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Stable Stabilization with H_2 and H_∞ Performance Constraints*

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Abstract

In this paper, we derive sufficient conditions for stable compensators with closed-loop H_2 and H_∞ performance criteria. We first generalize prior results in which H_2-suboptimal stable compensators were considered. Then using two different approaches, we derive conditions for constructing mixed-norm H_2/H_∞ stable compensators.

Key words: mixed-norm H_2/H_∞ control, strong stabilization, reduced-order control

1 Introduction

It was shown in [10] that a system is stabilizable by means of a stable compensator if and only if the real, unstable poles and zeros of the system satisfy the parity interlacing property. Consequently, certain plants can only be stabilized by unstable compensators. Issues related to strong stabilization were further studied by many authors, for example, [2] and [3]. In [7] and [8], the authors modified the closed-loop Lyapunov equation along with parameter optimization to guarantee controller stability. In this *a priori* approach controller stability is guaranteed prior to carrying out the optimization. An *a posteriori* approach, given in [5] and [9] is based upon the modified LQG Riccati equations to yield a stable compensator.

In this paper, we first generalize the results in [9] by using the *a posteriori* approach for H_2 strong stabilization. Specifically, we show that

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there exist several modifications to the LQG result that yield a stable compensator. Then using both the *a priori* and *a posteriori* approaches, we derive sufficient conditions for constructing stable compensators with the closed-loop system satisfying H_2 and H_∞ performance constraints.

It is important to note that the conditions given in this paper for stable stabilization are sufficient. The more difficult problem of determining conditions for stable stabilization that are both necessary and sufficient in the context of H_2 and $\mathrm{H}_2/\mathrm{H}_\infty$ control objectives remains an open problem for future research.

2 H_2-Suboptimal Strong Stabilization

In this section, we generalize the *a posteriori* approach of [9]. Consider the nth-order plant

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t), \tag{2.1}$$

$$y(t) = Cx(t) + D_2 w(t), \tag{2.2}$$

with performance variables

$$z(t) = E_1 x(t) + E_2 u(t), \tag{2.3}$$

where $w(t)$ is a standard white noise process. Using the n_cth-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{2.4}$$

$$u(t) = C_c x_c(t), \tag{2.5}$$

we obtain the closed-loop system

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t), \tag{2.6}$$

$$z(t) = \tilde{E}\tilde{x}(t), \tag{2.7}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix},$$

$$\tilde{E} \triangleq \begin{bmatrix} E_1 & E_2 C_c \end{bmatrix}.$$

The performance index is given by

$$J(A_c, B_c, C_c) = \lim_{t\to\infty} \mathcal{E}[x^{\mathrm{T}}(t) R_1 x(t) + u^{\mathrm{T}}(t) R_2 u(t)], \tag{2.8}$$

where '$\mathcal{E}$' denotes expectation and $R_1 \triangleq E_1^{\mathrm{T}} E_1$, $R_2 \triangleq E_2^{\mathrm{T}} E_2 > 0$. For convenience, we define $V_1 \triangleq D_1 D_1^{\mathrm{T}}$, $V_2 \triangleq D_2 D_2^{\mathrm{T}} > 0$, and $\Sigma \triangleq B R_2^{-1} B^{\mathrm{T}}$,

$\bar{\Sigma} \triangleq C^{\mathrm{T}}V_2^{-1}C$. For simplicity, we assume $R_{12} \triangleq E_1^{\mathrm{T}}E_2 = 0$ and $V_{12} \triangleq D_1D_2^{\mathrm{T}} = 0$.

The H_2 optimal control problem can be stated as follows: minimize the H_2 performance criterion given by (2.8), or, equivalently,

$$J(A_c, B_c, C_c) = \| \tilde{E}(sI - \tilde{A})^{-1}\tilde{D} \|_2^2 = \mathrm{tr}\, \tilde{Q}\tilde{R}, \tag{2.9}$$

subject to

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V}, \tag{2.10}$$

where

$$\tilde{R} \triangleq \tilde{E}^{\mathrm{T}}\tilde{E} = \begin{bmatrix} R_1 & 0 \\ 0 & C_c^{\mathrm{T}}R_2C_c \end{bmatrix}, \quad \tilde{V} \triangleq \tilde{D}\tilde{D}^{\mathrm{T}} = \begin{bmatrix} V_1 & 0 \\ 0 & B_cV_2B_c^{\mathrm{T}} \end{bmatrix}.$$

Throughout the paper we make the standard assumptions that both (A, B) and (A, D_1) are stabilizable and both (C, A) and (E_1, A) are detectable. In this case, it is well known that there exist nonnegative-definite matrices Q, P satisfying

$$0 = A^{\mathrm{T}}P + PA + R_1 - P\Sigma P, \tag{2.11}$$

$$0 = AQ + QA^{\mathrm{T}} + V_1 - Q\bar{\Sigma}Q, \tag{2.12}$$

such that the optimal full-order controller is given by

$$A_c = A + BC_c - B_cC, \tag{2.13}$$

$$B_c = QC^{\mathrm{T}}V_2^{-1}, \tag{2.14}$$

$$C_c = -R_2^{-1}B^{\mathrm{T}}P. \tag{2.15}$$

Although LQG controllers minimize the H_2 performance criterion (2.8) and guarantee closed-loop asymptotic stability, the resulting compensator is not necessarily either stable in the sense of Lyapunov or asymptotically stable. Hence, in this section we derive sufficient conditions that guarantee that the compensator is asymptotically stable along with suboptimal H_2 closed-loop performance.

For the full-order case $n_c = n$, Theorem 2.1 gives sufficient conditions for constructing stable controllers.

Theorem 2.1 *Let $R_1 > 0$ and let $\Omega(P, \hat{P}) \geq 0$ satisfy*

$$(A + \Sigma\hat{P})^{\mathrm{T}}P + P(A + \Sigma\hat{P}) + \Omega(P, \hat{P}) \geq 0, \tag{2.16}$$

for all $n \times n$ nonnegative-definite matrices P and $\hat{P}$. Furthermore, suppose there exist $n \times n$ nonnegative-definite matrices Q, P, and $\hat{P}$ satisfying

$$0 = AQ + QA^{\mathrm{T}} + V_1 - Q\bar{\Sigma}Q, \tag{2.17}$$

$$0 = A^{\mathrm{T}}P + PA + R_1 + \Omega(P, \hat{P}) - P\Sigma P, \tag{2.18}$$

$$0 = (A - Q\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P, \tag{2.19}$$

and let (A_c, B_c, C_c) be given by (2.13)-(2.15). Then A_c and $\tilde{A}$ are asymptotically stable.

Proof: Adding (2.18) to (2.19) and using (2.16) and $R_1 > 0$ yields $A_c^{\mathrm{T}}\hat{P} + \hat{P}A_c < 0$. Hence A_c is asymptotically stable. Now defining $\bar{R}_1 \triangleq R_1 + \Omega(P, \hat{P}) > 0$, it can be seen that (2.17) and (2.18) are in the form of the standard LQG Riccati equations. Thus $\tilde{A}$ is asymptotically stable. □

Note that by letting $\Omega(P, \hat{P}) = 0$, we recover the standard LQG result where A_c is not necessarily stable. Since the ordering induced by the cone of nonnegative-definite matrices is only a partial ordering, there does not exist a unique function $\Omega(\cdot, \cdot)$ satisfying (2.16). The next result gives nine such functions.

Proposition 2.1 *For arbitrary $\alpha, \beta > 0$, the following matrix functions $\Omega(P, \hat{P})$ satisfy (2.16) for all nonnegative-definite matrices $P, \hat{P}$:*

$$\begin{array}{ll}
(i) & \Omega(P,\hat{P}) = \alpha^2(A+\Sigma\hat{P})^{\mathrm{T}}(A+\Sigma\hat{P}) + \alpha^{-2}P^2, \\
(ii) & \Omega(P,\hat{P}) = [\alpha(A+\Sigma\hat{P}) - \alpha^{-1}P]^{\mathrm{T}}[\alpha(A+\Sigma\hat{P}) - \alpha^{-1}P], \\
(iii) & \Omega(P,\hat{P}) = [\alpha(A+\Sigma\hat{P}) - \alpha^{-1}I]^{\mathrm{T}}P[\alpha(A+\Sigma\hat{P}) - \alpha^{-1}I], \\
(iv) & \Omega(P,\hat{P}) = \alpha^2A^{\mathrm{T}}A + \alpha^{-2}P^2 + \beta^2P\Sigma^2P + \beta^{-2}\hat{P}^2, \\
(v) & \Omega(P,\hat{P}) = \alpha^2A^{\mathrm{T}}A + (\alpha^{-2}+\beta^{-2})P^2 + \beta^2\hat{P}\Sigma^2\hat{P}, \\
(vi) & \Omega(P,\hat{P}) = \alpha^2A^{\mathrm{T}}A + \alpha^{-2}P^2 + (\beta P - \beta^{-1}\hat{P})\Sigma(\beta P - \beta^{-1}\hat{P}), \\
(vii) & \Omega(P,\hat{P}) = (\alpha A - \alpha^{-1}I)^{\mathrm{T}}P(\alpha A - \alpha^{-1}I) + \beta^2P\Sigma^2P + \beta^{-2}\hat{P}^2, \\
(viii) & \Omega(P,\hat{P}) = (\alpha A - \alpha^{-1}I)^{\mathrm{T}}P(\alpha A - \alpha^{-1}I) + \beta^2\hat{P}\Sigma^2\hat{P} + \beta^{-2}P^2, \\
(ix) & \Omega(P,\hat{P}) = (\alpha A - \alpha^{-1}I)^{\mathrm{T}}P(\alpha A - \alpha^{-1}I) \\
 & \qquad\qquad + (\beta P - \beta^{-1}\hat{P})\Sigma(\beta P - \beta^{-1}\hat{P}).
\end{array}$$

Proof: The proof involves straightforward matrix manipulations and hence is omitted. □

For reduced-order dynamic compensation $n_c < n$, we recall from [6] the necessary conditions for H_2 optimality.

Theorem 2.2 *Let $n_c \le n$ and suppose (A_c, B_c, C_c) minimizes $J(A_c, B_c, C_c)$. Then there exist $n \times n$ nonnegative-definite matrices Q, $\hat{Q}$, P, $\hat{P}$ such that A_c, B_c, C_c are given by*

$$A_c = \Gamma(A - Q\bar{\Sigma} - \Sigma P)G^{\mathrm{T}}, \tag{2.20}$$

$$B_c = \Gamma QC^{\mathrm{T}}V_2^{-1}, \tag{2.21}$$

$$C_c = -R_2^{-1}B^{\mathrm{T}}PG^{\mathrm{T}}, \tag{2.22}$$

where Q, $\hat{Q}$, P, $\hat{P}$, Γ and G *satisfy*

$$0 = AQ + QA^{\mathrm{T}} + V_1 - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}}, \tag{2.23}$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^{\mathrm{T}} + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}}, \tag{2.24}$$

$$0 = A^{\mathrm{T}}P + PA + R_1 - P\Sigma P + \tau_\perp^{\mathrm{T}} P\Sigma P\tau_\perp, \tag{2.25}$$

$$0 = (A - Q\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P - \tau_\perp^{\mathrm{T}} P\Sigma P\tau_\perp, \tag{2.26}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c, \tag{2.27}$$

$$\hat{Q}\hat{P} = G^{\mathrm{T}}M\Gamma, \;\; \Gamma G^{\mathrm{T}} = I_{n_c}, \;\; M \in \mathcal{R}^{n_c\times n_c}, \tag{2.28}$$

$$\tau \triangleq G^{\mathrm{T}}\Gamma, \;\; \tau_\perp \triangleq I_n - \tau, \tag{2.29}$$

$$\hat{Q} = \tau\hat{Q}, \;\; \hat{P} = \hat{P}\tau. \tag{2.30}$$

Next, we modify Theorem 2.2 to construct reduced-order stable controllers.

Theorem 2.3 *Let* $R_1 > 0$ *and let* $\Omega(P,\hat{P}) \geq 0$ *satisfy (2.16) for all* $n \times n$ *nonnegative-definite matrices* P, $\hat{P}$. *Suppose there exist* $n \times n$ *nonnegative-definite matrices* Q, $\hat{Q}$, P, $\hat{P}$ *satisfying (2.27)-(2.30) and*

$$0 = AQ + QA^{\mathrm{T}} + V_1 - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}}, \tag{2.31}$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^{\mathrm{T}} + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}}, \tag{2.32}$$

$$0 = A^{\mathrm{T}}P + PA + R_1 + \Omega(P,\hat{P}) - P\Sigma P + \tau_\perp^{\mathrm{T}} P\Sigma P\tau_\perp, \tag{2.33}$$

$$0 = (A - Q\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P - \tau_\perp^{\mathrm{T}} P\Sigma P\tau_\perp. \tag{2.34}$$

Furthermore, let (A_c, B_c, C_c) *be given by (2.20)-(2.22) and assume that* $(\tilde{A}, \tilde{V})$ *is stabilizable and* (C_c, A_c) *is observable. Then* A_c *and* $\tilde{A}$ *are asymptotically stable.*

Proof: Adding (2.33) to (2.34) yields

$$0 = (A - Q\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + A^{\mathrm{T}}P + PA + R_1 + \Omega(P,\hat{P}),$$

which can be written as

$$\begin{aligned} 0 = {} & (A - Q\bar{\Sigma} - \Sigma P)^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma} - \Sigma P) + R_1 \\ & + (A + \Sigma\hat{P})^{\mathrm{T}}P + P(A + \Sigma\hat{P}) + \Omega(P,\hat{P}). \end{aligned}$$

Since (C_c, A_c) is observable, it follows that $P_2 \triangleq G\hat{P}G^{\mathrm{T}} > 0$ [6]. Using (2.16) and $P_2 > 0$ with the fact that $R_1 > 0$ and $\hat{P}\tau = \hat{P}$, it then follows that

$$A_c^{\mathrm{T}}P_2 + P_2A_c = -G[R_1 + \;(A + \Sigma\hat{P})^{\mathrm{T}}P + P(A + \Sigma\hat{P}) + \Omega(P,\hat{P})]G^{\mathrm{T}} < 0,$$

which implies that A_c is asymptotically stable. Now since $(\tilde{A}, \tilde{V})$ is stabilizable, it follows that $\tilde{A}$ is asymptotically stable. □

Remark 2.1 Setting $G = \Gamma = \tau = I$ in Theorem 2.3, which corresponds to $n_c = n$, we recover Theorem 2.1 for constructing full-order stable compensators.

Remark 2.2 Note that the modification $\Omega(P, \hat{P})$ for constructing stable compensators involves equations (2.18) and (2.33) for P and $\hat{P}$ as shown in both Theorem 2.1 and Theorem 2.3. By duality, we can similarly modify the Q and $\hat{Q}$ equations to obtain stable compensators.

3 Mixed-Norm $\mathrm{H}_2/\mathrm{H}_\infty$-Suboptimal Strong Stabilization

In this section, we generalize the results in the previous section to the case of $\mathrm{H}_2/\mathrm{H}_\infty$-suboptimal stable stabilization. The goal is to obtain an asymptotically stable compensator dynamics matrix A_c such that

(*i*) the closed-loop system (2.6), (2.7) is asymptotically stable;

(*ii*) the closed-loop transfer function $\tilde{G}_\infty(s) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{D} \\ \hline \tilde{E}_\infty & 0 \end{array}\right]$ from the disturbance $w(t)$ to performance variables $z(t) = E_{1\infty}x(t) + E_{2\infty}u(t)$, satisfies the constraint

$$\| \tilde{G}_\infty(s) \|_\infty \leq \gamma, \tag{3.1}$$

where $\gamma > 0$ is a given constant and $\tilde{E}_\infty \triangleq [E_{1\infty} \quad E_{2\infty}C_c]$; and

(*iii*) the H_2 performance measure (2.8) is minimized.

Following the *a posteriori* approach discussed in the previous section, we state the main result for reduced-order mixed-norm $\mathrm{H}_2/\mathrm{H}_\infty$ stable stabilization. For notational convenience, we define $R_{1\infty} \triangleq E_{1\infty}^{\mathrm{T}} E_{1\infty}$, $R_{2\infty} \triangleq E_{2\infty}^{\mathrm{T}} E_{2\infty} = \kappa R_2$ and $S \triangleq (I + \kappa\gamma^{-2}\hat{Q}P)^{-1}$, where $\kappa \geq 0$ is a given constant.

Theorem 3.1 *Let $R_1 > 0$ and let $\Phi(Q, \hat{Q}, P, \hat{P}) \geq 0$ satisfy*

$$\begin{aligned}[][A + \gamma^{-2}(Q+\hat{Q})R_{1\infty}]^{\mathrm{T}}P + P[A + \gamma^{-2}(Q + \hat{Q})R_{1\infty}]& \\ + S^{\mathrm{T}}P\Sigma\hat{P} + \hat{P}\Sigma PS + \Phi(Q, \hat{Q}, P, \hat{P}) \geq 0&,\end{aligned} \tag{3.2}$$

for all $n \times n$ nonnegative-definite matrices Q, $\hat{Q}$, P, $\hat{P}$. Suppose there exist $n \times n$ nonnegative-definite matrices Q, $\hat{Q}$, P, $\hat{P}$ satisfying (2.27)-(2.30) *and*

$$
\begin{aligned}
0 &= AQ + QA^{\mathrm{T}} + V_1 + \gamma^{-2}QR_{1\infty}Q - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}}, && (3.3)\\
0 &= (A - \Sigma PS + \gamma^{-2}QR_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma PS + \gamma^{-2}QR_{1\infty})^{\mathrm{T}} \\
&\quad + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}} + \gamma^{-2}\hat{Q}(R_{1\infty} + \kappa S^{\mathrm{T}}P\Sigma PS)\hat{Q}, && (3.4)\\
0 &= [A + \gamma^{-2}(Q + \hat{Q})R_{1\infty}]^{\mathrm{T}}P + P[A + \gamma^{-2}(Q + \hat{Q})R_{1\infty}] \\
&\quad + R_1 + \Phi(Q, \hat{Q}, P, \hat{P}) - S^{\mathrm{T}}P\Sigma PS + \tau_\perp^{\mathrm{T}}S^{\mathrm{T}}P\Sigma PS\tau_\perp, && (3.5)\\
0 &= (A - Q\bar{\Sigma} + \gamma^{-2}QR_{1\infty})^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma} + \gamma^{-2}QR_{1\infty}) \\
&\quad + S^{\mathrm{T}}P\Sigma PS - \tau_\perp^{\mathrm{T}}S^{\mathrm{T}}P\Sigma PS\tau_\perp. && (3.6)
\end{aligned}
$$

Furthermore, let

$$
\begin{aligned}
A_c &= \Gamma(A - Q\bar{\Sigma} - \Sigma PS + \gamma^{-2}QR_{1\infty})G^{\mathrm{T}}, && (3.7)\\
B_c &= \Gamma QC^{\mathrm{T}}V_2^{-1}, && (3.8)\\
C_c &= -R_2^{-1}B^{\mathrm{T}}PSG^{\mathrm{T}}, && (3.9)
\end{aligned}
$$

and assume that $(\tilde{A}, \tilde{V})$ is stabilizable and (C_c, A_c) is observable. Then $\tilde{A}$ and A_c are asymptotically stable, and $\| \tilde{G}_\infty(s) \|_\infty \leq \gamma$.

Proof: Defining

$$\hat{R} \triangleq R_1 + \Phi(Q, \hat{Q}, P, \hat{P}) > 0,$$

it can be seen that (3.3)-(3.6) correspond to the standard reduced-order $\mathrm{H}_2/\mathrm{H}_\infty$ controller [1] with R_1 replaced by $\hat{R}$ in (3.5) and hence $\| \tilde{G}_\infty(s) \|_\infty \leq \gamma$ is immediately satisfied. Using the fact that $(\tilde{A}, \tilde{V})$ is stabilizable, it follows that $\tilde{A}$ is asymptotically stable. To show that A_c is asymptotically stable, add (3.5) to (3.6) to obtain

$$
\begin{aligned}
0 &= (A - Q\bar{\Sigma} - \Sigma PS + \gamma^{-2}QR_{1\infty})^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma} - \Sigma PS + \gamma^{-2}QR_{1\infty}) \\
&\quad + R_1 + [A + \gamma^{-2}(Q + \hat{Q})R_{1\infty}]^{\mathrm{T}}P + P[A + \gamma^{-2}(Q + \hat{Q})R_{1\infty}] \\
&\quad + S^{\mathrm{T}}P\Sigma\hat{P} + \hat{P}\Sigma PS + \Phi(Q, \hat{Q}, P, \hat{P}).
\end{aligned}
$$

Since (C_c, A_c) is observable, it can be shown that $P_2 = G\hat{P}G^{\mathrm{T}} > 0$ [1]. Using (3.2), $\hat{P}\tau = \hat{P}$, and $P_2 > 0$, it follows that

$$
\begin{aligned}
&A_c^{\mathrm{T}}P_2 + P_2A_c = -G\{R_1 + [A + \gamma^{-2}(Q + \hat{Q})R_{1\infty}]^{\mathrm{T}}P + \\
&\quad P[A + \gamma^{-2}(Q + \hat{Q})R_{1\infty}] + S^{\mathrm{T}}P\Sigma\hat{P} + \hat{P}\Sigma PS + \Phi(Q, \hat{Q}, P, \hat{P})\}G^{\mathrm{T}} < 0,
\end{aligned}
$$

which implies that A_c is asymptotically stable. □

Proposition 3.1 *For arbitrary $\alpha, \beta > 0$, the following matrix functions $\Phi(Q, \hat{Q}, P, \hat{P})$ satisfy (3.2) for all nonnegative-definite matrices $Q, \hat{Q}, P, \hat{P}$:*

(i) $\Phi(Q,\hat{Q},P,\hat{P}) = \alpha^2[A+\gamma^{-2}(Q+\hat{Q})R_{1\infty}]^{\mathrm{T}}[A+\gamma^{-2}(Q+\hat{Q})R_{1\infty}]+ \alpha^{-2}P^2+\beta^2 S^{\mathrm{T}}P\Sigma^2 PS+\beta^{-2}\hat{P}^2$,

(ii) $\Phi(Q,\hat{Q},P,\hat{P}) = \alpha^2[A+\gamma^{-2}(Q+\hat{Q})R_{1\infty}]^{\mathrm{T}}[A+\gamma^{-2}(Q+\hat{Q})R_{1\infty}]+ \alpha^{-2}P^2+(\beta PS-\beta^{-1}\hat{P})^{\mathrm{T}}\Sigma(\beta PS-\beta^{-1}\hat{P})$,

(iii) $\Phi(Q,\hat{Q},P,\hat{P}) = [\alpha(A+\gamma^{-2}(Q+\hat{Q})R_{1\infty})-\alpha^{-1}I]^{\mathrm{T}}P[\alpha(A+\gamma^{-2}(Q+ \hat{Q})R_{1\infty})-\alpha^{-1}I]+\beta^2 S^{\mathrm{T}}P\Sigma^2 PS+\beta^{-2}\hat{P}^2$,

(iv) $\Phi(Q,\hat{Q},P,\hat{P}) = [\alpha(A+\gamma^{-2}(Q+\hat{Q})R_{1\infty})-\alpha^{-1}I]^{\mathrm{T}}P[\alpha(A+\gamma^{-2}(Q+ \hat{Q})R_{1\infty})-\alpha^{-1}I]+(\beta PS-\beta^{-1}\hat{P})^{\mathrm{T}}\Sigma(\beta PS-\beta^{-1}\hat{P})$.

Proof: The proof involves straightforward matrix manipulations and hence is omitted. □

Next, we specialize Theorem 3.1 to full-order dynamic compensation.

Corollary 3.1 (Full-Order $\mathbf{H_2/H_\infty}$ Strong Stabilization) *Let $R_1 > 0$ and let $\Phi(Q,\ \hat{Q},\ P,\ \hat{P}) \geq 0$ satisfy (3.2). Suppose there exist $n \times n$ nonnegative-definite matrices Q, $\hat{Q}$, P, $\hat{P}$ satisfying*

$$\begin{aligned}
0 &= AQ+QA^{\mathrm{T}}+V_1+\gamma^{-2}QR_{1\infty}Q-Q\bar{\Sigma}Q, && (3.10)\\
0 &= (A-\Sigma PS+\gamma^{-2}QR_{1\infty})\hat{Q}+\hat{Q}(A-\Sigma PS+\gamma^{-2}QR_{1\infty})^{\mathrm{T}}+Q\bar{\Sigma}Q\\
&\quad +\gamma^{-2}\hat{Q}(R_{1\infty}+\kappa S^{\mathrm{T}}P\Sigma PS)\hat{Q}, && (3.11)\\
0 &= [A+\gamma^{-2}(Q+\hat{Q})R_{1\infty}]^{\mathrm{T}}P+P[A+\gamma^{-2}(Q+\hat{Q})R_{1\infty}]+R_1\\
&\quad +\Phi(Q,\hat{Q},P,\hat{P})-S^{\mathrm{T}}P\Sigma PS, && (3.12)\\
0 &= (A-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty})^{\mathrm{T}}\hat{P}+\hat{P}(A-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty})\\
&\quad +S^{\mathrm{T}}P\Sigma PS. && (3.13)
\end{aligned}$$

Furthermore, let

$$\begin{aligned}
A_c &= A-Q\bar{\Sigma}-\Sigma PS+\gamma^{-2}QR_{1\infty}, && (3.14)\\
B_c &= QC^{\mathrm{T}}V_2^{-1}, && (3.15)\\
C_c &= -R_2^{-1}B^{\mathrm{T}}PS. && (3.16)
\end{aligned}$$

and assume that $(\tilde{A},\tilde{V})$ is stabilizable and (C_c, A_c) is observable. Then $\tilde{A}$ and A_c are asymptotically stable and $\|\tilde{G}_\infty(s)\|_\infty \leq \gamma$.

Proof: The result is a direct consequence of Theorem 3.1 by setting $n_c = n$ so that $\tau = \Gamma = G = I_n$. □

Remark 3.1 Note that by letting $\gamma \to \infty$ in Theorem 3.1 and Corollary 3.1, we recover the H_2 results obtained in Section 2 for both full- and reduced-order dynamic compensation. It is interesting to note that in contrast to the full-order H_2 case described in Section 2 and [9] which involves three matrix equations for constructing stable controllers, the full-order H_2/H_∞ problem involves four matrix equations for obtaining stable controllers.

Next, we use the cost modification approach proposed in [9] to design $\mathrm{H}_2/\mathrm{H}_\infty$ stable controllers. This approach seeks to minimize

$$\mathcal{J}(A_c, B_c, C_c) = \text{ tr } \mathcal{Q}\tilde{R} \tag{3.17}$$

subject to

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} + \tilde{V} + \Omega(\mathcal{Q}), \tag{3.18}$$

where

$$\tilde{R} \triangleq \begin{bmatrix} R_1 & 0 \\ 0 & C_c^{\mathrm{T}}R_2C_c \end{bmatrix}, \; \tilde{R}_\infty \triangleq \begin{bmatrix} R_{1\infty} & 0 \\ 0 & C_c^{\mathrm{T}}R_{2\infty}C_c \end{bmatrix},$$

$$\tilde{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & B_cV_2B_c^{\mathrm{T}} \end{bmatrix}.$$

As shown in [1], minimizing (3.17) subject to (3.18) with $\Omega(\mathcal{Q}) = 0$ guarantees closed-loop stability, H_∞ disturbance attenuation, and a worst-case bound on H_2 performance. To also guarantee stability of the compensator dynamics A_c, we set $\Omega(\mathcal{Q})$ to

$$\Omega(\mathcal{Q}) = \begin{bmatrix} 0 & 0 \\ 0 & Q_{12}^{\mathrm{T}}\bar{\Sigma}Q_{12} \end{bmatrix}, \tag{3.19}$$

as in [9], where $\mathcal{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2 \end{bmatrix}$. Now using the fixed-structure optimization approach outlined in [1] to minimize (3.17) subject to (3.18) with $\Omega(\mathcal{Q})$ given by (3.19) yields the following theorem.

Theorem 3.2 *Suppose there exist $n \times n$ nonnegative-definite Q, $\hat{Q}$, P, $\hat{P}$ satisfying (2.27)-(2.30) and*

$$\begin{aligned}
0 &= AQ + QA^{\mathrm{T}} + V_1 + \gamma^{-2}QR_{1\infty}Q - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}} + \hat{Q}\bar{\Sigma}\hat{Q}, && (3.20)\\
0 &= (A - \Sigma PS + \gamma^{-2}QR_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma PS + \gamma^{-2}QR_{1\infty})^{\mathrm{T}} + Q\bar{\Sigma}Q \\
&\quad - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}} - \hat{Q}\bar{\Sigma}\hat{Q} + \gamma^{-2}\hat{Q}(R_{1\infty} + \kappa S^{\mathrm{T}}P\Sigma PS)\hat{Q}, && (3.21)\\
0 &= [A + \gamma^{-2}(Q + \hat{Q})R_{1\infty}]^{\mathrm{T}}P + P[A + \gamma^{-2}(Q + \hat{Q})R_{1\infty}] + R_1 \\
&\quad + \hat{P}\hat{Q}\bar{\Sigma} + \bar{\Sigma}\hat{Q}\hat{P} - S^{\mathrm{T}}P\Sigma PS + \tau_\perp^{\mathrm{T}}S^{\mathrm{T}}P\Sigma PS\tau_\perp, && (3.22)\\
0 &= [A - (Q + \hat{Q})\bar{\Sigma} + \gamma^{-2}QR_{1\infty}]^{\mathrm{T}}\hat{P} + \hat{P}[A - (Q + \hat{Q})\bar{\Sigma} + \gamma^{-2}QR_{1\infty}] \\
&\quad + S^{\mathrm{T}}P\Sigma PS - \tau_\perp^{\mathrm{T}}S^{\mathrm{T}}P\Sigma PS\tau_\perp, && (3.23)
\end{aligned}$$

and let

$$\begin{aligned}
A_c &= \Gamma[A - (Q + \hat{Q})\bar{\Sigma} - \Sigma PS + \gamma^{-2}QR_{1\infty}]G^{\mathrm{T}}, && (3.24)\\
B_c &= \Gamma QC^{\mathrm{T}}V_2^{-1}, && (3.25)\\
C_c &= -R_2^{-1}B^{\mathrm{T}}PSG^{\mathrm{T}}. && (3.26)
\end{aligned}$$

If (A_c, B_c) is stabilizable, (C_c, A_c) is observable and $R_{1\infty} > 0$, then $\tilde{A}$ and A_c are asymptotically stable, $\| \tilde{G}_\infty(s) \|_\infty \leq \gamma$, and the H_2 *performance satisfies the bound*

$$J(A_c, B_c, C_c) \leq \text{ tr } [(Q + \hat{Q})R_1 + \hat{Q}S^{\mathrm{T}}P\Sigma PS]. \tag{3.27}$$

Proof: Since $\Omega(\mathcal{Q}) \geq 0$, it can be seen from (3.18) that if there exist nonnegative-definite Q and $\hat{Q}$ satisfying (3.18), then $\tilde{A}$ is asymptotically stable and $\| \tilde{G}_\infty(s) \|_\infty \leq \gamma$. Now applying the Lagrange multiplier method to minimize (3.17) subject to (3.18), yields

$$\begin{aligned} A_c Q_2 + Q_2 A_c^{\mathrm{T}} = -\Gamma[&(Q + \hat{Q})\bar{\Sigma}(Q + \hat{Q}) \\ &+ \gamma^{-2}\hat{Q}(R_{1\infty} + \kappa S^{\mathrm{T}}P\Sigma PS)\hat{Q}]\Gamma^{\mathrm{T}} < 0, \end{aligned}$$

which implies that A_c is asymptotically stable. Equations (3.20)-(3.26) follow from algebraic manipulations. See [1] for details of a similar proof. □

Remark 3.2 In the full-order case, set $n_c = n$ so that $\Gamma = G = \tau = I_n$ to obtain sufficient conditions for constructing stable $\mathrm{H}_2/\mathrm{H}_\infty$ compensators using the cost modification approach. Finally, relaxing the H_∞ constraint, i.e., $\gamma \to \infty$, Theorem 3.2 specializes to Theorem 4.2 of [9].

4 Illustrative Numerical Examples

We now use the numerical procedure proposed in [9] to solve the coupled matrix Riccati equations developed in the previous sections. This procedure involves first-order parameter variations of Q, $\hat{Q}$, P and $\hat{P}$ and is based upon Newton's method.

Example 1 Consider the two-mass system shown in Figure 1, where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

with disturbance weighting matrices given by

$$D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 68 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

and performance weighting matrices given by

$$E_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}.$$

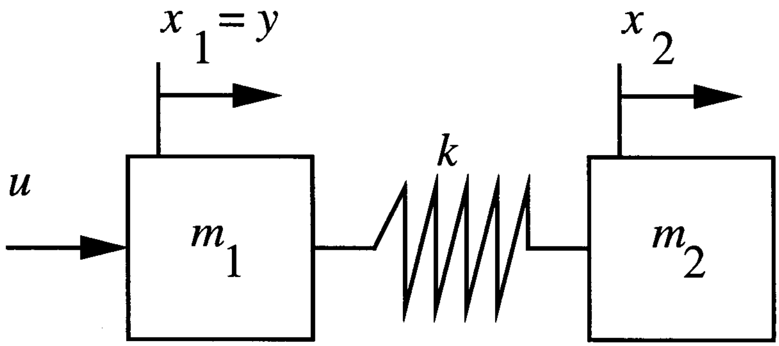

Figure 1: Two Mass System

Note that the results in [9] are based upon (vi) of Proposition 2.1 in which $\Omega(P, \hat{P})$ involves two parameters α and β. In this paper, we alternatively choose (i) of Proposition 2.1 to guarantee the asymptotic stability of A_c while implementing Theorem 2.1. Since there is only one parameter α preset in (i) of Proposition 2.1, the numerical procedure for obtaining an asymptotically stable compensator is somewhat simpler. The closed-loop poles are

$$\{-97.6052,\ -1.0277 \pm \jmath 2.8161,\ -2.4941 \pm \jmath 1.1604,\ -0.9728,\ -0.0033 \pm \jmath 1.0\}$$

while the eigenvalues of A_c are

$$\{-97.2507,\ -8.3768,\ -0.0003 \pm \jmath 1.0364\}.$$

The modified cost is 264.4246 while the LQG optimal cost is 261.6534. Hence the cost increment is 1.06%. Comparing to the cost increment 26.94 % of the modified design given in [9], we have a lower cost increment using (i) of Corollary 2.1 while implementing Theorem 2.1. In general, the choice of $\Omega(P, \hat{P})$ in Corollary 2.1 which leads to the minimal cost increment is problem dependent.

Our next example involves the design of a stable $\mathrm{H}_2/\mathrm{H}_\infty$ controller.

Example 2 Consider again the system given in Example 1 with plant dynamics A, B, C, disturbance matrix D_1 and sensor noise matrix D_2 with the performance weightings

$$E_1 = \begin{bmatrix} 10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

and H_∞ weighting matrices $R_{1\infty} = 0.05 R_1$ and $R_{2\infty} = 0$. Applying Corollary 3.1 with Φ given by (ii) in Proposition 3.1 and using the numerical

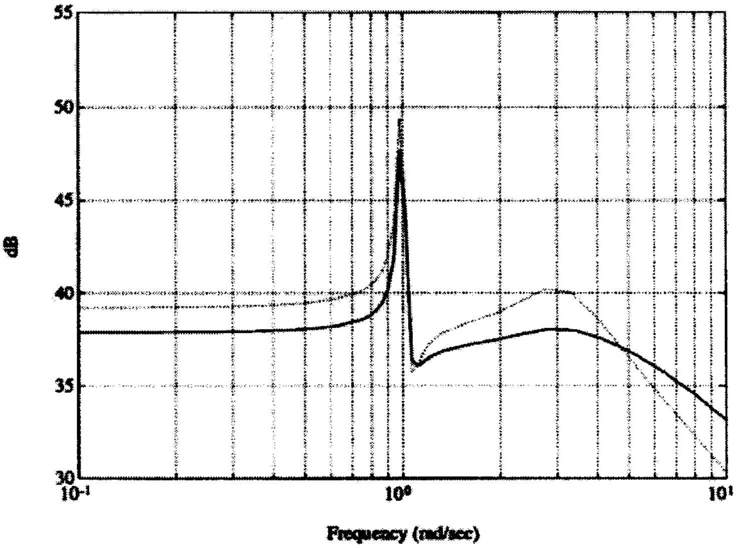

Figure 2: Maximum Singular Value of $\tilde{G}(s)$; LQG (Dotted Line) vs. H_2/H_∞ strong stabilization (Solid Line)

algorithm developed in [9], we obtain controllers for H_2/H_∞ strong stabilization. In the full-order case, the LQG cost of the H_2-optimal solution is 2.6165×10^4. The LQG design places the closed-loop poles at $\{-99.985, -2.4941 \pm \jmath 1.1604, -1.0277 \pm \jmath 2.8161, -1.0001, -0.0035 \pm \jmath 1.0\}$ while the poles of A_c are $\{-99.637, -8.399, .0003 \pm \jmath 1.0396\}$ which yields an unstable LQG controller. Now choosing $\alpha = 0.5$ and $\beta = 2$ in Proposition 3.1, the resulting stable H_2/H_∞ controller has closed-loop poles at $\{-96.3944, -5.575, -1.2085 \pm \jmath 2.5093, -2.6172, -1.0003, -0.0022 \pm \jmath 1.0\}$ with A_c poles $\{-95.7916, -12.2165, -0.0005 \pm \jmath 1.0282\}$. The total cost is 3.5453×10^4. Furthermore, the peak magnitude of the maximum singular value $\tilde{G}(s) = \tilde{E}(sI - \tilde{A})^{-1}\tilde{D}$ with the LQG gains is 49.368dB whereas the peak magnitude of $\| \tilde{G}(s) \|_\infty$ with the stable H_2/H_∞ gains is 47.62dB as shown in Figure 2.

5 Conclusion

Several alternative modifications of the full- and reduced-order H_2 optimality conditions have been shown to enforce the stability of the compensator. The modification terms, once selected, involve free parameters whose values can be determined by an iterative search.

This paper considered both *a priori* and *a posteriori* approaches, discussed in [9], to derive results for strong stabilization with H_2 and H_∞ performance measures. It is noted that in the full-order case with the *a posteriori* approach, the stable controller is characterized by three modified Riccati equations in the H_2 setting whereas four modified Riccati equations are needed to characterize the full-order stable mixed-norm H_2/H_∞ controller.

References

[1] D.S. Bernstein and W.M. Haddad. LQG control with an H_∞ performance bound: A Riccati equation approach, *IEEE Trans. Autom. Contr.* **34** (1989), 293-305.

[2] S. Boyd. Stable H_∞-Optimal Controllers, Technical Report No. L-104-87-1, Stanford University, 1987.

[3] G. Ganesh and J.B. Pearson. H^2-optimization with stable controllers, *Automatica* **25** (1989), 629-634.

[4] G. Ganesh and J.B. Pearson. Design of optimal control systems with stable feedback, *Proc. Amer. Contr. Conf.* Seattle, WA, (1986), 1969-1972.

[5] Y. Halevi, D.S. Bernstein and W.M. Haddad. On stable full-order and reduced-order LQG controllers, *Optimal Control Applications & Methods* **12** (1991), 163-172.

[6] D.C. Hyland and D.S. Bernstein. The optimal projection equations for fixed-order dynamic compensation, *IEEE Transactions on Automatic Control,* **29** (1984), 1034-1037.

[7] M.J. Jacobus. *Stable, Fixed-Order Dynamic Compensation with Applications to Positive Real and* H^∞*-Constrained Control Design.* Ph.D. Dissertation, University of New Mexico, 1990.

[8] M. Jacobus, M. Jamshidi, C. Abdallah, P. Dorato and D.S. Bernstein. Suboptimal strong stabilization using fixed-order dynamic compensation, *Proc. Amer. Contr. Conf.*, San Diego, CA (1990), 2659-2660.

[9] Y.W. Wang and D.S. Bernstein. H_2-suboptimal stable stabilization, *IEEE Contr. Dec. Conf.*, San Antonio, TX (1993), 1828-1829.

[10] D.C. Youla, J.J. Bongiorno, Jr. and C.N. Lu. Single-loop feedback-stabilization of linear multivariable dynamical plants, *Automatica* **10** (1974), 159-173.

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