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## Optimal nonlinear, but continuous, feedback control of systems with saturating actuators

DENNIS S. BERNSTEIN†

This paper presents a modification of the optimal saturating feedback control laws given by Frankena and Sivan (1979) and Ryan (1982 a) for asymptotically stable systems. Unlike their results, which involve bang-bang action and singular extremals, the modified control law is continuous. Specifically, the new control law is linear inside a cylinder set and saturated elsewhere. Using steady-state Hamilton–Jacobi–Bellman theory, this control law is shown to be optimal for a modified performance functional with a discontinuous integrand.

### 1. Introduction

Since all real actuators are subject to physical constraints, one of the most widespread and fundamental problems in control engineering is coping with actuator saturation. The literature on this problem is extensive and reflects considerable current activity, e.g. Keerthi and Gilbert (1987), Åström and Rundqwist (1989), Campo *et al.* (1989), Sontag and Sussmann (1990), Sznaier and Damborg (1990), Dolphus and Schmitendorf (1991), Sussmann and Yang (1991), Teel (1992), Tsirukis and Morari (1992), and Yang *et al.* (1992). The purpose of this paper is to approach this problem within an optimal nonlinear feedback control framework as in Frankena and Sivan (1979), Ryan (1982 a, b) but with certain crucial differences described below.

The starting point for our approach is Frankena and Sivan (1979) which considers the linear plant

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \quad (1)$$

with an ellipsoidal control constraint set

$$\Omega \triangleq \{u \in \mathbb{R}^m: u^T R_2 u \leq 1\} \quad (2)$$

where  $R_2 > 0$ . If the control were not confined to  $\Omega$ , then one could apply the usual linear-quadratic approach which involves minimizing the performance functional

$$J = \int_0^{\infty} [x^T R_1 x + u^T R_2 u] dt \quad (3)$$

where  $R_1 \geq 0$ . Assuming that  $(R_1, A)$  is detectable and  $(A, B)$  is stabilizable, minimizing  $J$  yields the linear feedback control law  $u = \phi(x)$ , where

$$\phi(x) = -R_2^{-1} B^T P x \quad (4)$$

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where  $P \geq 0$  satisfies the Riccati equation

$$0 = A^T P + PA + R_1 - PSP \quad (5)$$

where  $S \triangleq BR_2^{-1}B^T$ . Although the feedback control law (4) yields a globally asymptotically stable closed-loop system,  $u = \phi(x)$  can clearly violate the control constraint given by  $\Omega$ .

To account for the control constraint set  $\Omega$ , we confine our attention, as in Frankena and Sivan (1979) and Ryan (1982 a), to open-loop asymptotically stable systems. In this case, the approach in Frankena and Sivan (1979) involves the modified performance functional

$$J = \int_0^\infty [x^T R_1 x + 2(x^T P_0 S P_0 x)^{1/2}] dt \quad (6)$$

where  $P_0$  is the solution to the Lyapunov equation

$$0 = A^T P_0 + P_0 A + R_1 \quad (7)$$

Note that  $P_0$  exists and is non-negative definite since  $A$  is assumed to be asymptotically stable. Although the term  $2(x^T P_0 S P_0 x)^{1/2}$  in (6) is not quadratic, it serves to penalize further the deviation of the state  $x$  from the origin. Furthermore, although the presence of  $P_0$  in the performance functional (6) is somewhat contrived, this approach is completely consistent with the techniques used by Bass and Webber (1966), Speyer (1976) and Bernstein (1993). Specifically, the nonlinear control laws obtained in these papers were based upon non-quadratic performance functionals involving auxiliary terms whose presence was used to obtain closed-form solutions to the steady-state Hamilton–Jacobi–Bellman equation. In the case of (6), the auxiliary term  $2(x^T P_0 S P_0 x)^{1/2}$  appears in place of the usual quadratic control-weighting term  $u^T R_2 u$ .

As shown in Frankena and Sivan (1979), the optimal feedback control law  $u = \phi(x)$  for (6) is given by

$$\phi(x) = (x^T P_0 S P_0 x)^{-1/2} R_2^{-1} B^T P_0 x \quad (8)$$

if  $B^T P_0 x \neq 0$ . If, however,  $B^T P_0 x = 0$ , then (8) must be replaced by a singular control law which requires special analysis to determine whether the control constraint is satisfied. Nevertheless, in the non-singular case, that is,  $B^T P_0 x \neq 0$ , the control law (8) is bang-bang and is thus discontinuous. The purpose of this paper is to replace (8) with a continuous control law that is guaranteed to satisfy the control constraints. As will be seen, an additional advantage of this modified controller is that special treatment of the singular control is no longer needed. In addition to the ellipsoidal constraint set (2) we obtain analogous results for the rectangular control constraint set considered by Ryan (1982 a).

## 2. Continuous saturating controls

We now replace the performance functional (6) used by Frankena and Sivan (1979) with the modified performance functional

$$J = \int_0^\infty [x^T R_1 x + h(x, u)] dt \quad (9)$$

where

$$h(x, u) \triangleq \begin{cases} x^T P_0 S P_0 x + u^T R_2 u, & x^T P_0 S P_0 x < 1 \\ 2(x^T P_0 S P_0 x)^{1/2}, & x^T P_0 S P_0 x \geq 1 \end{cases} \quad (10)$$

and where  $P_0$  is given by (7), and we now assume  $R_1 > 0$ . Although this minor modification of (6) leaves the optimal feedback control (8) unchanged for  $x^T P_0 S P_0 x \geq 1$ , the control law for  $x^T P_0 S P_0 x < 1$  (and, in particular, for  $B^T P_0 x = 0$  in which case  $x^T P_0 S P_0 x = 0$ ) is now quite different. Specifically, applying Theorem A.1 of the Appendix yields the optimal stabilizing feedback control law

$$\phi(x) = \begin{cases} -R_2^{-1} B^T P_0 x, & x^T P_0 S P_0 x < 1 \\ -(x^T P_0 S P_0 x)^{-1/2} R_2^{-1} B^T P_0 x, & x^T P_0 S P_0 x \geq 1 \end{cases} \quad (11)$$

Closed-loop stability is guaranteed by the Lyapunov function  $V(x) = x^T P_0 x$  whose derivative is given by

$$\dot{V}(x) = \begin{cases} -x^T (R_1 + 2P_0 S P_0) x, & x^T P_0 S P_0 x < 1 \\ -x^T [R_1 + 2(x^T P_0 S P_0 x)^{-1/2} P_0 S P_0] x, & x^T P_0 S P_0 x \geq 1 \end{cases} \quad (12)$$

Details are given in the Appendix.

The form of  $h(x, u)$  in (10) can be viewed as a merging of (3) and (6) so that, in the region  $x^T P_0 S P_0 x < 1$ , (9) yields the usual linear LQR controller (4) with  $P = P_0$ , while, in the complementary region  $x^T P_0 S P_0 x \geq 1$ , (9) yields the saturated control law (8). Furthermore, these regions are chosen so that at their common boundary the control law is continuous. Note that in order to achieve this continuous control law the function  $h(x, u)$  was chosen to be a discontinuous function of  $x$  and  $u$ . This is permitted by Theorem A.1 of Appendix A given by Bernstein (1993).

In discussing (11), it is convenient to define the set

$$\mathcal{C} \triangleq \{x \in \mathbb{R}^n : x^T P_0 S P_0 x < 1\}$$

To see that  $\mathcal{C}$  is a cylinder set with ellipsoidal cross-section, apply an orthogonal coordinate transformation to  $P_0 S P_0$  so that in the new basis  $P_0 S P_0$  has the form

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

where the size of the positive diagonal matrix  $D$  is equal to the rank of  $P_0 S P_0$ . Thus,  $\mathcal{C}$  corresponds to the translation of an ellipsoid along the null space of  $P_0 S P_0$ .

It can now be seen that  $\phi(x)$  given by (11) is a continuous saturation-type control law that remains within the control constraint set  $\Omega$ . Outside of the cylinder set  $\mathcal{C}$ , the control  $\phi(x)$  is saturated, while inside  $\mathcal{C}$  the control is linear and thus the control effort diminishes gracefully to zero. In fact, during operation, the system is free to pass in and out of  $\mathcal{C}$  without large, discontinuous control variations. Note, however, that the linear portion of  $\phi(x)$  is different from the usual LQR control law (4) since  $P_0$  is given by the Lyapunov equation (7) rather than the Riccati equation (5). This construction is allowable since  $A$  is assumed to be asymptotically stable. Finally, since  $\phi(x)$  is defined globally, no special attention is needed for the case  $B^T P_0 x = 0$ , as in Frankena and Sivan (1979).

### 3. Rectangular control constraint set

We now consider the rectangular control constraint set

$$\Omega = \{u = [u_1 \ u_2 \ \dots \ u_m]^T \in \mathbb{R}^m : |u_i| \leq a_i, i = 1, \dots, m\} \quad (13)$$

where  $a_i > 0$ ,  $i = 1, \dots, m$ , in place of the ellipsoidal control constraint set (2). In most applications the rectangular control constraint set (13) provides a more realistic model of control constraints than the ellipsoidal control constraint set since each control signal  $u_i$  (and hence the corresponding actuator) saturates independently of the other control signals. Optimal, discontinuous controllers for this problem were given by Ryan (1982 a, b).

Applying the same ideas as in § 2, we consider the cost functional (9) but now with  $h(x, u)$  given by

$$h(x, u) = \sum_{i=1}^m h_i(x, u) \quad (14)$$

where, for  $i = 1, \dots, m$ ,

$$h_i(x, u) = \begin{cases} (B_i^T P_0 x)^2 + u_i^2, & |B_i^T P_0 x| < a_i \\ 2a_i |B_i^T P_0 x|, & |B_i^T P_0 x| \geq a_i \end{cases} \quad (15)$$

where  $B_i$  denotes the  $i$ th column of  $B$ . Note that if  $|B_i^T P_0 x| < a_i$  for all  $i = 1, \dots, m$ , then  $h(x, u) = [\sum_{i=1}^m x^T P_0 B_i B_i^T P_0 x] + u^T u = x^T P B B^T P x + u^T u$ , which is identical to (10) for the case  $x^T P_0 S P_0 x < 1$  with  $R_2$  equal to the identity matrix.

With (15) we consider the feedback control law  $\phi(x) = [\phi_1(x) \dots \phi_m(x)]^T$ , where, for  $i = 1, \dots, m$ ,  $\phi_i(x)$  is given by

$$\phi_i(x) = \begin{cases} -B_i^T P_0 x, & |B_i^T P_0 x| < a_i \\ \frac{-a_i B_i^T P_0 x}{|B_i^T P_0 x|}, & |B_i^T P_0 x| \geq a_i \end{cases} \quad (16)$$

As before, this control law is linear within a cylinder set which now has polygonal cross-section. Outside of this cylinder set at least one component of  $\phi(x)$  is saturated, while  $\phi(x)$  is continuous throughout  $\mathbb{R}^n$ .

Closed-loop stability is guaranteed by means of the Lyapunov function  $V(x) = x^T P_0 x$  whose derivative is given by

$$\dot{V}(x) = - \left[ x^T R_1 x + 2 \sum_{i \in I_{\text{lin}}} |B_i^T P_0 x|^2 + 2 \sum_{i \in I_{\text{sat}}} a_i |B_i^T P_0 x| \right] \quad (17)$$

where  $I_{\text{lin}}(x) \triangleq \{i: |B_i^T P_0 x| < a_i\}$  and  $I_{\text{sat}}(x) \triangleq \{i: |B_i^T P_0 x| \geq a_i\}$  denote the unsaturated and saturated components of  $\phi(x)$ , respectively. Finally, optimality is guaranteed by condition (A 8), which is of the form

$$H(x, V'^T(x), u) = \sum_{i \in I_{\text{lin}}} (B_i^T P_0 x + u_i)^2 + 2 \sum_{i \in I_{\text{sat}}} [a_i |B_i^T P_0 x| + B_i^T P_0 x u_i] \quad (18)$$

Finally, with (16), it follows that  $H(x, V'^T(x), \phi(x)) = 0$  so that (A 7) holds, which verifies Theorem A.1.

#### 4. Output feedback dynamic compensation

We now consider the more practical case in which only measurements are available for feedback. Hence, consider

$$\dot{x} = Ax + Bu \quad (19)$$

$$y = Cx \quad (20)$$

with the observer-based output feedback dynamic compensator

$$\dot{x}_c = Ax_c + Bu + B_c(y - Cx_c) \quad (21)$$

$$u = \phi(x_c) \quad (22)$$

where  $u \in \Omega \subset \mathbb{R}^m$ ,  $y \in \mathbb{R}^l$ ,  $x_c \in \mathbb{R}^n$  and  $\phi(0) = 0$ . The following result, which is based on the approach of Yang *et al.* (1992), guarantees closed-loop stability using the optimal saturated controllers given in § 2 and 3.

**Proposition 4.1:** *Assume that  $A$  and  $A - B_cC$  are asymptotically stable matrices, and let  $\phi(x)$  be given by either (11) or (16) with  $P_0$  given by (7) with  $R_1 > 0$ . Then, the zero solution of the closed-loop system (19)–(22) is globally asymptotically stable.*

**Proof:** Define  $e \triangleq x_c - x$  and write (19)–(22) as

$$\dot{x} = Ax + B\phi(x + e) \quad (23)$$

$$\dot{e} = (A - B_cC)e \quad (24)$$

Now let  $R_e$  be an  $n \times n$  positive-definite matrix such that

$$R_1 + 2P_0SP_0 > P_0SP_0R_e^{-1}P_0SP_0 \quad (25)$$

and let  $P_e > 0$  satisfy

$$0 = (A - B_cC)^T P_e + P_e(A - B_cC) + R_e \quad (26)$$

Now, by defining the positive-definite Lyapunov candidate  $V(x, e)$  by

$$V(x, e) = x^T P_0 x + e^T P_e e \quad (27)$$

it follows from the ellipsoidal control constraint set with control given by (11) that

$$\dot{V}(x, e) = \left. \begin{aligned} & - \begin{bmatrix} x \\ e \end{bmatrix}^T \begin{bmatrix} R_1 + 2P_0SP_0 & P_0SP_0 \\ P_0SP_0 & R_e \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}, \quad \beta(x, e) < 1 \\ & - \begin{bmatrix} x \\ e \end{bmatrix}^T \begin{bmatrix} R_1 + 2\beta(x, e)^{-1/2}P_0SP_0 & \beta(x, e)^{-1/2}P_0SP_0 \\ \beta(x, e)^{-1/2}P_0SP_0 & R_e \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \\ & \beta(x, e) \geq 1 \end{aligned} \right\} \quad (28)$$

where  $\beta(x, e) \triangleq (x + e)^T P_0 S P_0 (x + e)$ . For the rectangular control constraint set a slight modification of (28) is needed. From (25) it follows that

$$R_1 + 2\beta(x, e)^{-1/2}P_0SP_0 > \beta(x, e)^{-1}P_0SP_0R_e^{-1}P_0SP_0 \quad (29)$$

for both  $\beta(x, e) < 1$  and  $\beta(x, e) \geq 1$ . Thus, both matrices in (28) are positive definite. Consequently,  $\dot{V}(x, e)$  is negative definite, as required.  $\square$

**Remark 4.1:** The Lyapunov function proof of Proposition 4.1 takes the place of the CICS (converging-input converging-state) property used by Yang *et al.* (1992) to prove asymptotic stability of the observer-based dynamic compensator. While our results are limited to open-loop asymptotically stable plants, the results of Sontag and Sussmann (1990), Sussmann and Yang (1991) and Yang *et al.* (1991) apply to plants with poles on the imaginary axis. As shown by

Sussmann and Yang (1991), plants such as the multiple integrator are not stabilizable by means of a saturation of a linear control law, which is precisely the form of (11) and (15).  $\square$

**Remark 4.2:** Although the control law given by Proposition 4.1 is globally asymptotically stabilizing, it is not guaranteed to be optimal for some performance functional. Since the open-loop system is assumed to be stable, it is necessary to compare open and closed-loop performance in order to determine the benefits of implementing (21), (22).  $\square$

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#### Appendix

First we quote the steady-state Hamilton–Jacobi–Bellman result from Bernstein (1993). Let  $f: \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ , where  $0 \in \Omega \subset \mathbb{R}^m$ , and consider the system

$$\dot{x} = f(x, u), \quad x(0) = x_0, \quad t \geq 0 \quad (\text{A } 1)$$

where  $f(0, 0) = 0$ . A control  $u(\cdot)$  is *admissible* if it is measurable and  $u(t) \in \Omega$ ,  $t \geq 0$ . Furthermore, let  $L: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  and, for  $p \in \mathbb{R}^n$ , define  $H(x, p, u) \triangleq L(x, u) + p^T f(x, u)$ . The following result is proved in Bernstein (1993).

**Theorem A.1:** Consider the controlled system (A 1) with performance functional

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty L(x(t), u(t)) dt \quad (\text{A } 2)$$

Assume that there exist a  $C^1$  function  $V: \mathbb{R}^n \rightarrow \mathbb{R}$  and a function  $\phi: \mathbb{R}^n \rightarrow \Omega$  such that

$$V(0) = 0 \quad (\text{A } 3)$$

$$V(x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0 \quad (\text{A } 4)$$

$$\phi(0) = 0 \quad (\text{A } 5)$$

$$V'(x)f(x, \phi(x)) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0 \quad (\text{A } 6)$$

$$H(x, V'^T(x), \phi(x)) = 0, \quad x \in \mathbb{R}^n \quad (\text{A } 7)$$

$$H(x, V'^T(x), u) \geq 0, \quad x \in \mathbb{R}^n, \quad u \in \Omega \quad (\text{A } 8)$$

Furthermore, assume that  $V(x) \rightarrow \infty$  as  $\|x\| \rightarrow \infty$ . Then, with the feedback control law  $u(\cdot) = \phi(x(\cdot))$ , the solution  $x(t) = 0$ ,  $t \geq 0$ , of the closed-loop system is globally asymptotically stable, and

$$J(x_0, \phi(x(\cdot))) = V(x_0) \quad (\text{A } 9)$$

Finally, the feedback control  $u(\cdot) = \phi(x(\cdot))$  minimizes  $J(x_0, u(\cdot))$  in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{P}(x_0)} J(x_0, u(\cdot)) \quad (\text{A } 10)$$

where

$$\mathcal{S}(x_0) \triangleq \{u(\cdot): u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (A 1) satisfies} \\ \lim_{t \rightarrow \infty} V(x(t)) = 0\}$$

Now consider the problem of finding admissible  $u(\cdot)$  to minimize (9), where  $\Omega$  is given by (2). To do this let  $V(x) = x^T P_0 x$ , where  $P_0$  satisfies (7), and let  $\phi(\cdot)$  be given by (11). Then clearly  $\phi: \mathbb{R}^n \rightarrow \Omega$  and (A 3)–(A 5) are satisfied. For the case  $x^T P_0 S P_0 x < 1$ , we have

$$V'(x)f(x, \phi(x)) = -x^T(R_1 + 2P_0 S P_0)x$$

which verifies (A 6). Next, we obtain  $H(x, V'^T(x), u) = (u + R_2^{-1} B^T P_0 x)^T R_2^{-1} (u + R_2^{-1} B^T P_0 x)$ , which proves (A 7) and (A 8). For the case  $x^T P_0 S P_0 x \geq 1$  with  $\phi(x)$  given by (11), it follows that

$$V'(x)f(x, \phi(x)) = -x^T[R_1 + 2(x^T P_0 S P_0 x)^{-1/2} P_0 S P_0]x$$

which implies (A 6). To obtain (A 7) note that

$$H(x, V'^T(x), u) = 2[(x^T P_0 S P_0 x)^{1/2} - (-x^T P_0 B R_2^{-1/2})(R_2^{1/2} u)] \\ \geq 2(x^T P_0 S P_0 x)^{1/2}[1 - (u^T R_2 u)^{1/2}] \\ \geq 0$$

Thus, (A 8) is satisfied by (11) since  $\phi^T(x) R_2 \phi(x) = 1$ .

**Remark:** In the case  $x^T P_0 S P_0 x \geq 1$ , the control law  $\phi(x)$  must be chosen to satisfy  $x^T P_0 B \phi(x) \leq 0$  and  $\phi^T(x) R_2 \phi(x) = 1$ . Although  $\phi(x) = -(x^T P_0 S P_0 x)^{1/2} R_2^{-1} B^T P_0 x$  given by (11) satisfies these conditions, it is clearly not the only choice of  $\phi(x)$  to do so. However, with  $\phi(x) = -R_2^{-1} B^T P_0 x$  for the case  $x^T P_0 S P_0 x < 1$ , it follows that the resulting control law (11) is continuous on  $\mathbb{R}^n$ .  $\square$

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