Robust Controller Synthesis via Shifted Parameter-Dependent Quadratic Cost Bounds

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***Abstract*—Parameterized Lyapunov bounds and shifted quadratic guaranteed cost bounds are merged to develop shifted parameter-dependent quadratic cost bounds for robust stability and robust performance. Robust fixed-order (i.e., full- and reduced-order) controllers are developed based on new shifted parameter-dependent bounding functions. A numerical example is presented to demonstrate the effectiveness of the proposed approach.**

***Index Terms*—Fixed-structure controllers, real parameter uncertainty, shifted parameter-dependent bounding functions.**

Nomenclature

$\mathbb{R},\ \mathbb{R}^{r\times s},\ \mathbb{R}^r$	Real numbers, $r\times s$ real matrices, $\mathbb{R}^{r\times 1}$.
$(\,)^T,\ (\,)^{-1},\ \mathrm{tr}(\,),\ \mathbb{E}$	Transpose, inverse, trace, expectation.
$I_r,\ 0_r$	$r\times r$ identity matrix, $r\times r$ zero matrix.
$\mathbb{S}^r,\ \mathbb{N}^r,\ \mathbb{P}^r$	$r\times r$ symmetric, nonnegative-definite, positive-definite matrices.
$Z_1\le Z_2,\ Z_1<Z_2$	$Z_2-Z_1\in\mathbb{N}^r,\ Z_2-Z_1\in\mathbb{P}^r;\ Z_1,\ Z_2\in\mathbb{S}^r$.

I. Introduction

One of the principal objectives of robust control theory is to synthesize feedback controllers with *a priori* guarantees of robust stability and performance. In structured singular value synthesis [3], [9] these guarantees are achieved by means of bounds involving frequency-dependent scales and multipliers which account for the structure of the uncertainty as well as its real or complex nature. An alternative robustness approach involves bounding the effect of real or complex uncertain parameters on the H_2 performance of the closed-loop system [6], [11]. These guaranteed cost bounds take the form of modifications to the usual Lyapunov equation to provide bounds for robust stability and performance [1], [4]–[6].

A diverse collection of guaranteed cost bounds have been developed. Bounded-real-type guaranteed cost bounds were developed in [8] and [10], while positive-real-type bounds are discussed in [4]. More recently, parameter-dependent Popov guaranteed cost bounds [6] have provided links with frequency-dependent scales and multipliers while providing reliable bounds for the peak real structured singular value [6], [11]. Finally, the introduction of shift terms in [12] has been shown to reduce the conservatism of guaranteed cost bounds for structured real uncertainty without requiring frequency-dependent scales and multipliers.

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It can easily be seen that parameter-independent guaranteed cost bounds provide the means for obtaining solutions to the quadratic stability linear matrix inequality (LMI)

$$0>(A+B_0FC_0)^TP+P(A+B_0FC_0)+E^TE$$

for all admissible uncertainty F. The solution to this LMI then provides a bound for the worst case H_2 cost. It was shown in [12] that the inclusion of the shift terms in both the bounded-real and positive-real guaranteed cost bounds can reduce the conservatism of these bounds. Since the Popov guaranteed cost bound [6] also entails less conservatism than classical bounded-real and positive-real guaranteed cost bounds, the objective of this paper is to combine features of both the Popov bound and shifted quadratic bounds.

The bound we construct in this paper is the most general of its kind developed thus far, encompassing the Popov, positive-real, and shifted positive-real bounds as special cases. The benefits of this generalization are demonstrated by a numerical example involving robust controller synthesis. Specifically, our numerical results show that the combination of both the shift terms and the parameter-dependent terms provides reduced conservatism and improved robustness/performance tradeoffs as compared to either the Popov bound [6], [11] or the shifted positive-real bound [12] separately.

The contents of the paper are as follows. In Section II, we state the robust fixed-order dynamic compensation problem. In Section III, we restate a key theorem from [6] to provide sufficient conditions for robust stability and performance. In Section IV, we develop a novel shifted parameter-dependent bounding function for robust stability and performance. In Section V, we provide constructive sufficient conditions for robust stability and performance via fixed-order (i.e., full- and reduced-order) dynamic compensation. Section VI provides a numerical example to demonstrate the effectiveness of the newly developed bounds for robust controller synthesis. Finally, Section VII gives conclusions.

II. Robust Fixed-Order Dynamic Compensation

In this section, we introduce the robust stability and performance problem. This problem involves a set $\mathcal{U}\subset\mathbb{R}^{n\times n}$ of constant uncertain perturbations ΔA of the nominal system matrix A. The objective of the problem is to determine a fixed-order strictly proper dynamic compensator $(A_c,\ B_c,\ C_c)$ that stabilizes the plant for all variations in $\mathcal{U}$ and minimizes the worst case H_2 performance of the closed-loop system. In this and the following section, no explicit structure is assumed for the elements of $\mathcal{U}$. In Section IV, the structure of $\mathcal{U}$ will be specified.

A. Robust Dynamic Compensation Problem

Given the nth-order stabilizable and detectable plant

$$\dot{x}(t)=(A+\Delta A)x(t)+Bu(t)+D_1w(t),\qquad t\ge 0 \tag{1}$$

$$y(t)=Cx(t)+D_2w(t) \tag{2}$$

where $w(\cdot)$ denotes a unit-intensity white noise signal, determine an n_cth-order dynamic compensator

$$\dot{x}_c(t)=A_cx_c(t)+B_cy(t) \tag{3}$$

$$u(t)=C_cx_c(t) \tag{4}$$

that satisfies the following criteria:

1) the closed-loop system (1)–(4) is asymptotically stable for all $\Delta A\in\mathcal{U}$;

2) the performance functional

$$J(A_c, B_c, C_c) \triangleq \sup_{\Delta A \in \mathcal{U}} \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t z^T(s)z(s)\,ds; \qquad (5)$$

where $z(t) \triangleq E_1x(t) + E_2u(t)$ is minimized.

Note that for each uncertain variation $\Delta A \in \mathcal{U}$, the closed-loop system (1)–(4) can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{D}w(t), \qquad t \geq 0 \qquad (6)$$

$$z(t) = \tilde{E}\tilde{x}(t) \qquad (7)$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \qquad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}$$

$$\Delta\tilde{A} \triangleq \begin{bmatrix} \Delta A & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}$$

and (5) becomes

$$J(A_c, B_c, C_c) = \sup_{\Delta A \in \mathcal{U}} \limsup_{t\to\infty} \mathbb{E}[\tilde{x}^T(t)\tilde{R}\tilde{x}(t)] \qquad (8)$$

where

$$\tilde{R} \triangleq \begin{bmatrix} R_1 & 0 \\ 0 & C_c^TR_2C_c \end{bmatrix}, \qquad R_1 \triangleq E_1^TE_1$$

$$R_{12} \triangleq E_1^TE_2 = 0, \qquad R_2 \triangleq E_2^TE_2 > 0.$$

Furthermore, for a given compensator (A_c, B_c, C_c) such that $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$, the performance (5) is given by

$$J(A_c, B_c, C_c) = \sup_{\Delta A \in \mathcal{U}} \operatorname{tr} \tilde{P}_{\Delta\tilde{A}}\tilde{V} \qquad (9)$$

where

$$\tilde{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & B_cV_2B_c^T \end{bmatrix}, \qquad V_1 \triangleq D_1D_1^T$$

$$V_{12} \triangleq D_1D_2^T = 0, \qquad V_2 \triangleq D_2D_2^T > 0$$

and $\tilde{P}_{\Delta\tilde{A}} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ is the unique nonnegative-definite solution to

$$0 = (\tilde{A} + \Delta\tilde{A})^T\tilde{P}_{\Delta\tilde{A}} + \tilde{P}_{\Delta\tilde{A}}(\tilde{A} + \Delta\tilde{A}) + \tilde{R}. \qquad (10)$$

III. Sufficient Conditions for Robust Stability and Performance via Parameter-Dependent Bounding Functions

In this section, we restate a theorem from [6] to determine an upper bound for $J(A_c, B_c, C_c)$ given by (9). The key step in obtaining robust stability and performance is to bound the uncertain terms $\Delta\tilde{A}^T\tilde{P}_{\Delta\tilde{A}} + \tilde{P}_{\Delta\tilde{A}}\Delta\tilde{A}$ in the Lyapunov equation (10) by means of a *parameter-dependent* bounding function. As discussed in [6], a key aspect of this approach is the fact that it constrains the class of allowable time-varying uncertainties, thus reducing conservatism in the presence of constant real parameter uncertainty, hence providing sharper H_2 performance bounds. The following fundamental result provides the basis for all later developments.

Theorem 3.1 [6]: Let (A_c, B_c, C_c) be given, let $\Omega_0\colon \mathbb{N}^{\tilde{n}} \to \mathbb{S}^{\tilde{n}}$ and $\mathcal{P}_0\colon \mathcal{U} \to \mathbb{S}^{\tilde{n}}$ be such that

$$\begin{aligned} &\Delta\tilde{A}^T\mathcal{P} + \mathcal{P}\Delta\tilde{A} \leq \Omega(\mathcal{P}, \Delta\tilde{A}) \\ &\quad\triangleq \Omega_0(\mathcal{P}) - [(\tilde{A} + \Delta\tilde{A})^T\mathcal{P}_0(\Delta\tilde{A}) + \mathcal{P}_0(\Delta\tilde{A})(\tilde{A} + \Delta\tilde{A})], \\ &\qquad \Delta A \in \mathcal{U}, \qquad \mathcal{P} \in \mathbb{N}^{\tilde{n}} \end{aligned} \qquad (11)$$

and suppose there exists $\mathcal{P} \in \mathbb{N}^{\tilde{n}}$ satisfying

$$0 = \tilde{A}^T\mathcal{P} + \mathcal{P}\tilde{A} + \Omega_0(\mathcal{P}) + \tilde{R} \qquad (12)$$

and such that $\mathcal{P} + \mathcal{P}_0(\Delta\tilde{A})$ is nonnegative definite for all $\Delta A \in \mathcal{U}$. Then $(\tilde{A} + \Delta\tilde{A},\ \tilde{E})$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case

$$\tilde{P}_{\Delta\tilde{A}} \leq \mathcal{P} + \mathcal{P}_0(\Delta\tilde{A}), \qquad \Delta A \in \mathcal{U} \qquad (13)$$

where $\tilde{P}_{\Delta\tilde{A}}$ is given by (10). Consequently

$$J(A_c, B_c, C_c) \leq \operatorname{tr} \mathcal{P}\tilde{V} + \sup_{\Delta A \in \mathcal{U}} \operatorname{tr} \mathcal{P}_0(\Delta\tilde{A})\tilde{V}. \qquad (14)$$

If, in addition, there exists $\overline{\mathcal{P}}_0 \in \mathbb{S}^{\tilde{n}}$ such that

$$\mathcal{P}_0(\Delta\tilde{A}) \leq \overline{\mathcal{P}}_0, \qquad \Delta A \in \mathcal{U} \qquad (15)$$

then

$$J(A_c, B_c, C_c) \leq \operatorname{tr}[(\mathcal{P} + \overline{\mathcal{P}}_0)\tilde{V}]. \qquad (16)$$

IV. Uncertainty Structure and a Shifted Parameter-Dependent Bounding Function

We now assign explicit structure to the uncertainty set $\mathcal{U}$ and the parameter-dependent bounding function $\Omega(\mathcal{P}, \Delta\tilde{A})$. Specifically, the uncertainty set $\mathcal{U}$ is defined by

$$\mathcal{U} \triangleq \{\Delta A \in \mathbb{R}^{n\times n}\colon \Delta A = B_0FC_0,\ F \in \mathcal{F}\} \qquad (17)$$

where $\mathcal{F}$ satisfies

$$\mathcal{F} \subseteq \hat{\mathcal{F}} \triangleq \{F \in \mathbb{S}^{m_0}\colon M_1 \leq F \leq M_2\} \qquad (18)$$

$B_0 \in \mathbb{R}^{n\times m_0}$, $C_0 \in \mathbb{R}^{m_0\times n}$ are fixed matrices denoting the structure of uncertainty, $F \in \mathbb{S}^{m_0}$ is an uncertain symmetric matrix, and $M_1, M_2 \in \mathbb{S}^{m_0}$ are symmetric matrices such that $M \triangleq M_2 - M_1 \in \mathbb{P}^{m_0}$. Note that $M_1, M_2 \in \hat{\mathcal{F}}$. Furthermore, $\mathcal{F}$ may be a specified proper subset of $\hat{\mathcal{F}}$. For example, $\mathcal{F} \subseteq \hat{\mathcal{F}}$ may consist of block-structured matrices $F = \text{block-diag}(I_{l_1} \otimes F_1, I_{l_2} \otimes F_2, \cdots, I_{l_r} \otimes F_r)$ with possibly repeated blocks so that $l_i \geq 1$, $F_i \in \mathbb{R}^{m_{0_i}\times m_{0_i}}$, and $\sum_{i=1}^r l_im_{0_i} = m_0$ and where $\otimes$ denotes Kronecker product. Furthermore, we assume that $M_1, M_2 \in \mathcal{F}$. We restrict our attention to symmetric uncertainties F for convenience only. More general uncertainty sets as in [6] can also be considered.

With the uncertainty set $\mathcal{U}$ given by (17) the closed-loop system (6) has structured uncertainty of the form $\Delta\tilde{A} = \tilde{B}_0F\tilde{C}_0$ where

$$\tilde{B}_0 \triangleq \begin{bmatrix} B_0 \\ 0 \end{bmatrix}, \qquad \tilde{C}_0 \triangleq [C_0 \quad 0].$$

Next, define the sets of compatible scaling matrices $\mathcal{H}$ and $\mathcal{N}$ by

$$\mathcal{H} \triangleq \{H \in \mathbb{P}^{m_0}\colon FH = HF, \quad F \in \mathcal{F}\} \qquad (19)$$

$$\mathcal{N} \triangleq \{N \in \mathbb{R}^{m_0\times m_0}\colon FN = N^TF, \quad F \in \mathcal{F}\}. \qquad (20)$$

Finally, define the notation $\tilde{\tilde{A}} \triangleq \tilde{A} + \tilde{B}_0M_1\tilde{C}_0$. The following result provides a parameter-dependent bounding function $\Omega(\cdot, \cdot)$ satisfying (11).

Proposition 4.1: Let $X \in \mathbb{R}^{m_0\times m_0}$ and $\tilde{Y} \in \mathbb{N}^{\tilde{n}}$ be such that

$$\tilde{B}_0X^T(F - M_1)\tilde{C}_0 + \tilde{C}_0^T(F - M_1)X\tilde{B}_0^T \leq \tilde{Y}, \qquad F \in \mathcal{F} \quad (21)$$

and let $H \in \mathcal{H}$ and $N \in \mathcal{N}$ be such that

$$R_0 \triangleq [HM^{-1} - N\tilde{C}_0\tilde{B}_0] + [HM^{-1} - N\tilde{C}_0\tilde{B}_0]^T > 0. \qquad (22)$$

Furthermore, let $\mathcal{U}$ be given by (17) and define $\Omega_0(\mathcal{P})$ and $\mathcal{P}_0(F)$ by

$$\begin{aligned} \Omega_0(\mathcal{P}) \triangleq\ &(H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} + \tilde{B}_0^T\mathcal{P} - X\tilde{B}_0^T)^TR_0^{-1} \\ &\cdot (H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} + \tilde{B}_0^T\mathcal{P} - X\tilde{B}_0^T) \\ &+ \mathcal{P}\tilde{B}_0M_1\tilde{C}_0 + \tilde{C}_0^TM_1\tilde{B}_0^T\mathcal{P} + \tilde{Y} \end{aligned} \qquad (23)$$

$$\mathcal{P}_0(F) \triangleq \tilde{C}_0^T(F - M_1)N\tilde{C}_0. \qquad (24)$$

Then (11) is satisfied.

Proof: Recall that $M_1 \le F \le M_2$ for all $F \in \mathcal{F}$ if and only if [7]

$$(F - M_1) - (F - M_1)M^{-1}(F - M_1) \ge 0, \qquad F \in \mathcal{F}. \tag{25}$$

Next, since $H \in \mathcal{H}$, $F \in \mathcal{F}$, and $M_1, M_2 \in \mathcal{F}$, it follows that $(F - M_1)H = H(F - M_1)$ and $M^{-1}H = HM^{-1}$. Now noting that H commutes with the left-hand side of (25), it follows that $H[(F - M_1) - (F - M_1)M^{-1}(F - M_1)] \ge 0$ for all $F \in \mathcal{F}$. Hence, it follows that, for all $F \in \mathcal{F}$

$$\begin{aligned}
0 &\le [H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} + \tilde{B}_0^T\mathcal{P} - X\tilde{B}_0^T - R_0(F - M_1)\tilde{C}_0]^T \\
&\quad \cdot R_0^{-1}[H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} + \tilde{B}_0^T\mathcal{P} - X\tilde{B}_0^T \\
&\qquad - R_0(F - M_1)\tilde{C}_0] + 2\tilde{C}_0^T H \\
&\quad \cdot [(F - M_1) - (F - M_1)M^{-1}(F - M_1)]\tilde{C}_0 \\
&= (H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} + \tilde{B}_0^T\mathcal{P} - X\tilde{B}_0^T)^T R_0^{-1} \\
&\quad \cdot (H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} + \tilde{B}_0^T\mathcal{P} - X\tilde{B}_0^T) \\
&\quad - (H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} + \tilde{B}_0^T\mathcal{P} - X\tilde{B}_0^T)^T(F - M_1)\tilde{C}_0 \\
&\quad - \tilde{C}_0^T(F - M_1)(H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} + \tilde{B}_0^T\mathcal{P} - X\tilde{B}_0^T) \\
&\quad + \tilde{C}_0^T(F - M_1)[\{HM^{-1} - N\tilde{C}_0\tilde{B}_0\} \\
&\qquad + \{HM^{-1} - N\tilde{C}_0\tilde{B}_0\}^T](F - M_1)\tilde{C}_0 \\
&\quad + 2\tilde{C}_0^T H[(F - M_1) - (F - M_1)M^{-1}(F - M_1)]\tilde{C}_0 \\
&\le \Omega_0(\mathcal{P}) - \tilde{A}^T\tilde{C}_0^T N^T(F - M_1)\tilde{C}_0 - \tilde{C}_0^T(F - M_1) \\
&\quad \cdot N\tilde{C}_0\tilde{A} - \tilde{C}_0^T F\tilde{B}_0^T\tilde{C}_0^T N^T(F - M_1)\tilde{C}_0 - \tilde{C}_0^T \\
&\quad \cdot (F - M_1)N\tilde{C}_0\tilde{B}_0 F\tilde{C}_0 - \mathcal{P}\tilde{B}_0 F\tilde{C}_0 - \tilde{C}_0^T F\tilde{B}_0^T\mathcal{P} \\
&= \Omega_0(\mathcal{P}) - [\Delta\tilde{A}^T\mathcal{P} + \mathcal{P}\Delta\tilde{A} + (\tilde{A} + \Delta\tilde{A})^T\mathcal{P}_0(F) \\
&\qquad + \mathcal{P}_0(F)(\tilde{A} + \Delta\tilde{A})]
\end{aligned}$$

which proves (11) with $\mathcal{U}$ given by (17). □

Remark 4.1: To construct X and Y satisfying (21), note that $[\tilde{B}_0X^T(F - M_1)^{1/2} - \tilde{C}_0^T(F - M_1)^{1/2}][\tilde{B}_0X^T(F - M_1)^{1/2} - \tilde{C}_0^T(F - M_1)^{1/2}]^T \ge 0$ implies

$$\begin{aligned}
&\tilde{B}_0X^T(F - M_1)\tilde{C}_0 + \tilde{C}_0^T(F - M_1)X\tilde{B}_0^T \\
&\quad \le \tilde{B}_0X^T(F - M_1)X\tilde{B}_0^T + \tilde{C}_0^T(F - M_1)\tilde{C}_0 \\
&\quad \le \tilde{B}_0X^T(M_2 - M_1)X\tilde{B}_0^T + \tilde{C}_0^T(M_2 - M_1)\tilde{C}_0
\end{aligned}$$

which shows that

$$\tilde{Y} = \tilde{B}_0X^TMX\tilde{B}_0^T + \tilde{C}_0^TM\tilde{C}_0 \tag{26}$$

satisfies (21) for all $X \in \mathbb{R}^{m_0 \times m_0}$ and $F \in \mathcal{F}$. For the special case of diagonal uncertainty F it can be shown that $\tilde{Y} = \tilde{B}_0X^TX\tilde{B}_0^T + \tilde{C}_0^TM^2\tilde{C}_0$ also satisfies (21).

Note that with $N \in \mathcal{N}$, it follows from (18) that there exists $\mu \in \mathbb{N}^{m_0}$ such that $(F - M_1)N \le \mu$ for all $F \in \mathcal{F}$. Next, using Theorem 3.1 and Proposition 4.1 and defining the notation

$$\begin{aligned}
\mathcal{N}_+ \triangleq \{&N \in \mathbb{R}^{m_0 \times m_0}: (F - M_1)N = N^T(F - M_1) \ge 0 \\
&F \in \mathcal{F}\}
\end{aligned}$$

we have the following result.

Theorem 4.1: Let $H \in \mathcal{H}$ and $N \in \mathcal{N}_+$ be such that $R_0 > 0$, and let $X \in \mathbb{R}^{m_0 \times m_0}$ and $\tilde{Y} \in \mathbb{N}^{\tilde{n}}$ be such that (21) is satisfied. Furthermore, suppose there exists a nonnegative-definite matrix $\mathcal{P}$ satisfying

$$\begin{aligned}
0 &= \tilde{\tilde{A}}^T\mathcal{P} + \mathcal{P}\tilde{\tilde{A}} + (H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} + \tilde{B}_0^T\mathcal{P} - X\tilde{B}_0^T)^T R_0^{-1} \\
&\quad \cdot (H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} + \tilde{B}_0^T\mathcal{P} - X\tilde{B}_0^T) + \tilde{Y} + \tilde{R}.
\end{aligned} \tag{27}$$

Then $(\tilde{A} + \Delta\tilde{A}, \tilde{E})$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case

$$J(A_c, B_c, C_c) \le \mathrm{tr}[(\mathcal{P} + \tilde{C}_0^T\mu\tilde{C}_0)\tilde{V}]. \tag{28}$$

Proof: The result is a direct specialization of Theorem 3.1 using Proposition 4.1. We only note that $\mathcal{P}_0(\Delta\tilde{A})$ now has the form $\mathcal{P}_0(F) = \tilde{C}_0^T(F - M_1)N\tilde{C}_0$. Since by assumption $N \in \mathcal{N}_+$, it follows that $\mathcal{P} + \mathcal{P}_0(F)$ is nonnegative definite for all $F \in \mathcal{F}$ as required by Theorem 3.1. □

Remark 4.2: An equivalent form of (27) is

$$\begin{aligned}
0 &= \tilde{A}_s^T\mathcal{P} + \mathcal{P}\tilde{A}_s + (H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} - X\tilde{B}_0^T)^T R_0^{-1} \\
&\quad \cdot (H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} - X\tilde{B}_0^T) + \mathcal{P}\tilde{B}_0\tilde{R}_0^{-1}\tilde{B}_0^T\mathcal{P} + \tilde{Y} + \tilde{R}
\end{aligned} \tag{29}$$

where $\tilde{A}_s \triangleq \tilde{\tilde{A}} + \tilde{B}_0\tilde{R}_0^{-1}(H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} - X\tilde{B}_0^T)$ is a *shifted* dynamics matrix. Now, setting $X = 0$ and choosing $\tilde{Y} = 0$, (29) specializes to the Popov Riccati equation considered in [6]. Alternatively, setting $H = I$ and $N = 0$ (29) specializes to the positive-real-type shifted quadratic bound given in [12] with $\hat{M} = I$. Finally, if $X = 0$, $\tilde{Y} = 0$, $N = 0$, and $H = I$, then (29) reduces to the positive-real circle Riccati equation [5]

$$\begin{aligned}
0 &= \left[\tilde{A} + \tfrac{1}{2}(M_1 + M_2)\tilde{C}_0\right]^T\mathcal{P} + \mathcal{P}\left[\tilde{A} + \tfrac{1}{2}(M_1 + M_2)\tilde{C}_0\right] \\
&\quad + \tfrac{1}{2}\tilde{C}_0^TM\tilde{C}_0 + \tfrac{1}{2}\mathcal{P}\tilde{B}_0M\tilde{B}_0^T\mathcal{P} + \tilde{R}.
\end{aligned} \tag{30}$$

If, in addition, $M_2 = -M_1 = \gamma^{-1}I$, where $\gamma > 0$, then (30) yields the bounded-real Riccati equation

$$0 = \tilde{A}^T\mathcal{P} + \mathcal{P}\tilde{A} + \gamma^{-2}\mathcal{P}\tilde{B}_0\tilde{B}_0^T\mathcal{P} + \tilde{C}_0^T\tilde{C}_0 + \gamma\tilde{R}. \tag{31}$$

Remark 4.3: Consider a skew-symmetric structured uncertainty set, that is, $\tilde{B}_0F\tilde{C}_0 + \tilde{C}_0^TF\tilde{B}_0^T = 0$ for all $F \in \mathcal{F}$, with uncertainty bounds $M_1 = -M_2$. Furthermore, let $X = \alpha I_{m_0}$, where $\alpha \in \mathbb{R}$, so that $\tilde{B}_0X^T(F - M_1)\tilde{C}_0 + \tilde{C}_0^T(F - M_1)X\tilde{B}_0^T = 0$ and hence $\tilde{Y}$ satisfying (21) can be chosen as $\tilde{Y} = 0$. Finally, let $H = I$ and $N = 0$. Then (29) can be written as

$$\begin{aligned}
0 &= (\tilde{A} - \alpha\tilde{B}_0M_2\tilde{B}_0^T)^T\mathcal{P} + \mathcal{P}(\tilde{A} - \alpha\tilde{B}_0M_2\tilde{B}_0^T) \\
&\quad + \mathcal{P}\tilde{B}_0M_2\tilde{B}_0^T\mathcal{P} + \tilde{C}_0^TM_2\tilde{C}_0 + \alpha^2\tilde{B}_0M_2\tilde{B}_0 + \tilde{R}
\end{aligned} \tag{32}$$

which involves the shifted dynamics matrix $\tilde{A} - \alpha\tilde{B}_0M_2\tilde{B}_0^T$.

V. Robust Controller Synthesis via Shifted Parameter-Dependent Bounding Functions

In this section, we state constructive sufficient conditions for characterizing fixed-order (i.e., full- and reduced-order) robust controllers. As in [6], these results are obtained by minimizing the worst case H_2 cost bound (28). In order to state the main result of this section we require some additional notation and a lemma concerning a pair of nonnegative-definite matrices.

Lemma 5.1 [1]: Let $\hat{Q}$, $\hat{P}$ be $n \times n$ nonnegative-definite matrices and suppose that rank $\hat{Q}\hat{P} = n_c$. Then there exist $n_c \times n$ matrices G, Γ and an $n_c \times n_c$ invertible matrix $\hat{M}$, unique except for a change of basis in $\mathbb{R}^{n_c}$, such that

$$\hat{Q}\hat{P} = G^T\hat{M}\Gamma, \qquad \Gamma G^T = I_{n_c}. \tag{33}$$

Furthermore, the $n \times n$ matrices $\tau \triangleq G^T\Gamma$ and $\tau_\perp \triangleq I_n - \tau$ are idempotent and have rank n_c and $n - n_c$, respectively.

To apply Theorem 4.1 to fixed-order dynamic compensation, let $\tilde{Y}$ have the form

$$\tilde{Y} = \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} \tag{34}$$

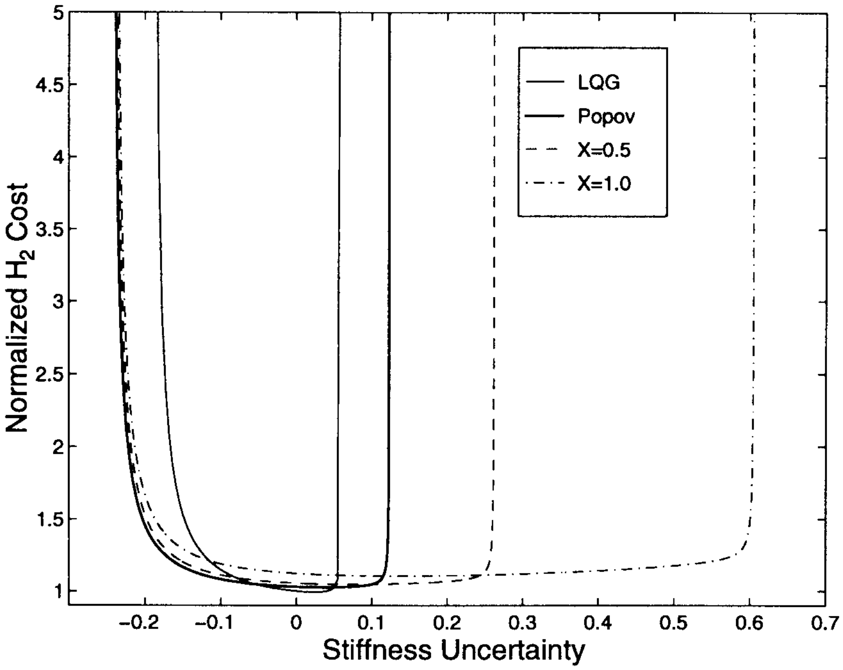

Fig. 1. Performance versus robustness tradeoffs for LQG and Theorem 5.1 controllers: Example 6.1.

where $Y \in \mathbb{N}^n$ satisfies

$$B_0X^T(F-M_1)C_0 + C_0^T(F-M_1)XB_0^T \le Y, \qquad F \in \mathcal{F}. \quad (35)$$

With $\tilde{Y}$ given by (34), it follows that (21) implies (35). Hence, it follows that (see Remark 4.1) one choice of Y satisfying (35) for all $X \in \mathbb{R}^{m_0\times m_0}$ and $F \in \mathcal{F}$ is given by $Y = B_0X^TMXB_0^T + C_0^TMC_0$.

For convenience, define the notation

$$\hat{A} \triangleq A + B_0M_1C_0, \; \overline{\Sigma} \triangleq C^TV_2^{-1}C$$
$$R_{2a} \triangleq R_2 + B^TC_0^TN^TR_0^{-1}NC_0B$$
$$P_a \triangleq B^TP + B^TC_0^TN^TR_0^{-1} \cdot (HC_0 + NC_0\hat{A} + B_0^TP - XB_0^T)$$
$$A_P \triangleq \hat{A} + B_0R_0^{-1}(HC_0 + NC_0\hat{A} - XB_0^T)$$
$$A_{\hat{P}} \triangleq A_P - Q\overline{\Sigma} + B_0R_0^{-1}B_0^TP$$
$$A_Q \triangleq A_P + B_0R_0^{-1}B_0^T(P+\hat{P})$$
$$A_{\hat{Q}} \triangleq A_P + B_0R_0^{-1}B_0^TP - (I + B_0R_0^{-1}NC_0)BR_{2a}^{-1}P_a$$

for arbitrary $P,\, Q,\, \hat{P} \in \mathbb{R}^{n\times n}$.

Theorem 5.1: Let $n_c \le n$, let $H \in \mathcal{H}$ and $N \in \mathcal{N}_+$ be such that $R_0 > 0$, and let $X \in \mathbb{R}^{m_0\times m_0}$ and $Y \in \mathbb{N}^n$ be such that (35) is satisfied. Furthermore, assume there exist $n \times n$ nonnegative-definite matrices $P,\, Q,\, \hat{P}$, and $\hat{Q}$ satisfying

$$0 = A_P^TP + PA_P + R_1 + Y + (HC_0 + NC_0\hat{A} - XB_0^T)^T \cdot R_0^{-1}(HC_0 + NC_0\hat{A} - XB_0^T) + PB_0R_0^{-1}B_0^TP - P_a^TR_{2a}^{-1}P_a + \tau_\perp^TP_a^TR_{2a}^{-1}P_a\tau_\perp \quad (36)$$
$$0 = A_QQ + QA_Q^T + V_1 - Q\overline{\Sigma}Q + \tau_\perp Q\overline{\Sigma}Q\tau_\perp^T \quad (37)$$
$$0 = A_{\hat{P}}^T\hat{P} + \hat{P}A_{\hat{P}} + \hat{P}B_0R_0^{-1}B_0^T\hat{P} + P_a^TR_{2a}^{-1}P_a - \tau_\perp^TP_a^TR_{2a}^{-1}P_a\tau_\perp \quad (38)$$
$$0 = A_{\hat{Q}}\hat{Q} + \hat{Q}A_{\hat{Q}}^T + Q\overline{\Sigma}Q - \tau_\perp Q\overline{\Sigma}Q\tau_\perp^T \quad (39)$$
$$\text{rank}\,\hat{Q} = \text{rank}\,\hat{P} = \text{rank}\,\hat{Q}\hat{P} = n_c \quad (40)$$

and let $A_c,\, B_c$, and C_c be given by

$$A_c = \Gamma[A_{\hat{Q}} - Q\overline{\Sigma}]G^T \quad (41)$$
$$B_c = \Gamma QC^TV_2^{-1} \quad (42)$$
$$C_c = -R_{2a}^{-1}P_aG^T. \quad (43)$$

Then $(\tilde{A} + \Delta\tilde{A},\, \tilde{E})$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case, the worst case H_2 performance criterion (9) satisfies the bound

$$J(A_c,\, B_c,\, C_c) \le \text{tr}[(P+\hat{P})V_1 + \hat{P}Q\overline{\Sigma}Q + C_0^T\mu C_0V_1]. \quad (44)$$

Proof: The proof is analogous to the proof of Theorem 6.1 given in [6]. □

Remark 5.1: In the full-order case, set $n_c = n$ so that $G = \Gamma = \tau = I_n$ and $\tau_\perp = 0$. In this case, the last term in each of (36)–(39) is zero and (39) is superfluous.

Remark 5.2:

When solving (36)–(39) numerically, the matrices $M_1,\; M_2,\; H,\; N$, and X and the structure matrices B_0 and C_0 appearing in the design equations can be adjusted to examine the tradeoffs between H_2 performance and robustness. As discussed in [6], to further reduce conservatism, one can view the matrices $H,\; N$, and X as free parameters and optimize the H_2 performance bound $\mathcal{J} \triangleq \text{tr}[(\mathcal{P} + \tilde{C}_0^T\mu\tilde{C}_0)\tilde{V}]$ with respect to $H,\; N$, and X. In particular, $\partial\mathcal{J}/\partial H$, $\partial\mathcal{J}/\partial N$, and $\partial\mathcal{J}/\partial X$ are given by

$$\frac{\partial\mathcal{J}}{\partial H} = R_0^{-1}[H\tilde{C}_0 + N\tilde{C}_0\tilde{\hat{A}} + \tilde{B}_0\mathcal{P} - X\tilde{B}_0^T]\mathcal{Q} \cdot [\tilde{C}_0^T - \{H\tilde{C}_0 + N\tilde{C}_0\tilde{\hat{A}} + \tilde{B}_0\mathcal{P} - X\tilde{B}_0^T\}R_0^{-1}M^{-1}] \quad (45)$$
$$\frac{\partial\mathcal{J}}{\partial N} = M\tilde{C}_0\tilde{V}\tilde{C}_0^T + R_0^{-1}(H\tilde{C}_0 + N\tilde{C}_0\tilde{\hat{A}} + \tilde{B}_0\mathcal{P} - X\tilde{B}_0^T)\mathcal{Q}[\tilde{\hat{A}} + \tilde{B}_0R_0^{-1}(H\tilde{C}_0 + N\tilde{C}_0\tilde{\hat{A}} + \tilde{B}_0\mathcal{P} - X\tilde{B}_0^T)]^T\tilde{C}_0^T \quad (46)$$
$$\frac{\partial\mathcal{J}}{\partial X} = R_0^{-1}[H\tilde{C}_0 + N\tilde{C}_0\tilde{\hat{A}} + \tilde{B}_0\mathcal{P} - X\tilde{B}_0^T]\mathcal{Q}\tilde{B}_0 + \frac{\partial}{\partial X}\,\text{tr}\,\tilde{Y}(X)\mathcal{Q} \quad (47)$$

where $\mathcal{Q}$ satisfies

$$0 = [\tilde{\tilde{A}} + \tilde{B}_0 R_0^{-1}(H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} + \tilde{B}_0^T\mathcal{P} - X\tilde{B}_0^T)]\mathcal{Q} + \mathcal{Q}[\tilde{\tilde{A}} + \tilde{B}_0 R_0^{-1}(H\tilde{C}_0 + N\tilde{C}_0\tilde{\tilde{A}} + \tilde{B}_0^T\mathcal{P} - X\tilde{B}_0^T)]^T + \tilde{V} \quad (48)$$

and $\tilde{Y}(X)$ satisfies (21). By using (45)–(47) within a numerical optimization algorithm, the optimal robust reduced-order controller and matrices H, N, and X can be determined simultaneously.

VI. Illustrative Numerical Example

In this section, we provide a numerical example to demonstrate Theorem 5.1. For simplicity, we consider the design of full-order dynamic output feedback controllers. In this paper, we employed a quasi-Newton optimization algorithm initialized with linear-quadratic-Gaussian (LQG) gains. The matrices H and N were initialized by solving an LMI feasibility problem. For given values of robustness bounds M_1 and M_2, the quasi-Newton optimization algorithm was used to find A_c, B_c, C_c, H, and N satisfying the necessary conditions. After each iteration, M_1 and M_2 were increased and the current values of (A_c, B_c, C_c) were used to find feasible H and N matrices which were then used as the starting point for the next iteration; for details of a similar algorithm, see [2].

Example 6.1: Consider the three-mass, two-spring system given in [6]. The nominal system dynamics and performance weighting matrices are

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad C = [1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$D_2 = [0 \quad 1], \qquad E_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The uncertainty in the dynamics matrix A corresponds to stiffness uncertainty in the second spring and is characterized by $\Delta A = B_0FC_0$, where $F \in \mathcal{F} \triangleq \{F: -\delta \le F \le \delta\}$, and B_0 and C_0 are given by

$$B_0 = [0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 1]^T, \qquad C_0 = [0 \quad 1 \quad -1 \quad 0 \quad 0 \quad 0].$$

Using Theorem 5.1 and design parameters $n_c = 6$, $\delta = 0.05$, and $Y = X^2B_0B_0^T + 4\delta^2C_0^TC_0$ (see Remark 4.1) several dynamic compensators were obtained for different values of X. Fig. 1 provides a comparison of robust stability and performance obtained from LQG theory and Theorem 5.1. Fig. 1 also shows the tradeoffs between robust performance and robust stability obtained from increasing X. Note that the tradeoff curve for $X = 0$ (with $Y = 0$) corresponds to the Popov-type controllers obtained in [6]. It can be seen that the controller obtained using nonzero value of X gives a significantly wider stability region than the LQG and Popov-type controllers with only slight degradation in cost.

VII. Conclusion

This paper combined the parameterized Lyapunov bounds and shifted quadratic guaranteed cost bounds to obtain a shifted parameter-dependent bound. The proposed shifted parameter-dependent bound was used to address the problem of robust stability and performance via fixed-order dynamic compensation. A quasi-Newton optimization algorithm was used to obtain numerical solutions for an illustrative numerical example. The design example considered demonstrated the effectiveness of the newly developed bounds.

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