Longitudinal Aircraft Dynamics and the Instantaneous Acceleration Center of Rotation

The case of the vanishing zeros

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Nonminimum-phase zeros, that is, closed-right-half-plane (CRHP) zeros, affect both the open- and closed-loop behavior of continuous-time linear systems in undesirable ways [1]. For example, an asymptotically stable linear system with an odd number of positive zeros experiences initial undershoot to a step input (see "Initial Undershoot"). Moreover, under the rules of root locus, zeros in the open-right-half plane (ORHP) attract closed-loop poles, which limits the controller gain and thus the performance of the closed-loop system. In linear quadratic Gaussian theory, closed-loop poles are attracted to the reflected locations of the open-loop ORHP zeros in the high-control-authority (that is, cheap-control) limit, thus constraining the achievable closed-loop bandwidth [2, p. 289].

Given the critical role of nonminimum-phase zeros, it is useful to identify physical characteristics that give rise to them. Although spatial separation between sensors and actuators is often postulated as a source of nonminimum-phase zeros, analysis of the transfer functions between separated masses in a serially connected structure shows that

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Initial Undershoot

Initial undershoot occurs when the step response of a transfer function initially moves in the direction opposite to the direction of its asymptotic value.

Let $G(s) \triangleq \beta(s)/(s^r\alpha(s))$ be a strictly proper transfer function with relative degree $d > 0$, where $r \geq 0$ and $\alpha(s)$ is asymptotically stable. Let $y(t)$ be the unit-step response of G. Then *initial undershoot* occurs at $t = 0$ if

$$y^{(d)}(0^+)y^{(r)}(\infty) < 0,$$

where $y^{(d)}(0^+) \triangleq \lim_{t\to 0^+} y^{(d)}(t)$ and $y^{(r)}(\infty) \triangleq \lim_{t\to\infty} y^{(r)}(t)$. The unit-step response has the initial curvature

$$y^{(d)}(0^+) = \lim_{t\to 0^+} y^{(d)}(t) = \lim_{s\to\infty} s(s^d\hat{y}(s)) = \lim_{s\to\infty} s^{d+1}\left(G(s)\frac{1}{s}\right) = \lim_{s\to\infty} s^d G(s) \neq 0,$$

as well as the asymptotic curvature

$$y^{(r)}(\infty) \triangleq \lim_{t\to\infty} y^{(r)}(t) = \lim_{s\to 0} s^{r+1}\left(G(s)\frac{1}{s}\right) = \frac{\beta(0)}{\alpha(0)}.$$

The initial direction of the step response depends on the sign of the product of the initial curvature $y^{(d)}(0^+)$ and the asymptotic curvature $y^{(r)}(\infty)$. The following result is discussed in [1].

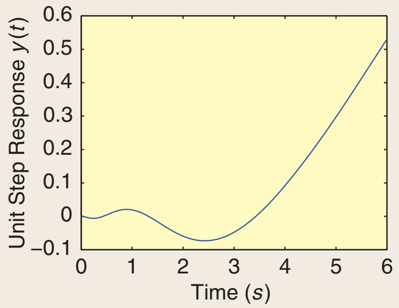

FIGURE S1 Unit step response of the transfer function $G(s) = -(s-1)(s-2)(s-3)/(s(s+1)(s+2)(s+3)(s+4))$. The step response of this system exhibits initial undershoot with three direction reversals due to its three positive zeros.

PROPOSITION S1

Let $G \triangleq \beta(s)/(s^r\alpha(s))$ be a strictly proper transfer function, where $r \geq 0$ and $\alpha(s)$ is asymptotically stable. Then the unit step response has an initial undershoot if and only if $G(s)$ has an odd number of positive zeros.

As an example, consider the transfer function $G(s) = -(s-1)(s-2)(s-3)/(s(s+1)(s+2)(s+3)(s+4))$. The unit step response exhibits initial undershoot with three direction reversals due to the three positive zeros, as shown in Figure S1.

this is not necessarily the case [3]. On the other hand, noncolocation in rotational motion may give rise to nonminimum-phase zeros [4], [5].

Aside from zero locations, the number of zeros determines the relative degree of the system, which impacts the asymptotic, that is, high frequency, phase of the transfer function. The relative degree of an asymptotically stable transfer function also plays a role in the initial behavior of the step response. This relationship is apparent from the initial value theorem applied to the derivative of the output. When the initial slope of the output is zero, higher order derivatives of the initial response, which determine the initial curvature of the output, can be evaluated to detect the possibility of initial undershoot. In particular, the sign of the first nonzero derivative of the output relative to the sign of the dc gain determines whether or not the step response exhibits initial undershoot. The number of derivatives that must be evaluated to determine the sign of the first nonzero derivative is equal to the relative degree of the system.

In aircraft dynamics, the instantaneous acceleration center of rotation (IACR) of an aircraft is the point on the aircraft that has zero instantaneous acceleration. For an aircraft that is perturbed from steady horizontal flight by an elevator step deflection, the IACR is the point at which the elevator-to-vertical-velocity transfer function and the elevator-to-horizontal-velocity transfer function both have at least one zero that vanishes.

For the elevator-to-vertical-velocity transfer function, the zero that vanishes typically corresponds to a nonminimum-phase zero aft of the IACR and a minimum-phase zero forward of the IACR. In this case, as the point p, at which the vertical-velocity response is determined, is moved forward from the tail to the IACR, a real nonminimum-phase zero moves toward ∞, where it vanishes. As p moves past the IACR, the zero "reappears" at $-\infty$ and moves toward an asymptotic location as a minimum-phase zero. Thus, the vertical-velocity measurement at each point along the aircraft between the tail and the IACR exhibits initial undershoot. This phenomenon plays a role in the literature on aircraft dynamics and control [6, pp. 313–316], [7]–[15]. Vanishing zeros are discussed in [16].

In the present article, we demonstrate the relationship between vanishing zeros and the response of the aircraft at the IACR. The IACR of a rigid body is related to, but distinct from, the center of rotation. See "Center of Rotation and Center of Percussion," which discusses the motion of a bar-like rigid body in response to an impact. A bar-like rigid body possesses a point, called the center of percussion, with the property that an impulsive force at this location leads to zero velocity at another point on the body, called the center of rotation, at the instant

Center of Rotation and Center of Percussion

Consider the free rigid body shown in Figure S2, with concentrated masses $m_1, \ldots, m_n$ at distances of $\ell_1, \ldots, \ell_n$, respectively, from the point O_B, which is the origin of the body-fixed frame F_B. The frame F_A is assumed to be an inertial frame. Consider a force $\vec{F}$ that impacts the structure at the point P and perpendicular to the body, and assume that R is the point on the body at which the velocity $\vec{v}_{R/O_A/A}$ of R relative to O_A with respect to F_A is zero at the instant immediately following the impact. The point R is the *center of rotation relative to* P; equivalently, P is the *center of percussion relative* to R. Let ℓ_R and ℓ_P denote the distances from the upper end of the body to R and P, respectively. The distance ℓ_c from the upper end of the body to the center of mass c is given by

$$\ell_c = \frac{\sum_{i=1}^{n} m_i \ell_i}{m_{\text{total}}},$$

where $m_{\text{total}} \triangleq \sum_{i=1}^{n} m_i$ is the total mass of the body.

Next, viewing O_A as an unforced particle, Newton's second law implies

$$\vec{F} = m_{\text{total}} \overset{A\bullet}{\vec{v}}_{c/O_A/A}, \tag{S1}$$

where $\vec{F} = F_0 \delta(t) \hat{\jmath}_A$, and $\vec{v}_{c/O_A/A}$ is the velocity of c relative to O_A with respect to F_A, which can be written as $\vec{v}_{c/O_A/A} = v_c(t)\hat{\jmath}_A$. Thus, it follows from (S1) that $F_0\delta(t) = m_{\text{total}}\dot{v}_c(t)$, which implies that the velocity after the impulse, that is, at $t = 0^+$, is given by

$$v_c(0^+) = \frac{F_0}{m_{\text{total}}}. \tag{S2}$$

Next, the moment $\vec{M}_{P/c}$ on P about c due to $\vec{F}$ is given by

$$\vec{M}_{P/c} = \vec{r}_{P/c} \times \vec{F} = I_c \overset{A\bullet}{\vec{\omega}}_{B/A}, \tag{S3}$$

where $\vec{\omega}_{B/A}$ is the angular velocity of F_B relative to F_A, $I_c \triangleq \sum_{i=1}^{n} m_i(\ell_i - \ell_c)^2$ is the moment of inertia of the body relative to c, and the position of P relative to c is given by $\vec{r}_{P/c} = (\ell_P - \ell_c)\hat{\imath}_B$. Since $\vec{F} = F_0\delta(t)\hat{\jmath}_A = F_0\delta(t)\hat{\jmath}_B$ and $\hat{k}_A$ is aligned with $\hat{k}_B$, it follows from (S3) that $F_0(\ell_P - \ell_c)\delta(t) = I_c\dot{\omega}(t)$, which implies that the angular velocity after the impulse, that is, at $t = 0^+$, is given by

$$\omega(0^+) = \frac{F_0(\ell_P - \ell_c)}{I_c}. \tag{S4}$$

Next, the velocity $\vec{v}_{R/O_A/A}$ of R relative to O_A with respect to F_A can be written as

$$\begin{aligned} \vec{v}_{R/O_A/A} &= \overset{A\bullet}{\vec{r}}_{R/O_A} \\ &= \overset{A\bullet}{\vec{r}}_{R/c} + \overset{A\bullet}{\vec{r}}_{c/O_A} \\ &= \vec{v}_{c/O_A/A} + \overset{B\bullet}{\vec{r}}_{R/c} + \vec{\omega}_{B/A} \times \vec{r}_{R/c} \\ &= \vec{v}_{c/O_A/A} + \vec{\omega}_{B/A} \times \vec{r}_{R/c} \\ &= v_c\hat{\jmath}_A + (\ell_R - \ell_c)\omega\hat{\jmath}_B. \end{aligned} \tag{S5}$$

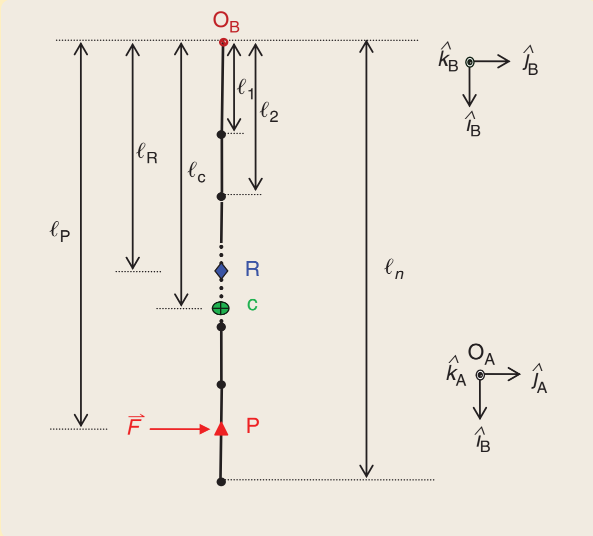

FIGURE S2 A free rigid body with nonuniform concentrated masses $m_1, \ldots, m_n$ at distances of $\ell_1, \ldots, \ell_n$ from the upper end O_B of the structure. The point R is the center of rotation relative to P, while the point P is the center of percussion relative to R.

Note that $\overset{B\bullet}{\vec{r}}_{R/c} = 0$ since R and c are fixed in the body. Since, at $t = 0^+$, $\hat{\jmath}_A$ is aligned with $\hat{\jmath}_B$, it follows from (S2), (S4), and (S5) that, for $t = 0^+$,

$$\vec{v}_{R/O_A/A} = F_0\left(\frac{1}{m_{\text{total}}} + \frac{(\ell_R - \ell_c)(\ell_P - \ell_c)}{I_c}\right)\hat{\jmath}_A.$$

Lastly, since R is the center of rotation, we have, for $t = 0^+$,

$$F_0\left(\frac{1}{m_{\text{total}}} + \frac{(\ell_R - \ell_c)(\ell_P - \ell_c)}{I_c}\right) = 0.$$

It follows that the location of R is given by

$$\ell_R = \ell_c - \frac{I_c}{m_{\text{total}}(\ell_P - \ell_c)}. \tag{S6}$$

Consequently, if at $t = t_0$ the force impacts the body at the center of percussion P relative to R, where P is located at ℓ_P, then the velocity $\vec{v}_{R/O_A/A}$ at the center of rotation located at ℓ_R given by (S6) is zero at $t = t_0^+$. In other words, (S6) characterizes the location of R.

Instantaneous Velocity Center of Rotation

Let $\mathcal{B}$ be a rigid body with body-fixed frame F_B, let F_A be a frame with origin O_A, and let $\vec{\omega}_{B/A}$ be the angular velocity of F_B relative to F_A. A point p that is fixed relative to $\mathcal{B}$ is an *instantaneous velocity center of rotation* (IVCR) of $\mathcal{B}$ relative to F_A at time t if $\vec{\omega}_{B/A}(t) \neq 0$ and $\vec{v}_{p/O_A/A}(t) = 0$ [S1, pp. 147–149], [S2, pp. 49–52]. For convenience, we omit the phrase "relative to F_A". The motion of $\mathcal{B}$ can be viewed as instantaneously rotating about p. See Figure S3.

Let q be a point that is fixed relative to $\mathcal{B}$. It follows from the definition of an IVCR and the transport theorem that p is an IVCR of $\mathcal{B}$ if and only if $\vec{\omega}_{B/A} \neq 0$ and

$$\vec{v}_{p/O_A/A} = \vec{\omega}_{B/A} \times \vec{r}_{p/q} + \vec{v}_{q/O_A/A} = 0. \quad \text{(S7)}$$

Resolving $\vec{v}_{q/O_A/A}$, $\vec{\omega}_{B/A}$, and $\vec{r}_{p/q}$ in F_B as

$$v \triangleq \vec{v}_{q/O_A/A}\Big|_B,\ \omega \triangleq \vec{\omega}_{B/A}\Big|_B,\ r \triangleq \vec{r}_{p/q}\Big|_B,$$

(S7) can be rewritten as

$$\omega^{\times} r + v = 0. \quad \text{(S8)}$$

The existence of an IVCR thus depends on the existence of a solution r to (S8). Since $\omega^{\times}$ is singular, (S8) has either zero or infinitely many solutions. Let $\mathcal{R}$ denote range.

FACT S1

The following statements hold:

i) If $v \notin \mathcal{R}(\omega^{\times})$, then $\mathcal{B}$ has no IVCR.

ii) If $v \in \mathcal{R}(\omega^{\times})$, then $\mathcal{B}$ has infinitely many IVCRs.

iii) Suppose $v \in \mathcal{R}(\omega^{\times})$. Then p is an IVCR if and only if there exists $\alpha \in \mathbb{R}$ such that

$$r = \alpha\omega - \frac{1}{|\omega|^2}\omega \times v. \quad \text{(S9)}$$

It follows from (S7) that, if p is an IVCR of $\mathcal{B}$ and q is fixed relative to $\mathcal{B}$, then $\vec{\omega}_{B/A} \cdot \vec{v}_{q/O_A/A} = \omega^{\mathrm{T}} v = -\omega^{\mathrm{T}}(\omega^{\times} r) = 0$. Hence, if $\vec{\omega}_{B/A} \cdot \vec{v}_{q/O_A/A} \neq 0$, then $\mathcal{B}$ has no IVCR. This situation occurs, for example, in bullet flight, where the translational velocity is parallel to its angular velocity.

FACT S2

p is an IVCR of $\mathcal{B}$ if and only if p satisfies the following conditions:

i) $\vec{\omega}_{B/A} \cdot \vec{v}_{q/O_A/A} = 0.$

ii) $\vec{\omega}_{B/A} \times \left(\vec{r}_{p/q} - \frac{1}{|\vec{\omega}_{B/A}|^2}\vec{\omega}_{B/A} \times \vec{v}_{q/O_A/A}\right) = 0.$

In this case,

$$\vec{r}_{p/q} = \frac{1}{|\vec{\omega}_{B/A}|^2}\vec{\omega}_{B/A} \times \vec{v}_{q/O_A/A} + \frac{\vec{\omega}_{B/A} \cdot \vec{r}_{p/q}}{|\vec{\omega}_{B/A}|^2}\vec{\omega}_{B/A}. \quad \text{(S10)}$$

PROOF

Assume that p is an IVCR of $\mathcal{B}$. Then it follows from (S7) that

$$\vec{\omega}_{B/A} \cdot \vec{v}_{q/O_A/A} = \vec{\omega}_{B/A} \cdot (-\vec{\omega}_{B/A} \times \vec{r}_{p/q}) = 0,$$

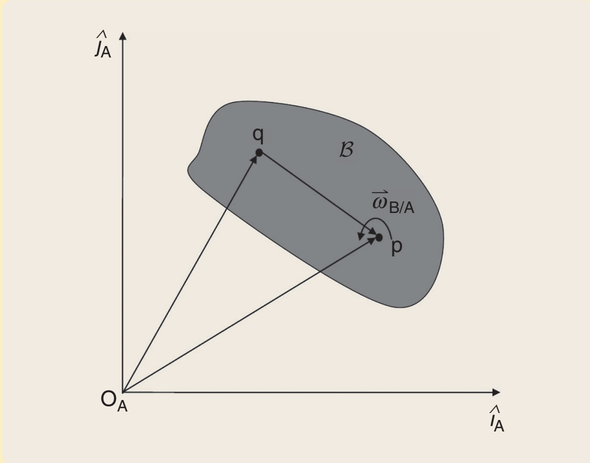

FIGURE S3 Instantaneous velocity center of rotation p. $\mathcal{B}$ is a rigid body, and the point q is fixed relative to $\mathcal{B}$. F_A is a frame with origin O_A, $\vec{\omega}_{B/A}$ is the angular velocity of F_B relative to F_A, and it is assumed that $\vec{\omega}_{B/A} \neq 0$. The point p, which is fixed relative to $\mathcal{B}$, has the property that, at time t, the velocity of p relative to O_A with respect to F_A is zero. Thus, $\mathcal{B}$ is instantaneously rotating about p.

which proves i). To prove ii), it follows from (S7) that

$$\vec{\omega}_{B/A} \times \left(\vec{r}_{p/q} - \frac{1}{|\vec{\omega}_{B/A}|^2}\vec{\omega}_{B/A} \times \vec{v}_{q/O_A/A}\right) = \vec{\omega}_{B/A} \times \vec{r}_{p/q} + \vec{v}_{q/O_A/A}$$
$$= 0.$$

Hence, ii) holds.

Conversely, it follows from ii) that there exists $\alpha \in \mathbb{R}$ such that $\vec{r}_{p/q} = (1/|\vec{\omega}_{B/A}|^2)\vec{\omega}_{B/A} \times \vec{v}_{q/O_A/A} + \alpha\vec{\omega}_{B/A}$. Using i) and ii), it follows that

$$\begin{aligned}
\vec{v}_{p/O_A/A} &= \vec{v}_{p/q/A} + \vec{v}_{q/O_A/A} \\
&= \vec{v}_{p/q/B} + \vec{\omega}_{B/A} \times \vec{r}_{p/q} + \vec{v}_{q/O_A/A} \\
&= \vec{\omega}_{B/A} \times \left(\frac{1}{|\vec{\omega}_{B/A}|^2}\vec{\omega}_{B/A} \times \vec{v}_{q/O_A/A} + \alpha\vec{\omega}_{B/A}\right) + \vec{v}_{q/O_A/A} \\
&= -\vec{v}_{q/O_A/A} + \vec{v}_{q/O_A/A} \\
&= 0.
\end{aligned}$$

To show (S10), assume p is an IVCR of $\mathcal{B}$. It follows from (S7) that

$$\vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{r}_{p/q} + \vec{v}_{q/O_A/A}) = 0,$$

which is equivalent to

$$(\vec{\omega}_{B/A} \cdot \vec{r}_{p/q})\,\vec{\omega}_{B/A} - |\vec{\omega}_{B/A}|^2\,\vec{r}_{p/q} + \vec{\omega}_{B/A} \times \vec{v}_{q/O_A/A} = 0. \quad \text{(S11)}$$

Solving for $\vec{r}_{p/q}$ in (S11) yields (S10). □

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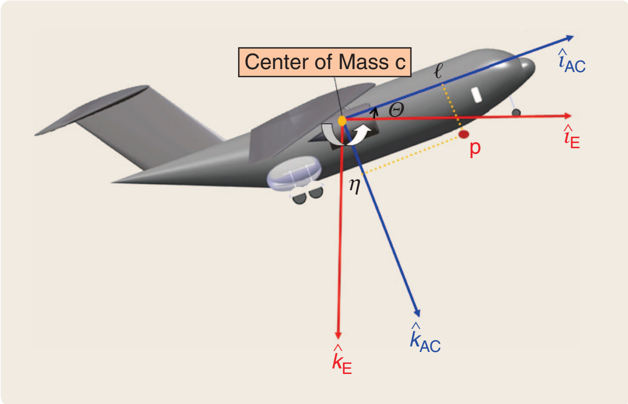

FIGURE 1 Aircraft and Earth frames. The aircraft frame is fixed to the aircraft, while the Earth frame is assumed to be an inertial frame. The signed quantities ℓ and η determine the location of the point p at which the output is defined relative to the center of mass c. The pitch angle Θ, which is positive as shown, is determined by the right-hand rule about the axis $\hat{\jmath}_{AC} = \hat{\jmath}_E$, which is not shown but is directed out of the page.

immediately following the impact. Another related notion is the instantaneous velocity center of rotation (IVCR), which is discussed in "Instantaneous Velocity Center of Rotation."

To demonstrate the relationship between vanishing zeros and the response of the aircraft at its IACR, we consider both the vertical-velocity response and the horizontal-velocity response of the aircraft to an elevator step deflection. In particular, we show that, at the IACR, the relative degree of the linearized transfer function from elevator deflection to vertical velocity (and thus to altitude) increases by at least one, and the relative degree of the linearized transfer function from elevator deflection to horizontal velocity increases by at least one. Moreover, we provide conditions under which the zeros that vanish at the IACR are nonminimum phase. Furthermore, we characterize the relationship between these vanishing zeros and the potential for initial undershoot in the aircraft's step response. For a business jet example, we show that each point on the aircraft that is aft of the IACR experiences initial undershoot in vertical velocity, whereas each point forward of the IACR does not experience initial velocity undershoot in the vertical direction.

To provide a tutorial development of the relevant transfer functions, we begin with the nonlinear equations of motion, show how these equations incorporate aerodynamic effects in terms of stability derivatives, and then arrive at the transfer functions for the linearized motion. This development provides an introduction to aircraft dynamics, which may be useful to readers who have not had the benefit of a course on flight dynamics. For further details on aircraft dynamics, see [6], [17], and [18].

AIRCRAFT KINEMATICS

The Earth frame F_E, whose orthogonal axes are labeled $\hat{\imath}_E$, $\hat{\jmath}_E$, and $\hat{k}_E$, is assumed to be an inertial frame, that is, a frame with respect to which Newton's second law is valid [19]. A hat denotes a dimensionless unit-length physical vector. The origin O_E of the Earth frame is any convenient point on the Earth. The axes $\hat{\imath}_E$ and $\hat{\jmath}_E$ are horizontal, while the axis $\hat{k}_E$ points downward; we assume the Earth is flat. The aircraft frame F_{AC}, whose axes are labeled $\hat{\imath}_{AC}$, $\hat{\jmath}_{AC}$, and $\hat{k}_{AC}$, is fixed to the aircraft. The center of mass c and frame vectors $\hat{\imath}_{AC}$ and $\hat{k}_{AC}$ are shown in Figure 1. The aircraft is assumed to be a three-dimensional rigid body.

In longitudinal flight, the aircraft moves in an inertially nonrotating vertical plane by translating along $\hat{\imath}_{AC}$ and $\hat{k}_{AC}$ and by rotating about $\hat{\jmath}_{AC}$. The direction of $\hat{\jmath}_{AC}$ is thus fixed with respect to F_E. For convenience, we assume that $\hat{\jmath}_{AC} = \hat{\jmath}_E$. The velocity and acceleration of the aircraft along $\hat{\jmath}_{AC}$ are thus identically zero for longitudinal flight, as are the roll and yaw components of the angular velocity of the aircraft relative to the Earth frame. The sign of the pitch angle Θ, which is the angle from $\hat{\imath}_E$ to $\hat{\imath}_{AC}$, is determined by the right-hand rule with the thumb pointing along $\hat{\jmath}_{AC}$ and with the fingers curled around $\hat{\jmath}_{AC}$. For example, the pitch angle Θ, shown in Figure 1, is positive.

Let p denote a point in the plane that is parallel to the $\hat{\imath}_{AC}$-$\hat{k}_{AC}$ plane and passes through c. The position of p relative to O_E can be written as

$$\vec{r}_{p/O_E} = r_{ph}\hat{\imath}_E + r_{pv}\hat{k}_E, \tag{1}$$

where a harpoon denotes a physical vector. The position of p relative to c is given by

$$\vec{r}_{p/c} = \vec{r}_{p/O_{AC}} + \vec{r}_{O_{AC}/c} = \vec{r}_{p/O_{AC}} - \vec{r}_{c/O_{AC}}, \tag{2}$$

which can be written as

$$\vec{r}_{p/c} = \ell\hat{\imath}_{AC} + \eta\hat{k}_{AC}, \tag{3}$$

where $\ell > 0$ indicates that p is forward of c, that is, toward the nose, and $\ell < 0$ denotes that p is aft of c, that is, toward the tail. Resolving $\vec{r}_{p/c}$ in F_{AC} yields

$$\vec{r}_{p/c}\Big|_{AC} = \begin{bmatrix} \ell \\ 0 \\ \eta \end{bmatrix}. \tag{4}$$

The distance between the aircraft center of mass c and the point p is given by

$$|\vec{r}_{p/c}| = \sqrt{\ell^2 + \eta^2}.$$

The orientation matrix, that is, the direction cosine matrix, of F_{AC} relative to F_E corresponding to the pitch angle Θ is

$$\mathcal{O}_{AC/E} \triangleq \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix}.$$

Therefore,

$$\mathcal{O}_{\mathrm{E/AC}} = \mathcal{O}_{\mathrm{AC/E}}^{\mathrm{T}} = \begin{bmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & \cos\Theta \end{bmatrix}. \tag{5}$$

Hence, using (4) we have

$$\vec{r}_{\mathrm{p/c}}\Big|_{\mathrm{E}} = \mathcal{O}_{\mathrm{E/AC}}\,\vec{r}_{\mathrm{p/c}}\Big|_{\mathrm{AC}} = \begin{bmatrix} \ell\cos\Theta + \eta\sin\Theta \\ 0 \\ -\ell\sin\Theta + \eta\cos\Theta \end{bmatrix}. \tag{6}$$

Since, in longitudinal flight, the aircraft rotates about $\hat{\jmath}_{\mathrm{AC}}$, the angular velocity of $\mathrm{F_{AC}}$ relative to $\mathrm{F_E}$ and resolved in $\mathrm{F_{AC}}$ is given by

$$\vec{\omega}_{\mathrm{AC/E}}\Big|_{\mathrm{AC}} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix}. \tag{7}$$

Note that $Q = \dot{\Theta}$ and that P and R are identically zero. Resolving $\vec{\omega}_{\mathrm{AC/E}}$ in $\mathrm{F_E}$, we have

$$\vec{\omega}_{\mathrm{AC/E}}\Big|_{\mathrm{E}} = \mathcal{O}_{\mathrm{E/AC}}\,\vec{\omega}_{\mathrm{AC/E}}\Big|_{\mathrm{AC}} = \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix}. \tag{8}$$

To change the frame with respect to which the physical vector $\vec{x}$ is differentiated, we use the transport theorem, which is given by the "ABBA rule"

$$\overset{\mathrm{A}\bullet}{\vec{x}} = \overset{\mathrm{B}\bullet}{\vec{x}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{x}, \tag{9}$$

where a labeled dot over a physical vector denotes the frame derivative with respect to the indicated frame. In particular, if $\vec{x} = x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}$, then $\overset{\mathrm{A}\bullet}{\vec{x}} = \dot{x}_1\hat{\imath}_{\mathrm{A}} + \dot{x}_2\hat{\jmath}_{\mathrm{A}} + \dot{x}_3\hat{k}_{\mathrm{A}}$. Hence,

$$\overset{\mathrm{E}\bullet}{\vec{\omega}}_{\mathrm{AC/E}} = \overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC/E}} + \vec{\omega}_{\mathrm{AC/E}} \times \vec{\omega}_{\mathrm{AC/E}} = \overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC/E}}, \tag{10}$$

and thus it follows from (7), (8), and (10) that

$$\overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC/E}}\Big|_{\mathrm{AC}} = \overset{\mathrm{E}\bullet}{\vec{\omega}}_{\mathrm{AC/E}}\Big|_{\mathrm{E}} = \overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC/E}}\Big|_{\mathrm{E}} = \overset{\mathrm{E}\bullet}{\vec{\omega}}_{\mathrm{AC/E}}\Big|_{\mathrm{AC}} = \begin{bmatrix} 0 \\ \ddot{\Theta} \\ 0 \end{bmatrix}.$$

Let $\vec{v}_{\mathrm{c/O_E/E}}$ and $\vec{a}_{\mathrm{c/O_E/E}}$ denote the velocity and acceleration of c relative to $\mathrm{O_E}$ with respect to $\mathrm{F_E}$, respectively, and let $\vec{v}_{\mathrm{p/O_E/E}}$ and $\vec{a}_{\mathrm{p/O_E/E}}$ denote the velocity and acceleration of p relative to $\mathrm{O_E}$ with respect to $\mathrm{F_E}$, respectively, that is,

$$\vec{v}_{\mathrm{c/O_E/E}} \triangleq \overset{\mathrm{E}\bullet}{\vec{r}}_{\mathrm{c/O_E}},$$
$$\vec{a}_{\mathrm{c/O_E/E}} \triangleq \overset{\mathrm{E}\bullet\bullet}{\vec{r}}_{\mathrm{c/O_E}},$$

and

$$\vec{v}_{\mathrm{p/O_E/E}} \triangleq \overset{\mathrm{E}\bullet}{\vec{r}}_{\mathrm{p/O_E}},$$
$$\vec{a}_{\mathrm{p/O_E/E}} \triangleq \overset{\mathrm{E}\bullet\bullet}{\vec{r}}_{\mathrm{p/O_E}}.$$

We resolve $\vec{v}_{\mathrm{c/O_E/E}}$ in $\mathrm{F_{AC}}$ as

$$\vec{v}_{\mathrm{c/O_E/E}}\Big|_{\mathrm{AC}} = \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} U \\ 0 \\ W \end{bmatrix}, \tag{11}$$

and note that V is identically zero for longitudinal flight. Next, it follows from (2) that

$$\vec{r}_{\mathrm{p/O_E}} = \vec{r}_{\mathrm{p/c}} + \vec{r}_{\mathrm{c/O_E}},$$

which implies that

$$\vec{v}_{\mathrm{p/O_E/E}} = \overset{\mathrm{E}\bullet}{\vec{r}}_{\mathrm{p/O_E}} = \overset{\mathrm{E}\bullet}{\vec{r}}_{\mathrm{p/c}} + \overset{\mathrm{E}\bullet}{\vec{r}}_{\mathrm{c/O_E}} = \vec{v}_{\mathrm{p/c/E}} + \vec{v}_{\mathrm{c/O_E/E}}, \tag{12}$$

where

$$\vec{v}_{\mathrm{p/c/E}} \triangleq \overset{\mathrm{E}\bullet}{\vec{r}}_{\mathrm{p/c}} = \vec{\omega}_{\mathrm{AC/E}} \times \vec{r}_{\mathrm{p/c}}. \tag{13}$$

Next, it follows from (5)–(8) and (11)–(13) that

$$\begin{aligned} \vec{v}_{\mathrm{p/O_E/E}}\Big|_{\mathrm{E}} &= \vec{v}_{\mathrm{c/O_E/E}}\Big|_{\mathrm{E}} + \left(\vec{\omega}_{\mathrm{AC/E}} \times \vec{r}_{\mathrm{p/c}}\right)\Big|_{\mathrm{E}} \\ &= \begin{bmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & \cos\Theta \end{bmatrix}\begin{bmatrix} U \\ 0 \\ W \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} \times \begin{bmatrix} \ell\cos\Theta + \eta\sin\Theta \\ 0 \\ -\ell\sin\Theta + \eta\cos\Theta \end{bmatrix} \\ &= \begin{bmatrix} v_{\mathrm{ph}} \\ 0 \\ v_{\mathrm{pv}} \end{bmatrix}, \end{aligned}$$

where

$$v_{\mathrm{ph}} \triangleq (\cos\Theta)U + (\sin\Theta)W - \ell(\sin\Theta)\dot{\Theta} + \eta(\cos\Theta)\dot{\Theta}, \tag{14}$$

$$v_{\mathrm{pv}} \triangleq -(\sin\Theta)U + (\cos\Theta)W - \ell(\cos\Theta)\dot{\Theta} - \eta(\sin\Theta)\dot{\Theta}. \tag{15}$$

Next, it follows from (9) and (11) that

$$\begin{aligned} \vec{a}_{\mathrm{c/O_E/E}}\Big|_{\mathrm{AC}} &= \overset{\mathrm{E}\bullet}{\vec{v}}_{\mathrm{c/O_E/E}}\Big|_{\mathrm{AC}} \\ &= \left(\overset{\mathrm{AC}\bullet}{\vec{v}}_{\mathrm{c/O_E/E}} + \vec{\omega}_{\mathrm{AC/E}} \times \vec{v}_{\mathrm{c/O_E/E}}\right)\Big|_{\mathrm{AC}} \\ &= \begin{bmatrix} \dot{U} \\ 0 \\ \dot{W} \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} \times \begin{bmatrix} U \\ 0 \\ W \end{bmatrix} \\ &= \begin{bmatrix} \dot{U} + \dot{\Theta}W \\ 0 \\ \dot{W} - \dot{\Theta}U \end{bmatrix}. \end{aligned} \tag{16}$$

Differentiating the transport theorem (9) yields

$$\overset{\mathrm{A}\bullet\bullet}{\vec{x}} = \overbrace{\overset{\mathrm{B}\bullet}{\vec{x}}}^{\mathrm{A}\bullet} + \overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{x} + \vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\vec{x}}$$

$$= \overset{B\bullet\bullet}{\vec{x}} + \vec{\omega}_{B/A} \times \overset{B\bullet}{\vec{x}} + \overset{A\bullet}{\vec{\omega}}_{B/A} \times \vec{x} + \vec{\omega}_{B/A} \times \left(\overset{B\bullet}{\vec{x}} + \vec{\omega}_{B/A} \times \vec{x} \right)$$
$$= \overset{B\bullet\bullet}{\vec{x}} + 2\vec{\omega}_{B/A} \times \overset{B\bullet}{\vec{x}} + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{x} + \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{x}), \quad (17)$$

which is the double transport theorem. Note that

$$\vec{a}_{p/O_E/E} \triangleq \overset{E\bullet\bullet}{\vec{r}}_{p/O_E} = \overset{E\bullet\bullet}{\vec{r}}_{p/c} + \overset{E\bullet\bullet}{\vec{r}}_{c/O_E} = \vec{a}_{p/c/E} + \vec{a}_{c/O_E/E}, \quad (18)$$

where

$$\vec{a}_{p/c/E} \triangleq \overset{E\bullet\bullet}{\vec{r}}_{p/c}. \quad (19)$$

Now, using (16)–(19), we have

$$\vec{a}_{p/O_E/E}\Big|_{AC} = \vec{a}_{p/c/E}\Big|_{AC} + \vec{a}_{c/O_E/E}\Big|_{AC}$$
$$= \left(\overset{AC\bullet\bullet}{\vec{r}}_{p/c} + 2\vec{\omega}_{AC/E} \times \overset{AC\bullet}{\vec{r}}_{p/c} + \overset{AC\bullet}{\vec{\omega}}_{AC/E} \times \vec{r}_{p/c} + \vec{\omega}_{AC/E} \times (\vec{\omega}_{AC/E} \times \vec{r}_{p/c}) \right)\Big|_{AC} + \vec{a}_{c/O_E/E}\Big|_{AC}$$
$$= \overset{AC\bullet}{\vec{\omega}}_{AC/E}\Big|_{AC} \times \vec{r}_{p/c}\Big|_{AC} + \vec{\omega}_{AC/E}\Big|_{AC} \times \left(\vec{\omega}_{AC/E}\Big|_{AC} \times \vec{r}_{p/c}\Big|_{AC} \right) + \vec{a}_{c/O_E/E}\Big|_{AC}$$
$$= \begin{bmatrix} 0 \\ \ddot{\Theta} \\ 0 \end{bmatrix} \times \begin{bmatrix} \ell \\ 0 \\ \eta \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} \times \begin{bmatrix} \ell \\ 0 \\ \eta \end{bmatrix} \right) + \begin{bmatrix} \dot{U} + \dot{\Theta} W \\ 0 \\ \dot{W} - \dot{\Theta} U \end{bmatrix}$$
$$= \begin{bmatrix} -\ell\dot{\Theta}^2 + \dot{U} + W\dot{\Theta} + \eta\ddot{\Theta} \\ 0 \\ -\ell\ddot{\Theta} + \dot{W} - U\dot{\Theta} - \eta\dot{\Theta}^2 \end{bmatrix}. \quad (20)$$

AIRCRAFT DYNAMICS

To apply Newton's second law for translational acceleration, we view O_E as an unforced particle [19] and all forces as acting at the aircraft's center of mass. We thus have

$$m\,\vec{a}_{c/O_E/E} = m\,\vec{g} + \vec{F}_A + \vec{F}_T, \quad (21)$$

where m is the mass of the aircraft, $\vec{g} = g\hat{k}_E$ is the acceleration due to gravity, $\vec{F}_A$ is the aerodynamic force, and $\vec{F}_T$ is the engine thrust force. Resolving (21) in F_{AC} yields

$$m\,\vec{a}_{c/O_E/E}\Big|_{AC} = m\,\vec{g}\Big|_{AC} + \vec{F}_A\Big|_{AC} + \vec{F}_T\Big|_{AC}, \quad (22)$$

where

$$\vec{g}\Big|_{AC} = \mathcal{O}_{AC/E}\,\vec{g}\Big|_{E} = \begin{bmatrix} -g\sin\Theta \\ 0 \\ g\cos\Theta \end{bmatrix}, \quad (23)$$

under longitudinal flight.

Next, the aerodynamic force $\vec{F}_A$ is given by

$$\vec{F}_A = -D\hat{\imath}_W - D_s\hat{\jmath}_W - L\hat{k}_W,$$

where $\hat{\imath}_W$, $\hat{\jmath}_W$, and $\hat{k}_W$ are the axes of the wind frame, which is a velocity-dependent frame defined such that $\hat{\imath}_W$ is aligned with $\vec{v}_{c/O_E/E}$; $\hat{k}_W$ is aligned with the stability-frame unit vector $\hat{k}_S$ defined below; and where D, D_s, and L denote the magnitudes of the drag, side drag, and lift forces, respectively. For simplicity, we assume $D_s = 0$, and thus

$$\vec{F}_A\Big|_W = \begin{bmatrix} -D \\ 0 \\ -L \end{bmatrix}.$$

The stability frame F_S with axes $\hat{\imath}_S$, $\hat{\jmath}_S$, and $\hat{k}_S$ is obtained by rotating the wind frame through the sideslip angle β, which is the angle from the $\hat{\imath}_{AC}$-$\hat{k}_{AC}$ plane to $\vec{v}_{c/O_E/E}$. Resolving $\vec{F}_A$ in the stability frame yields

$$\vec{F}_A\Big|_S = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -D \\ 0 \\ -L \end{bmatrix} = \begin{bmatrix} -D\cos\beta \\ -D\sin\beta \\ -L \end{bmatrix}.$$

Furthermore, resolving $\vec{F}_A$ in the aircraft frame yields

$$\vec{F}_A\Big|_{AC} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} -D\cos\beta \\ -D\sin\beta \\ -L \end{bmatrix}$$
$$= \begin{bmatrix} -D(\cos\beta)\cos\alpha + L\sin\alpha \\ -D\sin\beta \\ -D(\cos\beta)\sin\alpha - L\cos\alpha \end{bmatrix},$$

where α is the angle of attack of the aircraft, that is, the angle from $\hat{\imath}_S$ to $\hat{\imath}_{AC}$. Since we consider only longitudinal flight, it follows that β is identically zero, and thus

$$\vec{F}_A\Big|_{AC} = \begin{bmatrix} -D\cos\alpha + L\sin\alpha \\ 0 \\ -D\sin\alpha - L\cos\alpha \end{bmatrix}. \quad (24)$$

For the thrust force, we have

$$\vec{F}_T\Big|_{AC} = \begin{bmatrix} \cos\Phi_T & 0 & \sin\Phi_T \\ 0 & 1 & 0 \\ -\sin\Phi_T & 0 & \cos\Phi_T \end{bmatrix} \begin{bmatrix} F_T \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_T\cos\Phi_T \\ 0 \\ -F_T\sin\Phi_T \end{bmatrix}, \quad (25)$$

where $F_{\mathrm{T}} \triangleq |\vec{F}_{\mathrm{T}}|$ is the engine force magnitude and Φ_{T} is the angle from $\hat{\imath}_{\mathrm{AC}}$ to the engine force direction. We assume that the component of the engine thrust in the direction $\hat{\jmath}_{\mathrm{AC}}$ is zero.

Now, substituting (16), (23), (24), and (25) into (22) yields the surge and plunge equations

$$m(\dot{U} + W\dot{\Theta}) = -mg \sin\Theta - D\cos\alpha + L\sin\alpha + F_{\mathrm{T}}\cos\Phi_{\mathrm{T}}, \quad (26)$$

$$m(\dot{W} - U\dot{\Theta}) = mg\cos\Theta - D\sin\alpha - L\cos\alpha - F_{\mathrm{T}}\sin\Phi_{\mathrm{T}}. \quad (27)$$

The sway equation for $\dot{V}$ plays no role in longitudinal flight.

Note that differential equations (26) and (27) involve the variables U, W, Θ, and α. To eliminate W from (26) and (27), we derive a relationship among W, U, and α. Resolving $\vec{v}_{\mathrm{c/O_E/E}}$ in $\mathrm{F_S}$ yields

$$\vec{v}_{\mathrm{c/O_E/E}}\Big|_{\mathrm{S}} = \begin{bmatrix} \overline{U} \\ 0 \\ 0 \end{bmatrix},$$

where $\overline{U} \triangleq \sqrt{U^2 + W^2}$. Likewise, resolving $\vec{v}_{\mathrm{c/O_E/E}}$ in $\mathrm{F_{AC}}$ yields

$$\vec{v}_{\mathrm{c/O_E/E}}\Big|_{\mathrm{AC}} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}\begin{bmatrix} \overline{U} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \overline{U}\cos\alpha \\ 0 \\ \overline{U}\sin\alpha \end{bmatrix}. \quad (28)$$

It follows from (11) and (28) that

$$\begin{bmatrix} U \\ 0 \\ W \end{bmatrix} = \begin{bmatrix} \overline{U}\cos\alpha \\ 0 \\ \overline{U}\sin\alpha \end{bmatrix}.$$

Hence,

$$\frac{W}{U} = \tan\alpha. \quad (29)$$

For longitudinal flight, U is nonzero. Thus, it follows from (29) that

$$W = U\tan\alpha, \quad (30)$$

which implies

$$\dot{W} = \dot{U}\tan\alpha + U(\sec^2\alpha)\dot{\alpha}. \quad (31)$$

Finally, substituting (30) and (31) into (26) and (27) yields

$$m(\dot{U} + U(\tan\alpha)\dot{\Theta}) = -mg\sin\Theta - D\cos\alpha + L\sin\alpha + F_{\mathrm{T}}\cos\Phi_{\mathrm{T}}, \quad (32)$$

$$m(\dot{U}\tan\alpha + U(\sec^2\alpha)\dot{\alpha} - U\dot{\Theta}) = mg\cos\Theta - D\sin\alpha - L\cos\alpha - F_{\mathrm{T}}\sin\Phi_{\mathrm{T}}. \quad (33)$$

Next, the rotational momentum equation for the aircraft about its center of mass is given by Euler's equation

$$\vec{I}_{\mathrm{AC/c}} \overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC/E}} + \vec{\omega}_{\mathrm{AC/E}} \times \vec{I}_{\mathrm{AC/c}}\,\vec{\omega}_{\mathrm{AC/E}} = \vec{M}_{\mathrm{AC/c}}, \quad (34)$$

where the physical inertia matrix is defined by

$$\vec{I}_{\mathrm{AC/c}} \triangleq \int_{\mathrm{AC}} |\vec{r}_{\mathrm{d}m/\mathrm{c}}|^2 \vec{U} - \vec{r}_{\mathrm{d}m/\mathrm{c}}\,\vec{r}^{\,\prime}_{\mathrm{d}m/\mathrm{c}}\,\mathrm{d}m, \quad (35)$$

$\vec{r}_{\mathrm{d}m/\mathrm{c}}$ is the position of a mass element relative to c, $(\cdot)'$ denotes a physical covector [20, p. 269], and the physical identity matrix $\vec{U}$ is defined by

$$\vec{U} \triangleq \hat{\imath}_{\mathrm{AC}}\hat{\imath}^{\prime}_{\mathrm{AC}} + \hat{\jmath}_{\mathrm{AC}}\hat{\jmath}^{\prime}_{\mathrm{AC}} + \hat{k}_{\mathrm{AC}}\hat{k}^{\prime}_{\mathrm{AC}}. \quad (36)$$

Note that the integral in (35) is evaluated over the aircraft body. In (35) and (36), the notation $\vec{x}\,\vec{y}^{\,\prime}$ for vectors $\vec{x}$ and $\vec{y}$ denotes a second-order tensor, which operates on a vector $\vec{z}$ according to $(\vec{x}\,\vec{y}^{\,\prime})\vec{z} = \vec{x}\,\vec{y}^{\,\prime}\vec{z} = (\vec{y}\cdot\vec{z})\vec{x}$ [20]. Finally, $\vec{M}_{\mathrm{AC/c}}$ denotes the total thrust and aerodynamic moment acting on the aircraft relative to c.

Next, resolving $\vec{I}_{\mathrm{AC/c}}$ in $\mathrm{F_{AC}}$ yields

$$\vec{I}_{\mathrm{AC/c}}\Big|_{\mathrm{AC}} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}, \quad (37)$$

where

$$I_{xx} = \int_{\mathrm{AC}} (y^2 + z^2)\mathrm{d}m,$$

$$I_{xy} = \int_{\mathrm{AC}} xy\,\mathrm{d}m,$$

and likewise for the remaining entries. Assuming that $\hat{\imath}_{\mathrm{AC}}$-$\hat{k}_{\mathrm{AC}}$ is a plane of symmetry of the aircraft, it follows that

$$I_{xy} = I_{yz} = 0.$$

Thus, (37) becomes

$$\vec{I}_{\mathrm{AC/c}}\Big|_{\mathrm{AC}} = \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix}.$$

Now resolving Euler's equation (34) in the aircraft frame, that is,

$$\left(\vec{I}_{\mathrm{AC/c}} \overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC}}\right)\Big|_{\mathrm{AC}} + \left(\vec{\omega}_{\mathrm{AC/E}} \times \vec{I}_{\mathrm{AC/c}}\,\vec{\omega}_{\mathrm{AC/E}}\right)\Big|_{\mathrm{AC}} = \vec{M}_{\mathrm{AC/c}}\Big|_{\mathrm{AC}},$$

yields

TABLE 1 Aerodynamic parameters. These parameters characterize the basic features of the aircraft for steady longitudinal flight.	
S	Wing area
b	Wing tip-to-tip distance
$\bar{c}$	Wing mean chord
ρ	Air density
V_{AC}	Aircraft speed
p_d	Dynamic pressure $\frac{1}{2}\rho V_{AC}^2$
V_{AC_0}	U_0
p_{d_0}	$\frac{1}{2}\rho U_0^2$

$$\begin{bmatrix} 0 \\ I_{yy}\ddot{\Theta} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \dot{\Theta} \\ 0 & 0 & 0 \\ -\dot{\Theta} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ I_{yy}\dot{\Theta} \\ 0 \end{bmatrix} = \begin{bmatrix} L_{AC} \\ M_{AC} \\ N_{AC} \end{bmatrix},$$

where $\vec{M}_{AC/c}\Big|_{AC} \triangleq [L_{AC} \quad M_{AC} \quad N_{AC}]^T$. The pitch equation is thus given by

$$I_{yy}\ddot{\Theta} = M_{AC}. \tag{38}$$

LINEARIZING THE EQUATIONS OF MOTION

In steady horizontal longitudinal flight, the aircraft is assumed to fly at constant velocity $U = U_0$, constant angle of attack $\alpha = \alpha_0$, and constant pitch angle $\Theta = \Theta_0$, with $\vec{v}_{c/O_E/E}$ aligned with $\hat{\imath}_E$. To simplify the aerodynamic analysis, we choose F_{AC} so that $\Theta_0 = 0$. This choice is universally made in the literature [18, p. 67]. Since the steady flight-path angle is zero, this choice of F_{AC} implies that the steady angle of attack α_0 is zero, that is, in steady flight, $\hat{\imath}_S$ is aligned with $\hat{\imath}_{AC}$. Linearizing the surge, plunge, and pitch equations (32), (33), and (38) about $(U_0, \alpha_0, \Theta_0)$ using the first-order approximations $U \approx U_0 + u$, $\alpha \approx \alpha_0 + \delta\alpha$, and $\Theta \approx \Theta_0 + \theta$, where $\alpha_0 = \Theta_0 = 0$, and dividing the linearized equations by the mass m and inertia I_{yy} to solve for the linear and angular acceleration, yields

$$\dot{u} = -g\theta + f_{A_x} + f_{T_x}, \tag{39}$$

$$U_0\delta\dot{\alpha} = U_0 q + f_{A_z}, \tag{40}$$

$$\dot{q} = m_{AC}, \tag{41}$$

$$\dot{\theta} = q, \tag{42}$$

where

$$f_{A_x} \triangleq X_{u_0}u + X_{\alpha_0}\delta\alpha + X_{\delta e_0}\delta e, \tag{43}$$

$$f_{T_x} \triangleq X_{T_{u_0}}u, \tag{44}$$

$$f_{A_z} \triangleq Z_{u_0}u + Z_{\alpha_0}\delta\alpha + Z_{\dot{\alpha}_0}\delta\dot{\alpha} + Z_{q_0}q + Z_{\delta e_0}\delta e, \tag{45}$$

TABLE 2 Force stability derivatives. The aerodynamic parameters are given in Table 1. These lift and drag stability derivatives model the aerodynamic forces applied to the aircraft due to perturbations from steady longitudinal flight. This table is based on [17, Table 6.1].	
$C_L(u, q, \delta\alpha, \delta\dot{\alpha}, \delta e)$	$C_{L_0} + \frac{1}{U_0}C_{L_{u_0}}u + \frac{\bar{c}}{2U_0}C_{L_{q_0}}q + C_{L_{\alpha_0}}\delta\alpha + \frac{\bar{c}}{2U_0}C_{L_{\dot{\alpha}_0}}\delta\dot{\alpha} + C_{L_{\delta e_0}}\delta e$
C_{L_0}	$\frac{L}{p_{d_0}S}$
$C_{L_{u_0}}$	$\frac{\partial C_L}{\partial\left(\frac{u}{U_0}\right)}\Big\vert_0$
$C_{L_{q_0}}$	$\frac{\partial C_L}{\partial\left(\frac{\bar{c}q}{2U_0}\right)}\Big\vert_0$
$C_{L_{\alpha_0}}$	$\frac{\partial C_L}{\partial\delta\alpha}\Big\vert_0$
$C_{L_{\dot{\alpha}_0}}$	$\frac{\partial C_L}{\partial\left(\frac{\bar{c}\delta\dot{\alpha}}{2U_0}\right)}\Big\vert_0$
$C_{L_{\delta e_0}}$	$\frac{\partial C_L}{\partial\delta e}\Big\vert_0$
$C_D(u, q, \delta\alpha, \delta\dot{\alpha}, \delta e)$	$C_{D_0} + \frac{1}{U_0}C_{D_{u_0}}u + \frac{\bar{c}}{2U_0}C_{D_{q_0}}q + C_{D_{\alpha_0}}\delta\alpha + C_{D_{\dot{\alpha}_0}}\delta\dot{\alpha} + C_{D_{\delta e_0}}\delta e$
C_{D_0}	$\frac{D}{p_{d_0}S}$
$C_{D_{u_0}}$	$2KC_{L_0}C_{L_{u_0}}$
$C_{D_{q_0}}$	$2KC_{L_0}C_{L_{q_0}}$
$C_{D_{\alpha_0}}$	$2KC_{L_0}C_{L_{\alpha_0}}$
$C_{D_{\dot{\alpha}_0}}$	$2KC_{L_0}C_{L_{\dot{\alpha}_0}}$
$C_{D_{\delta e_0}}$	$2KC_{L_0}C_{L_{\delta e_0}}$

$$m_{AC} \triangleq M_{u_0}u + M_{\alpha_0}\delta\alpha + M_{\dot{\alpha}_0}\delta\dot{\alpha} + M_{q_0}q + M_{\delta e_0}\delta e + M_{T_{u_0}}u + M_{T_{\alpha_0}}\delta\alpha, \tag{46}$$

and δe denotes the elevator perturbation from its trim deflection. Note that f_{A_x} and f_{A_z} are the perturbations of $\vec{F}_A$ in the direction of $\hat{\imath}_{AC}$ and $\hat{k}_{AC}$, respectively. Furthermore, f_{T_x} is the perturbation of $\vec{F}_T$ in the direction of $\hat{\imath}_{AC}$, and m_{AC} is the perturbation of M_{AC}. The stability parameters X_{u_0}, X_{α_0}, $X_{\delta e_0}$, $X_{T_{u_0}}$, Z_{u_0}, Z_{α_0}, $Z_{\dot{\alpha}_0}$, Z_{q_0}, $Z_{\delta e_0}$, M_{u_0}, M_{α_0}, $M_{\dot{\alpha}_0}$, M_{q_0}, $M_{\delta e_0}$, $M_{T_{u_0}}$, and $M_{T_{\alpha_0}}$ are combinations of aerodynamic parameters and stability derivatives, which are defined in Table 1, Table 2, and Table 3. The stability parameters are defined in Table 4.

It follows from (39)–(46) that the linearized surge, plunge, and pitch equations are given by

$$\dot{u} = (X_{u_0} + X_{T_{u_0}})u + X_{\alpha_0}\delta\alpha - g\theta + X_{\delta e_0}\delta e, \tag{47}$$

$$U_0\delta\dot{\alpha} = Z_{u_0}u + Z_{\alpha_0}\delta\alpha + (U_0 + Z_{q_0})q + Z_{\dot{\alpha}_0}\delta\dot{\alpha} + Z_{\delta e_0}\delta e, \tag{48}$$

TABLE 3 Moment stability derivatives. The aerodynamic parameters are given in Table 1. These pitch stability derivatives model the aerodynamic moments applied to the aircraft due to perturbations from steady longitudinal flight. This table is based on [17, Table 6.1].

$C_m(u, q, \delta\alpha, \delta\dot{\alpha}, \delta e)$	$\frac{1}{U_0}(2C_{m_0} + C_{m_{u_0}})u + \frac{\bar{c}}{2U_0}C_{m_{q_0}}q + C_{m_{\alpha_0}}\delta\alpha + \frac{\bar{c}}{2U_0}C_{m_{\dot{\alpha}_0}}\delta\dot{\alpha} + C_{m_{\delta e_0}}\delta e$
C_{m_0}	$\frac{M_A}{p_{d_0}S\bar{c}}$
$C_{m_{u_0}}$	$\left.\frac{\partial C_m}{\partial(\frac{u}{U_0})}\right\vert_0$
$C_{m_{q_0}}$	$\left.\frac{\partial C_m}{\partial(\frac{\bar{c}q}{2U_0})}\right\vert_0$
$C_{m_{\alpha_0}}$	$\left.\frac{\partial C_m}{\partial \delta\alpha}\right\vert_0$
$C_{m_{\dot{\alpha}_0}}$	$\left.\frac{\partial C_m}{\partial(\frac{\bar{c}\delta\dot{\alpha}}{2U_0})}\right\vert_0$
$C_{m_{\delta e_0}}$	$\left.\frac{\partial C_m}{\partial \delta e}\right\vert_0$

$$\dot{q} = (M_{u_0} + M_{T_{u_0}})u + (M_{\alpha_0} + M_{T_{\alpha_0}})\delta\alpha + M_{q_0}q + M_{\dot{\alpha}0}\delta\dot{\alpha} + M_{\delta e_0}\delta e, \tag{49}$$

$$\dot{\theta} = q. \tag{50}$$

LAPLACE TRANSFORM ANALYSIS

Taking the Laplace transform of (47)–(50) and assuming that the initial conditions of the perturbations $(u, \delta\alpha, \theta)$ are zero yields

$$\begin{bmatrix} s-(X_{u_0}+X_{T_{u_0}}) & -X_{\alpha_0} & g \\ -Z_{u_0} & s(U_0-Z_{\dot{\alpha}_0})-Z_{\alpha_0} & -(U_0+Z_{q_0})s \\ -(M_{u_0}+M_{T_{u_0}}) & -(M_{\dot{\alpha}_0}s+M_{\alpha_0}+M_{T\alpha_0}) & s^2-M_{q_0}s \end{bmatrix} \cdot \begin{bmatrix} \hat{u}(s) \\ \delta\hat{\alpha}(s) \\ \hat{\theta}(s) \end{bmatrix} = \begin{bmatrix} X_{\delta e_0} \\ Z_{\delta e_0} \\ M_{\delta e_0} \end{bmatrix} \delta\hat{e}(s),$$

where hat in this context denotes the Laplace transform of a scalar function of time. The transfer functions from $\delta\hat{e}(s)$ to $\hat{u}(s)$, $\delta\hat{\alpha}(s)$, and $\hat{\theta}(s)$ are thus given by

$$\begin{bmatrix} G_{\hat{u}/\delta\hat{e}}(s) \\ G_{\delta\hat{\alpha}/\delta\hat{e}}(s) \\ G_{\hat{\theta}/\delta\hat{e}}(s) \end{bmatrix} \triangleq \begin{bmatrix} \frac{\hat{u}(s)}{\delta\hat{e}(s)} \\ \frac{\delta\hat{\alpha}(s)}{\delta\hat{e}(s)} \\ \frac{\hat{\theta}(s)}{\delta\hat{e}(s)} \end{bmatrix}$$

$$= \begin{bmatrix} s-(X_{u_0}+X_{T_{u0}}) & -X_{\alpha_0} & g \\ -Z_{u_0} & s(U_0-Z_{\dot{\alpha}_0})-Z_{\alpha 0} & -(U_0+Z_{q_0})s \\ -(M_{u_0}+M_{T_{u_0}}) & -(M_{\dot{\alpha}_0}s+M_{\alpha 0}+M_{T\alpha_0}) & s^2-M_{q_0}s \end{bmatrix}^{-1} \cdot \begin{bmatrix} X_{\delta e_0} \\ Z_{\delta e_0} \\ M_{\delta e_0} \end{bmatrix}.$$

TABLE 4 Stability parameters. These parameters are functions of the aircraft parameters and stability derivatives given in Table 2. This table is based on [17, Table 6.3].

Stability Parameter	Definition	Units
X_{u_0}	$-\frac{p_{d_0}S}{mU_0}(2C_{D_0} + C_{D_{u_0}})$	1/s
$X_{T_{u_0}}$	$\frac{p_{d_0}S}{mU_0}(2C_{Tx_0} + C_{Tx_{u_0}})$	1/s
X_{α_0}	$\frac{p_{d_0}S}{m}(C_{L_0} - C_{D_{\alpha_0}})$	ft/s^2-rad
$X_{\delta e_0}$	$\frac{p_{d_0}S}{m}C_{D_{\delta e_0}}$	ft/s^2-rad
Z_{u_0}	$-\frac{p_{d_0}S}{mU_0}(2C_{L_0} + C_{L_{u_0}})$	1/s
Z_{α_0}	$\frac{p_{d_0}S}{m}(C_{L_{\alpha_0}} - C_{D_0})$	ft/s^2-rad
$Z_{\dot{\alpha}_0}$	$-\frac{p_{d_0}S\bar{c}}{2mU_0}C_{L_{\dot{\alpha}_0}}$	ft/s-rad
Z_{q_0}	$-\frac{p_{d_0}S\bar{c}}{2mU_0}C_{L_{q_0}}$	ft/s-rad
$Z_{\delta e_0}$	$-\frac{p_{d_0}S}{m}C_{L_{\delta e_0}}$	ft/s^2-rad
M_{u_0}	$\frac{p_{d_0}S\bar{c}}{I_{yy}U_0}(2C_{m_0} + C_{m_{u_0}})$	rad/ft-s
$M_{T_{u_0}}$	$\frac{p_{d_0}S\bar{c}}{I_{yy}U_0}(2C_{Tm_0} + C_{Tm_{u_0}})$	1/ft-s
M_{α_0}	$\frac{p_{d_0}S\bar{c}}{I_{yy}}C_{m_{\alpha_0}}$	1/s^2
$M_{T_{\alpha_0}}$	$\frac{p_{d_0}S\bar{c}}{I_{yy}}C_{Tm_{\alpha_0}}$	1/s^2
$M_{\dot{\alpha}_0}$	$\frac{p_{d_0}S\bar{c}^2}{2I_{yy}U_0}C_{m_{\dot{\alpha}_0}}$	1/s
M_{q_0}	$\frac{p_{d_0}S\bar{c}^2}{2I_{yy}U_0}C_{m_{q_0}}$	1/s
$M_{\delta e_0}$	$\frac{p_{d_0}S\bar{c}}{I_{yy}}C_{m_{\delta e_0}}$	1/s^2

Consequently,

$$G_{\hat{u}/\delta\hat{e}}(s) = \frac{A_u s^3 + B_u s^2 + C_u s + D_u}{Es^4 + Fs^3 + Gs^2 + Hs + I}, \tag{51}$$

$$G_{\delta\hat{\alpha}/\delta\hat{e}}(s) = \frac{A_\alpha s^3 + B_\alpha s^2 + C_\alpha s + D_\alpha}{Es^4 + Fs^3 + Gs^2 + Hs + I}, \tag{52}$$

$$G_{\hat{\theta}/\delta\hat{e}}(s) = \frac{A_\theta s^2 + B_\theta s + C_\theta}{Es^4 + Fs^3 + Gs^2 + Hs + I}, \tag{53}$$

TABLE 5 Transfer function numerator coefficients. These coefficients appear in the transfer functions from the elevator deflection $\delta\hat{e}(s)$ to $\hat{u}(s)$, $\delta\hat{\alpha}(s)$, $\hat{\theta}(s)$, $\delta\hat{v}_{\mathrm{ph}}(s)$, and $\delta\hat{v}_{\mathrm{pv}}(s)$.

A_u	$X_{\delta e_0}(U_0 - Z_{\dot{\alpha}_0})$
B_u	$-X_{\delta e_0}[(U_0 - Z_{\dot{\alpha}_0})M_{q_0} + Z_{\alpha_0} + M_{\dot{\alpha}_0}(U_0 + Z_{q_0}) + Z_{\delta e_0}X_{\alpha_0}]$
C_u	$X_{\delta e_0}[M_{q_0}Z_{\alpha_0} - (M_{\alpha_0} + M_{\mathrm{T}_{\alpha_0}})(U_0 + Z_{q_0})] - Z_{\delta e_0}[M_{\dot{\alpha}_0}g + X_{\alpha_0}M_{q_0}] + M_{\delta e_0}[X_{\dot{\alpha}_0}(U_0 + Z_{q_0}) - (U_0 - Z_{\dot{\alpha}_0})g]$
D_u	$-Z_{\delta e_0}M_{\alpha_0}g + M_{\delta e_0}Z_{\alpha_0}g$
A_α	$Z_{\delta e_0}$
B_α	$X_{\delta e_0}Z_{u_0} + Z_{\delta e_0}[-M_{q_0} - (X_{u_0} + X_{\mathrm{T}_{u_0}})] + M_{\delta e_0}(U_0 + Z_{q_0})$
C_α	$X_{\delta e_0}[(U_0 + Z_{q_0})(M_{u_0} + M_{\mathrm{T}_{u_0}}) - M_{q_0}Z_{u_0}] + Z_{\delta e_0}M_{q_0}(X_{u_0} + X_{\mathrm{T}_{u_0}}) - M_{\delta e_0}(U_0 + Z_{q_0})(X_{u_0} + X_{\mathrm{T}_{u_0}})$
D_α	$Z_{\delta e_0}(M_{u_0} + M_{\mathrm{T}_{u_0}})g - M_{\delta e_0}Z_{u_0}g$
A_θ	$M_{\delta e_0}(U_0 - Z_{\dot{\alpha}_0}) + Z_{\delta e_0}M_{\dot{\alpha}_0}$
B_θ	$X_{\delta e_0}[Z_{u_0}M_{\dot{\alpha}_0} + (U_0 - Z_{\dot{\alpha}_0})(M_{u_0} + M_{\mathrm{T}_{u_0}})] + Z_{\delta e_0}[(M_{\alpha_0} + M_{\mathrm{T}_{\alpha_0}}) - M_{\dot{\alpha}_0}(X_{u_0} + X_{\mathrm{T}_{u_0}})] + M_{\delta e_0}[-Z_{\alpha_0} - (U_0 - Z_{\dot{\alpha}_0})(X_{u_0} + X_{\mathrm{T}_{u_0}})]$
C_θ	$X_{\delta e_0}[(M_{\alpha_0} + M_{\mathrm{T}_{\alpha_0}})Z_{u_0} - Z_{\alpha_0}(M_{u_0} + M_{\mathrm{T}_{u_0}})] + M_{\delta e_0}[Z_{\alpha_0}(X_{u_0} + X_{\mathrm{T}_{u_0}}) - X_{\alpha_0}Z_{u_0}] + Z_{\delta e_0}[-(M_{\alpha_0} + M_{\mathrm{T}_{\alpha_0}})(X_{u_0} + X_{\mathrm{T}_{u_0}}) + X_{\alpha_0}(M_{u_0} + M_{\mathrm{T}_{u_0}})]$
A_{v}	$-\ell A_\theta + U_0A_\alpha$
B_{v}	$-\ell B_\theta - U_0A_\theta + U_0B_\alpha$
C_{v}	$-\ell C_\theta - U_0B_\theta + U_0C_\alpha$
D_{v}	$-U_0C_\theta + U_0D_\alpha$
A_{h}	$\eta A_\theta + A_u$
B_{h}	$\eta B_\theta + B_u$
C_{h}	$\eta C_\theta + C_u$
D_{h}	D_u

where the coefficients of (51)–(53) are defined in tables 5 and 6. Note that the relative degree of (53) is two. For details, see "Markov Parameters and Relative Degree."

Next, we find the transfer function from the elevator perturbation to the vertical-velocity perturbation. It follows from (15) and (30) that

$$v_{\mathrm{pv}} = -(\sin\Theta)U + (\cos\Theta)U(\tan\alpha) - \ell(\cos\Theta)\dot{\Theta} - \eta(\sin\Theta)\dot{\Theta}. \quad (54)$$

Letting v_{pv_0} denote the vertical velocity in steady horizontal longitudinal flight, it follows from (54) that

$$v_{\mathrm{pv}_0} = 0.$$

TABLE 6 Transfer function denominator coefficients. These coefficients appear in the transfer functions from the elevator deflection $\delta\hat{e}(s)$ to $\hat{u}(s)$, $\delta\hat{\alpha}(s)$, $\hat{\theta}(s)$, $\delta\hat{v}_{\mathrm{ph}}(s)$, and $\delta\hat{v}_{\mathrm{pv}}(s)$.

E	$U_0 - Z_{\dot{\alpha}_0}$
F	$-(U_0 - Z_{\dot{\alpha}_0})(X_{u_0} - X_{\mathrm{T}_{u_0}} + M_{q_0}) - Z_{\alpha_0} - M_{\dot{\alpha}_0}(U_0 + Z_{q_0})$
G	$(X_{u_0} - X_{\mathrm{T}_{u_0}})[M_{q_0}(U_0 - Z_{\dot{\alpha}_0}) + Z_{\alpha_0} - M_{\dot{\alpha}_0}(U_0 + Z_{q_0})] + M_{q_0}Z_{\alpha_0} - Z_{u_0}X_{\alpha_0} - (M_{\alpha_0} + M_{\mathrm{T}_{\alpha_0}})(U_0 + Z_{q_0})$
H	$g[Z_{u_0}M_{\dot{\alpha}_0} + (M_{u_0} + M_{\mathrm{T}_{u_0}})(U_0 - Z_{\dot{\alpha}_0})] + (M_{u_0} + M_{\mathrm{T}_{u_0}})[-X_{\alpha_0}(U_0 + Z_{q_0})] + Z_{u_0}X_{\alpha_0}M_{q_0} + (X_{u_0} + X_{\mathrm{T}_{u_0}})[(M_{\alpha_0} + M_{\mathrm{T}_{\alpha_0}})(U_0 + Z_{q_0}) - M_{q_0}Z_{\alpha_0}]$
I	$g[(M_{\alpha_0} + M_{\mathrm{T}_{\alpha_0}})Z_{u_0} - Z_{\alpha_0}(M_{u_0} + M_{\mathrm{T}_{u_0}})]$

Linearizing (54) about $(U_0, \alpha_0, \Theta_0) = (U_0, 0, 0)$ using the first-order approximations $v_{\mathrm{pv}} \approx v_{\mathrm{pv}_0} + \delta v_{\mathrm{pv}}$, $U \approx U_0 + u$, $\alpha \approx \delta\alpha$, and $\Theta \approx \theta$ yields

$$v_{\mathrm{pv}_0} + \delta v_{\mathrm{pv}} = -(\sin\theta)(U_0 + u) + (\cos\theta)(U_0 + u)(\tan\delta\alpha) - \ell(\cos\theta)\dot{\theta} - \eta(\sin\theta)\dot{\theta},$$

where δv_{pv} is the first-order approximation of the vertical-velocity perturbation. Neglecting products of perturbation variables, and approximating $\cos\theta \approx 1$, $\sin\theta \approx \theta$, and $\tan\delta\alpha \approx \delta\alpha$ yields

$$\delta v_{\mathrm{pv}} = U_0\delta\alpha - U_0\theta - \ell\dot{\theta}. \quad (55)$$

Next, taking the Laplace transform of (55) and assuming that the initial conditions of the perturbations $(u, \delta\alpha, \theta)$ are zero yields

$$\delta\hat{v}_{\mathrm{pv}}(s) = U_0\delta\hat{\alpha}(s) - (U_0 + \ell s)\hat{\theta}(s). \quad (56)$$

It follows from (52), (53), and (56) that the transfer function from $\delta\hat{e}(s)$ to $\delta\hat{v}_{\mathrm{pv}}(s)$ is given by

$$G_{\delta\hat{v}_{\mathrm{pv}}/\delta\hat{e}}(s) = \frac{A_{\mathrm{v}}s^3 + B_{\mathrm{v}}s^2 + C_{\mathrm{v}}s + D_{\mathrm{v}}}{Es^4 + Fs^3 + Gs^2 + Hs + I}, \quad (57)$$

where the numerator coefficients are defined in Table 5 and the denominator coefficients are defined in Table 6.

Next, to find the transfer function from the elevator perturbation to the horizontal-velocity perturbation, it follows from (14) and (30) that

$$v_{\mathrm{ph}} = (\cos\Theta)U + (\sin\Theta)(\tan\alpha)U - \ell(\sin\Theta)\dot{\Theta} + \eta(\cos\Theta)\dot{\Theta}. \quad (58)$$

Letting v_{ph_0} denote the horizontal velocity in steady horizontal longitudinal flight, it follows from (58) that

$$v_{\mathrm{ph}_0} = U_0.$$

Markov Parameters and Relative Degree

Consider

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t),$$
$$\dot{y}(t) = \tilde{C}x(t) + \tilde{D}u(t),$$

whose Laplace form is given by

$$s\hat{x}(s) - x(0) = \tilde{A}\hat{x}(s) + \tilde{B}\hat{u}(s),$$
$$\hat{y}(s) = \tilde{C}\hat{x}(s) + \tilde{D}\hat{u}(s).$$

Then,

$$\hat{y}(s) = \tilde{C}(sI - \tilde{A})^{-1}x(0) + [\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}]\hat{u}(s),$$

where

$$G(s) \triangleq \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}.$$

Expanding $G(s)$ in a Laurent series about infinity yields

$$G(s) = \frac{1}{s}\tilde{C}\left(I - \frac{1}{s}\tilde{A}\right)^{-1}\tilde{B} + \tilde{D}$$
$$= \tilde{D} + \frac{1}{s}\tilde{C}\tilde{B} + \frac{1}{s^2}\tilde{C}\tilde{A}\tilde{B} + \frac{1}{s^3}\tilde{C}\tilde{A}^2\tilde{B} + \cdots. \quad \text{(S12)}$$

We now consider $G_{\hat{\theta}/\delta\hat{e}}(s)$ given by (53). Using (S12), we obtain

$$\lim_{s\to\infty} sG_{\hat{\theta}/\delta\hat{e}}(s) = \tilde{C}\tilde{B}.$$

Writing (47)–(49) in state-space form with elevator-deflection input and setting $Z_{\dot{\alpha}_0} = 0$ and $M_{\dot{\alpha}_0} = 0$ for convenience yields

$$\begin{bmatrix} \dot{u} \\ \delta\dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \tilde{A}\begin{bmatrix} u \\ \delta\alpha \\ q \\ \theta \end{bmatrix} + \tilde{B}\delta e. \quad \text{(S13)}$$

where

$$\tilde{A} \triangleq \begin{bmatrix} X_{u_0} + X_{T_{u_0}} & X_{\alpha_0} & X_{q_0} & -g \\ \dfrac{Z_{u_0}}{U_0} & \dfrac{Z_{\alpha_0}}{U_0} & \dfrac{U_0 + Z_{q_0}}{U_0} & 0 \\ M_{u_0} + M_{T_{u_0}} & M_{\alpha_0} + M_{T_{\alpha_0}} & M_{q_0} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\tilde{B} \triangleq \begin{bmatrix} X_{\delta e_0} \\ \dfrac{Z_{\delta e_0}}{U_0} \\ M_{\delta e_0} \\ 0 \end{bmatrix}, \quad \tilde{C} \triangleq [0 \quad 0 \quad 0 \quad 1].$$

Note that

$$\tilde{C}\tilde{B} = 0.$$

Since $\tilde{D} = 0$ and $\tilde{C}\tilde{B} = 0$, it follows from (S12) that

$$\lim_{s\to\infty} s^2G(s) = \tilde{C}\tilde{A}\tilde{B}. \quad \text{(S14)}$$

Therefore,

$$\lim_{s\to\infty} s^2G_{\hat{\theta}/\delta\hat{e}}(s) = \frac{A_\theta}{E}, \quad \text{(S15)}$$

where A_θ is the coefficient of s^2 in the numerator of (53). From (S14) and (S15) it follows that $A_\theta/E = \tilde{C}\tilde{A}\tilde{B} = M_{\delta e_0}$ for $Z_{\dot{\alpha}_0} = 0$ and $M_{\dot{\alpha}_0} = 0$. It thus follows that the numerator of $G_{\hat{\theta}/\delta\hat{e}}(s)$ in (53) is of second order.

Linearizing (58) about $(U_0, \alpha_0, \Theta_0) = (U_0, 0, 0)$ using the first-order approximations $v_{\rm ph} \approx v_{\rm ph_0} + \delta v_{\rm ph}$, $U \approx U_0 + u$, $\alpha \approx \delta\alpha$, and $\Theta \approx \theta$ yields

$$v_{\rm ph_0} + \delta v_{\rm ph} = (\cos\theta)(U_0 + u) + (\sin\theta)(U_0 + u)(\tan\delta\alpha)$$
$$- \ell(\sin\theta)\dot{\theta} + \eta(\cos\theta)\dot{\theta},$$

where $\delta v_{\rm ph}$ is the first-order approximation of the horizontal-velocity perturbation. Neglecting products of perturbation variables, and approximating $\cos\theta \approx 1$, $\sin\theta \approx \theta$, and $\tan\delta\alpha \approx \delta\alpha$ yields

$$\delta v_{\rm ph} = u + \eta\dot{\theta}. \quad (59)$$

Next, taking the Laplace transform of (59) and assuming that the initial conditions of the perturbations $(u, \delta\alpha, \theta)$ are zero yields

$$\delta\hat{v}_{\rm ph}(s) = \hat{u}(s) + \eta s\hat{\theta}(s). \quad (60)$$

It follows from (51), (53), and (60) that the transfer function from $\delta\hat{e}(s)$ to $\delta\hat{v}_{\rm ph}(s)$ is given by

$$G_{\delta\hat{v}_{\rm ph}/\delta\hat{e}}(s) = \frac{A_{\rm h}s^3 + B_{\rm h}s^2 + C_{\rm h}s + D_{\rm h}}{Es^4 + Fs^3 + Gs^2 + Hs + I}, \quad (61)$$

where the numerator coefficients are defined in Table 5, and the denominator coefficients are defined in Table 6.

INSTANTANEOUS VELOCITY CENTER OF ROTATION

The point $p_{\rm IVCR}$ is an IVCR of the aircraft at time t_0 if $p_{\rm IVCR}$ is fixed relative to the aircraft and, at time t_0, the angular velocity of the aircraft relative to F_E is not zero and the velocity of $p_{\rm IVCR}$ relative to O_{AC} with respect to F_E is zero. For details, see "Instantaneous Velocity Center of Rotation." It follows that the location of the unique $p_{\rm IVCR}$ whose coordinate along $\hat{\jmath}_{AC}$ is zero, if it exists, has the form

$$\vec{r}_{\mathrm{p_{IVCR}/c}}\Big|_{\mathrm{AC}} = \begin{bmatrix} \ell_{\mathrm{IVCR}} \\ 0 \\ \eta_{\mathrm{IVCR}} \end{bmatrix}. \tag{62}$$

It thus follows from (S10) that

$$\vec{r}_{\mathrm{p_{IVCR}/c}} = \frac{1}{|\vec{\omega}_{\mathrm{AC/E}}|^2}\vec{\omega}_{\mathrm{AC/E}} \times \vec{v}_{\mathrm{c/O_E/E}} + \frac{\vec{\omega}_{\mathrm{AC/E}} \cdot \vec{r}_{\mathrm{p_{IVCR}/c}}}{|\vec{\omega}_{\mathrm{AC/E}}|^2}\vec{\omega}_{\mathrm{AC/E}}. \tag{63}$$

Note that the second term in (63) is zero since $\vec{\omega}_{\mathrm{AC/E}}$ is aligned with $\hat{\jmath}_{\mathrm{AC}}$ and the component of $\vec{r}_{\mathrm{p_{IVCR}/c}}$ along $\hat{\jmath}_{\mathrm{AC}}$ is zero. Thus, (63) can be written as

$$\begin{aligned} \vec{r}_{\mathrm{p_{IVCR}/c}} &= \frac{1}{|\vec{\omega}_{\mathrm{AC/E}}|^2}\vec{\omega}_{\mathrm{AC/E}} \times \vec{v}_{\mathrm{c/O_E/E}} \\ &= \frac{1}{\dot{\Theta}^2}[\dot{\Theta}\,\hat{\jmath}_{\mathrm{AC}} \times (U\hat{\imath}_{\mathrm{AC}} + W\hat{k}_{\mathrm{AC}})] \\ &= \frac{W}{\dot{\Theta}}\hat{\imath}_{\mathrm{AC}} - \frac{U}{\dot{\Theta}}\hat{k}_{\mathrm{AC}}. \end{aligned} \tag{64}$$

Therefore,

$$\ell_{\mathrm{IVCR}} = \frac{W}{\dot{\Theta}} = \frac{U\tan\alpha}{\dot{\Theta}}, \tag{65}$$

$$\eta_{\mathrm{IVCR}} = -\frac{U}{\dot{\Theta}}. \tag{66}$$

Since $\dot{\Theta}_0 = 0$, it follows that ℓ_{IVCR} and η_{IVCR} are infinite for steady flight, and thus no IVCR exists in steady flight.

Next, for the elevator step deflection $\delta e(t) = \varepsilon\mathbf{1}(t - t_0)$, where $\varepsilon \neq 0$, we approximate ℓ_{IVCR} and η_{IVCR} at t_0^+ using the first-order approximations $U \approx U_0 + u$, $\alpha \approx \delta\alpha$, and $\Theta \approx \theta$. Thus,

$$\ell_{\mathrm{IVCR}}(t_0^+) \approx \frac{(U_0 + u(t_0^+))(\tan\delta\,\alpha(t_0^+))}{\dot{\theta}(t_0^+)}, \tag{67}$$

$$\eta_{\mathrm{IVCR}}(t_0^+) \approx -\frac{U_0 + u(t_0^+)}{\dot{\theta}(t_0^+)}, \tag{68}$$

where it follows from the initial value theorem that

$$\begin{aligned} \theta(t_0^+) &= \lim_{s\to\infty} s\hat{\theta}(s) \\ &= \lim_{s\to\infty} sG_{\hat{\theta}/\delta\hat{e}}(s)\frac{\varepsilon}{s} \\ &= \lim_{s\to\infty} \frac{\varepsilon(A_\theta s^2 + B_\theta s + C_\theta)}{Es^4 + Fs^3 + Gs^2 + Hs + I} \\ &= 0, \end{aligned} \tag{69}$$

$$\begin{aligned} \dot{\theta}(t_0^+) &= \lim_{s\to\infty} s[s\hat{\theta}(s) - \theta(t_0^+)] \\ &= \lim_{s\to\infty} s^2\, G_{\hat{\theta}/\delta\hat{e}}(s)\frac{\varepsilon}{s} \\ &= \lim_{s\to\infty} \frac{\varepsilon(A_\theta s^3 + B_\theta s^2 + C_\theta s)}{Es^4 + Fs^3 + Gs^2 + Hs + I} \\ &= 0, \end{aligned} \tag{70}$$

$$\begin{aligned} \delta\alpha(t_0^+) &= \lim_{s\to\infty} s\delta\hat{\alpha}(s) \\ &= \lim_{s\to\infty} sG_{\delta\hat{\alpha}/\delta\hat{e}}(s)\frac{\varepsilon}{s} \\ &= \lim_{s\to\infty} \frac{\varepsilon(A_\alpha s^3 + B_\alpha s^2 + C_\alpha s + D_\alpha)}{Es^4 + Fs^3 + Gs^2 + Hs + I} \\ &= 0, \end{aligned} \tag{71}$$

$$\begin{aligned} u(t_0^+) &= \lim_{s\to\infty} s\hat{u}(s) \\ &= \lim_{s\to\infty} sG_{\hat{u}/\delta\hat{e}}(s)\frac{\varepsilon}{s} \\ &= \lim_{s\to\infty} \frac{\varepsilon(A_u s^3 + B_u s^2 + C_u s + D_u)}{Es^4 + Fs^3 + Gs^2 + Hs + I} \\ &= 0. \end{aligned} \tag{72}$$

Thus it follows from (67)–(72) that

$$\ell_{\mathrm{IVCR}}(t_0^+) \approx \frac{U_0\tan\alpha_0}{\dot{\theta}(t_0^+)} = \infty,$$

$$\eta_{\mathrm{IVCR}}(t_0^+) \approx -\frac{U_0}{\dot{\theta}(t_0^+)} = \infty.$$

Therefore, no IVCR exists for an elevator step deflection.

INSTANTANEOUS ACCELERATION CENTER OF ROTATION

The point $\mathrm{p_{IACR}}$ is an IACR of the aircraft at time t_0 if $\mathrm{p_{IACR}}$ is fixed relative to the aircraft and, at time t_0, the acceleration of $\mathrm{p_{IACR}}$ relative to $\mathrm{O_{AC}}$ with respect to $\mathrm{F_E}$ is zero. For details, see "Instantaneous Acceleration Center of Rotation." It follows that the location of the unique $\mathrm{p_{IACR}}$ whose coordinate along $\hat{\jmath}_{\mathrm{AC}}$ is zero, if it exists, has the form

$$\vec{r}_{\mathrm{p_{IACR}/c}}\Big|_{\mathrm{AC}} = \begin{bmatrix} \ell_{\mathrm{IACR}} \\ 0 \\ \eta_{\mathrm{IACR}} \end{bmatrix}. \tag{73}$$

It thus follows from (20) and the definition of the IACR that

$$\vec{a}_{\mathrm{p_{IACR}/O_E/E}}\Big|_{\mathrm{AC}} = \begin{bmatrix} -\ell_{\mathrm{IACR}}\dot{\Theta}^2 + \dot{U} + W\dot{\Theta} + \eta_{\mathrm{IACR}}\ddot{\Theta} \\ 0 \\ -\ell_{\mathrm{IACR}}\ddot{\Theta} + \dot{W} - U\dot{\Theta} - \eta_{\mathrm{IACR}}\dot{\Theta}^2 \end{bmatrix} = 0,$$

which implies

$$\ell_{\mathrm{IACR}} = \frac{W\dot{\Theta}^3 + \dot{U}\dot{\Theta}^2 - U\dot{\Theta}\ddot{\Theta} + \dot{W}\ddot{\Theta}}{\dot{\Theta}^4 + \ddot{\Theta}^2}, \tag{74}$$

$$\eta_{\mathrm{IACR}} = \frac{-U\dot{\Theta}^3 + \dot{W}\dot{\Theta}^2 + W\dot{\Theta}\ddot{\Theta} - \dot{U}\ddot{\Theta}}{\dot{\Theta}^4 - \ddot{\Theta}^2}. \tag{75}$$

Alternatively, using (S27) yields

$$\begin{aligned} \vec{r}_{\mathrm{p_{IACR}/c}} &= \frac{|\vec{\omega}_{\mathrm{AC/E}}|^2\,\vec{a}_{\mathrm{c/O_E/E}} + \overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC/E}} \times \vec{a}_{\mathrm{c/O_E/E}}}{|\vec{\omega}_{\mathrm{AC/E}}|^4 + |\overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC/E}}|^2} \\ &= \frac{\dot{\Theta}^2\vec{a}_{\mathrm{c/O_E/E}} + \overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC/E}} \times \vec{a}_{\mathrm{c/O_E/E}}}{\dot{\Theta}^4 + \ddot{\Theta}^2}. \end{aligned}$$

Therefore,

In aircraft dynamics, the instantaneous acceleration center of rotation of an aircraft is the point on the aircraft that has zero instantaneous acceleration.

$$\vec{r}_{\mathrm{P_{IACR}}/c}\Big|_{\mathrm{AC}} = \frac{1}{\dot{\Theta}^4+\ddot{\Theta}^2}\left(\dot{\Theta}^2\begin{bmatrix}\dot{U}+W\dot{\Theta}\\ 0\\ \dot{W}-U\dot{\Theta}\end{bmatrix}+\begin{bmatrix}0\\ \ddot{\Theta}\\ 0\end{bmatrix}\times\begin{bmatrix}\dot{U}+W\dot{\Theta}\\ 0\\ \dot{W}-U\dot{\Theta}\end{bmatrix}\right)$$

$$= \begin{bmatrix}\dfrac{W\dot{\Theta}^3+\dot{U}\dot{\Theta}^2-U\dot{\Theta}\ddot{\Theta}+\dot{W}\ddot{\Theta}}{\dot{\Theta}^4+\ddot{\Theta}^2}\\ 0\\ \dfrac{-U\dot{\Theta}^3+\dot{W}\dot{\Theta}^2+W\dot{\Theta}\ddot{\Theta}-\dot{U}\ddot{\Theta}}{\dot{\Theta}^4+\ddot{\Theta}^2}\end{bmatrix},$$

which agrees with (73)–(75).

Next, it follows from (30), (31), (74), and (75) that

$$\ell_{\mathrm{IACR}} = \frac{U(\tan\alpha)\dot{\Theta}^3+\dot{U}\dot{\Theta}^2-U\dot{\Theta}\ddot{\Theta}+(\dot{U}\tan\alpha+U(\sec^2\alpha)\dot{\alpha})\ddot{\Theta}}{\dot{\Theta}^4+\ddot{\Theta}^2}, \tag{76}$$

$$\eta_{\mathrm{IACR}} = \frac{-U\dot{\Theta}^3+(\dot{U}\tan\alpha+U(\sec^2\alpha)\dot{\alpha})\dot{\Theta}^2+U(\tan\alpha)\dot{\Theta}\ddot{\Theta}-\dot{U}\ddot{\Theta}}{\dot{\Theta}^4+\ddot{\Theta}^2}. \tag{77}$$

Since $\dot{\Theta}_0 = 0$ and $\ddot{\Theta}_0 = 0$, it follows that ℓ_{IACR} and η_{IACR} are infinite for steady flight.

Next, for the elevator step deflection $\delta e(t) = \varepsilon\mathbf{1}(t-t_0)$, where $\varepsilon \neq 0$, we approximate ℓ_{IACR} and η_{IACR} at t_0^+ using the first-order approximations $U \approx U_0 + u$, $\alpha \approx \delta\alpha$, and $\Theta \approx \theta$. Thus,

$$\begin{aligned}\ell_{\mathrm{IACR}}(t_0^+) \approx \frac{1}{\ddot{\theta}^2(0^+)+\dot{\theta}^4(0^+)}\big([U_0+u(t_0^+)](\tan\delta\alpha(t_0^+))\dot{\theta}^3(t_0^+)\\ +\dot{u}(t_0^+)\dot{\theta}^2(t_0^+)+[\dot{u}(t_0^+)(\tan\delta\alpha(t_0^+))\\ +[U_0+u(t_0^+)](\sec^2\delta\alpha(t_0^+))\delta\dot{\alpha}(t_0^+)]\ddot{\theta}(t_0^+)\\ -[U_0+\dot{u}(t_0^+)]\dot{\theta}(t_0^+)\ddot{\theta}(t_0^+)\big),\end{aligned} \tag{78}$$

$$\begin{aligned}\eta_{\mathrm{IACR}}(t_0^+) \approx \frac{1}{\ddot{\theta}^2(0^+)+\dot{\theta}^4(0^+)}\big([U_0+u(t_0^+)](\tan\delta\alpha(t_0^+))\dot{\theta}(t_0^+)\ddot{\theta}(t_0^+)\\ -\dot{u}(t_0^+)\ddot{\theta}(t_0^+)+[\dot{u}(t_0^+)(\tan\delta\alpha(t_0^+))\\ +[U_0+u(t_0^+)](\sec^2\delta\alpha(t_0^+))\delta\dot{\alpha}(t_0^+)]\dot{\theta}^2(t_0^+)\\ -[U_0+u(t_0^+)]\dot{\theta}^3(t_0^+)\big),\end{aligned} \tag{79}$$

where the initial value theorem implies that

$$\begin{aligned}\delta\dot{\alpha}(t_0^+) &= \lim_{s\to\infty} s[s\delta\hat{\alpha}(s)-\delta\alpha(t_0^+)]\\ &= \lim_{s\to\infty} s^2 G_{\delta\hat{\alpha}/\delta\hat{e}}(s)\frac{\varepsilon}{s}\\ &= \lim_{s\to\infty}\frac{\varepsilon(A_\alpha s^4+B_\alpha s^3+C_\alpha s^2+D_\alpha)}{Es^4+Fs^3+Gs^2+Hs+I}\\ &= \frac{\varepsilon A_\alpha}{E},\end{aligned} \tag{80}$$

$$\begin{aligned}\ddot{\theta}(t_0^+) &= \lim_{s\to\infty} s[s^2\hat{\theta}(s)-s\theta(t_0^+)-\dot{\theta}(t_0^+)]\\ &= \lim_{s\to\infty} s^3 G_{\hat{\theta}/\delta\hat{e}}(s)\frac{\varepsilon}{s}\\ &= \lim_{s\to\infty}\frac{\varepsilon(A_\theta s^4+B_\theta s^3+C_\theta s^2)}{Es^4+Fs^3+Gs^2+Hs+I}\\ &= \frac{\varepsilon A_\theta}{E},\end{aligned} \tag{81}$$

$$\begin{aligned}\dot{u}(t_0^+) &= \lim_{s\to\infty} s[s\hat{u}(s)-u(t_0^+)]\\ &= \lim_{s\to\infty} s^2 G_{\delta\hat{u}/\delta\hat{e}}(s)\frac{\varepsilon}{s}\\ &= \lim_{s\to\infty}\frac{\varepsilon(A_u s^4+B_u s^3+C_u s^2+D_u s)}{Es^4+Fs^3+Gs^2+Hs+I}\\ &= \frac{\varepsilon A_u}{E}.\end{aligned} \tag{82}$$

It thus follows from (69)–(72), (78)–(82), and the expressions given in Table 5 that

$$\begin{aligned}\ell_{\mathrm{IACR}}(t_0^+) &\approx \frac{U_0 A_\alpha}{A_\theta}\\ &= \frac{U_0 Z_{\delta e_0}}{Z_{\delta e_0}M_{\dot{\alpha}_0}+M_{\delta e_0}(U_0-Z_{\dot{\alpha}_0})}\end{aligned} \tag{83}$$

and

$$\begin{aligned}\eta_{\mathrm{IACR}}(t_0^+) &\approx -\frac{A_u}{A_\theta}\\ &= -\frac{X_{\delta e_0}(U_0-Z_{\dot{\alpha}_0})}{Z_{\delta e_0}M_{\dot{\alpha}_0}+M_{\delta e_0}(U_0-Z_{\dot{\alpha}_0})}.\end{aligned} \tag{84}$$

INITIAL SLOPE AND QUADRATIC CURVATURE OF THE VERTICAL- AND HORIZONTAL-VELOCITY PERTURBATIONS AT THE IACR FOR AN ELEVATOR STEP DEFLECTION

The vertical-velocity perturbation $\delta v_{\mathrm{pv}}(t_0^+)$ at p due to the elevator step deflection $\delta e(t) = \varepsilon\mathbf{1}(t-t_0)$, where $\varepsilon \neq 0$, is given by

$$\begin{aligned}\delta v_{\mathrm{pv}}(t_0^+) &= \lim_{s\to\infty} s\delta\hat{v}_{\mathrm{pv}}(s)\\ &= \lim_{s\to\infty} sG_{\delta\hat{v}_{\mathrm{pv}}/\delta\hat{e}}(s)\frac{\varepsilon}{s}\\ &= \lim_{s\to\infty}\frac{\varepsilon(A_v s^3+B_v s^2+C_v s+D_v)}{Es^4+Fs^3+Gs^2+Hs+I}\\ &= 0\end{aligned}$$

while the initial slope $\delta\dot{v}_{\mathrm{pv}}(t_0^+)$ of the vertical-velocity perturbation is given by

Instantaneous Acceleration Center of Rotation

Let $\mathcal{B}$ be a rigid body with body-fixed frame F_B, let F_A be a frame with origin O_A, and let $\vec{\omega}_{B/A}$ be the angular velocity of F_B relative to F_A. A point p that is fixed relative to $\mathcal{B}$ is an *instantaneous acceleration center of rotation* (IACR) of $\mathcal{B}$ relative to F_A at time t if $\vec{a}_{p/O_A/A}(t) = 0$ [S1, pp. 150–155], [S3, pp. 336–338]. For convenience, we omit the phrase "relative to F_A."

To characterize this property, let q be a point fixed relative to the rigid body $\mathcal{B}$. It follows from the definition of an IACR and the transport theorem that p is an IACR if and only if

$$\vec{a}_{p/O_A/A} = \overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{r}_{p/q} + \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{r}_{p/q}) + \vec{a}_{q/O_A/A} = 0. \tag{S16}$$

Resolving $\vec{a}_{q/O_A/A}$, $\overset{B\bullet}{\vec{\omega}}_{B/A}$, $\vec{\omega}_{B/A}$, and $\vec{r}_{p/q}$ in F_B as

$$a \triangleq \vec{a}_{q/O_A/A}\Big|_B,\ \omega \triangleq \vec{\omega}_{B/A}\Big|_B,\ \dot{\omega} \triangleq \overset{B\bullet}{\vec{\omega}}_{B/A}\Big|_B,\ r \triangleq \vec{r}_{p/q}\Big|_B,$$

(S16) can be rewritten as

$$(\dot{\omega}^{\times} + \omega^{\times 2})r + a = 0. \tag{S17}$$

The existence of an IACR thus depends on the existence of a solution r to (S17). Furthermore, (S17) can yield zero, one, or infinitely many IACRs.

Note that the determinant of $\dot{\omega}^{\times} + \omega^{\times 2}$ is given by

$$\begin{aligned}\det(\dot{\omega}^{\times} + \omega^{\times 2}) &= (\vec{\omega}_{B/A} \cdot \overset{B\bullet}{\vec{\omega}}_{B/A})^2 - (\vec{\omega}_{B/A} \cdot \vec{\omega}_{B/A})(\overset{B\bullet}{\vec{\omega}}_{B/A} \cdot \overset{B\bullet}{\vec{\omega}}_{B/A}) \\ &= -\,|\vec{\omega}_{B/A}|^2 |\overset{B\bullet}{\vec{\omega}}_{B/A}|^2 \sin^2\theta,\end{aligned} \tag{S18}$$

where

$$\theta \triangleq \cos^{-1}\frac{\vec{\omega}_{B/A}\,\overset{B\bullet}{\vec{\omega}}_{B/A}}{|\vec{\omega}_{B/A}||\overset{B\bullet}{\vec{\omega}}_{B/A}|}. \tag{S19}$$

FACT S3

There exists a unique IACR if and only if θ/π is not an integer, $\vec{\omega}_{B/A} \neq 0$, and $\overset{B\bullet}{\vec{\omega}}_{B/A} \neq 0$.

PROOF

Suppose (S17) has a unique solution. Therefore, $\dot{\omega}^{\times} + \omega^{\times 2}$ is nonsingular, and thus the determinant of $\dot{\omega}^{\times} + \omega^{\times 2}$ is nonzero. Hence, it follows from (S18) that

$$\det(\dot{\omega}^{\times} + \omega^{\times 2}) = -\,|\vec{\omega}_{B/A}|^2 |\overset{B\bullet}{\vec{\omega}}_{B/A}|^2 \sin^2\theta \neq 0,$$

which implies that θ/π is not an integer, $\vec{\omega}_{B/A} \neq 0$, and $\overset{B\bullet}{\vec{\omega}}_{B/A} \neq 0$.

Conversely, since θ/π is not an integer, $\vec{\omega}_{B/A} \neq 0$, and $\overset{B\bullet}{\vec{\omega}}_{B/A} \neq 0$, it follows from (S18) that $\det(\dot{\omega}^{\times} + \omega^{\times 2}) = -\,|\vec{\omega}_{B/A}|^2 |\overset{B\bullet}{\vec{\omega}}_{B/A}|^2 \sin^2\theta \neq 0$, which implies that (S17) has a unique solution. □

FACT S4

Assume $\vec{\omega}_{B/A} = 0$, $\overset{B\bullet}{\vec{\omega}}_{B/A} \neq 0$, and $\vec{a}_{q/O_A/A} \neq 0$. Then p is an IACR if and only if p satisfies the following conditions:

i) $\overset{B\bullet}{\vec{\omega}}_{B/A} \cdot \vec{a}_{q/O_A/A} \neq 0$.

ii) $\overset{B\bullet}{\vec{\omega}}_{B/A} \times \left(\vec{r}_{p/q} - \dfrac{1}{|\overset{B\bullet}{\vec{\omega}}_{B/A}|^2}\overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{a}_{q/O_A/A}\right) = 0.$

In this case, p satisfies

$$\vec{r}_{p/q} = \frac{1}{|\overset{B\bullet}{\vec{\omega}}_{B/A}|^2}\overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{a}_{q/O_A/A} + \frac{\overset{B\bullet}{\vec{\omega}}_{B/A} \cdot \vec{r}_{p/q}}{|\overset{B\bullet}{\vec{\omega}}_{B/A}|^2}\overset{B\bullet}{\vec{\omega}}_{B/A}. \tag{S20}$$

PROOF

Assume p is an IACR. Since $\vec{\omega}_{B/A} = 0$, it follows from (S16) that

$$\begin{aligned}\overset{B\bullet}{\vec{\omega}}_{B/A} \cdot \vec{a}_{q/O_A/A} &= \overset{B\bullet}{\vec{\omega}}_{B/A} \cdot \left(-\overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{r}_{p/q} - \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{r}_{p/q})\right) \\ &= -\overset{B\bullet}{\vec{\omega}}_{B/A} \cdot (\overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{r}_{p/q}) \\ &= 0,\end{aligned}$$

which proves i). To prove ii), it follows from (S16) that

$$\begin{aligned}\overset{B\bullet}{\vec{\omega}}_{B/A} \times \left(\vec{r}_{p/q} - \frac{1}{|\overset{B\bullet}{\vec{\omega}}_{B/A}|^2}\overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{a}_{q/O_A/A}\right) &= \overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{r}_{p/q} + \vec{a}_{q/O_A/A} \\ &= 0.\end{aligned}$$

Hence, ii) holds.

Conversely, it follows from ii) that there exists $\alpha \in \mathbb{R}$ such that

$$\vec{r}_{p/q} = \frac{1}{|\overset{B\bullet}{\vec{\omega}}_{B/A}|^2}\overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{a}_{q/O_A/A} + \alpha\,\overset{B\bullet}{\vec{\omega}}_{B/A}. \tag{S21}$$

Using i) and (S21), it follows that

$$\begin{aligned}\vec{a}_{p/O_A/A} &= \overset{A\bullet\bullet}{\vec{r}}_{p/O_A} \\ &= \overset{A\bullet\bullet}{\vec{r}}_{p/q} + \overset{A\bullet\bullet}{\vec{r}}_{q/O_A} \\ &= \overset{B\bullet\bullet}{\vec{r}}_{p/q} + 2\vec{\omega}_{B/A} \times \overset{B\bullet}{\vec{r}}_{p/q} + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{r}_{p/q} \\ &\quad + \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{r}_{p/q}) + \vec{a}_{q/O_A/A} \\ &= \overset{B\bullet}{\vec{\omega}}_{B/A} \times \left(\frac{1}{|\overset{B\bullet}{\vec{\omega}}_{B/A}|^2}\overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{a}_{q/O_A/A} + \alpha\,\overset{B\bullet}{\vec{\omega}}_{B/A}\right) + \vec{a}_{q/O_A/A} \\ &= \frac{\overset{B\bullet}{\vec{\omega}}_{B/A} \cdot \vec{a}_{q/O_A/A}}{|\overset{B\bullet}{\vec{\omega}}_{B/A}|^2}\overset{B\bullet}{\vec{\omega}}_{B/A} - \vec{a}_{q/O_A/A} + \vec{a}_{q/O_A/A} \\ &= 0,\end{aligned}$$

and thus p is an IACR.

To show (S20), assume p is an IACR. It follows from (S16) that

$$\overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{a}_{p/O_A/A} = \overset{B\bullet}{\vec{\omega}}_{B/A} \times \left(\overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{r}_{p/q} + \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{r}_{p/q}) + \vec{a}_{q/O_A/A}\right) = 0,$$

which implies that

$$(\overset{B\bullet}{\vec{\omega}}_{B/A} \cdot \vec{r}_{p/q})\overset{B\bullet}{\vec{\omega}}_{B/A} - (\overset{B\bullet}{\vec{\omega}}_{B/A} \cdot \overset{B\bullet}{\vec{\omega}}_{B/A})\vec{r}_{p/q} + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{a}_{q/O_A/A} = 0. \tag{S22}$$

Hence, solving for $\vec{r}_{p/q}$ in (S22) yields (S20). □

FACT S5

Assume $\overset{B\bullet}{\vec{\omega}}_{B/A}=0$, $\vec{\omega}_{B/A}\neq 0$, and $\vec{a}_{q/O_A/A}\neq 0$. Then p is an IACR if and only if p satisfies the following conditions:

i) $\vec{\omega}_{B/A}\cdot\vec{a}_{q/O_A/A}=0$.

ii) $\vec{\omega}_{B/A}\times\left(\vec{r}_{p/q}-\dfrac{\vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^2}\right)=0$.

In this case, p satisfies

$$\vec{r}_{p/q}=\frac{\vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^2}+\frac{\vec{\omega}_{B/A}\cdot\vec{r}_{p/q}}{|\vec{\omega}_{B/A}|^2}\vec{\omega}_{B/A}. \tag{S23}$$

PROOF

Assume p is an IACR. Since $\overset{B\bullet}{\vec{\omega}}_{B/A}=0$, it follows from (S16) that

$$\begin{aligned}\vec{\omega}_{B/A}\cdot\vec{a}_{q/O_A/A}&=\vec{\omega}_{B/A}\cdot\left(-\overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{r}_{p/q}-\vec{\omega}_{B/A}\times(\vec{\omega}_{B/A}\times\vec{r}_{p/q})\right)\\&=-\vec{\omega}_{B/A}\cdot\left(\vec{\omega}_{B/A}\times(\vec{\omega}_{B/A}\times\vec{r}_{p/q})\right)\\&=0,\end{aligned}$$

which proves i). To prove ii), it follows from (S16) that

$$\begin{aligned}\vec{\omega}_{B/A}\times\left(\vec{r}_{p/q}-\frac{\vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^2}\right)&=\vec{\omega}_{B/A}\times\vec{r}_{p/q}-\vec{\omega}_{B/A}\\&\quad\times\frac{-\overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{r}_{p/q}-\vec{\omega}_{B/A}\times(\vec{\omega}_{B/A}\times\vec{r}_{p/q})}{|\vec{\omega}_{B/A}|^2}\\&=\vec{\omega}_{B/A}\times\vec{r}_{p/q}+\vec{\omega}_{B/A}\times\frac{\vec{\omega}_{B/A}\times(\vec{\omega}_{B/A}\times\vec{r}_{p/q})}{|\vec{\omega}_{B/A}|^2}\\&=\vec{\omega}_{B/A}\times\vec{r}_{p/q}-\vec{\omega}_{B/A}\times\vec{r}_{p/q}\\&=0.\end{aligned}\tag{S24}$$

Hence, ii) holds.

Conversely, it follows from ii) that there exists $\alpha\in\mathbb{R}$ such that

$$\vec{r}_{p/q}=\frac{\vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^2}+\alpha\vec{\omega}_{B/A}. \tag{S25}$$

Using i) and (S25), it follows that

$$\begin{aligned}\vec{a}_{p/O_A/A}&=\overset{A\bullet\bullet}{\vec{r}}_{p/O_A}\\&=\overset{A\bullet\bullet}{\vec{r}}_{p/q}+\overset{A\bullet\bullet}{\vec{r}}_{q/O_A}\\&=\overset{B\bullet\bullet}{\vec{r}}_{p/q}+2\vec{\omega}_{B/A}\times\overset{B\bullet}{\vec{r}}_{p/q}+\overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{r}_{p/q}\\&\quad+\vec{\omega}_{B/A}\times(\vec{\omega}_{B/A}\times\vec{r}_{p/q})+\vec{a}_{q/O_A/A}\\&=\vec{\omega}_{B/A}\times\left(\vec{\omega}_{B/A}\times\left(\frac{\vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^2}+\alpha\vec{\omega}_{B/A}\right)\right)+\vec{a}_{q/O_A/A}\\&=-\vec{a}_{q/O_A/A}+\vec{a}_{q/O_A/A}\\&=0.\end{aligned}$$

To show (S23), assume p is an IACR. It follows from (S16) that

$$\begin{aligned}\overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{r}_{p/q}+\vec{\omega}_{B/A}\times(\vec{\omega}_{B/A}\times\vec{r}_{p/q})+\vec{a}_{q/O_A/A}&=\vec{\omega}_{B/A}\times(\vec{\omega}_{B/A}\times\vec{r}_{p/q})+\vec{a}_{q/O_A/A}\\&=(\vec{\omega}_{B/A}\cdot\vec{r}_{p/q})\vec{\omega}_{B/A}-(\vec{\omega}_{B/A}\cdot\vec{\omega}_{B/A})\vec{r}_{p/q}+\vec{a}_{q/O_A/A}\\&=0.\end{aligned}\tag{S26}$$

Solving (S26) for $\vec{r}_{p/q}$ yields (S23). □

FACT S6

Assume $\overset{B\bullet}{\vec{\omega}}_{B/A}=0$ and $\vec{\omega}_{B/A}=0$. Then every point p that is fixed relative to $\mathcal{B}$ is an IACR if and only if

$$\vec{a}_{q/O_A/A}=0.$$

PROOF

Assuming p is an IACR, it follows from (S16) that

$$\begin{aligned}0&=\overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{r}_{p/q}+\vec{\omega}_{B/A}\times(\vec{\omega}_{B/A}\times\vec{r}_{p/q})+\vec{a}_{q/O_A/A}\\&=\vec{a}_{q/O_A/A}.\end{aligned}$$

Conversely,

$$\begin{aligned}\vec{a}_{p/O_A/A}&=\overset{A\bullet\bullet}{\vec{r}}_{p/O_A}\\&=\overset{A\bullet\bullet}{\vec{r}}_{p/q}+\overset{A\bullet\bullet}{\vec{r}}_{q/O_A}\\&=\overset{B\bullet\bullet}{\vec{r}}_{p/q}+2\vec{\omega}_{B/A}\times\overset{B\bullet}{\vec{r}}_{p/q}+\overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{r}_{p/q}\\&\quad+\vec{\omega}_{B/A}\times(\vec{\omega}_{B/A}\times\vec{r}_{p/q})+\vec{a}_{q/O_A/A}\\&=\vec{a}_{q/O_A/A}\\&=0.\end{aligned}$$

FACT S7

Assume $\overset{B\bullet}{\vec{\omega}}_{B/A}$ and $\vec{\omega}_{B/A}$ are colinear, and let $\kappa\triangleq(\overset{B\bullet}{\vec{\omega}}_{B/A}\cdot\vec{\omega}_{B/A})/|\vec{\omega}_{B/A}|^2$. Then p is an IACR if and only if p satisfies the following conditions:

i) $\vec{\omega}_{B/A}\cdot\vec{a}_{q/O_A/A}=0$.

ii) $\vec{\omega}_{B/A}\times\left(\vec{r}_{p/q}-\dfrac{|\vec{\omega}_{B/A}|^2\vec{a}_{q/O_A/A}+\overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^4+|\overset{B\bullet}{\vec{\omega}}_{B/A}|^2}\right)=0$.

In this case, p satisfies

$$\begin{aligned}\vec{r}_{p/q}&=\frac{|\vec{\omega}_{B/A}|^2\vec{a}_{q/O_A/A}+\overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^4+|\overset{B\bullet}{\vec{\omega}}_{B/A}|^2}\\&\quad+\frac{|\vec{\omega}_{B/A}|^2(\vec{\omega}_{B/A}\cdot\vec{r}_{p/q})+\kappa\,\overset{B\bullet}{\vec{\omega}}_{B/A}\cdot\vec{r}_{p/q}}{|\vec{\omega}_{B/A}|^4+|\overset{B\bullet}{\vec{\omega}}_{B/A}|^2}\vec{\omega}_{B/A}.\end{aligned}\tag{S27}$$

PROOF

Assume p is an IACR. It follows from (S16) that $\vec{\omega}_{B/A}\cdot\vec{a}_{q/O_A/A}=0$, which proves i). To prove ii), note that, since p is an IACR, it follows from (S16) that

$$\begin{aligned}0&=\overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{r}_{p/q}+\vec{\omega}_{B/A}\times(\vec{\omega}_{B/A}\times\vec{r}_{p/q})+\vec{a}_{q/O_A/A}\\&=\overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{r}_{p/q}+(\vec{\omega}_{B/A}\cdot\vec{r}_{p/q})\vec{\omega}_{B/A}-(\vec{\omega}_{B/A}\cdot\vec{\omega}_{B/A})\vec{r}_{p/q}+\vec{a}_{q/O_A/A}.\end{aligned}\tag{S28}$$

Next, the cross product of $\overset{B\bullet}{\vec{\omega}}_{B/A}$ and (S28) can be expressed as

$$\begin{aligned}0&=\overset{B\bullet}{\vec{\omega}}_{B/A}\times\Big(\overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{r}_{p/q}+(\vec{\omega}_{B/A}\cdot\vec{r}_{p/q})\vec{\omega}_{B/A}\\&\quad-(\vec{\omega}_{B/A}\cdot\vec{\omega}_{B/A})\vec{r}_{p/q}+\vec{a}_{q/O_A/A}\Big)\\&=(\overset{B\bullet}{\vec{\omega}}_{B/A}\cdot\vec{r}_{p/q})\overset{B\bullet}{\vec{\omega}}_{B/A}-|\overset{B\bullet}{\vec{\omega}}_{B/A}|^2\vec{r}_{p/q}\\&\quad-|\vec{\omega}_{B/A}|^2(\overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{r}_{p/q})+\overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{a}_{q/O_A/A}.\end{aligned}\tag{S29}$$

It follows from (S28) that

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{p/q}} = -\left(\vec{\omega}_{\mathrm{B/A}} \cdot \vec{r}_{\mathrm{p/q}}\right)\vec{\omega}_{\mathrm{B/A}} + \left(\vec{\omega}_{\mathrm{B/A}} \cdot \vec{\omega}_{\mathrm{B/A}}\right)\vec{r}_{\mathrm{p/q}} - \vec{a}_{\mathrm{q/O_A/A}}. \quad \text{(S30)}$$

Substituting (S30) into (S29) yields

$$\begin{aligned} 0 &= (\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \cdot \vec{r}_{\mathrm{p/q}})\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} - |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2\vec{r}_{\mathrm{p/q}} + |\vec{\omega}_{\mathrm{B/A}}|^2(\vec{\omega}_{\mathrm{B/A}} \cdot \vec{r}_{\mathrm{p/q}})\vec{\omega}_{\mathrm{B/A}} \\ &\quad - |\vec{\omega}_{\mathrm{B/A}}|^4\vec{r}_{\mathrm{p/q}} + |\vec{\omega}_{\mathrm{B/A}}|^2\vec{a}_{\mathrm{q/O_A/A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{\mathrm{q/O_A/A}} \\ &= [\kappa\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \cdot \vec{r}_{\mathrm{p/q}} + |\vec{\omega}_{\mathrm{B/A}}|^2(\vec{\omega}_{\mathrm{B/A}} \cdot \vec{r}_{\mathrm{p/q}})]\vec{\omega}_{\mathrm{B/A}} \\ &\quad + |\vec{\omega}_{\mathrm{B/A}}|^2\vec{a}_{\mathrm{q/O_A/A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{\mathrm{q/O_A/A}} - (|\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2 + |\vec{\omega}_{\mathrm{B/A}}|^4)\, r_{\mathrm{p/q}}. \end{aligned} \quad \text{(S31)}$$

Now, solving (S31) for $\vec{r}_{\mathrm{p/q}}$ yields (S27), which implies that ii) is satisfied.

Conversely, it follows from ii) that there exists $\alpha \in \mathbb{R}$ such that

$$\vec{r}_{\mathrm{p/q}} = \frac{|\vec{\omega}_{\mathrm{B/A}}|^2\vec{a}_{\mathrm{q/O_A/A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{\mathrm{q/O_A/A}}}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} + \alpha\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}. \quad \text{(S32)}$$

Using i) and (S32), $\vec{a}_{\mathrm{p/O_A/A}}$ is given by

$$\begin{aligned} \vec{a}_{\mathrm{p/O_A/A}} &= \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{\mathrm{p/O_A}} \\ &= \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{\mathrm{p/q}} + \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{\mathrm{q/O_A}} \\ &= \overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{\mathrm{p/q}} + 2\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{r}}_{\mathrm{p/q}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{p/q}} \\ &\quad + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{p/q}}) + \vec{a}_{\mathrm{q/O_A/A}} \\ &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \left(\frac{|\vec{\omega}_{\mathrm{B/A}}|^2\vec{a}_{\mathrm{q/O_A/A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{\mathrm{q/O_A/A}}}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} + \alpha\vec{\omega}_{\mathrm{B/A}}\right) \\ &\quad + \vec{\omega}_{\mathrm{B/A}} \times \left(\vec{\omega}_{\mathrm{B/A}} \times \left(\frac{|\vec{\omega}_{\mathrm{B/A}}|^2\vec{a}_{\mathrm{q/O_A/A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{\mathrm{q/O_A/A}}}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} + \alpha\vec{\omega}_{\mathrm{B/A}}\right)\right) + \vec{a}_{\mathrm{q/O_A/A}} \\ &= -\vec{a}_{\mathrm{q/O_A/A}} + \vec{a}_{\mathrm{q/O_A/A}} \\ &= 0. \qquad \square \end{aligned}$$

REFERENCES

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[S4] J. M. R. Martinez and J. Duffy, "Determination of the acceleration center of a rigid body in spatial motion," *Eur. J. Mech. A Solids*, vol. 17, pp. 969–977, 1998.

[S5] K. E. Bisshopp, "Note on acceleration in laminar motion," *J. Mech.*, vol. 4, pp. 235–242, 1969.

$$\begin{aligned} \delta\dot{v}_{\mathrm{pv}}(t_0^+) &= \lim_{s\to\infty} s[s\delta\hat{v}_{\mathrm{pv}}(s) - \delta v_{\mathrm{pv}}(t_0^+)] \\ &= \lim_{s\to\infty} s^2 G_{\delta\hat{v}_{\mathrm{pv}}/\delta\hat{e}}(s)\frac{\varepsilon}{s} \\ &= \lim_{s\to\infty}\frac{\varepsilon(A_{\mathrm{v}}s^4 + B_{\mathrm{v}}s^3 + C_{\mathrm{v}}s^2 + D_{\mathrm{v}}s)}{Es^4 + Fs^3 + Gs^2 + Hs + I} \\ &= \frac{\varepsilon A_{\mathrm{v}}}{E}. \end{aligned} \quad (85)$$

Hence, if $\varepsilon A_{\mathrm{v}}/E \neq 0$, then the vertical-velocity perturbation has a slope discontinuity due to the elevator step deflection. Note that the initial slope $\delta\dot{v}_{\mathrm{pv}}(t_0^+)$ of the vertical-velocity perturbation is the initial value of the vertical-acceleration perturbation.

Next, it follows from the expression for A_{v} given in Table 5 that

$$A_{\mathrm{v}} = -\ell A_\theta + U_0 A_\alpha. \quad (86)$$

Therefore, $A_{\mathrm{v}} = 0$ if and only if

$$\ell = \frac{U_0 A_\alpha}{A_\theta}. \quad (87)$$

Hence, it follows from (85) that $\delta\dot{v}_{\mathrm{pv}}(t_0^+) = 0$ if and only if ℓ satisfies (87). For details, see "The Initial Curvature Theorem and Unit-Step Response."

Similarly, the horizontal-velocity perturbation $\delta v_{\mathrm{ph}}(t_0^+)$ at p due to the elevator step deflection $\delta e(t) = \varepsilon\mathbf{1}(t - t_0)$, where $\varepsilon \neq 0$, is given by

$$\begin{aligned} \delta v_{\mathrm{ph}}(t_0^+) &= \lim_{s\to\infty} s\delta\hat{v}_{\mathrm{ph}}(s) \\ &= \lim_{s\to\infty} sG_{\delta\hat{v}_{\mathrm{ph}}/\delta\hat{e}}(s)\frac{\varepsilon}{s} \\ &= \lim_{s\to\infty}\frac{\varepsilon(A_{\mathrm{h}}s^3 + B_{\mathrm{h}}s^2 + C_{\mathrm{h}}s + D_{\mathrm{h}})}{Es^4 + Fs^3 + Gs^2 + Hs + I} \\ &= 0, \end{aligned}$$

while the initial slope $\delta\dot{v}_{\mathrm{ph}}(t_0^+)$ of the horizontal-velocity perturbation is given by

$$\begin{aligned} \delta\dot{v}_{\mathrm{ph}}(t_0^+) &= \lim_{s\to\infty} s[s\delta\hat{v}_{\mathrm{ph}}(s) - \delta v_{\mathrm{ph}}(t_0^+)] \\ &= \lim_{s\to\infty} s^2 G_{\delta\hat{v}_{\mathrm{ph}}/\delta\hat{e}}(s)\frac{\varepsilon}{s} \\ &= \lim_{s\to\infty}\frac{\varepsilon(A_{\mathrm{h}}s^4 + B_{\mathrm{h}}s^3 + C_{\mathrm{h}}s^2 + D_{\mathrm{h}}s)}{Es^4 + Fs^3 + Gs^2 + Hs + I} \\ &= \frac{\varepsilon A_{\mathrm{h}}}{E}. \end{aligned} \quad (88)$$

Next, it follows from the expression for A_{h} given in Table 5 that

$$A_{\mathrm{h}} = \eta A_\theta + A_u. \quad (89)$$

Therefore, $A_{\mathrm{h}} = 0$ if and only if

$$\eta = -\frac{A_u}{A_\theta}. \quad (90)$$

Hence, it follows from (88) that $\delta\dot{v}_{\mathrm{ph}}(t_0^+) = 0$ if and only if η satisfies (90).

The Initial Curvature Theorem and the Unit-Step Response

INITIAL SLOPE THEOREM

Let $\hat{y}(s)$ denote the Laplace transform of $y(t)$. Then the initial slope of $y(t)$ is given by

$$y'(0^+) \triangleq \lim_{t\to 0^+} y'(t) = \lim_{s\to\infty} s[s\hat{y}(s) - y(0^+)].$$

To illustrate the initial slope theorem, we consider the unit-step response of the asymptotically stable, strictly proper transfer function G with relative degree $d \geq 1$. The unit-step response has the initial value $y(0^+) \triangleq \lim_{t\to 0^+} y(t) = \lim_{s\to\infty} s(G(s)1/s) = G(\infty) = 0$. The initial slope of $y(t)$ is thus given by

$$y'(0^+) = \lim_{s\to\infty} s^2\, \hat{y}(s) = \lim_{s\to\infty} sG(s).$$

Consequently, if $d = 1$, then $y'(0^+) \neq 0$, whereas, if $d \geq 2$, then $y'(0^+) = 0$. These results are illustrated in Figure S4 and Figure S5.

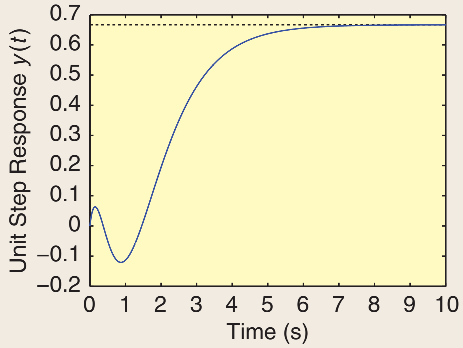

FIGURE S4 The unit step response of the asymptotically stable transfer function $G(s) = (s-2)^2/((s+1)(s+2)(s+3))$ with relative degree $d = 1$. The initial slope $y'(0^+)$ of the unit step response is one.

INITIAL CURVATURE THEOREM

Let $\hat{y}(s)$ denote the Laplace transform of $y(t)$. Then the initial curvature of $y(t)$ is given by

$$y^{(d)}(0^+) \triangleq \lim_{t\to 0^+} y^{(d)}(t) = \lim_{s\to\infty} s^{d+1}\hat{y}(s),$$

where $y^{(d)}$ denotes the dth derivative of y, and d is the smallest integer such that $y^{(d)}(0^+) \neq 0$.

We now consider the unit-step response of the asymptotically stable, strictly proper transfer function G with relative degree $d \geq 1$, where

$$G(s) = \frac{\beta_{n-d}s^{n-d} + \beta_{n-d-1}s^{n-d-1} + \cdots + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0}.$$

The initial derivatives of the unit step response are thus given by

$$\begin{aligned} y^{(i)}(0^+) &= \lim_{s\to\infty} s^{i+1}\hat{y}(s) \\ &= \lim_{s\to\infty} s^{i+1}G(s)\frac{1}{s} \\ &= \lim_{s\to\infty} s^{i}G(s) \\ &= \begin{cases} 0, & i = 1, \ldots, d-1, \\ \beta_{n-d}, & i = d. \end{cases} \end{aligned}$$

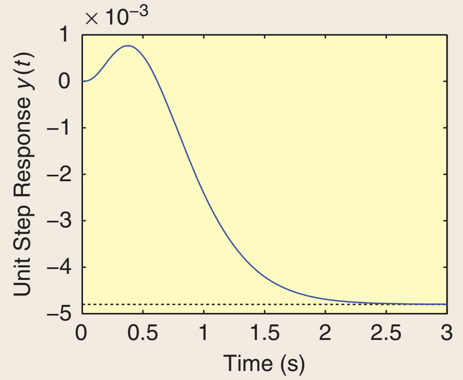

FIGURE S5 The unit step response of the asymptotically stable transfer function $G(s) = (s-3)/(s+5)^4$, whose relative degree is three. The initial slope $y'(0^+)$ of the unit step response is zero, whereas the initial curvature $y''(0^+)$ of the unit step response is one.

Therefore, the initial curvature of the unit step response is $y^{(d)}(0^+) = \beta_{n-d}$.

Next, it follows from (83) and (84) that $\mathrm{p}_{\mathrm{IACR}}$ for an elevator step deflection satisfies both (87) and (90). Therefore, $A_{\mathrm{v}} = 0$ and $A_{\mathrm{h}} = 0$ if and only if $(\ell, \eta) = (\ell_{\mathrm{IACR}}, \eta_{\mathrm{IACR}})$. Thus, evaluating (85) and (88) at $\mathrm{p}_{\mathrm{IACR}}$ for the elevator step deflection $\delta e(t) = \varepsilon \mathbf{1}(t - t_0)$, where $\varepsilon \neq 0$, yields $\delta\dot{v}_{\mathrm{pv}}(t_0^+) = 0$ and $\delta\dot{v}_{\mathrm{ph}}(t_0^+) = 0$. Therefore, at the IACR, the initial slopes of the vertical- and horizontal-velocity perturbations are zero. Equivalently, despite the step discontinuity in the elevator deflection, the initial values of the vertical- and horizontal-acceleration perturbations are zero. Therefore, the initial value of the acceleration measured by a body-fixed accelerometer whose direction of measurement is orthogonal to $\hat{\jmath}_{\mathrm{AC}}$ is zero [6, pp. 313–316], [7]–[15].

Since $A_{\mathrm{v}} = 0$ at the IACR, it follows that the transfer function $G_{\delta\hat{v}_{\mathrm{pv}}/\delta\hat{e}}(s)$ at the IACR becomes

$$G_{\delta\hat{v}_{\mathrm{pv}}/\delta\hat{e}}(s) = \frac{B_{\mathrm{v}}s^2 + C_{\mathrm{v}}s + D_{\mathrm{v}}}{Es^4 + Fs^3 + Gs^2 + Hs + I}.$$

Next, at the IACR, it follows from the expression for B_v given in Table 5 that

$$B_\text{v} = -\ell_\text{IACR}B_\theta - U_0A_\theta + U_0B_\alpha = -\left(\frac{A_\alpha B_\theta}{A_\theta} + A_\theta - B_\alpha\right)U_0.$$

Consequently, if $B_\text{v} \neq 0$, then the relative degree of $G_{\delta\hat{v}_\text{pv}/\delta\hat{e}}(s)$ increases from one to two, and thus one of the zeros of $G_{\delta\hat{v}_\text{pv}/\delta\hat{e}}(s)$ vanishes at the IACR.

Similarly, at the IACR, $A_\text{h} = 0$. Thus, if $B_\text{h} \neq 0$, then the relative degree of $G_{\delta\hat{v}_\text{ph}/\delta\hat{e}}(s)$ increases from one to two, and thus one of the zeros of $G_{\delta\hat{v}_\text{ph}/\delta\hat{e}}(s)$ vanishes at the IACR. The vanishing zeros are a consequence of the fact that the initial slope of the vertical-velocity perturbation and the horizontal-velocity perturbation are zero at the IACR. Note that ℓ_IACR and η_IACR depend on the speed U_0 and the stability derivatives $Z_{\delta e_0}$, $Z_{\dot{\alpha}_0}$, $X_{\delta e_0}$, $M_{\dot{\alpha}_0}$, and $M_{\delta e_0}$. Vanishing zeros are discussed in [16].

INITIAL UNDERSHOOT OF THE VERTICAL VELOCITY FOR AN ELEVATOR STEP DEFLECTION

Let $G(s) \triangleq \beta(s)/(s^r\alpha(s))$ be a strictly proper transfer function with relative degree $d > 0$, where $r \geq 0$ and $\alpha(s)$ is asymptotically stable. Let $y(t)$ denote the response of G to the step command $\mathbf{1}(t - t_0)$. Then *initial undershoot* occurs at time t_0 if the step response initially moves in the direction opposite to its asymptotic direction, that is,

$$y^{(d)}(t_0^+)y^{(r)}(\infty) < 0. \tag{91}$$

To determine whether the vertical-velocity perturbation $\delta v_\text{pv}(t)$ to the elevator step deflection $\delta e(t) = \varepsilon\mathbf{1}(t - t_0)$ exhibits initial undershoot, we investigate (91) with $G(s) = G_{\delta\hat{v}_\text{pv}/\delta\hat{e}}(s)$, $r = 0$, and $y(t) = \delta v_\text{pv}(t)$.

First, the asymptotic direction of the step response is given by the sign of

$$\begin{aligned}\delta v_\text{pv}(\infty) &= \lim_{s\to 0} s\delta\hat{v}_\text{pv}(s)\\ &= \lim_{s\to 0} sG_{\delta\hat{v}_\text{pv}/\delta\hat{e}}(s)\frac{\varepsilon}{s}\\ &= \lim_{s\to 0}\frac{\varepsilon(A_\text{v}s^3 + B_\text{v}s^2 + C_\text{v}s + D_\text{v})}{Es^4 + Fs^3 + Gs^2 + Hs + I}\\ &= \frac{\varepsilon D_\text{v}}{I}.\end{aligned} \tag{92}$$

It follows from Table 5 and Table 6 that $\delta v_\text{pv}(\infty)$ does not depend on the location of p, that is, the value of (ℓ, η).

Next, the initial direction of the step response is given by the sign of

$$\begin{aligned}\delta v_\text{pv}^{(d)}(t_0^+) &= \lim_{s\to\infty} s[s^d\delta\hat{v}_\text{pv}(s) - s^{d-1}\delta v_\text{pv}(t_0^+) - \cdots - \delta v_\text{pv}^{(d-1)}(t_0^+)]\\ &= \lim_{s\to\infty} s^{d+1}\delta\hat{v}_\text{pv}(s)\\ &= \lim_{s\to\infty} s^{d+1}G_{\delta\hat{v}_\text{pv}/\delta\hat{e}}(s)\frac{\varepsilon}{s}\\ &= \varepsilon s^d\left(\frac{A_\text{v}s^3 + B_\text{v}s^2 + C_\text{v}s + D_\text{v}}{Es^4 + Fs^3 + Gs^2 + Hs + I}\right)\\ &= \begin{cases}\frac{\varepsilon A_\text{v}}{E}, & \text{if } d = 1, \text{ (that is, } A_\text{v} \neq 0),\\ \frac{\varepsilon B_\text{v}}{E}, & \text{if } d = 2, \text{ (that is, } A_\text{v} = 0, B_\text{v} \neq 0),\\ \frac{\varepsilon C_\text{v}}{E}, & \text{if } d = 3, \text{ (that is, } A_\text{v} = B_\text{v} = 0, C_\text{v} \neq 0),\\ \frac{\varepsilon D_\text{v}}{E}, & \text{if } d = 4, \text{ (that is, } A_\text{v} = B_\text{v} = C_\text{v} = 0, D_\text{v} \neq 0).\end{cases}\end{aligned} \tag{93}$$

Thus, for $d = 1$, $\delta v_\text{pv}(t)$ exhibits initial undershoot if and only if $\delta\dot{v}_\text{pv}(t_0^+)\delta v_\text{pv}(\infty) = A_\text{v}D_\text{v}/(EI) < 0$; for $d = 2$, $\delta v_\text{pv}(t)$ exhibits initial undershoot if and only if $\delta\ddot{v}_\text{pv}(t_0^+)\delta v_\text{pv}(\infty) = B_\text{v}D_\text{v}/(EI) < 0$; for $d = 3$, $\delta v_\text{pv}(t)$ exhibits initial undershoot if and only if $\delta v_\text{pv}^{(3)}(t_0^+)\delta v_\text{pv}(\infty) = C_\text{v}D_\text{v}/(EI) < 0$. Furthermore, for $d = 4$, $\delta v_\text{pv}(t)$ does not exhibit initial undershoot since $\delta v_\text{pv}^{(4)}(t_0^+)\delta v_\text{pv}(\infty) = D_\text{v}^2/(EI) \geq 0$.

The following results follow from (87) and (91)–(93) along with Proposition S1.

Proposition 1

Assume that ℓ does not satisfy (87). Then the following statements hold:

i) The relative degree of $G_{\delta\hat{v}_\text{pv}/\delta\hat{e}}(s)$ is one, and thus $A_\text{v} \neq 0$.
ii) $\delta v_\text{pv}(t)$ exhibits initial undershoot if and only if $A_\text{v}D_\text{v} < 0$.
iii) $\delta v_\text{pv}(t)$ exhibits initial undershoot if and only if $G_{\delta\hat{v}_\text{pv}/\delta\hat{e}}(s)$ has either exactly one or exactly three real nonminimum-phase zeros.

Proposition 2

Assume that ℓ satisfies (87) and $B_\text{v} \neq 0$. Then the following statements hold:

i) The relative degree of $G_{\delta\hat{v}_\text{pv}/\delta\hat{e}}(s)$ is two, and thus $A_\text{v} = 0$.
ii) $\delta v_\text{pv}(t)$ exhibits initial undershoot if and only if $B_\text{v}D_\text{v} < 0$.
iii) $\delta v_\text{pv}(t)$ exhibits initial undershoot if and only if $G_{\delta\hat{v}_\text{pv}/\delta\hat{e}}(s)$ has exactly one real nonminimum-phase zero.

Following the same procedure for $\delta r_\text{pv}(t)$ yields identical results, that is, $\delta r_\text{pv}(t)$ exhibits initial undershoot if and only if $\delta v_\text{pv}(t)$ exhibits initial undershoot.

INITIAL UNDERSHOOT OF THE HORIZONTAL VELOCITY FOR AN ELEVATOR STEP DEFLECTION

To determine whether the horizontal-velocity perturbation $\delta v_\text{ph}(t)$ to the elevator step deflection $\delta e(t) = \varepsilon\mathbf{1}(t - t_0)$ exhibits initial undershoot, we investigate (91) with $G(s) = G_{\delta\hat{v}_\text{ph}/\delta\hat{e}}(s)$, $r = 0$, and $y(t) = \delta v_\text{ph}(t)$.

First, the asymptotic direction of the step response is given by the sign of

$$
\begin{aligned}
\delta v_{\rm ph}(\infty) &= \lim_{s\to 0} s\delta\hat{v}_{\rm ph}(s)\\
&= \lim_{s\to 0} sG_{\delta\hat{v}_{\rm ph}/\delta\hat{e}}(s)\frac{\varepsilon}{s}\\
&= \lim_{s\to 0} \frac{\varepsilon(A_{\rm h}s^3+B_{\rm h}s^2+C_{\rm h}s+D_{\rm h})}{Es^4+Fs^3+Gs^2+Hs+I}\\
&= \frac{\varepsilon D_{\rm h}}{I}. \qquad (94)
\end{aligned}
$$

It follows from Table 5 and Table 6 that $\delta v_{\rm ph}(\infty)$ does not depend on the location of p, that is, the value of (ℓ, η).

Next, the initial direction of the step response is given by the sign of

$$
\begin{aligned}
\delta v_{\rm ph}^{(d)}(t_0^+) &= \lim_{s\to\infty} s[s^d\delta\hat{v}_{\rm ph}(s) - s^{d-1}\delta v_{\rm ph}(t_0^+) - \cdots - \delta v_{\rm ph}^{(d-1)}(t_0^+)]\\
&= \lim_{s\to\infty} s^{d+1}\delta\hat{v}_{\rm ph}(s)\\
&= \lim_{s\to\infty} s^{d+1}G_{\delta\hat{v}_{\rm ph}/\delta\hat{e}}(s)\frac{\varepsilon}{s}\\
&= \varepsilon s^d\left(\frac{A_{\rm h}s^3+B_{\rm h}s^2+C_{\rm h}s+D_{\rm h}}{Es^4+Fs^3+Gs^2+Hs+I}\right)\\
&= \begin{cases} \frac{\varepsilon A_{\rm h}}{E}, & \text{if } d=1, \text{ (that is, } A_{\rm h}\neq 0),\\ \frac{\varepsilon B_{\rm h}}{E}, & \text{if } d=2, \text{ (that is, } A_{\rm h}=0, B_{\rm h}\neq 0),\\ \frac{\varepsilon C_{\rm h}}{E}, & \text{if } d=3, \text{ (that is, } A_{\rm h}=B_{\rm h}=0, C_{\rm h}\neq 0),\\ \frac{\varepsilon D_{\rm h}}{E}, & \text{if } d=4, \text{ (that is, } A_{\rm h}=B_{\rm h}=C_{\rm h}=0, D_{\rm h}\neq 0). \end{cases} \qquad (95)
\end{aligned}
$$

Thus, for $d=1$, $\delta v_{\rm ph}(t)$ exhibits initial undershoot if and only if $\delta\dot{v}_{\rm ph}(t_0^+)\delta v_{\rm ph}(\infty) = A_{\rm h}D_{\rm h}/(EI) < 0$; for $d=2$, $\delta v_{\rm ph}(t)$ exhibits initial undershoot if and only if $\delta\ddot{v}_{\rm ph}(t_0^+)\delta v_{\rm ph}(\infty) = B_{\rm h}D_{\rm h}/(EI) < 0$; for $d=3$, $\delta v_{\rm ph}(t)$ exhibits initial undershoot if and only if $\delta v_{\rm ph}^{(3)}(t_0^+)\delta v_{\rm ph}(\infty) = C_{\rm h}D_{\rm h}/(EI) < 0$; Furthermore, for $d=4$, $\delta v_{\rm ph}(t)$ does not exhibit initial undershoot since $\delta v_{\rm ph}^{(4)}(t_0^+)\delta v_{\rm ph}(\infty) = {D_{\rm h}}^2/(EI) \geq 0$.

The following results follow from (90), (91), (94), and (95) along with Proposition S1.

Proposition 3

Assume that η does not satisfy (90). Then the following statements hold:

i) The relative degree of $G_{\delta\hat{v}_{\rm ph}/\delta\hat{e}}(s)$ is one, and thus $A_{\rm h}\neq 0$.
ii) $\delta v_{\rm ph}(t)$ exhibits initial undershoot if and only if $A_{\rm h}D_{\rm h} < 0$.
iii) $\delta v_{\rm ph}(t)$ exhibits initial undershoot if and only if $G_{\delta\hat{v}_{\rm ph}/\delta\hat{e}}(s)$ has either exactly one or exactly three real nonminimum-phase zeros.

Proposition 4

Assume that η satisfies (90) and $B_{\rm h}\neq 0$. Then the following statements hold:

i) The relative degree of $G_{\delta\hat{v}_{\rm ph}/\delta\hat{e}}(s)$ is two, and thus $A_{\rm h}=0$.
ii) $\delta v_{\rm ph}(t)$ exhibits initial undershoot if and only if $B_{\rm h}D_{\rm h} < 0$.
iii) $\delta v_{\rm ph}(t)$ exhibits initial undershoot if and only if $G_{\delta\hat{v}_{\rm ph}/\delta\hat{e}}(s)$ has exactly one real nonminimum-phase zero.

The following result is a special case of propositions 2 and 4, where we consider the response at the IACR.

Proposition 5

Assume that $(\ell,\eta) = (\ell_{\rm IACR}, \eta_{\rm IACR})$, $B_{\rm v}\neq 0$, and $B_{\rm h}\neq 0$. Then the following statements hold:

i) The relative degrees of $G_{\delta\hat{v}_{\rm pv}/\delta\hat{e}}(s)$ and $G_{\delta\hat{v}_{\rm ph}/\delta\hat{e}}(s)$ are two. Thus, $A_{\rm v}=0$ and $A_{\rm h}=0$.
ii) $\delta v_{\rm pv}(t)$ exhibits initial undershoot if and only if $B_{\rm v}D_{\rm v} < 0$.
iii) $\delta v_{\rm ph}(t)$ exhibits initial undershoot if and only if $B_{\rm h}D_{\rm h} < 0$.
iv) $\delta v_{\rm pv}(t)$ exhibits initial undershoot if and only if $G_{\delta\hat{v}_{\rm pv}/\delta\hat{e}}(s)$ has exactly one real nonminimum-phase zero.
v) $\delta v_{\rm ph}(t)$ exhibits initial undershoot if and only if $G_{\delta\hat{v}_{\rm ph}/\delta\hat{e}}(s)$ has exactly one real nonminimum-phase zero.

BUSINESS JET EXAMPLE

To illustrate the instantaneous acceleration center of rotation, the initial slope of the vertical velocity and horizontal velocity, and vanishing zeros, we consider a business jet in cruise whose numerical data are given in Table 7, which is a reproduction of data provided in [18, p. 330].

For all expressions below, the units of ℓ and η are feet. Using the data given in Table 7 as well as the expressions given in Table 5 and Table 6, and (51), (52), (53), and (57), the transfer functions from $\delta\hat{e}(s)$ to $\hat{u}(s)$, $\delta\hat{\alpha}(s)$, and $\hat{\theta}(s)$ are

$$
G_{\hat{u}/\delta\hat{e}}(s) = \frac{-378.85s^2+2.72e5s+2.40e5}{675.99(s^4+2.01s^3+8.05s^2+0.085s+0.068)} \text{ ft/(s-rad)},
$$

$$
G_{\delta\hat{\alpha}/\delta\hat{e}}(s) = \frac{42.20s^3+11939.02s^2+88.5773s+79.30}{675.99(s^4+2.01s^3+8.05s^2+0.085s+0.068)},
$$

$$
G_{\hat{\theta}/\delta\hat{e}}(s) = \frac{-11930.17s^2-7652.06s-78.52}{675.99(s^4+2.01s^3+8.05s^2+0.085s+0.068)}.
$$

Furthermore, the transfer functions from $\delta\hat{e}(s)$ to $\delta\hat{v}_{\rm pv}$ and $\delta\hat{v}_{\rm ph}$ are shown in (96) and (97), found at the bottom of the next page. Next, with $U_0 = 675.12$ ft/s, $A_\alpha = -42.20$ 1/s, $A_u = 0$ m/s^2, $E = 675.99$ 1/s, $\varepsilon = 1$ deg-s $= 0.017$ rad-s, and $A_\theta = 11930.17$ 1/s^2, it follows from (83) and (84) that

$$
\ell_{\rm IACR} \approx -\frac{(675.12)(42.20)}{11930.17} \text{ ft} = -2.3881 \text{ ft},
$$

$$
\eta_{\rm IACR} \approx -\frac{0}{11930.17} \text{ ft} = 0 \text{ ft}.
$$

TABLE 7 Stability parameter values. These data for a business jet are given in [18, p. 330].

Stability Parameter	Value	Units
Θ_0	0.0000	rad
U_0	400.0000	kt
X_{u_0}	−0.0074	1/s
$X_{T_{u_0}}$	0.0000	1/s
X_{α_0}	8.9782	ft-rad/s^2
$X_{\delta e_0}$	0.0000	ft-rad/s^2
Z_{u_0}	0.1390	1/s
Z_{α_0}	−445.7224	ft-rad/s^2
$Z_{\dot{\alpha}_0}$	−0.8705	ft-rad/s
Z_{q_0}	−1.8598	ft-rad/s
$Z_{\delta e_0}$	−42.1968	ft-rad/s^2
M_{u_0}	0.0011	rad/ft-s
$M_{T_{u_0}}$	−0.0002	1/ft-s
M_{α_0}	−7.4416	1/s^2
$M_{T_{\alpha_0}}$	0.0000	1/s^2
$M_{\dot{\alpha}_0}$	−0.4062	1/s
M_{q_0}	−0.9397	1/s
$M_{\delta e_0}$	− 17.6737	1/s^2

Next, using (96), the initial vertical-velocity slope response due to the 1-deg elevator step deflection $\delta e(t) = (0.017)\mathbf{1}(t - t_0^+)$ is given by

$$\delta\dot{v}_{\rm pv}(t_0^+) = 42.15 + 17.65\ell.$$

It follows that, at $\ell = \ell_{\rm IACR}$, $\delta\dot{v}_{\rm pv}(t_0^+) = 0$, and the number of zeros of the transfer function $G_{\delta\hat{v}_{\rm pv}/\delta\hat{e}}(s)$ decreases from three to two.

Likewise, using (97), the initial horizontal-velocity slope response due to the 1-deg step elevator deflection $\delta e(t) = (0.017)\mathbf{1}(t - t_0^+)$ is given by

$$\delta\dot{v}_{\rm ph}(t_0^+) = 17.65\eta.$$

It follows that $\eta = \eta_{\rm IACR}$, $\delta\dot{v}_{\rm ph}(t_0^+) = 0$, and the number of zeros of the transfer function $G_{\delta\hat{v}_{\rm ph}/\delta\hat{e}}(s)$ decreases from three to two.

To demonstrate the initial vertical-velocity perturbation $\delta v_{\rm pv}$ and initial horizontal-velocity perturbation $\delta v_{\rm ph}$ forward and aft of the IACR, we simulate $\delta v_{\rm pv}$ and $\delta v_{\rm ph}$ with the 1-deg elevator step deflection $\delta e(t) = (0.017)\mathbf{1}(t - t_0^+)$ for several values of ℓ and η. Figure 2 shows that, for $\ell = -20$ ft, $\delta v_{\rm pv}$ experiences initial undershoot, whereas, for $\eta = 20$ ft, $\delta v_{\rm ph}$ experiences initial undershoot, as defined in [1] and "Initial Undershoot." This initial undershoot is a consequence of the fact that, for all $\ell < \ell_{\rm IACR}$, the transfer function $G_{\delta\hat{v}_{\rm pv}/\delta\hat{e}}(s)$ has one nonminimum-phase zero, while, for all $\eta > \eta_{\rm IACR}$, the transfer function $G_{\delta\hat{v}_{\rm ph}/\delta\hat{e}}(s)$ has one nonminimum-phase zero. On the other hand, for all $\ell > \ell_{\rm IACR}$, the initial slope $\delta\dot{v}_{\rm pv}(0^+)$ is in the direction of the asymptotic vertical velocity, while, for all $\eta < \eta_{\rm IACR}$, the initial slope $\delta\dot{v}_{\rm ph}(0^+)$ is in the direction of the asymptotic horizontal velocity. Finally, for $\ell = \ell_{\rm IACR}$, the initial slope $\delta\dot{v}_{\rm pv}(0^+)$ is zero, and, for $\eta = \eta_{\rm IACR}$, the initial slope $\delta\dot{v}_{\rm ph}(0^+)$ is zero. Note that, at $p_{\rm IACR}$, the initial slopes of both $\delta\dot{v}_{\rm pv}(0^+)$ and $\delta\dot{v}_{\rm pv}(0^+)$ are zero, as a consequence of the definition of the IACR. Simulations over a longer time interval are shown in Figure 3.

Next, we apply the Routh test to determine the locations of the poles and zeros of (96); for details, see "Routh Test for Third- and Fourth-Order Polynomials." Following the same procedure for the horizontal-velocity perturbation transfer function (97) yields similar results. Thus, we analyze the vertical-velocity perturbation transfer function (96) as an example. Writing the denominator of (96) as $p(s)$, where $p(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$ is defined by

$$p(s) = s^4 + 2.01s^3 + 8.05s^2 + 0.085s + 0.068,$$

it follows that

$$a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 = 1.2353\ 1/\text{s}^6 > 0,$$

where the units 1/s^6 reflect the assumption that the leading coefficient of the monic polynomial $p(s)$ is dimensionless. Consequently, all of the poles of $G_{\delta\hat{v}_{\rm pv}/\delta\hat{e}}(s)$ are in the open left-half plane (OLHP).

To determine the zeros of the transfer function from the elevator deflection $\delta\hat{e}(s)$ to the vertical-velocity perturbation $\delta\hat{v}_{\rm pv}(s)$, we apply the Routh test to the numerator of (96). Defining the polynomial $q(s) = s^3 + a_2 s^2 + a_1 s + a_0$ by

$$G_{\delta\hat{v}_{\rm pv}/\delta\hat{e}}(s) = \frac{(42.15 + 17.65\ell)s^3 + (23854.0 + 11.3\ell)s^2 + (7740.6 + 0.1\ell)s + 157.2}{s^4 + 2.01s^3 + 8.05s^2 + 0.085s + 0.068}\ \text{ft/(s-rad)} \tag{96}$$

$$G_{\delta\hat{v}_{\rm ph}/\delta\hat{e}}(s) = -\frac{17.65\eta s^3 + (11.32\eta - 0.56)s^2 - (402.4 - 0.12\eta)s + 355.0}{s^4 + 2.01s^3 + 8.05s^2 + 0.085s + 0.068}\ \text{ft/(s-rad)} \tag{97}$$

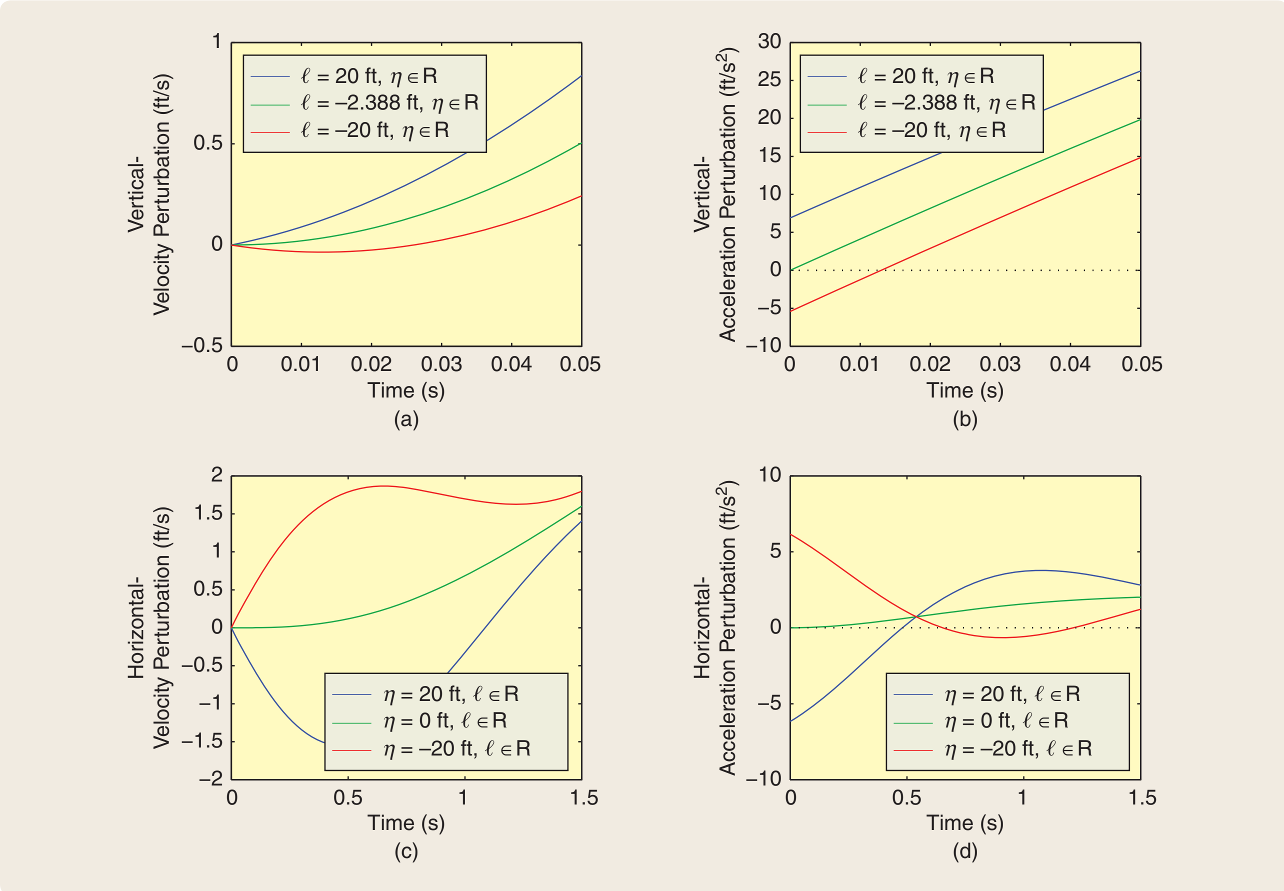

FIGURE 2 The response of the vertical-velocity perturbation $\delta v_{pv}(t)$ and the horizontal-velocity perturbation $\delta v_{ph}(t)$ of a typical business jet to the 1-deg elevator step deflection $\delta e(t) = 0.0171(t - t_0)$ at $t_0 = 0$ based on the data given in [18]. In (a) and (b), for $\ell < \ell_{IACR} = -2.388$ ft and $\eta \in \mathbb{R}$, where ℓ_{IACR} is the component of $\vec{r}_{p_{IACR}/c}$ along $\hat{k}_{AC}$. the transfer function $G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s)$ has one positive zero. For $\ell = \ell_{IACR}$ and all $\eta \in \mathbb{R}$, the initial slope of the vertical-velocity perturbation is zero, that is, the vertical-acceleration perturbation at t_0^+ is zero. In (c) and (d), for $\ell \in \mathbb{R}$ and $\eta > \eta_{IACR} = 0$ ft, where η_{IACR} is the component of $\vec{r}_{p_{IACR}/c}$ along $\hat{\imath}_{AC}$, the transfer function $G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s)$ has one positive zero. For $\ell \in \mathbb{R}$ and $\eta = \eta_{IACR}$, the initial slope of the horizontal-velocity perturbation is zero, that is, the horizontal-acceleration perturbation is zero at t_0^+, which indicates that $(\ell, \eta) = (\ell_{IACR}, \eta_{IACR})$ is the location of the IACR. This point is characterized by the vanishing zero, which, because of the increase in relative degree, yields zero initial velocity-perturbation slopes in both directions $\hat{\imath}_{AC}$ and $\hat{k}_{AC}$. Figure 3 shows the same simulations over a longer time interval.

$$q(s) \triangleq s^3 + \frac{177307 + 84.13\ell}{313.3 + 131.2\ell} s^2 + \frac{57535.6 + 0.8608\ell}{313.3 + 131.2\ell} s + \frac{1168.6}{313.3 + 131.2\ell},$$

it follows that

$$a_1 a_2 - a_0 = \left(\frac{57535.6 + 0.8608\ell}{313.3 + 131.2\ell}\right)\left(\frac{177307 + 84.13\ell}{313.3 + 131.2\ell}\right) - \frac{1168.6}{313.3 + 131.2\ell} = \frac{g(\ell)}{(313.3 + 131.2\ell)(0.11\ell + 0.27)} \text{ ft/s}, \quad (98)$$

where $g(\ell) \triangleq \ell^2 + 457.36\ell + 0.88$ ft^2. For $\ell > \ell_{IACR}$, it follows that $313.3 + 131.2\ell$, $0.11\ell + 0.27$, and $g(\ell)$ are positive, and thus (98) is positive. Therefore, for all $\ell > \ell_{IACR}$, all of

Routh Test for Third- and Fourth-Order Polynomials

All three roots of the cubic polynomial of $p(s) = s^3 + a_2 s^2 + a_1 s + a_0$ are in the open left-half plane (OLHP) if and only if

$$a_0, a_1, a_2 > 0$$

and

$$a_0 < a_1 a_2.$$

All four roots of the quartic polynomial $p(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$ are in the OLHP if and only if

$$a_0, a_1, a_2, a_3 > 0$$

and

$$a_0 a_3^2 + a_1^2 < a_1 a_2 a_3.$$

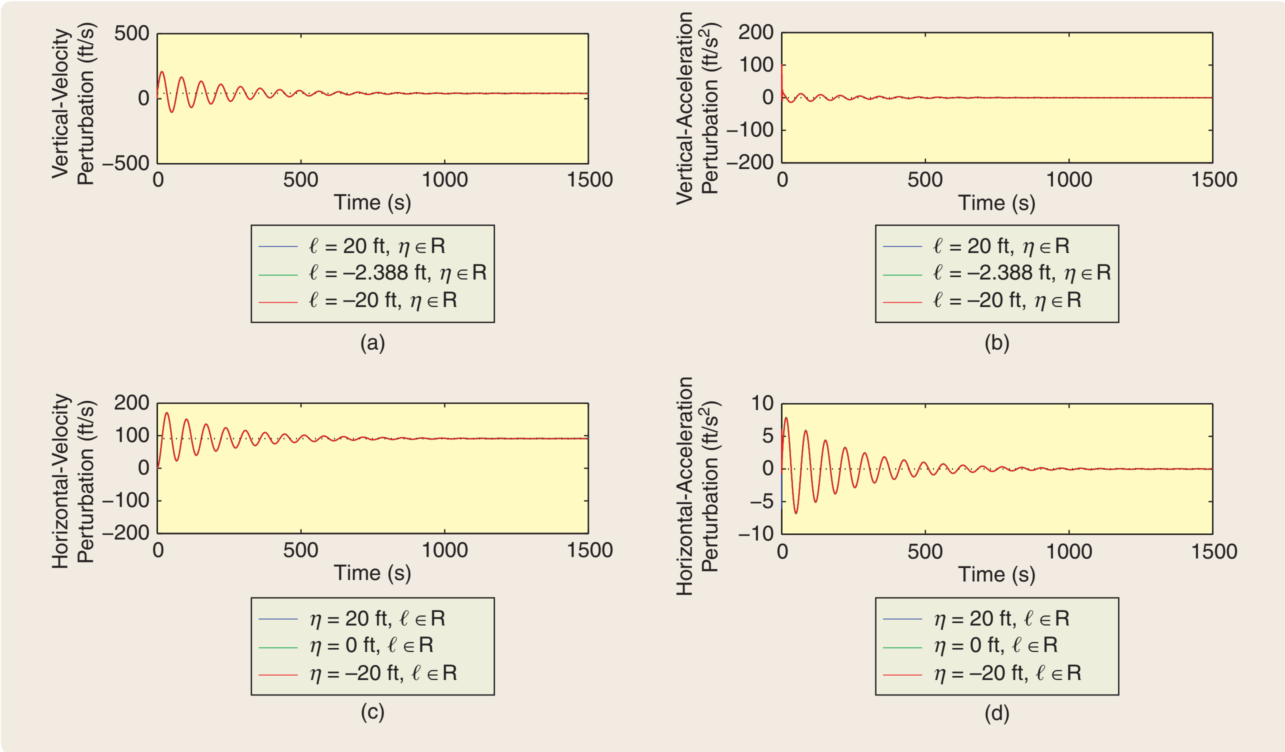

FIGURE 3 The responses of the vertical-velocity perturbation $\delta v_{\rm pv}(t)$, the vertical-acceleration perturbation $\delta \dot{v}_{\rm pv}(t)$, the horizontal-velocity perturbation $\delta v_{\rm ph}(t)$, and the horizontal-acceleration perturbation $\delta \dot{v}_{\rm ph}(t)$ of a typical business jet to the 1-deg elevator step deflection $\delta e(t) = 0.0171\mathbf{1}(t - t_0^+)$ at $t_0 = 0$ based on the aircraft parameters given in [18]. The asymptotic values are denoted by the dotted lines. Note that, for all values of (ℓ, η), the poles in (96) and (97) are close to the imaginary axis. Thus, $\delta v_{\rm pv}(t)$, $\delta \dot{v}_{\rm pv}(t)$, $\delta v_{\rm ph}(t)$, and $\delta \dot{v}_{\rm ph}(t)$ reach their asymptotic values slowly. As shown in Figure 2, the initial curvatures of $\delta v_{\rm pv}(t)$ and $\delta v_{\rm ph}(t)$ are different for different values of (ℓ, η). However, for all values of (ℓ, η), the vertical-velocity perturbation and the horizontal-velocity perturbation approach nonzero constants, while both acceleration perturbations approach zero. Note that, due to the magnitude of the transients, the traces in each plot are indistinguishable.

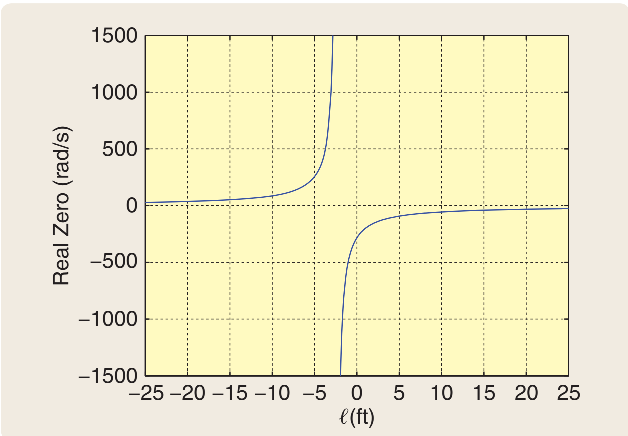

FIGURE 4 The real zero of a business jet based on data given in [18]. This plot shows the location of one of the real zeros of the numerator of the transfer function $G_{\delta\hat{v}_{\rm pv}/\delta\hat{e}}(s)$ from the elevator input δe to the vertical velocity $\delta v_{\rm pv}$ of the aircraft at p as a function of the component ℓ of the location of p along the direction $\hat{k}_{\rm AC}$. Note that negative values of ℓ correspond to locations of p aft of the aircraft's center of mass, that is, toward the tail of the aircraft. As ℓ is increased from -25 ft to $\ell_{\rm IACR} = -2.3$ ft, the zero moves along the real axis from 59.383 rad/s to ∞. This zero vanishes at $\ell_{\rm IACR}$. As ℓ is increased from $\ell_{\rm IACR}$ to 25 ft, the zero reappears at $-\infty$ and moves along the real axis to -49.606 rad/s. Figure 5 shows the locations of the remaining real zeros.

the roots of $q(s)$ are in the OLHP. On the other hand, for all $\ell < \ell_{\rm IACR}$, one zero of $G_{\delta\hat{v}_{\rm pv}/\delta\hat{e}}(s)$ is in the ORHP and two zeros are in the OLHP. This result follows from the first row of the Routh table, where one sign change appears.

Figure 4 shows that a real zero approaches ∞ as ℓ increases toward $\ell_{\rm IACR}$, whereas a real zero approaches $-\infty$ as ℓ decreases toward $\ell_{\rm IACR}$. This zero thus vanishes at the IACR. For $\ell \in [-25, 25]$ ft, Figure 5 shows the locations of the two remaining zeros of $G_{\delta\hat{v}_{\rm pv}/\delta\hat{e}}(s)$, which are real and do not vanish at the IACR.

For the horizontal-velocity perturbation $\delta v_{\rm ph}$, Figure 6(a) and (b) shows that, as η increases toward $\eta_{\rm IACR}$, one zero approaches $-\infty$, one zero approaches 717.7 rad/s, and the remaining zero approaches 0.88 rad/s. Figure 6(c) and (d) shows that, as η decreases toward $\eta_{\rm IACR}$, one zero approaches ∞, one zero approaches 717.7 rad/s, and the remaining zero approaches 0.88 rad/s. In Figure 6, (b) and (d) are zoom-in views near the origins of (a) and (c), respectively.

CONCLUSIONS

In this article, we used transfer function techniques to analyze the response of an aircraft to an elevator step deflection. We showed that the aircraft's initial response to an

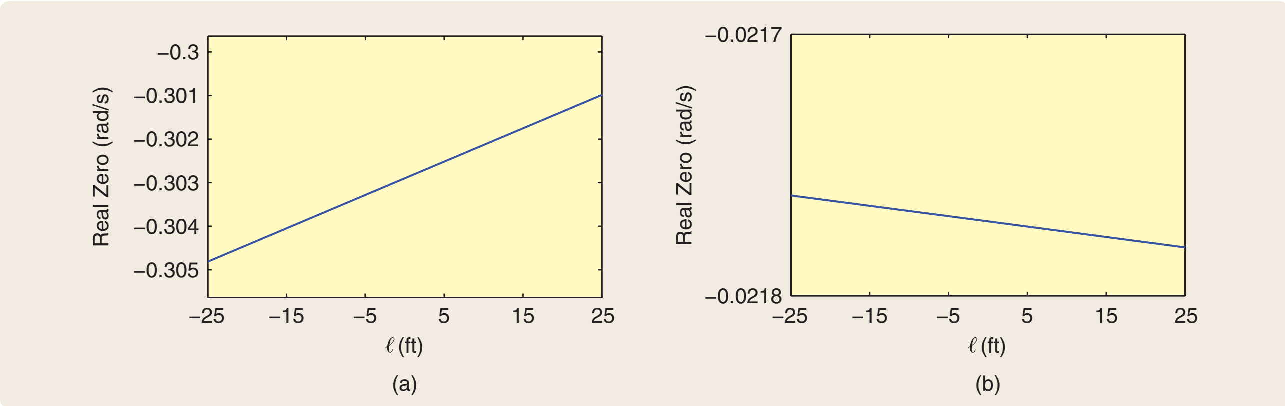

FIGURE 5 Zeros of the transfer function $G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s)$ For $\ell \in [-25, 25]$ ft, these plots show the locations of the two remaining zeros of $G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s)$, which are real and do not vanish at the IACR.

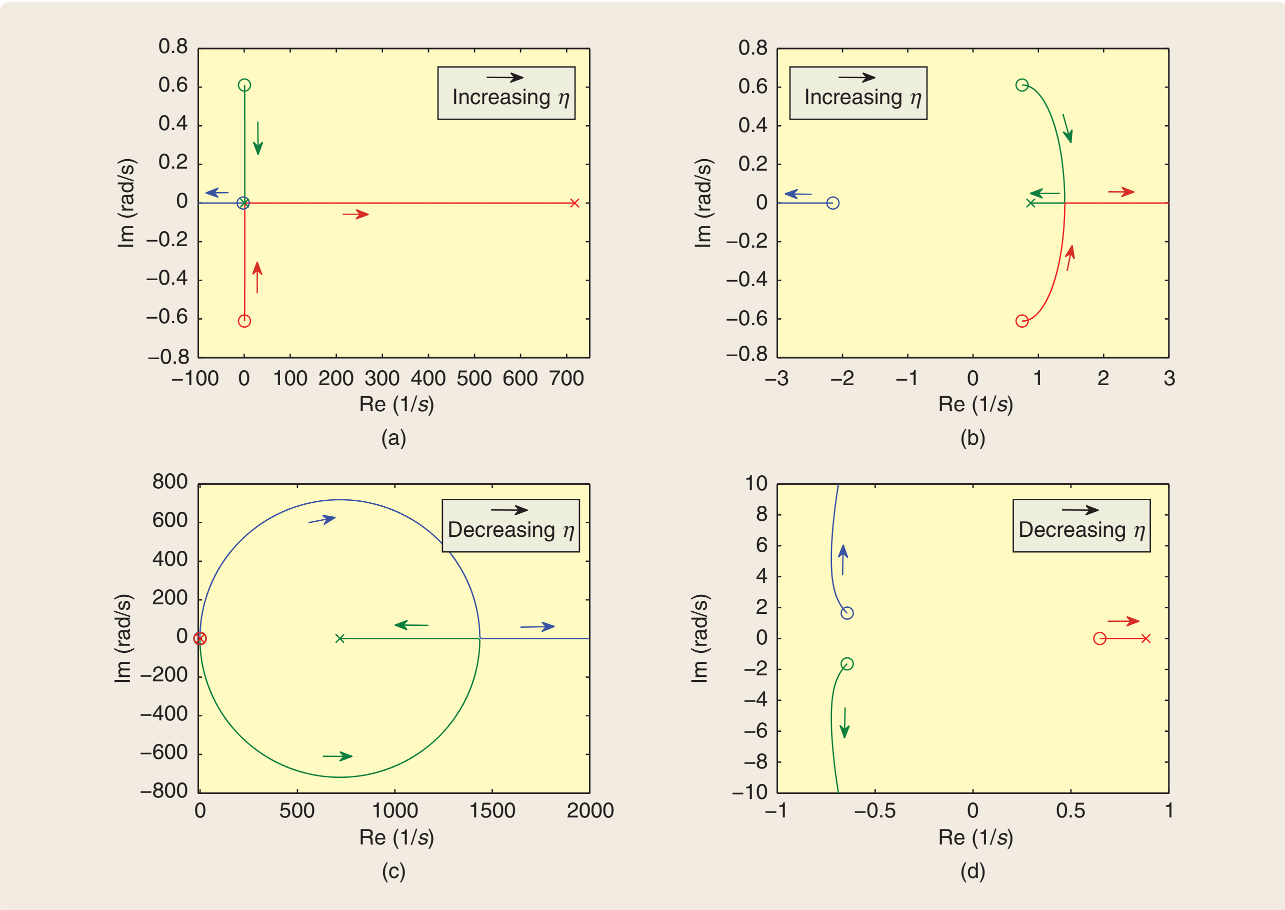

FIGURE 6 Zeros of the transfer function $G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s)$. (a) and (b) show the locations of the zeros of $G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s)$ for each location of p along $\hat{k}_{AC}$ parameterized by $\eta \in [-10 \text{ ft}, 0 \text{ ft})$, where $\eta_{IACR} = 0$ ft. The circles denote the zeros for $\eta = -10$ ft, while the crosses denote the asymptotic locations of the finite zeros as η increases toward η_{IACR}. As η increases toward η_{IACR}, one of the zeros approaches $-\infty$, while the finite zeros approach 0.88 rad/s and 717.7 rad/s. (c) and (d) show the locations of the zeros of $G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s)$ for each location of p along $\hat{k}_{AC}$ parameterized by $\eta \in [0 \text{ ft}, -10\text{ft})$, where $\eta_{IACR} = 0$ ft. The circles denote the zeros for $\eta = 10$ ft, while the crosses denote the asymptotic locations of the finite zeros as η decreases toward η_{IACR}. As η decreases toward η_{IACR}, one of the zeros approaches ∞, while the finite zeros approach 0.88 rad/s and 717.7 rad/s.

elevator step command is characterized by the IACR, which is the point at which the acceleration relative to O_{AC} with respect to F_E is zero. This point, which depends on the inertia and aerodynamics of the aircraft, is determined by deriving the linearized longitudinal equations of motion and evaluating the location of the IACR to first order. The initial vertical-velocity and horizontal-velocity response at the IACR to an elevator step deflection corresponds to an

The initial vertical-velocity and horizontal-velocity response at the IACR to an elevator step deflection corresponds to an increase in relative degree of the associated transfer functions at the IACR. This increase in relative degree reflects, in turn, the fact that zeros vanish at the IACR.

increase in relative degree of the associated transfer functions at the IACR. This increase in relative degree reflects, in turn, the fact that zeros vanish at the IACR.

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