State Space Modeling of an Acoustic Duct With an End-Mounted Speaker*

R. Venugopal

D. S. Bernstein

Department of Aerospace Engineering,
The University of Michigan,
Ann Arbor, MI 48109-2118
{ravinder, dsbaero}@umich.edu

This paper develops a state space model of the dynamics of an acoustic duct with end-mounted speakers. The initial model formulation includes the forcing term as part of the boundary conditions. The shifted particle velocity is then defined to transform the nonhomogeneous boundary conditions into homogeneous boundary conditions and thus develop the state space model. It is shown that the speaker and acoustic dynamics interact by means of feedback in which the speaker creates an acoustic field, which, in turn, affects the motion of the speaker cone. This interaction is studied using positive real closed-loop feedback analysis, and shifts in the modal frequencies of the duct due to the presence of the end-mounted speaker are predicted.

1 Introduction

The feasibility of active noise control technology is facilitated by the availability of affordable high performance actuators, that is, audio speakers. As in all control applications, the dynamical properties of the electromechanical actuator, such as frequency response, amplitude limits, and back EMF, must be accounted for in the design process. An additional consideration in real-world implementation that is not readily apparent is the two-way coupling of the acoustic dynamics and the actuator dynamics. Once the speaker properties have been characterized, it is standard practice to model the acoustic-speaker system as the cascade, or one-way interaction, of two dynamical systems. In reality, however, this interaction is two-way since the speaker also reacts to the acoustic back pressure. Consequently, the acoustic duct dynamics and the speaker dynamics are coupled by means of a feedback mechanism. The purpose of this paper is to derive a state space model of this feedback interaction.

The basic problem we consider involves a one-dimensional acoustic duct with an end-mounted speaker. Unlike the case of a side-mounted speaker [1], [2], [3], the end-mounted speaker case, which was studied in [4], requires careful consideration since the speaker cone velocity enters the partial differential equation through the boundary condition. To arrive at a state space model for control design, we define a shifted particle velocity as in [10], pp. 284–287, that satisfies standard, control-free, boundary conditions. In the case of an ideal speaker having commandable cone velocity, the duct with open-speaker and closed-speaker boundary conditions is shown to possess the same modal frequencies and mode shapes as in the case of a side-mounted speaker. We then introduce voltage-driven second-order speaker dynamics and show that in this case the speaker dynamics interact with the duct dynamics in a feedback loop. We show that this feedback interaction leads to the shifting of the duct modal frequencies in an intuitively expected way. Specifically, at high frequencies, where the speaker has high impedance, the modal frequencies correspond closely to those of a closed-end boundary condition, whereas, at low frequencies and especially near the speaker's resonance frequency, the modal frequencies correspond closely to those of an open-end boundary condition. These results are demonstrated for both open-speaker and closed-speaker boundary conditions. The method is then extended to the case of a duct with speakers mounted on both ends.

Our modal analysis of the dynamics of an end-mounted speaker complements the duct termination impedance analysis of [5] (pp. 119–120) and [6] (pp. 210–212) by providing an exact characterization of the modal frequencies of a duct with an end-speaker and yields a state space model for use in active noise control.

* This research was supported in part by the Air Force Office of Scientific Research under grant F49620-95-1-0019 and the University of Michigan Office of the Vice President for Research.

Contributed by the Technical Committee on Vibration and Sound for publication in the JOURNAL OF VIBRATION AND ACOUSTICS. Manuscript received Mar. 1996; revised Mar. 1997. Associate Technical Editor: G. Koopmann.

2 Duct Dynamics With Commandable Speaker Cone Velocity

2.1 Open-Speaker Boundary Conditions. Consider a one-dimensional acoustic duct of length l. Let x be the spatial coordinate along the length of the duct with one end at $x = 0$ and the other at $x = l$. A speaker is mounted at $x = 0$ and the end $x = l$ is open. Assuming that there is no mean flow in the duct, the wave equation that describes the acoustic dynamics in the duct is

$$\frac{\partial^2 u(x, t)}{\partial t^2} - c^2 \frac{\partial^2 u(x, t)}{\partial x^2} = 0, \tag{1}$$

where $u(x, t)$ is the longitudinal velocity of a particle in the acoustic medium at location x and time t, and c is the acoustic wave velocity in the medium. The boundary conditions are

$$u(0, t) = v_{0s}(t), \quad \left.\frac{\partial u(x, t)}{\partial x}\right|_{x=l} = 0, \tag{2}$$

where $v_{0s}(t)$ is the cone velocity of the speaker located at $x = 0$. The first boundary condition neglects the radiation effects due to the nonuniform flow and nonuniform pressure distribution across the open end of the duct. These effects can be neglected by assuming that the diameter of the duct is less than one tenth of the smallest wavelength of the acoustic waves [7] (pp. 471–474). If this assumption is not satisfied, additional mass and dissipative terms which represent the acoustic termination impedance must be included.

Let the shifted particle velocity $v(x, t)$ be defined by

$$v(x, t) \triangleq u(x, t) - v_{0s}(t). \tag{3}$$

Then from (2) it follows that

$$v(0, t) = 0, \quad \left.\frac{\partial v(x, t)}{\partial x}\right|_{x=l} = 0. \tag{4}$$

Substituting $u(x, t)$ from (3) into (1) yields

$$\frac{\partial^2 v(x, t)}{\partial t^2} - c^2 \frac{\partial^2 v(x, t)}{\partial x^2} = -\ddot{v}_{0s}(t), \tag{5}$$

where a dot over a variable denotes its time derivative. Letting

$$v(x, t) = \sum_{i=0}^{\infty} H_i(x) q_i(t), \tag{6}$$

the eigenfunctions of (5) with boundary conditions (4) are given by

$$H_i(x) = \sqrt{\frac{2}{l}} \sin \frac{(2i+1)\pi x}{2l}, \quad i = 0, 1, \ldots \tag{7}$$

From (5), (6) and (7) it thus follows that

$$\sum_{i=0}^{\infty} \left[\sqrt{\frac{2}{l}} \sin \frac{(2i+1)\pi x}{2l} \ddot{q}_i(t) + \left(\frac{c(2i+1)\pi}{2l} \right)^2 \sqrt{\frac{2}{l}} \sin \frac{(i+1)\pi x}{2l} q_i(t) \right] = -\ddot{v}_{0s}(t). \tag{8}$$

Multiplying (8) by $H_j(x)$ and integrating over the length $0 \le x \le l$ yields

$$\ddot{q}_j(t) + \omega_{nj}^2 q_j(t) = -\frac{2l}{(2j+1)\pi} \sqrt{\frac{2}{l}} \ddot{v}_{0s}(t), \quad j = 0, 1, \ldots, \tag{9}$$

where the natural frequency of the jth mode is $\omega_{nj} = c(2j+1)\pi/2l$. Retaining r modes and defining the modal state vector

$$q(t) \triangleq [q_1(t)\ \dot{q}_1(t)\ q_2(t)\ \dot{q}_2(t)\ \ldots\ q_r(t)\ \dot{q}_r(t)]^T, \tag{10}$$

Equation (9) can be written in the form

$$\dot{q}(t) = \hat{A} q(t) + \hat{B} \ddot{v}_{0s}(t), \tag{11}$$

where

$$\hat{A} \triangleq \text{block} - \text{diag} \left(\begin{bmatrix} 0 & 1 \\ -\omega_{n1}^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -\omega_{n2}^2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ -\omega_{nr}^2 & 0 \end{bmatrix} \right), \tag{12}$$

$$\hat{B} \triangleq \frac{2\sqrt{2l}}{\pi} \left[0 \quad -1 \quad 0 \quad -\frac{1}{3} \quad \ldots \quad 0 \quad -\frac{1}{2r-1} \right]^T. \tag{13}$$

Next, the rate of change of acoustic pressure is given by ([5], p. 15)

$$\dot{P}(x, t) = -\rho_0 c^2 \frac{\partial u(x, t)}{\partial x}, \tag{14}$$

where ρ_0 is the density of the acoustic medium. From (3) and (6) it follows that

$$\frac{\partial u(x, t)}{\partial x} = \sum_{i=0}^{\infty} H_i'(x) q_i(t), \tag{15}$$

where $H_i'(x) = \partial H_i(x)/\partial x$. Retaining r modes and using (14), (15) we obtain

$$\dot{P}_0(t) = -\rho_0 c^2 \sum_{i=0}^{r} H_i'(0)\, q_i(t), \tag{16}$$

where $P_0(t) \triangleq P(0, t)$. Using (7) this yields

$$\dot{P}_0(t) = \hat{C} q(t), \tag{17}$$

where

$$\hat{C} \triangleq \frac{-\rho_0 c^2 \pi}{\sqrt{2l^3}} [1 \quad 0 \quad 3 \quad 0 \quad \ldots \quad 2r-1 \quad 0]. \tag{18}$$

Now define the $(2r+1)$-dimensional state vector

$$x(t) \triangleq \begin{bmatrix} q(t) - \hat{B}\dot{v}_{0s}(t) - \hat{A}\hat{B}v_{0s}(t) \\ P_0(t) \end{bmatrix}, \tag{19}$$

and, noting that $\hat{C}\hat{B} = 0$, we rewrite (11) and (17) in state space form as

$$\dot{x}(t) = Ax(t) + Bv_{0s}(t), \tag{20}$$

$$P_0(t) = Cx(t), \tag{21}$$

where

$$A \triangleq \begin{bmatrix} \hat{A} & 0 \\ \hat{C} & 0 \end{bmatrix}, \quad B \triangleq \begin{bmatrix} \hat{A}^2\hat{B} \\ \hat{C}\hat{A}\hat{B} \end{bmatrix}, \quad C \triangleq [0 \quad \ldots \quad 0 \quad 1]. \tag{22}$$

We note that the zero eigenvalue of the system (22) comes from integrating the pressure rate $\dot{P}_0(t)$ in (17) to obtain the pressure measurement $P_0(t)$ in (21).

2.2 Closed-Speaker Boundary Conditions. For the case in which the end of the duct at $x = l$ is closed, the acoustic dynamics are given by (1) with the boundary conditions

$$u(0, t) = v_{0s}(t), \quad u(l, t) = 0. \tag{23}$$

Defining the shifted particle velocity $v(x, t)$ by

$$v(x, t) \triangleq u(x, t) - \left(1 - \frac{x}{l}\right) v_{0s}(t), \tag{24}$$

and substituting $u(x, t)$ from (24) into (1) yields

$$\frac{\partial^2 v(x, t)}{\partial t^2} - c^2 \frac{\partial^2 v(x, t)}{\partial x^2} = -\left(1 - \frac{x}{l}\right) \ddot{v}_{0s}(t), \tag{25}$$

with the boundary conditions (23) now written as

$$v(0, t) = 0, \quad v(l, t) = 0. \tag{26}$$

We note that the transformation (24), which converts the nonhomogeneous boundary conditions (23) to the homogeneous boundary conditions (26), is not unique. Although the shifted particle velocity $v(x, t)$ depends on this transformation, which involves $1 - (x/l)$ in this case, the particle velocity $u(x, t)$ is independent of (24).

Using (6), the eigenfunctions of (25) with the boundary conditions (26) are given by

$$H_i(x) = \sqrt{\frac{2}{l}} \sin \frac{i\pi x}{l}, \quad i = 1, 2, \ldots, \tag{27}$$

and from (25), (6) and (27) we obtain

$$\sum_{i=1}^{\infty} \left[\sqrt{\frac{2}{l}} \sin \frac{i\pi x}{l} \ddot{q}_i(t) + \left(\frac{ci\pi}{l} \right)^2 \sqrt{\frac{2}{l}} \sin \frac{i\pi x}{l} q_i(t) \right] = -\left(1 - \frac{x}{l}\right) \ddot{v}_{0s}(t). \tag{28}$$

Multiplying (28) by $\sqrt{2/l}\sin j\pi x/l$ and integrating over the length $0 \le x \le l$ yields

$$\ddot{q}_j(t) + \omega_{nj}^2 q_j(t) = -\frac{l}{j\pi}\sqrt{\frac{2}{l}}\,\ddot{v}_{0s}(t), \quad j = 1, 2, \ldots, \tag{29}$$

where the natural frequency of the jth mode is $\omega_{nj} = cj\pi/l$. Retaining r modes and defining the modal state vector as in (10), Eq. (29) can be rewritten in the form (11), where

$$\hat{A} \triangleq \text{block}-\text{diag}\left(\begin{bmatrix} 0 & 1 \\ -\omega_{n1}^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -\omega_{n2}^2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ -\omega_{nr}^2 & 0 \end{bmatrix}\right), \tag{30}$$

$$\hat{B} \triangleq \frac{\sqrt{2l}}{\pi}\begin{bmatrix} 0 & -1 & 0 & -\frac{1}{2} & \ldots & 0 & -\frac{1}{r} \end{bmatrix}^T. \tag{31}$$

Now, retaining r modes and using (14), (24) and (6) we obtain

$$\dot{P}_0(t) = -\rho_0 c^2\left(\sum_{i=1}^{r} H_i'(0)q_i(t) - \frac{v_{0s}(t)}{l}\right), \tag{32}$$

which implies that

$$\dot{P}_0(t) = \hat{C}q(t) + \hat{D}v_{0s}(t), \tag{33}$$

where

$$\hat{C} \triangleq -\rho_0 c^2 \pi\sqrt{\frac{2}{l^3}}\,[1 \quad 0 \quad 2 \quad 0 \quad \ldots \quad r \quad 0],$$

$$\hat{D} \triangleq \frac{\rho_0 c^2}{l}. \tag{34}$$

Noting that $\hat{C}\hat{B} = 0$ and defining $x(t)$ as in (19), we can rewrite (11) and (33) in state space form (20), (21), where

$$A \triangleq \begin{bmatrix} \hat{A} & 0 \\ \hat{C} & 0 \end{bmatrix}, \quad B \triangleq \begin{bmatrix} \hat{A}^2\hat{B} \\ \hat{C}\hat{A}\hat{B} + \hat{D} \end{bmatrix},$$

$$C \triangleq [0 \quad \ldots \quad 0 \quad 1]. \tag{35}$$

2.3 Speaker-Speaker Boundary Conditions. We now consider the case of speakers mounted on both ends of the duct. In this case the dynamics of the duct are given by (1) with the boundary conditions

$$u(0, t) = v_{0s}(t), \quad u(l, t) = v_{ls}(t), \tag{36}$$

where $v_{0s}(t)$ is the cone velocity of the speaker located at $x = 0$, and $v_{ls}(t)$ is the cone velocity of the speaker located at $x = l$. Defining the shifted particle velocity $v(x, t)$ by

$$v(x, t) \triangleq u(x, t) - \left(1 - \frac{x}{l}\right)v_{0s}(t) - \frac{x}{l}v_{ls}(t) \tag{37}$$

and substituting $u(x, t)$ from (37) into (1) yields

$$\frac{\partial^2 v(x, t)}{\partial t^2} - c^2\frac{\partial^2 v(x, t)}{\partial x^2} = -\left(1 - \frac{x}{l}\right)\ddot{v}_{0s}(t) - \frac{x}{l}\ddot{v}_{ls}(t), \tag{38}$$

with the nonhomogeneous boundary conditions (36) now converted to the homogeneous form (26). Using (6), the eigenfunctions of (38) with the boundary conditions (26) are given by (27), and from (38) and (6) we obtain

$$\sum_{i=1}^{\infty}\left[\sqrt{\frac{2}{l}}\sin\frac{i\pi x}{l}\,\ddot{q}_i(t) + \left(\frac{ci\pi}{l}\right)^2\sqrt{\frac{2}{l}}\sin\frac{i\pi x}{l}\,q_i(t)\right] = -\left(1 - \frac{x}{l}\right)\ddot{v}_{0s}(t) - \frac{x}{l}\ddot{v}_{ls}(t). \tag{39}$$

Multiplying (39) by $\sqrt{2/l}\sin j\pi x/l$ and integrating over the length $0 \le x \le l$ yields

$$\ddot{q}_j(t) + \omega_{nj}^2 q_j(t) = -\frac{l}{j\pi}\sqrt{\frac{2}{l}}\,\ddot{v}_{0s}(t) + \frac{l}{j\pi}\sqrt{\frac{2}{l}}(-1)^j\ddot{v}_{ls}(t), \quad j = 1, 2, \ldots, \tag{40}$$

where the natural frequency of the jth mode is $\omega_{nj} = cj\pi/l$. Retaining r modes and defining the modal state vector as in (10), Eq. (40) can be rewritten in the form

$$\dot{q}(t) = \hat{A}q(t) + \hat{B}\ddot{v}_{0ls}(t), \tag{41}$$

where

$$\hat{A} \triangleq \text{block}-\text{diag}\left(\begin{bmatrix} 0 & 1 \\ -\omega_{n1}^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -\omega_{n2}^2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ -\omega_{nr}^2 & 0 \end{bmatrix}\right), \tag{42}$$

$$\hat{B} \triangleq \frac{\sqrt{2l}}{\pi}\begin{bmatrix} 0 & -1 & 0 & -\frac{1}{2} & \cdots & 0 & -\frac{1}{r} \\ 0 & -1 & 0 & \frac{1}{2} & \cdots & 0 & \frac{(-1)^r}{r} \end{bmatrix}^T, \tag{43}$$

$$v_{0ls}(t) \triangleq \begin{bmatrix} v_{0s}(t) \\ v_{ls}(t) \end{bmatrix}. \tag{44}$$

Now, retaining r modes and using (14), (37) and (6), we obtain

$$\dot{P}_0(t) = -\rho_0 c^2\left(\sum_{i=1}^{r} H_i'(0)q_i(t) - \frac{v_{0s}(t)}{l} + \frac{v_{ls}(t)}{l}\right), \tag{45}$$

$$\dot{P}_l(t) = -\rho_0 c^2\left(\sum_{i=1}^{r} H_i'(l)q_i(t) - \frac{v_{0s}(t)}{l} + \frac{v_{ls}(t)}{l}\right), \tag{46}$$

where $P_l(t) \triangleq P(l, t)$. Letting $P(t) \triangleq \begin{bmatrix} P_0(t) \\ P_l(t) \end{bmatrix}$ and defining $q(t)$ as in (10), (45) and (46) implies

$$\dot{P}(t) = \hat{C}q(t) + \hat{D}v_{0ls}(t), \tag{47}$$

where

$$\hat{C} \triangleq -\rho_0 c^2\pi\sqrt{\frac{2}{l^3}}\begin{bmatrix} 1 & 0 & 2 & 0 & \cdots & r & 0 \\ -1 & 0 & 2 & 0 & \cdots & (-1)^r r & 0 \end{bmatrix},$$

$$\hat{D} \triangleq \rho_0 c^2\begin{bmatrix} 1/l & -1/l \\ 1/l & -1/l \end{bmatrix}. \tag{48}$$

Defining the $2r + 2$ dimensional state vector

$$x(t) \triangleq \begin{bmatrix} q(t) - \hat{B}\dot{v}_{12s}(t) - \hat{A}\hat{B}v_{12s}(t) \\ P(t) \end{bmatrix}, \tag{49}$$

and using $\hat{C}\hat{B} = 0$, we can rewrite (41) and (47) in the state space form

$$\dot{x}(t) = Ax(t) + Bv_{0Is}(t), \tag{50}$$

$$P(t) = Cx(t), \tag{51}$$

where A and B are of the form (35) and

$$C \triangleq \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}. \tag{52}$$

3 Coupled Acoustic and Speaker Dynamics

3.1 Open-Speaker and Closed-Speaker Boundary Conditions. In this section we consider an acoustic system comprised of a duct with either open-speaker or closed-speaker boundary conditions, and with speaker dynamics. To begin with, we assume that the dynamics of the speaker can be modeled as [8]

$$I_{0s}\dot{v}_{0s}(t) + \left(R_{0s} + \frac{\mu_0^2}{R_{0c}}\right)v_{0s}(t) + K_{0s}x_{0s}(t)$$

$$= \frac{\mu_0}{R_{0c}} V_0(t) - S_0 P_0(t), \tag{53}$$

where I_{0s} is the mechanical inertia of the speaker, R_{0s} is the viscous friction of the speaker, μ_0 is the electromagnetic coupling factor defined to be the electromagnetic force generated by the speaker coil per unit measure of current input to the coil, R_{0c} is the resistance of the speaker coil, K_{0s} is the stiffness of the speaker, $x_{0s}(t)$ is the speaker cone displacement, $V_0(t)$ is the voltage input to the speaker, S_0 is the speaker cone area and $P_0(t)$ is the acoustic pressure at $x = 0$.

Equation (53) can be written in state space form as

$$\dot{\xi}_s(t) = A_s\xi_s(t) + B_{Ps}P_0(t) + B_{Vs}V_0(t), \tag{54}$$

$$v_{0s}(t) = C_s\xi_s(t), \tag{55}$$

where $\xi_s \triangleq [x_{0s}(t)\ v_{0s}(t)]^T$ and

$$A_s \triangleq \begin{bmatrix} 0 & 1 \\ -\dfrac{K_{0s}}{I_{0s}} & -\dfrac{1}{I_{0s}}\left(R_{0s} + \dfrac{\mu_0^2}{R_{0c}}\right) \end{bmatrix}, \quad B_{Ps} \triangleq \begin{bmatrix} 0 \\ -\dfrac{S_0}{I_{0s}} \end{bmatrix},$$

$$B_{Vs} \triangleq \begin{bmatrix} 0 \\ \dfrac{\mu_0}{I_{0s}R_{0c}} \end{bmatrix}, \quad C_s \triangleq [0\ 1]. \tag{56}$$

Note that the speaker cone velocity $v_{0s}(t)$ appears in (2), (23) and (36).

Next, by combining (54), (55) with (20), (21) and by defining the augmented state vector

$$x_a(t) \triangleq \begin{bmatrix} x(t) \\ \xi_s(t) \end{bmatrix}, \tag{57}$$

we obtain the state space representation of the closed-loop system

$$\dot{x}_a = \mathcal{A}x_a(t) + \mathcal{B}V_0(t), \tag{58}$$

where

$$\mathcal{A} \triangleq \begin{bmatrix} A & BC_s \\ B_{Ps}C & A_s + B_{Ps}DC_s \end{bmatrix}, \quad \mathcal{B} \triangleq \begin{bmatrix} 0 \\ B_{Vs} \end{bmatrix}. \tag{59}$$

The combined system described by (58) is equivalent to the series connection of acoustic impedances as described in [6] (pp. 210–212). This series connection is equivalent to the feedback interconnection shown in Fig. 1. As shown in the Appendix, the lossless duct is positive real and the speaker is strictly positive real, and thus their feedback interconnection is asymptotically stable. The above analysis is valid for both open-speaker and closed-speaker boundary conditions.

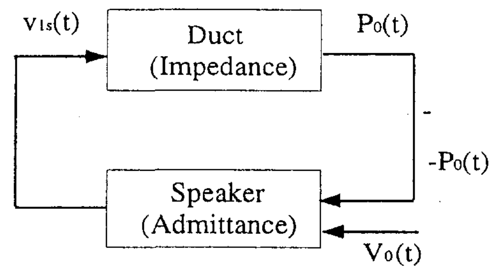

Fig. 1 Block diagram representation of coupled duct and end speaker dynamics

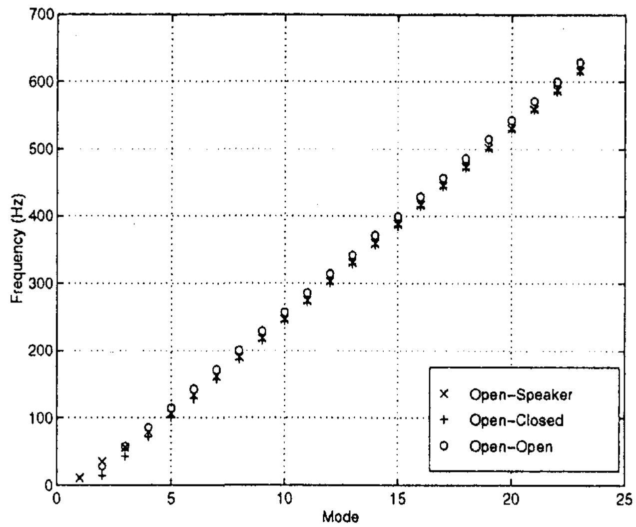

Fig. 2 Modal frequency comparison for open-speaker, open-closed and open-open boundary conditions

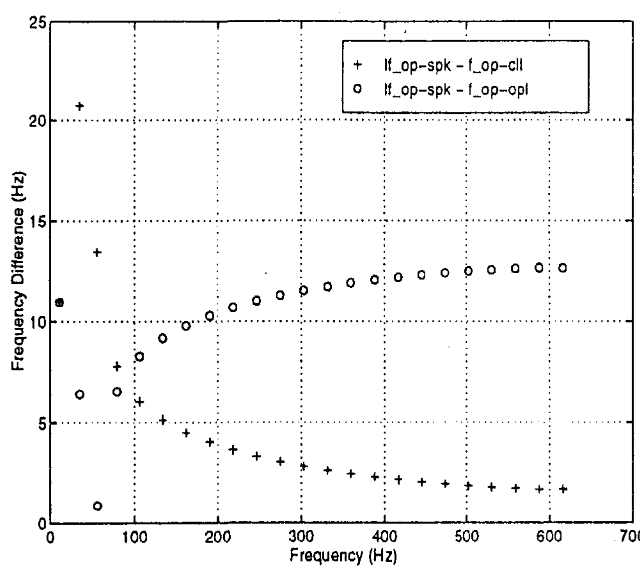

Fig. 3 Modal frequency difference between open-speaker and open-closed boundary conditions and between open-speaker and open-open boundary conditions

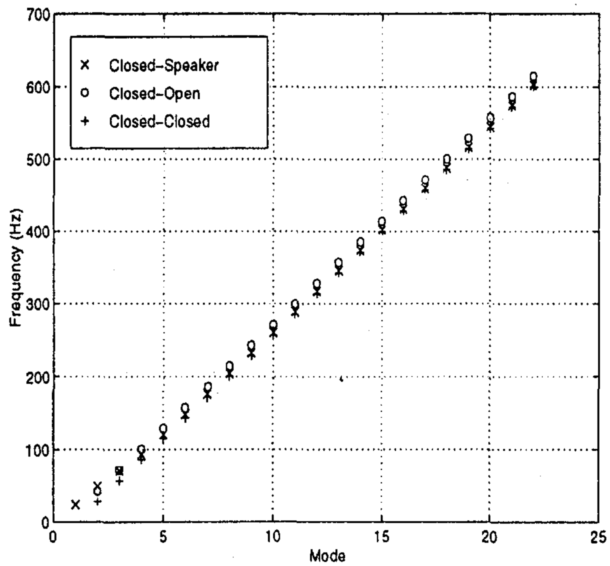

Fig. 4 Modal frequency comparison for closed-speaker, closed-open and closed-closed boundary conditions

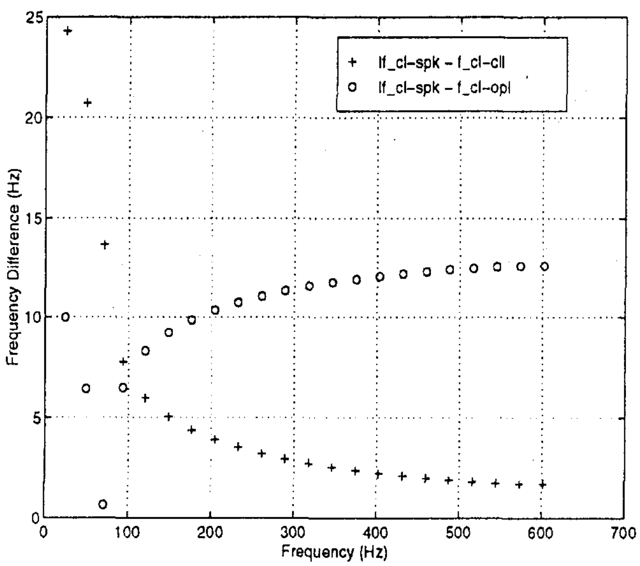

Fig. 5 Modal frequency difference between closed-speaker and closed-closed boundary conditions and between closed-speaker and closed-open boundary conditions

Figures 2 and 3 compare the modal frequencies of a 6 m duct for open-speaker, open-closed and open-open boundary conditions. The speaker parameters are $I_{0s} = 0.0149$ kg, $R_{0s} = 1.2931$ kg/s, $K_{0s} = 1.47 \times 10^3$ kg/s^2, $S_0 = 0.0206$ m^2, $\mu_0 = 6.3396$ N/amp and $R_{0c} = 5.89$ ohms. These parameters are taken from the specification sheet of a commercially available speaker. The free air resonance frequency of this speaker is 50 Hz. The diameter of the duct is approximately 16 cm. From Figs. 2 and 3, we observe that at lower frequencies, especially near the free air resonance frequency of the speaker, the open-speaker duct behavior closely resembles that of an open-open duct. Near the free air resonance frequency, the ratio of the diameter of the duct to the acoustic wavelength is approximately 0.02, and thus, radiation effects are negligible. At higher frequencies, however, where the speaker has high impedance, the open-speaker duct behavior is closer to that of an open-closed duct. Similarly, Figs. 4 and 5 compare the modal frequencies of the same duct-speaker combination with closed-speaker, closed-open and closed-closed boundary conditions. Again, we notice that the speaker end behaves more like an open end at low frequencies, particularly near the speaker resonance frequency, and more like a closed end at higher frequencies.

3.2 Speaker-Speaker Boundary Conditions. We now consider the two-speaker system described in Section 2.3 with one speaker located at $x = 0$ and the other located at $x = l$. The subscript 1 is used to denote the speaker at $x = 0$ and the subscript 2 is used to denote the speaker at $x = l$. Using the speaker dynamics model (53), we obtain a state space model of the two-speaker system of the form

$$\dot{\xi}_s(t) = A_s\xi_s(t) + B_{Ps}P(t) + B_{Vs}V(t), \quad (60)$$

$$v_{0ls}(t) = C_s\xi_s(t), \quad (61)$$

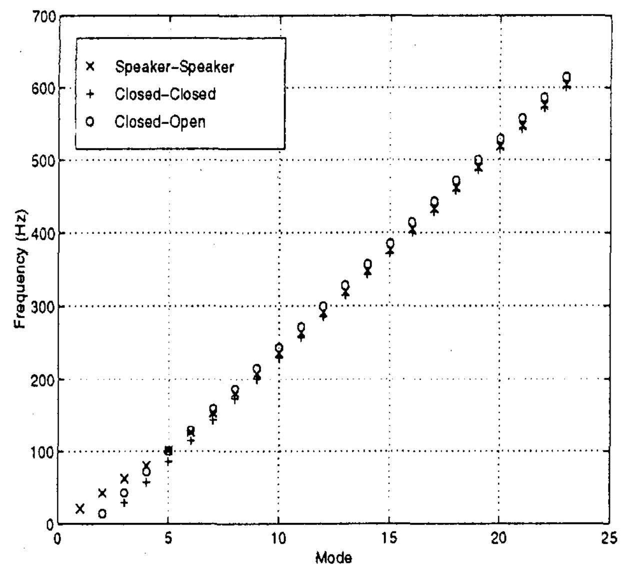

Fig. 6 Modal frequency comparison for speaker-speaker, closed-open and closed-closed boundary conditions

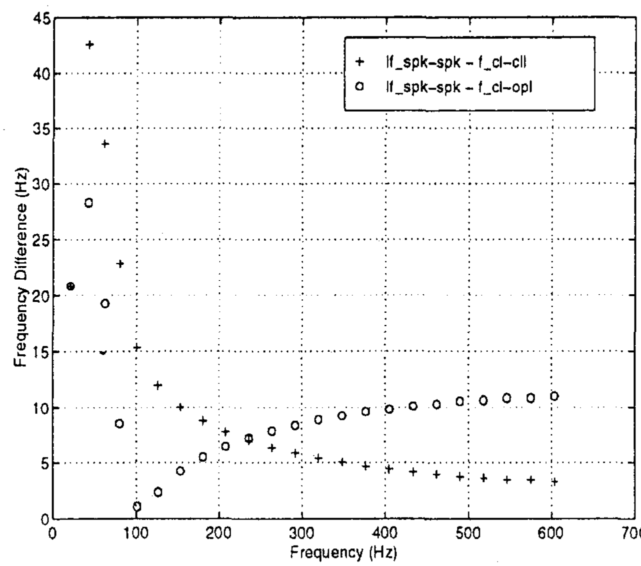

Fig. 7 Modal frequency difference between speaker-speaker and closed-closed boundary conditions and between speaker-speaker and closed-open boundary conditions

where

$$
A_s \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_{0s}}{I_{0s}} & -\frac{1}{I_{0s}}\left(R_{0s}+\frac{\mu_0^2}{R_{0c}}\right) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{K_{ls}}{I_{ls}} & -\frac{1}{I_{ls}}\left(R_{ls}+\frac{\mu_l^2}{R_{lc}}\right) \end{bmatrix},
$$

$$
B_{Ps} \triangleq \begin{bmatrix} 0 & 0 \\ -\frac{S_0}{I_{0s}} & 0 \\ 0 & 0 \\ 0 & \frac{S_l}{I_{ls}} \end{bmatrix}, \tag{62}
$$

$$
B_{Vs} \triangleq \begin{bmatrix} 0 & 0 \\ \frac{\mu_0}{I_{0s}R_{0c}} & 0 \\ 0 & 0 \\ 0 & -\frac{\mu_l}{I_{ls}R_{lc}} \end{bmatrix},
$$

$$
C_s \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad V(t) \triangleq \begin{bmatrix} V_0(t) \\ V_l(t) \end{bmatrix}, \tag{63}
$$

and $v_{0ls}(t)$ and $P(t)$ are defined in Section 2.3. We now combine (50), (51) and (60), (61) in feedback by defining the augmented state vector

$$
x_a(t) = \begin{bmatrix} x(t) \\ \xi_s(t) \end{bmatrix}, \tag{64}
$$

to obtain the closed-loop state space equation

$$
\dot{x}_a(t) = \mathscr{A} x_a(t) + \mathscr{B} V(t), \tag{65}
$$

where $\mathscr{A}$ and $\mathscr{B}$ are defined in (59).

Figures 6 and 7 compare the modal frequencies of the duct with speakers on both ends. The duct and speakers have the same parameters as those used for the numerical comparisons in Section 3.1. We observe that at lower frequencies the duct with two end-mounted speakers behaves neither like an open-closed duct nor like a closed-closed duct. At intermediate frequencies the modes correspond to those of an open-closed duct, while at higher frequency the modes are closer to those of a closed-closed duct.

4 Discussion

From the derivations in the previous sections, we note that the speaker dynamics enter the wave equation as a forcing term. The boundary conditions (4) correspond to those of a closed-open duct while the boundary conditions (26) correspond to those of a closed-closed duct. Hence the natural frequencies obtained for the open-speaker case are identical to those of an open-closed duct while the natural frequencies obtained for the closed-speaker case are identical to those of a closed-closed duct. However, if we account for the effect of acoustic back pressure and speaker dynamics, we see from (58), (65) and (59) that the plant modal frequencies are altered, with the speaker tending to behave more like an open end at low frequencies, especially near the free air resonance of the speaker, and like a closed end at higher frequencies. For the speaker-speaker boundary conditions the low frequency behavior is not clear, however, at high frequencies both speakers behave like closed ends.

References

1 Doak, P. E., "Excitation, Transmission and Radiation of Sound from Source Distribution in Hard-Walled Ducts of Finite Length," *Journal of Sound and Vibration*, Vol. 31, pp. 1–72, 1973.

2 Hull, A. J., Radcliffe, C. J., and Southward, S. C., "Global Active Noise Control of a One Dimensional Acoustic Duct Using a Feedback Controller," ASME *Journal of Dynamic Systems, Measurement and Control*, Vol. 115, pp. 488–494, September 1993.

3 Hong, J., Akers, J. C., Venugopal, R., Lee, M. N., Sparks, A. G., Washabaugh, P. D., and Bernstein, D. S., "Modeling, Identification and Feedback Control of Noise in an Acoustic Duct," *IEEE Transactions on Control System Technology*, Vol. 4, pp. 283–291, 1996.

4 Hull, A. J., Radcliffe, C. J., Miklavcic, M., and MacCluer, C. R., "State Space Representation of the Nonself-Adjoint Acoustic Duct System," ASME Journal of Vibration and Acoustics, Vol. 112, pp. 483–488, 1990.

5 Pierce, A. D., *Acoustics: An Introduction to Its Physical Principles and Applications*, Acoustical Society of America, NY, 1991.

6 Kinsler, L. E., Frey, A. R., Collins, A. B., and Saunders, J. V., *Fundamentals of Acoustics*, John Wiley, NY, 1982.

7 Morse, P. M., and Ingard, K. U., *Theoretical Acoustics*, Princeton University Press, Princeton 1986.

8 Radcliffe, C. J., and Gogate, S. D., "Identification and Modeling of Speaker Dynamics for Acoustic Control Applications," *Active Control of Noise and Vibrations*, ASME, Vol. 38, pp. 295–300, 1992.

9 Joshi, S. M., and Gupta, S., "On a Class of Marginally Stable Positive-Real Systems," *IEEE Transactions on Automatic Control*, Vol. 41, pp. 152–155, 1996.

10 Myint-U, T., and Debnath, L., *Partial Differential Equations for Scientists and Engineers*, North Holland, NY, 1987.

APPENDIX A

Consider the duct dynamics described by (20) and (21) represented in the form of a transfer function as

$$
G(s) = \frac{P_0(s)}{v_{1s}(s)} = \frac{2}{l}\rho_0 c^2 \sum_{i=0}^{r} \frac{s}{s^2+\omega_{ni}^2}. \tag{66}
$$

Since $\mathrm{Re}[G(j\omega)] = 0$ ($j = \sqrt{-1}$) for all real ω, the acoustic duct dynamics are positive real.

The speaker transfer function from pressure to velocity is

$$
G_s(s) = \frac{v_{1s}(s)}{P_0(s)} = \frac{S_1 s}{I_{1s}s^2 + \left(R_{1s}+\frac{\mu_1^2}{R_{1c}}\right)s + K_{1s}}. \tag{67}
$$

Since S_1, R_{1s}, μ_1^2 and R_{1c} are all positive, it follows that

$$
\mathrm{Re}[G_s(j\omega)] = \frac{S_1\left(R_{1s}+\frac{\mu_1^2}{R_{1c}}\right)\omega^2}{(K_{1s}-I_{1s}\omega^2)^2 + \left(R_{1s}+\frac{\mu_1^2}{R_{1c}}\right)^2\omega^2} > 0. \tag{68}
$$

Thus, $G_s(s)$ is strictly positive real. Consequently, the negative feedback interconnection of $G(s)$ and $G_s(s)$ is asymptotically stable [9].