal{U}}$ since $\tilde{\mathcal{U}}$ is convex. This observation applies to $\mathcal{U}_3$ in the sense that if $\mathcal{U}_3$ is a subset of $\tilde{\mathcal{U}}$ with regions $\mathcal{R} = \hat{\mathcal{R}}$ and $\mathcal{R} = \hat{\hat{\mathcal{R}}}$ separately, then $\mathcal{U}_3$ is also a subset with $\mathcal{R}$ equal to the convex hull of the union of $\hat{\mathcal{R}}$ and $\hat{\hat{\mathcal{R}}}$. Note that these observations follow from the convexity of $\tilde{\mathcal{U}}$ and do not contradict the fact that the set of asymptotically stable matrices is not convex.

Remark 5.4: The requirement that Ω be of the form ωI_n is not a constraint in applying Proposition 5.1. Indeed, it is only required that Ω be positive definite. To see this let invertible $\phi \in \mathbb{R}^{n\times n}$ be such that $\phi\Omega\phi^T = I_n$. Then Proposition 5.1 can be applied with suitable transformations of ΔA, Q, W, and A_i.

Remark 5.5: As in [2]–[4], [10], the sets $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ are defined in terms of strict inequalities. In this case $\mathcal{U}_1$, $\mathcal{U}_2$, and $\mathcal{U}_3$ consist of elements of $\tilde{\mathcal{U}}$ satisfying $\Delta AQ + Q\Delta A^T < \Omega$ so that (3.9) is satisfied. Thus, by Proposition 3.1, the stabilizability condition (3.3) is automatically satisfied without reference to V.

Remark 5.6: In the special case $p = 1$ it is clear that $\mathcal{R}_1 = \mathcal{R}_2$. Furthermore, in this case $\mathcal{R}_3$ is always a subset of $\mathcal{R}_1$ and $\mathcal{R}_2$, and hence leads to a more conservative stability region. The largest possible set of perturbations ΔA of the form $\sigma_i A_i$ contained in $\tilde{\mathcal{U}}$ is given by $\mathcal{R}_4$.

Remark 5.7: It is shown in [10, Remark 2.12] that $\mathcal{U}_2$ can be obtained as a consequence of $\mathcal{U}_3$ with $\mathcal{R} = \mathcal{R}_3$ and a suitable choice of A_i. Hence, $\mathcal{U}_2$ need not actually be considered separately. Our assumption that $W \gg 0$ (and not $W \geq\geq 0$) is for convenience only.

Remark 5.8: Note that all of the subsets of $\tilde{\mathcal{U}}$ given by Proposition 5.1 are symmetric except for $\mathcal{U}_3$ with $\mathcal{R} = \mathcal{R}_4$. When the actual stability region is highly asymmetric, it follows that a symmetric robust stability region is necessarily highly conservative. This observation is illustrated by an example in Section VI.

Remark 5.9: The regions given by $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ correspond, respectively, to 1-norm 2-norm, and ∞-norm neighborhoods. These results can easily be extended to include more general regions. For example, in the definition of $\mathcal{R}_2$ replace σ_i by σ_i/a_i and $A_iQ + QA_i^T$ by $a_i(A_iQ + QA_i^T)$, where a_i is an arbitrary positive constant, $i = 1, \cdots, p$. With this modification $\mathcal{R}_2$ corresponds to an elliptical robust stability region. Detailed investigation of such regions is beyond the scope of this note.

Remark 5.10: When each interval $\mathcal{I}_i$ is finite, or when only a finite interval is of interest, $\mathcal{R}_4$ can be expressed as the convex hull of a finite number of points. Specifically, letting $\mathcal{I}_i = [a_i, b_i]$, $i = 1, \cdots, p$, it follows that

$$\mathcal{R}_4 = \text{co } \{a_1e_1^{(p)}, b_1e_1^{(p)}, \cdots, a_pe_p^{(p)}, b_pe_p^{(p)}\}.$$

This set is illustrated by means of an example in the next section.

VI. Examples

As a first example we adopt Example 2 of [10]. This example, which involves two uncertain parameters, was used in [10] to illustrate the robust stability regions $\mathcal{R}_1'$, $\mathcal{R}_2'$, and $\mathcal{R}_3'$. The problem was originally cast in the form of a static output feedback controller with uncertain gains. Here for convenience in discussing robust performance we reformulate the example to involve uncertainty in the control input matrix. Hence, consider the control system

$$\dot{x}(t) = A_0x(t) + B_0u(t), \tag{6.1}$$

$$y(t) = C_0x(t), \tag{6.2}$$

$$u(t) = Ky(t) \tag{6.3}$$

where

$$A_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad C_0 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and the uncertainty ΔB_0 in B_0 is given by

$$\Delta B_0 = \begin{bmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_2 \\ -\sigma_1 & -\sigma_2 \end{bmatrix}.$$

The closed-loop dynamics matrix is then given by

$$A + \Delta A = \begin{bmatrix} -2+\sigma_1 & 0 & -1+\sigma_1 \\ 0 & -3+\sigma_2 & 0 \\ -1+\sigma_1 & -1+\sigma_2 & -4+\sigma_1 \end{bmatrix}$$

where $\Delta A = \sigma_1 A_1 + \sigma_2 A_2$ and A_1, A_2 have the evident definitions. It can easily be shown that the exact stability region is given by $\sigma_1 \in (-\infty, 1.75)$ and $\sigma_2 \in (-\infty, 3)$. Thus, the nominal dynamics matrix corresponding to $\sigma_1 = \sigma_2 = 0$ lies in the upper right-hand corner of the exact stability region so that, as noted in Remark 5.8, a high degree of conservatism can be expected using symmetric robustness regions. To consider robust stability alone, set $V = R = 0$ and $\omega = \lambda = 2$. In this case regions $\mathcal{R}_1'$, $\mathcal{R}_2'$, and $\mathcal{R}_3'$, as computed in [10], are shown in Fig. 1. Region $\mathcal{R}_4'$ for this problem is given (see Remark 5.10) by

$$\mathcal{R}_4' = \text{co}\left\{\begin{pmatrix} -29.6 \\ 0 \end{pmatrix}, \begin{pmatrix} 1.65 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -20.5 \end{pmatrix}, \begin{pmatrix} 0 \\ 2.85 \end{pmatrix}\right\}$$

which accounts somewhat better for the asymmetry of the stability region. The regions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ were found to be smaller than the corresponding dual regions, while $\mathcal{R}_4$ is given by

$$\mathcal{R}_4 = \text{co}\left\{\begin{pmatrix} -31.1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1.64 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -10.4 \end{pmatrix}, \begin{pmatrix} 0 \\ 2.63 \end{pmatrix}\right\}$$

which yields slight improvement in σ_1.

To evaluate robust performance replace (6.1) by

$$\dot{x}(t) = A_0x(t) + B_0u(t) + w(t) \tag{6.4}$$

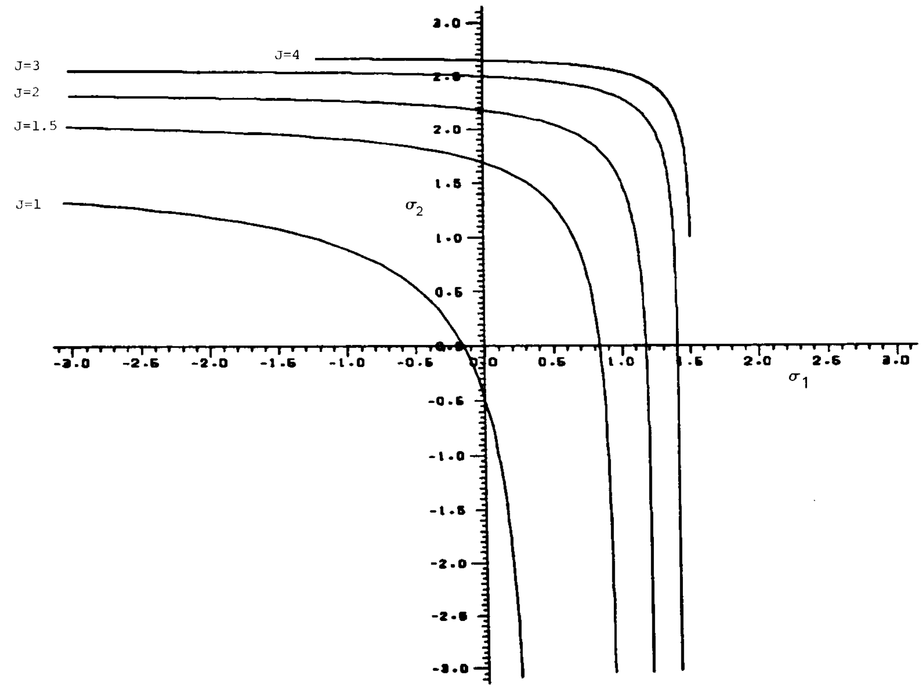

Fig. 2.

and define

$$J=\lim_{t\to\infty} \mathcal{E}[x^T(t)R_1x(t)+u^T(t)R_2u(t)]$$

which corresponds to (2.3) with $R = R_1 + C_0^TK^TR_2KC_0$. Hence, setting $R_1 = I_3$ and $R_2 = I_2$ yields

$$R=\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

We also set $V = I_3$ and $\omega = \lambda = 2$. The resulting stability region for these values of V and R is given by

$$\mathcal{R}_1'=\{(\sigma_1,\sigma_2): |\sigma_1|/0.70+|\sigma_2|/1.46<1\},$$

$$\mathcal{R}_2'=\{(\sigma_1,\sigma_2): \sigma_1^2+\sigma_2^2<(0.70)^2\},$$

$$\mathcal{R}_3'=\{(\sigma_1,\sigma_2): |\sigma_i|<0.68,\ i=1,2\},$$

$$\mathcal{R}_4'=\mathrm{co}\left\{\begin{pmatrix}-20.5\\0\end{pmatrix},\begin{pmatrix}0.70\\0\end{pmatrix},\begin{pmatrix}0\\-13.7\end{pmatrix},\begin{pmatrix}0\\1.46\end{pmatrix}\right\}.$$

Over these combined regions the performance bound was computed to be tr $PV = 2.26$. The primal result produced the regions

$$\mathcal{R}_1=\{(\sigma_1,\sigma_2): |\sigma_1|/1.09+|\sigma_2|/1.75<1\},$$

$$\mathcal{R}_2=\{(\sigma_1,\sigma_2): \sigma_1^2+\sigma_2^2<(1.08)^2\},$$

$$\mathcal{R}_3=\{(\sigma_1,\sigma_2): |\sigma_i|<1.0,\ i=1,2\},$$

$$\mathcal{R}_4=\mathrm{co}\left\{\begin{pmatrix}-20.8\\0\end{pmatrix},\begin{pmatrix}1.09\\0\end{pmatrix},\begin{pmatrix}0\\-6.93\end{pmatrix},\begin{pmatrix}0\\1.75\end{pmatrix}\right\}.$$

Over these regions the performance bound was computed to be tr $QR = 3.18$. Contour plots of actual performance for perturbed values of σ_1 and σ_2 are shown in Fig. 2. Note that when determining robust performance Theorems 3.1 and 4.1 yield performance bounds over robust stability regions which are generally smaller than the robust stability regions determined with $R = 0$ and $V = 0$. This mechanism represents the natural tradeoff between stability and performance. In general, to determine the largest stability regions, V and R should be set to zero initially.

As a second example we consider the control system given in [19] to demonstrate the lack of a guaranteed gain margin for LQG controllers. Hence, consider

$$\dot{x}_0(t)=A_0x_0(t)+B_0u(t)+w_1(t), \tag{6.5}$$

$$y(t)=C_0x_0(t)+w_2(t) \tag{6.6}$$

with controller

$$\dot{x}_c(t)=A_cx_c(t)+B_cy(t), \tag{6.7}$$

$$u(t)=C_cx_c(t) \tag{6.8}$$

and performance

$$J=\lim_{t\to\infty}\mathcal{E}[x_0^T(t)R_1x_0(t)+u^T(t)R_2u(t)]. \tag{6.9}$$

The data are

$$A_0=\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix},\ B_0=\begin{bmatrix}0\\1\end{bmatrix},\ C_0=[1\ \ 0],$$

$$V_1=R_1=\rho\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix},\ V_2=R_2=1$$

where V_1 and V_2 are the intensities of $w_1(t)$ and $w_2(t)$, respectively. Uncertainty ΔB_0 in B_0 is thus represented by σ_1B_1, where $B_1 = [0,\ 1]^T$. Thus, the closed-loop system corresponds to

$$A=\begin{bmatrix}A_0 & B_0C_c\\B_cC_0 & A_c\end{bmatrix},\ A_1=\begin{bmatrix}0 & B_1C_c\\0 & 0\end{bmatrix},$$

$$R=\begin{bmatrix}R_1 & 0\\0 & 0\end{bmatrix},\ V=\begin{bmatrix}V_1 & 0\\0 & B_cV_2B_c^T\end{bmatrix}$$

where the zero in the (2, 2) block of R denotes the fact that we are considering the robust performance bound for the state regulation cost

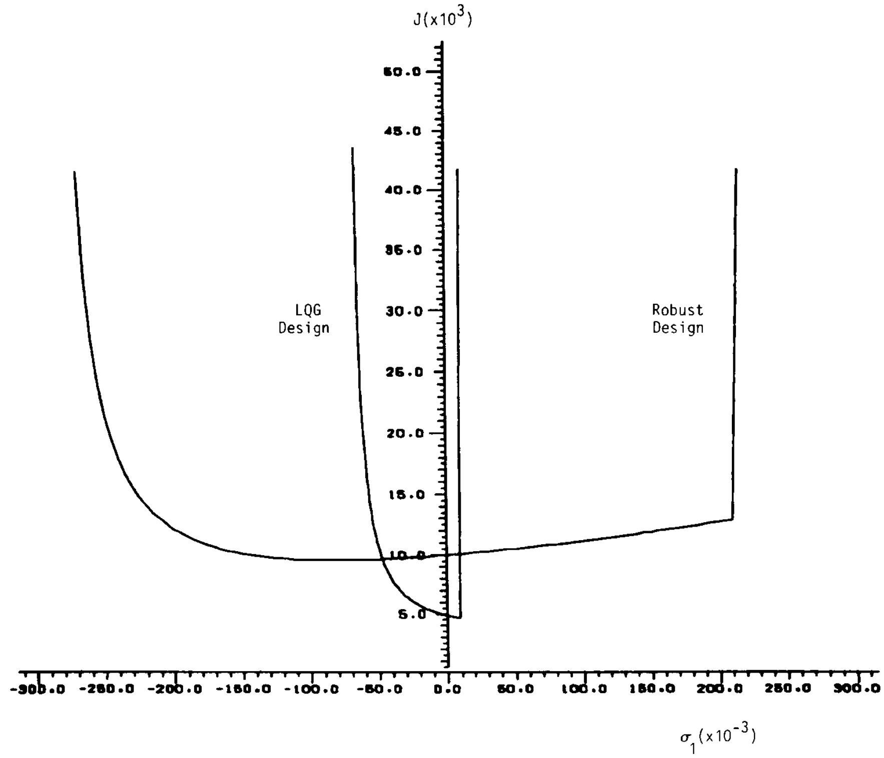

Fig. 3.

only. Choosing $\rho = 60$, it follows that the LQG gains are given by

$$A_c=\begin{bmatrix} -9 & 1 \\ -20 & -9 \end{bmatrix}, B_c=\begin{bmatrix} 10 \\ 10 \end{bmatrix}, C_c=[-10 \ -10].$$

For this controller the actual stability region corresponds to $\sigma_1 \in (-0.07, 0.01)$ (see Fig. 3). Applying the results of Section V with $V = R = 0$ (for robust stability only) and $\omega = \lambda = 2$, we obtain

$$\mathcal{R}_1=\mathcal{R}_2=\mathcal{R}_3=(-0.000242, 0.000242), \ \mathcal{R}_4=(-0.000242, 0.000728),$$

$$\mathcal{R}_1'=\mathcal{R}_2'=(-0.0000247, 0.0000247), \ \mathcal{R}_3'=(-0.0000219, 0.0000219),$$

$$\mathcal{R}_4'=(-0.0000247, 0.0000265).$$

Note that although the primal results are better than the dual results by an order of magnitude, they are conservative by two orders of magnitude with respect to the actual gain margin. For robust performance we again set $\omega = \lambda = 2$ and, using R and V given above, we obtained the bound tr $QR = 7633$ over the stability region $\mathcal{R}_4 = (-0.000192, 0.000613)$. The nominal performance was given by tr Q_0R = tr $P_0V = 4875$, while the dual performance bound was tr $PV = 10510$ over $\mathcal{R}_4' = (-0.0000222, 0.0000238)$.

Robustified controllers for the example of [19] were obtained in [20] using the approach discussed in [13]. As shown in Fig. 3 (see also [20]), the closed-loop system with the controller

$$A_c=\begin{bmatrix} -10.69 & 1 \\ -32.97 & -5.295 \end{bmatrix}, B_c=\begin{bmatrix} 11.69 \\ 26.67 \end{bmatrix}, C_c=[-6.245 \ -6.245]$$

is stable over the range $\sigma_1 \in (-0.28, 0.21)$. Hence, we wish to determine whether the robust stability tests are capable of detecting this increase in gain margin. Applying Theorems 3.1 and 4.1 with $\omega = \lambda = 2$ and $V = R = 0$ yields stability for σ_1 in the regions $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}_3 = (-0.0115, 0.0115)$ and $\mathcal{R}_4 = (-0.0115, 0.057)$. This guarantee of stability is two orders of magnitude greater than the guarantee for the LQG design but is still an order of magnitude conservative with respect to the actual stability region for this controller. Note, however, that for $\sigma_1 > 0$ the guaranteed gain margin for the robustified design given by $\mathcal{R}_4$ (i.e., 0.057) is greater than the *actual* gain margin of the LQG design (0.01). Hence, the robustness test given by the $\mathcal{R}_4$ was able to detect a factor of 5 stability augmentation provided by the robustified design compared to the LQG controller. Finally, the robust performance bound for this controller was computed to be tr $QR = 11185$ over the region $\mathcal{R}_4 = (-0.00165, 0.00493)$, while the dual bound was found to be tr $PV = 11223$ over $\mathcal{R}_4' = (-0.000724, 0.00123)$. For this problem the nominal performance is tr Q_0R = tr $P_0V = 9997$.

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Robustness of Pole-Assignment in a Specified Region

YAU-TARNG JUANG, ZUU-CHANG HONG, AND YI-TARNG WANG

Abstract—**This note deals with the analysis of pole-assignment for systems under linear time-invariant perturbations. Based upon the Lyapunov approach, new techniques to calculate allowable element bounds for highly-structured perturbations are presented. Under these allowable perturbations, both stability robustness and certain performance robustness will thus be ensured. Examples are given to illustrate proposed methods.**

NOTATION

$\|\|M\|\|$	l_2-norm for $M \in \mathbb{C}^{n\times n}$, i.e., the maximal singular value of M
M^*	The complex conjugate transpose of M
M_H	$\triangleq (M + M^*)/2$, the Hermitian part of M
$\|M\|$	$\{\|m_{ij}\|\}$, for $M = \{m_{ij}\} \in \mathbb{C}^{n\times n}$
$M_1 < M_2$	$m_{ij,1} < m_{ij,2}$, $\forall i, j = 1, 2, \cdots, n$, $M_1, M_2 \in \mathbb{R}^{n\times n}$.

I. INTRODUCTION

For maintaining stability of uncertain systems, allowable perturbations are discussed in [1]-[5]. These results are concerned only with stability robustness, they do not deal with the robustness analysis for maintaining a certain performance. References [6] and [7] have considered locating the poles of a system in a specified region, a vertical strip or a circular region, to shape the dynamic response. However, quantitative measures for the allowable perturbations have not been given in [6] and [7].

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Y.-T. Juang is with the Department of Electrical Engineering, National Central University, Chung-Li, Taiwan, Republic of China.

Z.-C. Hong and Y.-T. Wang are with the Department of Mechanical Engineering, National Central University, Chung-Li, Taiwan, Republic of China.

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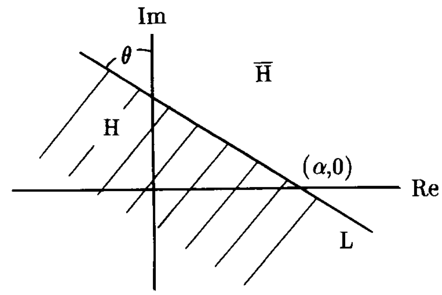

Fig. 1.

In this note, we propose new methods to analyze the pole-assignment robustness of dynamic systems under linear time-invariant highly-structured perturbations. Assigning eigenvalues of systems under perturbations in a specified region will not only ensure stability robustness but also achieve certain performance robustness in the linear time-invariant case. Based on the Lyapunov approach, the upper bounds on perturbations are obtained to retain system eigenvalues located within an arbitrarily-chosen region in the complex plane. In Section II, the main results are developed. Illustrative examples are given in Section III. Conclusions are given in Section IV.

II. MAIN RESULTS

This section presents the criteria to calculate the allowable bounds on highly-structured perturbations for maintaining the poles of a system in a specified region. Consider a line L which separates the complex plane into two open half-planes, namely H and $\bar{H}$ as shown in Fig. 1. The line L intersects the real axis at $(\alpha, 0)$ and makes an angle θ with respect to the positive imaginary axis, where θ is assumed positive in a counterclockwise sense and $-\pi < \theta \le \pi$.

Lemma 1: All the eigenvalues of a constant matrix $A \in \mathbb{C}^{n\times n}$ lie in the H region if and only if the matrix $e^{-j\theta}(A - \alpha I)$ is stable.

Proof: That the constant matrix $e^{-j\theta}(A-\alpha I)$ is stable means the eigenvalues of the matrix $e^{-j\theta}(A-\alpha I)$ lie in the open left-half complex plane. And after rotation and translation, this also means the eigenvalues of the matrix A lie in the H region. Q.E.D.

Theorem 1: If all eigenvalues of a constant matrix A lie in the H region as shown in Fig. 1, then all eigenvalues of the matrix $A + \Delta A$ will remain in the H region if

$$\frac{|\Delta a_{ij}|}{e_{ij}} < \frac{1}{\|(|P|E)_H\|} \triangleq \mu \tag{1}$$

where E is a nonnegative matrix representing the highly-structured information for the additive perturbation ΔA, and P is the unique positive definite Hermitian solution of the equation

$$e^{j\theta}A^*P + e^{-j\theta}PA - 2\alpha P\cos\theta = -2I. \tag{2}$$

Proof: Since the eigenvalues of matrix A lie in the H region, by Lemma 1 we know $e^{-j\theta}(A-\alpha I)$ is stable. From the Lyapunov theorem [8], we know the equation

$$[e^{-j\theta}(A-\alpha I)]^*P + P[e^{-j\theta}(A-\alpha I)] = -2I \tag{3}$$

has a unique positive definite Hermitian solution P. Because (3) is equivalent to (2), the unique positive definite Hermitian solution of (2) always exists. Now, consider the system

$$\dot{X} = \{e^{-j\theta}(A+\Delta A-\alpha I)\}X \tag{4}$$

and choose $V = X^*PX$ as a Lyapunov function for the system described by (4), where P is the solution of (2). Then after some manipulations we have

$$\dot{V} = X^*(-2I + e^{j\theta}\Delta A^*P + P\Delta A e^{-j\theta})X.$$