Adaptive Stabilization of Linear Second-Order Systems with Bounded Time-Varying Coefficients

ALEXANDER V. ROUP
Vehicle Control Technologies, Inc., 11180 Sunrise Valley Drive, Suite 350, Reston, VA 20191, USA

DENNIS S. BERNSTEIN
Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109-2140, USA

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Abstract: We consider adaptive stabilization for a class of linear time-varying second-order systems. Interpreting the system states as position and velocity, the system is assumed to have unknown, non-parametric, bounded time-varying damping and stiffness coefficients. The coefficient bounds need not be known to implement the adaptive controller. Lyapunov methods are used to prove global convergence of the system states. For illustration, the controller is used to stabilize several example systems.

Key Words: Adaptive, stabilization, time varying, control

1. INTRODUCTION

There are many applications of control in which a reliable model of the dynamical system is not available. This can occur if the system possesses unknown or changing parameters, if the system is not amenable to analytical modeling due to unknown physics, or if identification is not feasible due to instability, disturbance and sensor noise, poor repeatability, unpredictable changes, or high cost. Under such high levels of uncertainty, robust control may be ineffective and adaptive control may be viable.

Stabilization of time-varying systems is complicated by the fact that stability cannot be determined by the frozen-time system eigenvalues (Wu, 1974). Hence some control methods require explicit knowledge of the time variation of the system parameters for feedback synthesis (Wolovich, 1968; Wu, 1975; Cheng, 1979; Tadmor, 1992). In addition, some control strategies require a sufficiently small rate of time variation in order to guarantee stability (Kamen et al., 1989). The stability of systems with periodic time variation, including the classical Hill and Mathieu equations, is analyzed in Richards (1983).

In this paper we consider the problem of adaptive stabilization for a class of second-order time-varying systems under full-state feedback. In Section 2, we present a fixed-gain controller that facilitates the development of the adaptive controller. Then, in Section 3, we present the adaptive controller and prove convergence of the plant states. In Section 4, we present three example problems that demonstrate the breadth of the admissible system class.

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Interpreting the system states as position and velocity, the system is assumed to have unknown, time-varying damping and stiffness coefficients, which are assumed only to belong to a class of piecewise continuous functions and to be bounded. Furthermore, the coefficient bounds need not be known. The novel aspect of the controller is the fact that global stability and convergence is guaranteed under non-parametric assumptions about the time variation. The form of the adaptive controller is similar to direct adaptive controllers developed for linear time-invariant systems. Related theory can be found in Åström and Wittenmark (1995), Ioannou and Sun (1996), Kaufman et al. (1998), Krstic et al. (1995), Narendra and Annaswamy (1989), and Sastry and Bodson (1989), where the emphasis is on model following control. For adaptive stabilization of linear systems, a self-contained treatment of the relevant ideas and techniques is given in Hong and Bernstein (2001), where stability of the closed-loop system is proven for linear time-invariant plants. In Roup and Bernstein (2001), the same controller is applied to second-order nonlinear plants with position-dependent stiffness and damping coefficients.

Since we assume full-state feedback in companion coordinates, i.e. position and velocity measurements, the controller is a direct adaptive controller, and thus parameter estimates are not needed. In addition, full-state feedback availability avoids the need for positivity assumptions. Extensions to nonlinear time-varying uncertainty, output feedback, and model reference adaptive control will be considered in future work.

2. FIXED-GAIN STABILIZATION

We wish to determine a fixed-gain feedback control law for the linear time-varying system

$$m\,\ddot{q}(t) + g(t)\,\dot{q}(t) + f(t)\,q(t) = bu(t) + d, \tag{1}$$

where $t \in [0, \infty), f : [0, \infty) \to \mathbb{R}$, $g : [0, \infty) \to \mathbb{R}$, and $m, b, d \in \mathbb{R}$, such that $q(t) \to 0$ and $\dot{q}(t) \to 0$ as $t \to \infty$. We assume that f and g are piecewise continuous and bounded. Additionally, we assume that $m > 0$ and $b \neq 0$. Implementation of the controller requires that m, b, d, and upper and lower bounds on f and g be known. Explicit knowledge of f and g is not required for implementation.

Consider the control law

$$u(t) = k_1 q(t) + k_2 \dot{q}(t) + \phi\,, \tag{2}$$

where k_1, k_2, $\phi \in \mathbb{R}$. Define the state

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \triangleq \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}, \tag{3}$$

and the gain matrix

$$K \triangleq [k_1 \qquad k_2]\,. \tag{4}$$

Dynamic variables will henceforth be written without a time-dependence argument. Equations (1) and (2) can be written in state form as

$$\dot{x} = \begin{bmatrix} x_2 \\ (1/m)\,[bKx + b\phi + d - g(t)x_2 - f(t)x_1] \end{bmatrix}. \tag{5}$$

Let $P \triangleq \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$ be positive definite with $p_{12} > 0$. Let

$$\alpha_1 \triangleq \inf_{t\in[0,\infty)} f(t),\ \ \alpha_2 \triangleq \sup_{t\in[0,\infty)} f(t),\ \ \beta_1 \triangleq \inf_{t\in[0,\infty)} g(t),\ \ \beta_2 \triangleq \sup_{t\in[0,\infty)} g(t), \tag{6}$$

and define

$$\gamma_1 \triangleq \frac{p_{12}}{m}\beta_1 + \frac{p_2}{m}\alpha_1, \qquad \gamma_2 \triangleq \frac{p_{12}}{m}\beta_2 + \frac{p_2}{m}\alpha_2, \qquad \gamma \triangleq \max(|\gamma_1|, |\gamma_2|). \tag{7}$$

Define the set

$$\begin{aligned} \mathcal{K}_s(P) \ \triangleq \ & \Big\{ [k_1\ \ k_2] : bk_1 < \alpha_1, \quad \frac{p_{12}}{m}\,bk_2 + \frac{p_2}{m}\,bk_1 + p_1 < 0, \\ & \frac{p_{12}}{m}(\alpha_1 - bk_1)\left[\frac{p_2}{m}(\beta_1 - bk_2) - p_{12}\right] \ > \ \frac{1}{4}\gamma^2 \Big\}. \end{aligned} \tag{8}$$

We will show that the elements of the set $\mathcal{K}_s(P)$ are constant feedback gains K associated with P that stabilize the origin of equation (5).

Definition 1. (Definition 3.5 in Khalil, 1996) Consider the time-varying nonlinear differential equation

$$\dot{x} = f(t, x) \tag{9}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and assume $f(t, 0) = 0$ for all $t \geq 0$. The equilibrium $x = 0$ is globally exponentially stable if there exist $k > 0$ and $\gamma > 0$ such that, for all $t_0 \geq 0$ and for all initial states $x(t_0) \in \mathbb{R}^n$,

$$\|x(t)\| \leq k\|x(t_0)\|\mathrm{e}^{-\gamma\,(t-t_0)}, \qquad t > t_0. \tag{10}$$

Lemma 1. $\mathcal{K}_s(P)$ is not empty. Furthermore, let $K_s \in \mathcal{K}_s(P)$. Then, with $K = K_s$ and $\phi = -d/b$, the origin of equation (5) is globally exponentially stable.

Proof. The constraint inequality $bk_1 < \alpha_1$ in the definition (8) of $\mathcal{K}_s(P)$ provides an upper bound on bk_1. The remaining inequalities can be restated as upper bounds on bk_2 given by

$$bk_2 < -\frac{m}{p_{12}}\left(\frac{p_2}{m}\,bk_1 + p_1\right), \qquad bk_2 < \beta_1 - \frac{m}{p_2}\left[\frac{m\gamma^2}{4p_{12}(\alpha_1 - b\,k_1)} + p_{12}\right]. \tag{11}$$

Since bk_1 and bk_2 are bounded from above only it follows that $\mathcal{K}_s(P)$ is not empty.

Write $K_s = [k_{1s} \;\; k_{2s}]$ and define $\tilde{f} : [0, \infty) \to \mathbb{R}$ and $\tilde{g} : [0, \infty) \to \mathbb{R}$ by

$$\tilde{f}(t) \triangleq f(t) - bk_{1s}\,, \qquad \tilde{g}(t) \triangleq g(t) - bk_{2s}\,. \tag{12}$$

The closed-loop system (5) can be written in the form

$$\dot{x} = \begin{bmatrix} x_2 \\ (1/m)\left[-\tilde{g}(t)x_2 - \tilde{f}(t)x_1\right] \end{bmatrix}. \tag{13}$$

Define

$$\delta_p \triangleq -\left(p_1 + \frac{p_{12}}{m}bk_{2s} + \frac{p_2}{m}bk_{1s}\right) \tag{14}$$

and $h : [0, \infty) \to \mathbb{R}$ by

$$h(t) \triangleq \frac{p_{12}}{m}\tilde{g}(t) + \frac{p_2}{m}\tilde{f}(t) - p_1 - \delta_p. \tag{15}$$

Define

$$R_1 \triangleq \begin{bmatrix} \frac{p_{12}}{m}(\alpha_1 - bk_{1s}) & \frac{1}{2}\gamma_1 \\ \frac{1}{2}\gamma_1 & \frac{p_2}{m}(\beta_1 - bk_{2s}) - p_{12} \end{bmatrix}, \tag{16}$$

$$R_2 \triangleq \begin{bmatrix} \frac{p_{12}}{m}(\alpha_1 - bk_{1s}) & \frac{1}{2}\gamma_2 \\ \frac{1}{2}\gamma_2 & \frac{p_2}{m}(\beta_1 - bk_{2s}) - p_{12} \end{bmatrix}. \tag{17}$$

Note that since $K_s \in \mathcal{K}_s(P)$ it follows that $\tilde{f}(t) > 0$, $\tilde{g}(t) > 0$, and $\gamma_1 \leq h(t) \leq \gamma_2$ for all $t \in [0, \infty)$. Furthermore, $\delta_p > 0$, and R_1 and R_2 are positive definite. Let $\xi > 0$ satisfy

$$\xi \leq \lambda_{min}(R_1), \tag{18}$$

$$\xi \leq \lambda_{min}(R_2), \tag{19}$$

where $\lambda_{min}(\cdot)$ denotes the smallest eigenvalue of the argument matrix.

Consider the Lyapunov candidate for equation (5) given by

$$V(x) = \frac{1}{2}x^T P x + \frac{1}{2}\delta_p x_1^2, \tag{20}$$

which is positive definite, continuously differentiable, and radially unbounded. The time derivative $\dot{V}(t)$ of $V(t) \triangleq V(x(t))$ along the system trajectory is given by

$$\begin{aligned} \dot{V}(t) &= x^T P\dot{x} + \delta_p x_1 \dot{x}_1 \\ &= (x_1 p_1 + x_2 p_{12})x_2 - \frac{1}{m}(x_1 p_{12} + x_2 p_2)\left[x_2\tilde{g}(t) + x_1\tilde{f}(t)\right] + \delta_p x_1 x_2 \end{aligned}$$

$$
\begin{aligned}
&= -\frac{p_{12}}{m}\tilde{f}(t)x_1^2 - \left(\frac{p_{12}}{m}\tilde{g}(t) + \frac{p_2}{m}\tilde{f}(t) - p_1 - \delta_p\right)x_1x_2 - \left(\frac{p_2}{m}\tilde{g}(t) - p_{12}\right)x_2^2 \\
&= -\frac{p_{12}}{m}\tilde{f}(t)x_1^2 - h(t)x_1x_2 - \left(\frac{p_2}{m}\tilde{g}(t) - p_{12}\right)x_2^2 \\
&\le -\frac{p_{12}}{m}(\alpha_1 - bk_{1s})x_1^2 - h(t)x_1x_2 - \left[\frac{p_2}{m}(\beta_1 - bk_{2s}) - p_{12}\right]x_2^2 \\
&\le \begin{cases} -\frac{p_{12}}{m}(\alpha_1 - bk_{1s})x_1^2 - \gamma_1 x_1x_2 - \left[\frac{p_2}{m}(\beta_1 - bk_{2s}) - p_{12}\right]x_2^2, & x_1x_2 \ge 0, \\ -\frac{p_{12}}{m}(\alpha_1 - bk_{1s})x_1^2 - \gamma_2 x_1x_2 - \left[\frac{p_2}{m}(\beta_1 - bk_{2s}) - p_{12}\right]x_2^2, & x_1x_2 \le 0 \end{cases} \\
&= \begin{cases} -x^{\mathrm{T}}R_1x, & x_1x_2 \ge 0, \\ -x^{\mathrm{T}}R_2x, & x_1x_2 \le 0 \end{cases} \\
&\le -\xi x^T x. \qquad (21)
\end{aligned}
$$

Since $\xi x^T x$ is positive definite, it follows from Corollary 3.4 in Khalil (1996) that $x = 0$ is globally exponentially stable. □

Let $\mathcal{K}_s$ denote the set of constant gains K that render the equilibrium $x = 0$ of the closed-loop system (5) globally asymptotically stable with $\phi = -d/b$. Hence, $\mathcal{K}_s(P) \subseteq \mathcal{K}_s$.

3. ADAPTIVE STABILIZATION

Next we consider an adaptive stabilizer for equation (1) which operates with reduced parametric information. We assume that the functions f and g are globally piecewise uniformly continuous (see Appendix A) and bounded, m is positive, and b is non-zero. The globally piecewise uniformly continuous function class, a subset of the piecewise continuous functions class, includes certain discontinuous functions such as piecewise constant functions. For this case, the only parametric information required is knowledge of $\mathrm{sign}(b)$. Unlike the fixed-gain case, implementation of the adaptive algorithm does not require knowledge of m, b, d, and upper and lower bounds on f and g.

The control law

$$u(t) = k_1(t)\,q(t) + k_2(t)\,\dot{q}(t) + \phi(t), \qquad (22)$$

where the gains $k_1(t)$, $k_2(t)$ and the parameter $\phi(t)$ are adapted, will be used to obtain $q(t) \to 0$ and $\dot{q}(t) \to 0$ as $t \to \infty$.

Define the state x as in equation (3), and the gain matrix

$$K(t) \triangleq [k_1(t) \qquad k_2(t)]\,. \qquad (23)$$

Dynamic variables will henceforth be written without a time-dependence argument. Equations (1) and (22) written in state form yield equation (5), where now K and ϕ are time-dependent.

Next let $P \triangleq \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$ be positive definite with $p_{12} > 0$ and consider the adaptation law

$$\dot{K} = -B_0^{\mathrm{T}} P x x^{\mathrm{T}} \Lambda \tag{24}$$

$$\dot{\phi} = -B_0^{\mathrm{T}} P x \lambda \tag{25}$$

where $\Lambda \in \mathbb{R}^{2\times 2}$ is positive definite, and $\lambda \in \mathbb{R}$ is positive. Define $B_o \triangleq \begin{bmatrix} 0 \\ \mathrm{sign}\,(b) \end{bmatrix}$

The equilibrium set of equations (5), (24), and (25) is

$$\mathcal{E} = \{(x, K, \phi) \in \mathbb{R}^2 \times \mathbb{R}^{1\times 2} \times \mathbb{R} : \; x = [0, 0]^{\mathrm{T}}, \; K \in \mathbb{R}^{1\times 2}, \; \phi = -d/b\}.$$

Define the subsets of equilibria

$$\mathcal{E}_s \triangleq \{(x, K, \phi) \in \mathcal{E} : \; K \in \mathcal{K}_s\}$$

and

$$\mathcal{E}_s(P) \triangleq \{(x, K, \phi) \in \mathcal{E}_s : \; K \in \mathcal{K}_s(P)\}.$$

Theorem 1. Every element of $\mathcal{E}_s(P)$ is a uniformly Lyapunov stable equilibrium (Definition 3.2 in Khalil, 1996) of the closed-loop system (5), (24), and (25). Furthermore, let (x, K, ϕ) be a solution of equations (5), (24), and (25). Then $x(t) \to [0, 0]^{\mathrm{T}}$, $K(t)$ converges, and $\phi(t) \to -d/b$ as $t \to \infty$.

Proof. Let $\left([0 \;\; 0]^{\mathrm{T}}, K_s, -d/b\right) \in \mathcal{E}_s(P)$, where $K_s = [k_{1s} \;\; k_{2s}] \in \mathcal{K}_s$. Define

$$\tilde{k}_1 \triangleq k_1 - k_{1s}, \quad \tilde{k}_2 \triangleq k_2 - k_{2s}, \quad \tilde{K} \triangleq K - K_s, \quad \tilde{\phi} \triangleq \phi + d/b, \tag{26}$$

and $\tilde{f} : [0, \infty) \to \mathbb{R}$ and $\tilde{g} : [0, \infty) \to \mathbb{R}$ by equation (12). The closed-loop system (5), (24), and (25) can be written in the form

$$\dot{x} = \begin{bmatrix} x_2 \\ (1/m)\left[b\tilde{K}x + b\tilde{\phi} - \tilde{g}(t)x_2 - \tilde{f}(t)x_1\right] \end{bmatrix}, \tag{27}$$

$$\dot{\tilde{K}} = -B_0^{\mathrm{T}} P x x^{\mathrm{T}} \Lambda, \tag{28}$$

$$\dot{\tilde{\phi}} = -B_0^{\mathrm{T}} P x \lambda . \tag{29}$$

Let δ_p be given by equation (14), and let $h : [0, \infty) \to \mathbb{R}$ be given by equation (15). Let $R_1, \; R_2 \in \mathbb{R}^{2\times 2}$ be given by equations (16) and (17). As in Lemma 1, note that since $K_s \in \mathcal{K}_s(P)$ it follows that $\tilde{f}(t) > 0$ and $\tilde{g}(t) > 0$ for all $t \in [0, \infty)$, $\delta_p > 0$, and R_1 and R_2 are positive definite. Let $\xi > 0$ satisfy equations (18) and (19).

Consider the Lyapunov candidate for the system (27)–(29) given by

$$V(x,\tilde{K},\tilde{\phi}) = \frac{1}{2}x^{T}Px + \frac{1}{2}\delta_p x_1^2 + \frac{|b|}{2m}\operatorname{tr}\tilde{K}\Lambda^{-1}\tilde{K}^{\mathrm{T}} + \frac{|b|}{2m}\operatorname{tr}\tilde{\phi}\lambda^{-1}\tilde{\phi}^{\mathrm{T}}, \tag{30}$$

which is positive definite, continuously differentiable, and radially unbounded. The time derivative $\dot{V}(t)$ of $V(t) \triangleq V(x(t),\tilde{K}(t),\phi(t))$ along the system trajectory is given by

$$\begin{aligned}
\dot{V}(t) &= x^{\mathrm{T}}P\dot{x} + \delta_p x_1\dot{x}_1 + \frac{|b|}{m}\operatorname{tr}\tilde{K}\Lambda^{-1}\dot{\tilde{K}}^{\mathrm{T}} + \frac{|b|}{m}\operatorname{tr}\tilde{\phi}\lambda^{-1}\dot{\tilde{\phi}}^{\mathrm{T}} \\
&= (x_1p_1 + x_2p_{12})x_2 + \frac{1}{m}(x_1p_{12}+x_2p_2)\left(b\tilde{K}x + b\tilde{\phi} - x_2\tilde{g}(t) - x_1\tilde{f}(t)\right) \\
&\quad + \delta_p x_1x_2 + \frac{|b|}{m}\operatorname{tr}\tilde{K}\Lambda^{-1}\dot{\tilde{K}}^{\mathrm{T}} + \frac{|b|}{m}\operatorname{tr}\tilde{\phi}\lambda^{-1}\dot{\tilde{\phi}}^{\mathrm{T}} \\
&= (x_1p_1 + x_2p_{12})x_2 - \frac{1}{m}(x_1p_{12}+x_2p_2)\left(x_2\tilde{g}(t) + x_1\tilde{f}(t)\right) + \delta_p x_1x_2 \\
&\quad + \frac{|b|}{m}\operatorname{tr}\tilde{K}\left(xx^{\mathrm{T}}PB_0 + \Lambda^{-1}\dot{\tilde{K}}^{\mathrm{T}}\right) + \frac{|b|}{m}\operatorname{tr}\tilde{\phi}\left(x^{\mathrm{T}}PB_0 + \lambda^{-1}\dot{\tilde{\phi}}^{\mathrm{T}}\right) \\
&= -\frac{p_{12}}{m}\tilde{f}(t)x_1^2 - \left(\frac{p_{12}}{m}\tilde{g}(t) + \frac{p_2}{m}\tilde{f}(t) - p_1 - \delta_p\right)x_1x_2 - \left(\frac{p_2}{m}\tilde{g}(t) - p_{12}\right)x_2^2 \\
&= -\frac{p_{12}}{m}\tilde{f}(t)x_1^2 - h(t)x_1x_2 - \left(\frac{p_2}{m}\tilde{g}(t) - p_{12}\right)x_2^2 \\
&\le -\frac{p_{12}}{m}(\alpha_1 - b\,k_{1s})x_1^2 - h(t)x_1x_2 - \left[\frac{p_2}{m}(\beta_1 - b\,k_{2s}) - p_{12}\right]x_2^2 \\
&\le \begin{cases} -\frac{p_{12}}{m}(\alpha_1 - b\,k_{1s})x_1^2 - \gamma_1x_1x_2 - \left[\frac{p_2}{m}(\beta_1 - b\,k_{2s}) - p_{12}\right]x_2^2, & x_1x_2 \ge 0 \\ -\frac{p_{12}}{m}(\alpha_1 - b\,k_{1s})x_1^2 - \gamma_2x_1x_2 - \left[\frac{p_2}{m}(\beta_1 - b\,k_{2s}) - p_{12}\right]x_2^2, & x_1x_2 \le 0 \end{cases} \\
&= \begin{cases} -x^{\mathrm{T}}R_1x, & x_1x_2 \ge 0 \\ -x^{\mathrm{T}}R_2x, & x_1x_2 \le 0 \end{cases} \\
&\le -\xi x^{\mathrm{T}}x \\
&\le 0.
\end{aligned} \tag{31}$$

Hence $0 \le V(t) \le V(0)$ for all $t > 0$. Let $\|\cdot\|_2$ denote the Euclidean norm. Since V is quadratic and positive definite, there exist $\sigma_1,\sigma_2 > 0$ such that

$$\sigma_1\left\|\left[x^{\mathrm{T}},\tilde{K},\phi\right]^{\mathrm{T}}\right\|_2^2 \le V(x,\tilde{K},\tilde{\phi}) \le \sigma_2\left\|\left[x^{\mathrm{T}},\tilde{K},\phi\right]^{\mathrm{T}}\right\|_2^2, \tag{32}$$

for all $(x,\tilde{K},\phi) \in \mathbb{R}^n \times \mathbb{R}^{1\times n} \times \mathbb{R}$. Hence, for all $t > 0$,

$$\left\|\left[x^{\mathrm{T}}(t),\tilde{K}(t),\phi(t)\right]^{\mathrm{T}}\right\|_2^2 \le \left(\tfrac{1}{\sigma_1}\right)V(t) \le \left(\tfrac{1}{\sigma_1}\right)V(0) \le \left(\tfrac{\sigma_2}{\sigma_1}\right)\left\|\left[x^{\mathrm{T}}(0),\tilde{K}(0),\phi(0)\right]^{\mathrm{T}}\right\|_2, \tag{33}$$

which implies

$$\left\| \left[x^{\mathrm{T}}(t), \tilde{K}(t), \phi(t) \right]^{\mathrm{T}} \right\|_2 \leq \left(\frac{\sigma_2}{\sigma_1} \right) \left\| \left[x^{\mathrm{T}}(0), \tilde{K}(0), \phi(0) \right]^{\mathrm{T}} \right\|_2. \tag{34}$$

It follows from Lemma 3.3 in Khalil (1996) that $\left([0 \;\; 0]^{\mathrm{T}}, K_s, -d/b \right)$ is a uniformly Lyapunov stable equilibrium of equations (5), (24), and (25). Furthermore, since $V(t)$ is bounded and $V(x, \tilde{K}, \tilde{\phi})$ is radially unbounded, it follows that $x(t)$, $K(t)$, and $\phi(t)$ are bounded. Theorem 4.4 in Khalil (1996) implies that $\xi x^{\mathrm{T}}(t)\, x(t) \rightarrow 0$ as $t \rightarrow \infty$. Hence $x(t) \rightarrow []^{\mathrm{T}}$ as $t \rightarrow \infty$.

Since $x(t) \rightarrow 0$ as $t \rightarrow \infty$ it follows from equation (24) that $\dot{K}(t) \rightarrow 0$ as $t \rightarrow \infty$. Since $K(t)$ is bounded for all $t > 0$ it follows that $K(t)$ converges as $t \rightarrow \infty$.

Since x, K, ϕ, f, and g are bounded, it follows from equation (5) that $\dot{x}$ is bounded. Hence, since x is continuous, it follows that x is uniformly continuous. Since x is uniformly continuous and $x(t) \rightarrow 0$ as $t \rightarrow \infty$, it follows that $\lim_{t\rightarrow\infty} \int_0^t \dot{x}(\tau)\, \mathrm{d}\tau$ exists. Since f and g are globally piecewise uniformly continuous, it follows from equation (5) that $\dot{x}$ is globally piecewise uniformly continuous. Hence, Lemma 2 (see Appendix B) implies that $\dot{x}(t) \rightarrow 0$ as $t \rightarrow \infty$. Since $x(t) \rightarrow 0$ and $\dot{x}(t) \rightarrow 0$ as $t \rightarrow \infty$, and $K(t), f(t)$, and $g(t)$ are bounded, it follows from equation (5) that $\phi(t) \rightarrow -d/b$ as $t \rightarrow \infty$. □

Note that the bounds α_1, α_2, β_1, and β_2 for f and g defined by equation (6) are used only in the proof of Theorem 1 and need not be known in order to implement the adaptive controller (24), (25).

For the case in which equation (1) is time invariant, Theorem 1 specializes to Corollary 3.1 in Hong and Bernstein (2001), where the matrix P was obtained as the solution to the Lyapunov equation $A_s^{\mathrm{T}} P + P A_s + R = 0$, and where $A_s = A + B K_s$ and R is an arbitrary positive-definite matrix. It can be seen that when A_s is in companion form, the $(1, 2)$ entry of P is always positive. Hence, the requirement $p_{12} > 0$ represents no loss of generality when Theorem 1 is applied to linear time-invariant plants.

4. NUMERICAL EXAMPLES

Example 1. We begin with a simple introductory example problem to demonstrate the controller. Consider the system (1) with sinusoidally varying stiffness $f(t) = 5\sin(15t)$ and damping $g(t) = 3\sin(3\pi t)$ and with parameters $m = 1$, $b = 1$, and constant disturbance $d = 1$. Choose the controller parameters

$$P = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \lambda = 1 \tag{35}$$

Figure 1 shows the system trajectory in the $q, \dot{q}$ plane and Figure 2 shows the time history of k_1, k_2, and ϕ. The system is first simulated uncontrolled, and then the adaptive controller is activated at $t = 10$. The controlled response indicates that q and $\dot{q}$ converge to zero, the gains k_1 and k_2 converge, and ϕ converges to -1.

Example 2. Next we consider a system that is non-periodic and continuous with discontinuous derivative. This demonstrates that the controller does not require a basis

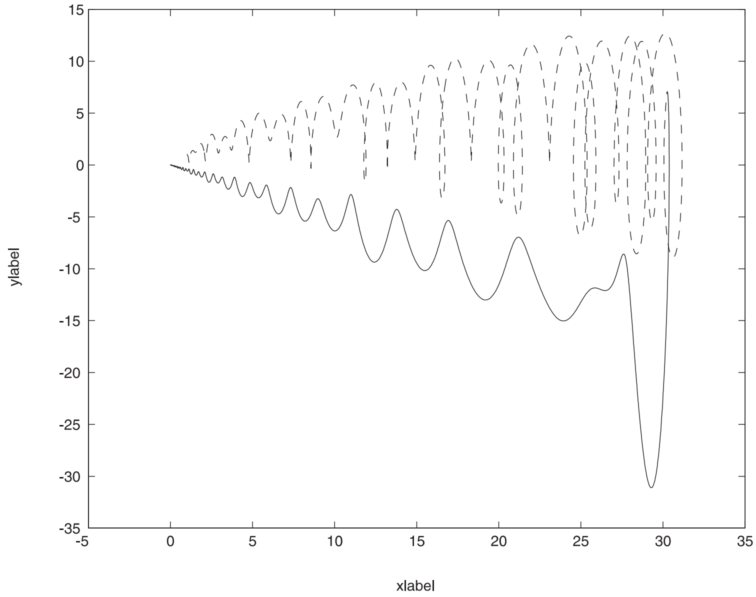

Figure 1. Trajectory in the $q, \dot{q}$ plane for Example 1. Initial conditions are $q = 1$, $\dot{q} = 1$, $k_1 = 0$, $k_2 = 0$, and $\phi = 0$. The uncontrolled response is shown by the dashed line, and the controlled response is shown by the solid line.

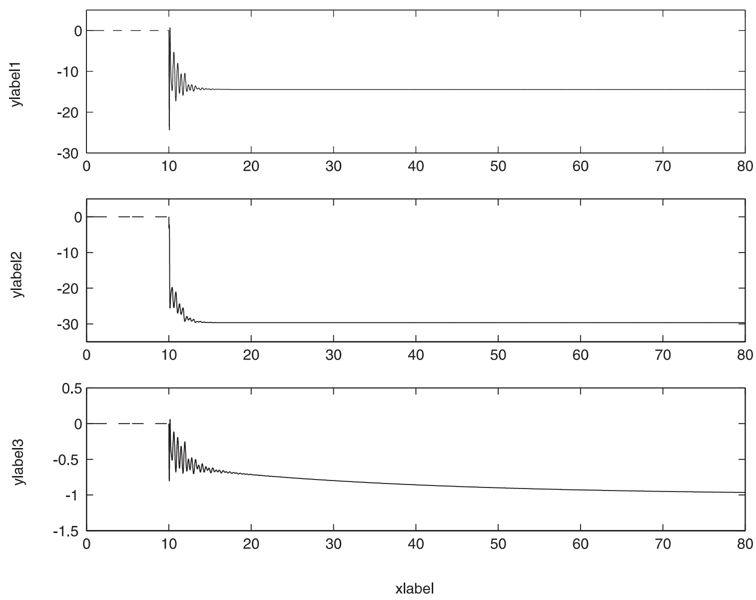

Figure 2. Time history of k_1, k_2, and ϕ for Example 1. The control system is activated at $t = 10$.

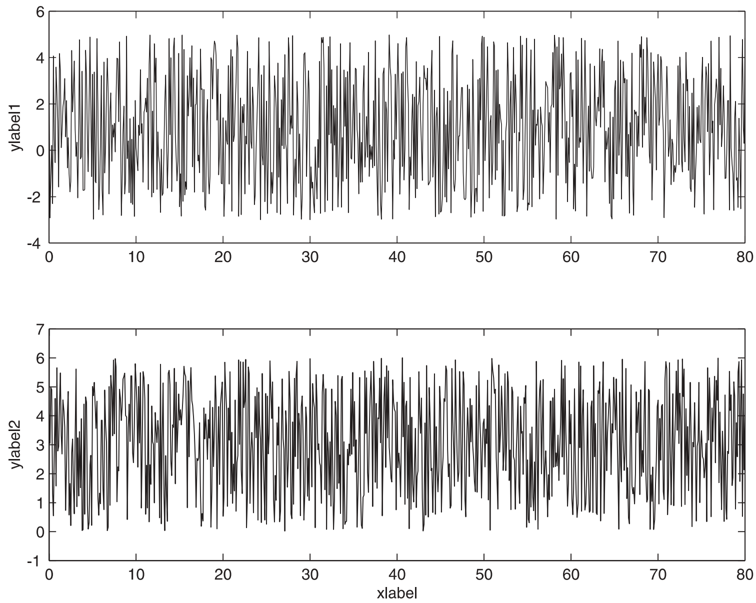

Figure 3. Time-varying stiffness and damping coefficients for Example 2.

expansion for the time variation. Consider the system given by equation (1) where the continuous piecewise linear functions $f(t)$ and $g(t)$ are shown in Figure 3. In this example the stiffness $f(t)$ varies between -3 and 5 and the damping $g(t)$ varies between 0 and 6. Let $m = 1$, $b = 1$, and $d = 1$. Choose controller parameters as in equation (35). Figure 4 shows the system trajectory in the $q, \dot{q}$ plane and Figure 5 shows the time history of k_1, k_2, and ϕ. Following activation of the adaptive controller at $t = 10$, q and $\dot{q}$ converge to zero, the gains k_1 and k_2 converge, and ϕ converges to -1.

Example 3. Finally, we consider a system with discontinuous time variation to demonstrate that continuity is not required for stability and convergence. Consider the system given by equation (1) with discontinuous coefficient functions $f(t) = 5 + 3\,\mathrm{sign}(\sin(t))$ and $g(t) = 3 + 5\,\mathrm{sign}(\cos(\pi t))$ shown in Figure 6. Let $m = 1$, $b = 1$, and $d = 1$. Again, choose controller parameters as in equation (35). Figure 7 shows the system trajectory in the q, $\dot{q}$ plane and Figure 8 shows the time history of k_1, k_2, and ϕ. As before, we allow the system to propagate uncontrolled until $t = 10$ in order to demonstrate the uncontrolled behavior of the system. After the control is activated at $t = 10$, q and $\dot{q}$ converge to zero, the gains k_1 and k_2 converge, and ϕ converges to -1.

5. CONCLUSION

In this paper we have adaptively stabilized a class of second-order linear time-varying systems. We have proven closed-loop Lyapunov stability and convergence of the plant states. Future work will include extension to higher-order plants, more general basis representations, and nonlinear time-varying plants.

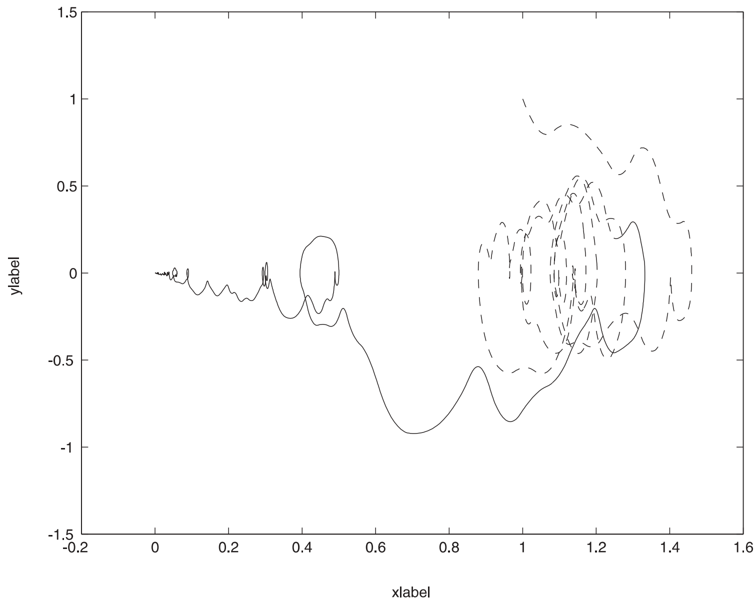

Figure 4. Trajectory in the $q, \dot{q}$ plane for Example 2. Initial conditions are $q = 1$, $\dot{q} = 1$, $k_1 = 0$, $k_2 = 0$, and $\phi = 0$. The uncontrolled response is shown by the dashed line, and the controlled response is shown by the solid line.

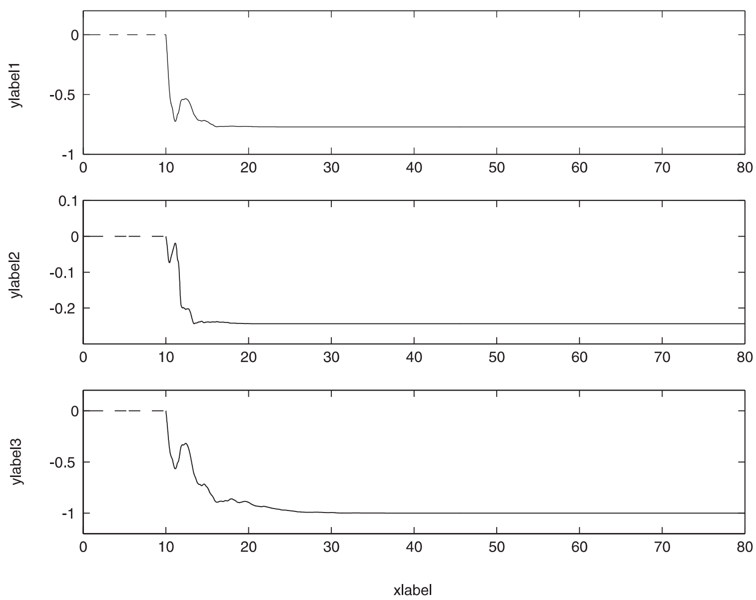

Figure 5. Time history of k_1, k_2, and ϕ for Example 2. The control system is activated at $t = 10$.

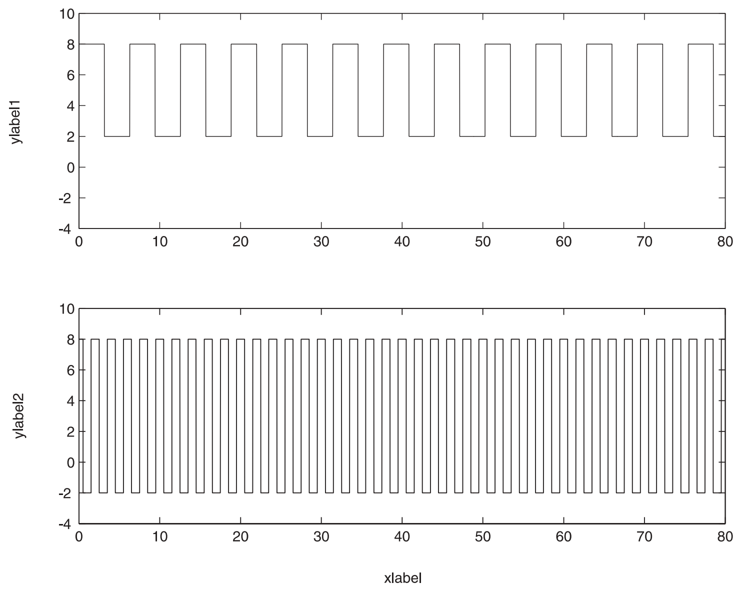

Figure 6. Discontinuous stiffness and damping coefficient functions for Example 3.

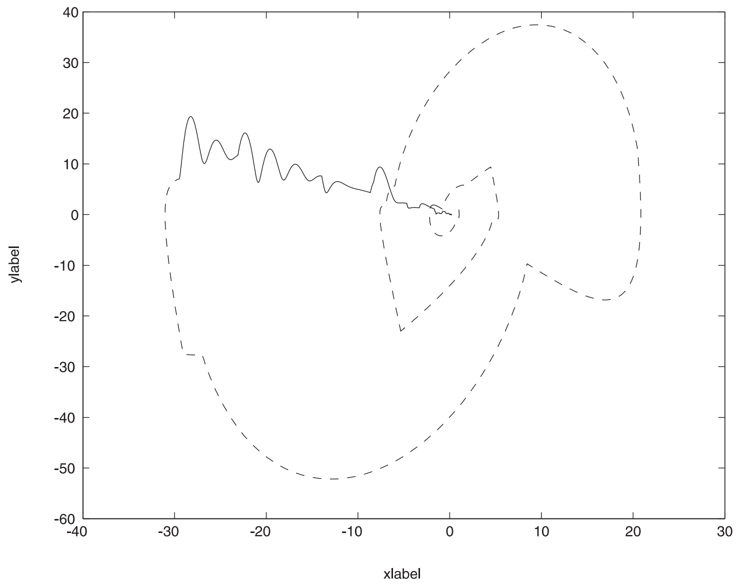

Figure 7. Trajectory in the $q, \dot{q}$ plane for Example 3. Initial conditions are $q = 1$, $\dot{q} = 1$, $k_1 = 0$, $k_2 = 0$, and $\phi = 0$. The uncontrolled response is shown by the dashed line, and the controlled response is shown by the solid line.

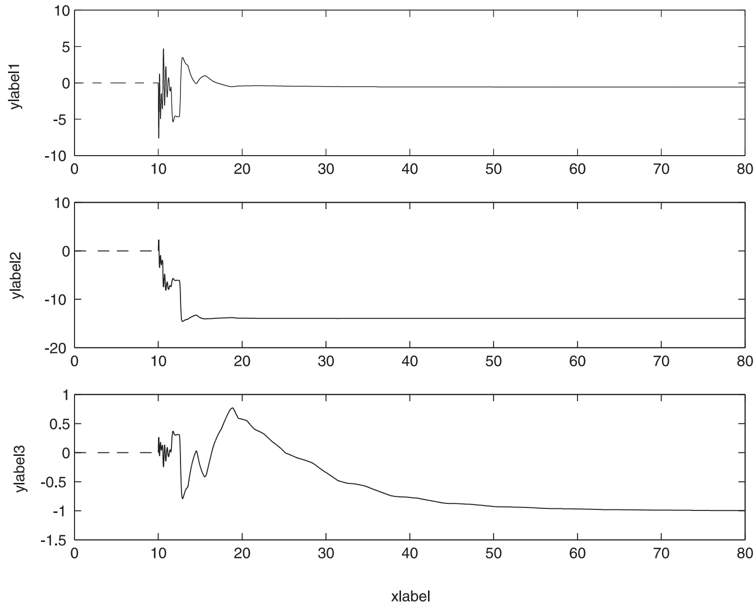

Figure 8. Time history of k_1, k_2, and ϕ for Example 3. The control system is activated at $t = 10$.

APPENDIX A

Definition 2. Let $\psi : [0, \infty) \to \mathbb{R}^n$ and let $\{t_i\}_{i=1}^{\infty}$ be an increasing sequence in $[0, \infty)$ such that $\inf_{i\in\mathcal{Z}_+}(t_{i+1} - t_i) > 0$. Let $\|\cdot\|$ denote a norm on $\mathbb{R}^n$. Then ψ is *globally piecewise uniformly continuous* if there exists $\delta : [0, \infty) \to [0, \infty)$ such that, for every $i \in \mathcal{Z}_+$ and $\varepsilon > 0$, $\|\psi(t) - \psi(t')\| < \varepsilon$ for all $t, t' \in [t_i, t_{i+1})$ such that $|t - t'| < \delta(\varepsilon)$.

APPENDIX B

Lemma 2. Let $\psi : [0, \infty) \to \mathbb{R}^n$ be globally piecewise uniformly continuous and assume that $\int_0^\infty \psi(\tau)\,\mathrm{d}\tau$ exists. Then $\psi(t) \to 0$ as $t \to \infty$.

Proof. It suffices to consider $n = 1$. Let $\{t_i\}_{i=1}^{\infty}$ be the increasing sequence of points of discontinuity of ψ as in Definition 2. Define $T \triangleq \inf_{i\in\mathcal{Z}_+}(t_{i+1} - t_i)$, and let $j : [0, \infty) \to \mathcal{Z}_+$ be given by

$$j(t) \triangleq i, \qquad t_i \le t < t_{i+1},\ i \in \mathcal{Z}_+. \tag{36}$$

The function $j(t)$ gives the index of the interval in which t resides. Next, we note that

$$\int_0^\infty \psi(\tau)\,\mathrm{d}\tau = \sum_{i=1}^{\infty} \Theta_i, \tag{37}$$

where

$$\Theta_i \triangleq \int_{t_i}^{t_{i+1}} \psi(\tau)\,\mathrm{d}\tau, \qquad i \in \mathcal{Z}_+. \tag{38}$$

Since $\sum_{i=1}^{\infty} \Theta_i$ exists, it follows that $\Theta_i \to 0$ as $i \to \infty$.

Next, define the piecewise constant function $\theta : [0,\infty) \to \mathbb{R}$ by

$$\theta(t) \triangleq \frac{\Theta_{j(t)}}{t_{j(t)+1} - t_{j(t)}}, \qquad t \in [0,\infty). \tag{39}$$

Note that

$$\int_0^{t_i} \theta(\tau)\,\mathrm{d}\tau = \int_0^{t_i} \psi(\tau)\,\mathrm{d}\tau, \qquad i \in \mathcal{Z}_+. \tag{40}$$

Define the piecewise linear function $\vartheta : [0,\infty) \to \mathbb{R}$ by

$$\vartheta(t) \triangleq \Theta_{j(t)} \frac{t - t_{j(t)}}{t_{j(t)+1} - t_{j(t)}}, \tag{41}$$

and note that

$$\int_0^{t} \theta(\tau)\,\mathrm{d}\tau = \int_0^{t_{j(t)}} \theta(\tau)\,\mathrm{d}\tau + \vartheta(t), \qquad t \in [0,\infty). \tag{42}$$

Since $0 \le \vartheta(t) \le \Theta_{j(t)}$ and $0 \le T\theta(t) \le \Theta_{j(t)}$ for all $t \in [0,\infty)$, and $\lim_{t\to\infty} \Theta_{j(t)} = \lim_{i\to\infty} \Theta_i = 0$, it follows that

$$\lim_{t\to\infty} \vartheta(t) = 0, \qquad \lim_{t\to\infty} \theta(t) = 0. \tag{43}$$

Hence, equations (43), (42), and (36) imply

$$\begin{aligned} \lim_{t\to\infty} \int_0^{t} \theta(\tau)\,\mathrm{d}\tau &= \lim_{t\to\infty} \left(\int_0^{t_{j(t)}} \theta(\tau)\,\mathrm{d}\tau + \vartheta(t) \right) \\ &= \lim_{t\to\infty} \int_0^{t_{j(t)}} \theta(\tau)\,\mathrm{d}\tau = \lim_{i\to\infty} \int_0^{t_i} \theta(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{44}$$

It follows from equations (44), (39), and (37) that

$$\lim_{t\to\infty} \int_0^{t} \theta(\tau)\,\mathrm{d}\tau = \lim_{i\to\infty} \int_0^{t_i} \theta(\tau)\,\mathrm{d}\tau = \sum_{i=1}^{\infty} \Theta_i = \lim_{t\to\infty} \int_0^{t} \psi(\tau)\,\mathrm{d}\tau, \tag{45}$$

which implies

$$\int_0^{\infty} \left(\psi(\tau) - \theta(\tau)\right)\,\mathrm{d}\tau = 0. \tag{46}$$

Define $v : [0, \infty) \to \mathbb{R}$ by

$$v(t) \triangleq \psi(t) - \theta(t), \tag{47}$$

so that $\int_0^\infty v(\tau)\,\mathrm{d}\tau = 0$. We wish to show that $v(t) \to 0$ as $t \to \infty$. Since ψ is globally piecewise uniformly continuous and θ is piecewise constant, it follows that v is globally piecewise uniformly continuous. Hence, there exists $\delta : [0, \infty) \to [0, \infty)$ such that, for every $i \in \mathcal{Z}_+$ and $\varepsilon_0 > 0$, $|v(t) - v(t')| < \varepsilon_0$ for all $t, t' \in [t_i, t_{i+1})$ such that $|t - t'| < \delta(\varepsilon_0)$.

Choose $\varepsilon > 0$. Let $\varepsilon' > 0$ satisfy $\varepsilon' < \varepsilon/4$, and let $\sigma > 0$ satisfy $\sigma < \min(\delta(\varepsilon'), T)$. Since $\int_0^\infty v(\tau)\,\mathrm{d}\tau = 0$, it follows that there exists $T_1 > 0$ such that

$$\left| \int_0^t v(\tau)\,\mathrm{d}\tau \right| < \frac{\sigma\varepsilon}{4} \qquad \text{for all } t > T_1. \tag{48}$$

Hence,

$$\left| \int_t^{t+\sigma} v(\tau)\,\mathrm{d}\tau \right| < \frac{\sigma\varepsilon}{2} \qquad \text{for all } t > T_1. \tag{49}$$

Let $i \in \mathcal{Z}_+$, $i \geq j(T_1)$ and let $t > T_1$ satisfy $t_i \leq t \leq t_{i+1} - \sigma$. Note that

$$|v(t)| - \varepsilon' \leq |v(t')| \leq |v(t)| + \varepsilon' \qquad \text{for all } t \leq t' \leq t + \sigma. \tag{50}$$

Suppose $|v(t)| > \varepsilon'$. Then equations (50) and (49) imply

$$\int_t^{t+\sigma} (|v(t)| - \varepsilon')\,\mathrm{d}\tau \leq \left| \int_t^{t+\sigma} v(\tau)\,\mathrm{d}\tau \right| < \frac{\sigma\varepsilon}{2}. \tag{51}$$

Hence,

$$|v(t)| < \frac{\varepsilon}{2} + \varepsilon', \tag{52}$$

and it follows from equation (50) that

$$|v(t')| < 2\left(\frac{\varepsilon}{4} + \varepsilon'\right) < \varepsilon \qquad \text{for all } t \leq t' \leq t + \sigma. \tag{53}$$

On the other hand, suppose $|v(t)| \leq \varepsilon'$. Then

$$|v(t')| \leq 2\varepsilon' < \varepsilon \qquad \text{for all } t \leq t' \leq t + \sigma. \tag{54}$$

It follows from equations (53) and (54) that $v(t) < \varepsilon$ for all $t > T_1$. Hence $v(t) \to 0$ as $t \to \infty$. Since $\theta(t) \to 0$ as $t \to \infty$, it follows that $\psi(t) \to 0$ as $t \to \infty$. $\square$

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