On the Zeros, Initial Undershoot, and Relative Degree of Collinear Lumped-Parameter Structures

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This paper considers collinear lumped-parameter structures where each mass in the structure has a single degree of freedom. Specifically, we analyze the zeros and relative degree of the single-input, single-output (SISO) transfer function from the force applied to an arbitrary mass to the position, velocity, or acceleration of another mass. In particular, we show that every SISO force-to-motion transfer function of a collinear lumped-parameter structure has no positive (real open-right-half-plane) zeros. In addition, every SISO force-to-position transfer function of a spring-connected collinear lumped-parameter structure has no non-negative (real closed-right-half-plane) zeros. As a consequence, the step response does not exhibit initial undershoot. In addition, we derive an expression for the relative degree of SISO force-to-position transfer functions. The formula depends on the placement of springs and dashpots, but is independent of the values of the spring constants and damping coefficients. Next, we consider the special case of serially connected collinear lumped-parameter structures. In this case, we show that every SISO force-to-position transfer function of a serially connected collinear lumped-parameter structure is minimum phase, that is, has no closed-right-half-plane zeros. The proofs of these results rely heavily on graph-theoretic techniques.
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1 Introduction

One of the main impediments to achievable performance in linear time-invariant control systems is the presence of nonminimum phase zeros, which contribute to peaking in the sensitivity function and thus limit gain margins for robust stability [1–3].

The role of nonminimum phase zeros in limiting both achievable performance and robust stability suggests the importance of understanding the mechanisms that give rise to such zeros in flexible structures. This issue is discussed in [4], where it is shown that nonminimum phase zeros arise in noncolocated transfer functions for beam models when multiple mechanisms are involved in energy transfer, for example, bending and torsion. Furthermore, it is shown in [5,6] that nonminimum phase zeros arise in noncolocated transfer functions for beam models when the dynamics are dispersive, as occurs in bending. In addition, [7,8] present pole-zero interlacing results for multi-input, multi-output mass-spring-dashpot structures, while [9,10] characterizes the transmission zeros of lumped and distributed parameter structural systems as the resonant frequencies of a constrained subsystem.

For noise and vibration control applications, stability robustness benefits from sensor/actuator colocation, although achievable performance can be improved by separating the control input from the measurement signal [11]. For colocated hardware, it is well known that the transfer function is minimum phase; in fact, force-to-velocity transfer functions are positive real. However, for a noncolocated arrangement of control hardware it is of interest to know whether the resulting transfer function is minimum or nonminimum phase. In [12,13], a class of noncolocated mass-spring-dashpot structures are shown to be minimum phase. In particular, [12,13] considers the system $M\ddot{q}+C\dot{q}+Kq=Bu$ and $y=C_pq$, where $M\in\mathbb{R}^{N\times N}$ is positive definite, $C\in\mathbb{R}^{N\times N}$ and $K\in\mathbb{R}^{N\times N}$ are positive semidefinite, $B\in\mathbb{R}^{N\times l_u}$ has full row rank, $C\in\mathbb{R}^{l_y\times N}$ has full column rank, and $l_u\le l_y$. In this case, the system is minimum phase, that is, the transmission zeros from u to y are in the closed-left-half-plane, if there exists $\Gamma\in\mathbb{R}^{l_y\times l_u}$ such that $B=C^T\Gamma$ [12,13]. However, this condition is equivalent to sensor/actuator colocation in the single-input single-output (SISO) $l_u=l_y=1$ case. Thus, [12,13] do not consider the zero properties of noncolocated SISO mass-spring-dashpot structures. In the present paper, we use graph-theoretic tools to address the zero properties of noncolocated SISO mass-spring-dashpot structures. Finally, the robustness of the condition $B=C^T\Gamma$ is examined in [14], and minimum-phase discrete-time mass-spring-dashpot systems are considered in [15,16].

Graph theory can provide a systematic framework for analyzing structures and dynamical systems [17–22]. In particular, [17] uses graph theory to derive expressions for the component forces at the ends of individual structural members. In [18–20], the dynamic equations of motion for a class of rigid body systems are derived using graph-theoretic tools.

In the present paper, we use graph-theoretic results to examine the zeros and relative degree of collinear lumped-parameter structures. In particular, we consider lumped-parameter structures in which each mass has a single degree of freedom with arbitrary spring and dashpot connections to the remaining masses. For these structures, we show that every SISO force-to-motion transfer function has no positive (real open-right-half-plane) zeros. Furthermore, we show that every SISO force-to-position transfer function of a spring-connected collinear lumped-parameter structure has no non-negative (real closed-right-half-plane) zeros. As a consequence of this result, the step response of every asymptotically stable SISO force-to-position transfer function of a spring-connected collinear lumped-parameter structure does not exhibit initial undershoot. We also derive a formula for the relative degree of every SISO force-to-motion transfer function. The formula depends on the placement of springs and dashpots, but does not depend on the specific values of the spring constants and damping coefficients.

As a special case of the general collinear lumped-parameter structure, we consider asymptotically stable serially connected collinear lumped-parameter structures. More specifically, we analyze the zeros and relative degree of a string of masses intercon-

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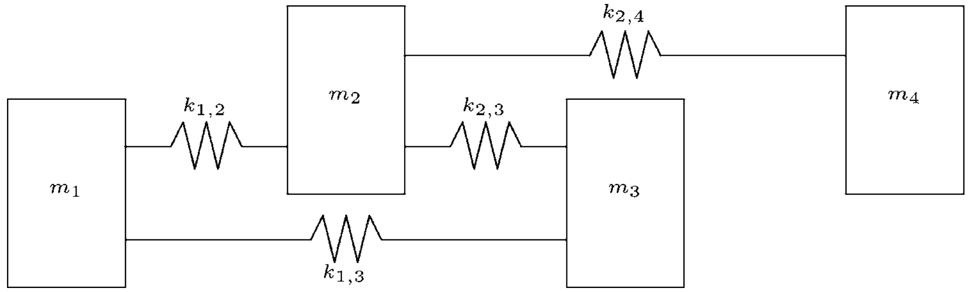

Fig. 1 4-mass structure with spring interconnections

nected by springs and dashpots. This structural configuration approximates a beam in compression, and is also useful for modeling the dynamics of a string of vehicles with pairwise control loops and for determining string stability for a convoy of automated vehicles [23–25]. In this special case, we show that all SISO force-to-position transfer functions of serially connected collinear lumped-parameter structures are minimum phase, that is, have neither real nor complex zeros in the closed-right-half-plane. We also obtain a specialization of the expression for the relative degree.

The contents of the paper are as follows. In Sec. 2, we present some basic graph-theoretic results used in later sections to analyze collinear lumped-parameter structures. In Sec. 3, we review the dynamics and stability properties of an N-mass collinear lumped-parameter structure. Section 4 presents results concerning the existence of positive and non-negative zeros in collinear lumped-parameter structures. The existence of complex non-minimum phase zeros in collinear lumped-parameter structures is considered in Sec. 5. Section 6 examines initial undershoot in collinear lumped-parameter structures, and an example is given in Sec. 7. In Sec. 8, we derive an expression for the relative degree of a SISO force-to-position transfer function. In Sec. 9, we consider the zeros and relative degree of asymptotically stable serially connected collinear lumped-parameter structures. Conclusions are given in Sec. 10. For the remainder of this paper, we only consider collinear lumped-parameter structures and refer to them as lumped-parameter structures.

2 Graph Theory Preliminaries

There is a natural relationship between lumped-parameter structures and graphs. The masses of a lumped-parameter structure represent the vertices of a graph, while the springs and dashpots connecting the masses represent the edges of the graph. For example, the 4-mass structure in Fig. 1 is represented by the 4-vertex graph in Fig. 2. Furthermore, the stiffnesses and damping coefficients can determine the weights associated with the edges. For example, the stiffnesses of the springs in Fig. 1 are the weights associated with the edges in Fig. 2. In this section, we present definitions and basic results that are useful for analyzing the zeros of lumped-parameter structures.

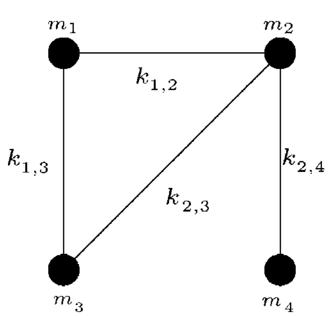

Fig. 2 4-vertex graph representing the 4-mass structure shown in Fig. 1

Let $V=\{v_1,v_2,\ldots,v_N\}$. The N elements of V are *vertices*, and V is the *vertex set*. Define $\mathcal{E}\triangleq\{\{v_i,v_j\}:v_i,v_j\in V, i\neq j\}$, and let $E\subseteq\mathcal{E}$. The elements of E are *edges*, and E is the *edge set*. Since the elements of E are sets and thus are unordered, the edges do not have directions. Thus all graphs considered in this paper are undirected graphs. Furthermore, we do not consider multiple edges since the elements of E are distinct, and we do not consider loops since, for all $i=1,\ldots,N,\{v_i,v_i\}\notin E$. The restriction to nonrepeated edges presents no loss of generality when analyzing lumped-parameter structures since multiple springs or dashpots connecting a pair of masses can be replaced by a single equivalent spring or dashpot. As discussed later, springs and dashpots are viewed as edges of different graphs.

DEFINITION 2.1. *$\mathcal{G}=(V,E)$ is a graph. If, in addition, for all $\{v_i,v_j\}\in E$, a weight $w_{i,j}>0$ is assigned to the edge $\{v_i,v_j\}$, then $\mathcal{G}$ is a weighted graph.*

DEFINITION 2.2. *Let $\mathcal{G}=(V,E)$ be a graph, and let $v_{n_0},v_{n_l}\in V$ be distinct. A* walk of length *l from v_{n_0} to v_{n_l} is the $(l+1)$-tuple $(v_{n_0},v_{n_1},\ldots,v_{n_l})\in V\times\cdots\times V$ such that, for all $i=1,2,\ldots,l,,\{v_{n_{i-1}},v_{n_i}\}\in E$.*

DEFINITION 2.3. *The graph $\mathcal{G}=(V,E)$ is connected if, for all distinct α, $\beta\in V$, there exists a walk between α and β.*

The *weighted adjacency matrix* $A_{\mathcal{G}}\in\mathbb{R}^{N\times N}$ associated with the weighted graph $\mathcal{G}=(V,E)$ is defined as

$$A_{\mathcal{G}}\triangleq\begin{bmatrix} 0 & w_{1,2} & w_{1,2} & \cdots & w_{1,N} \\ w_{2,1} & 0 & w_{2,3} & \cdots & w_{2,N} \\ w_{3,1} & w_{3,2} & 0 & & w_{3,N} \\ \vdots & \vdots & & \ddots & \vdots \\ w_{N,1} & w_{N,2} & w_{N,3} & \cdots & 0 \end{bmatrix}$$

where, for all $\{v_i,v_j\}\in E$, $w_{i,j}=w_{j,i}>0$ is the weight assigned to the edge $\{v_i,v_j\}$ and, for all $\{v_i,v_j\}\notin E$, $w_{i,j}=0$.

The *Laplacian matrix* $L_{\mathcal{G}}\in\mathbb{R}^{N\times N}$ associated with the weighted graph $\mathcal{G}=(V,E)$ is defined as

$$L_{\mathcal{G}}\triangleq D_{\mathcal{G}}-A_{\mathcal{G}}$$

where

$$D_{\mathcal{G}}\triangleq\operatorname{diag}\left(\sum_{i=2}^{N}w_{1,i},\sum_{i=1,i\neq 2}^{N}w_{2,i},\sum_{i=1,i\neq 3}^{N}w_{3,i},\ldots,\sum_{i=1}^{N-1}w_{N,i}\right)$$

The following definitions for Z-matrices and M-matrices can be found in [26].

DEFINITION 2.4. *$A\in\mathbb{R}^{n\times n}$ is a Z-matrix if every off-diagonal entry of A is nonpositive.*

DEFINITION 2.5. *$A\in\mathbb{R}^{n\times n}$ is an M-matrix if A can be written as $A=aI-B$, where $B\in\mathbb{R}^{n\times n}$ is a non-negative matrix and $a\in\mathbb{R}$ is greater than or equal to the spectral radius of B.*

Since the spectral radius of B is also an eigenvalue of B, it follows immediately that the M-matrix A is nonsingular if and only if a is greater than the spectral radius of B.

Next, we present two results concerning the Laplacian matrix. These results can be found in [27, p. 144] and [28, Theorem 3.16], respectively. In this paper, a matrix is positive semidefinite if it is symmetric with all nonnegative eigenvalues. Furthermore, a matrix is positive definite if it is symmetric with all positive eigenvalues.

LEMMA 2.1. *The Laplacian matrix $L_{\mathcal{G}}$ is a singular, positive-semidefinite M-matrix.*

LEMMA 2.2. *The graph $\mathcal{G}=(V,E)$ is connected if and only if its Laplacian matrix $L_{\mathcal{G}}$ is irreducible.*

The following result, which concerns nonsingular M-matrices, is given by [26, Theorem 2.7].

LEMMA 2.3. *Let $A_M \in \mathbb{R}^{N\times N}$ be an irreducible Z-matrix. Then A_M is a nonsingular M-matrix if and only if every entry, A_M^{-1} is positive*.

The next result of this section, which follows immediately from Lemmas 2.1–2.3, is used to analyze the zeros of lumped-parameter structures.

LEMMA 2.4. *Assume the graph $\mathcal{G}=(V,E)$ is connected. Then the Laplacian matrix $L_{\mathcal{G}}$ is an irreducible, singular, positive-semidefinite M-matrix. Furthermore, let $D\in\mathbb{R}^{N\times N}$, be positive definite and diagonal. Then $D+L_{\mathcal{G}}$ is an irreducible, nonsingular M-matrix, and thus every entry of $(D+L_{\mathcal{G}})^{-1}$ is positive.*

Now, we present results regarding the weighted adjacency matrices of two different graphs having the same vertex set; these results are used to analyze the relative degree of the transfer functions for lumped-parameter structures. Let $E_1\subseteq\mathcal{E}$ and $E_2\subseteq\mathcal{E}$, and consider the weighted graphs $\mathcal{G}_1=(V,E_1)$ and $\mathcal{G}_2=(V,E_2)$. Let $A_{\mathcal{G}_1}\in\mathbb{R}^{N\times N}$ and $A_{\mathcal{G}_2}\in\mathbb{R}^{N\times N}$ be the weighted adjacency matrices associated with the weighted graph $\mathcal{G}_1=(V,E_1)$ and $\mathcal{G}_2=(V,E_2)$, respectively.

LEMMA 2.5. *Consider the graph $\mathcal{G}_{12}=(V,E_1\cup E_2)$, and let $v_{n_0},v_{n_1}\in V$ be distinct. Then there exists a walk $(v_{n_0},v_{n_1},\ldots,v_{n_l})$ of length l on the graph $\mathcal{G}_{12}$ between v_{n_0} and v_{n_l} if and only if*

$$e_{n_l}^T\Theta_l\Theta_{l-1}\cdots\Theta_1 e_{n_0}>0 \tag{2.1}$$

where, for all $i=1,\ldots,l$, $\Theta_i\in\mathbb{R}^{N\times N}$ satisfies

$$\Theta_i\begin{cases}=A_{\mathcal{G}_1} & \text{if } \{v_{n_{i-1}},v_{n_i}\}\in E_1 \text{ and } \{v_{n_{i-1}},v_{n_i}\}\notin E_2\\ =A_{\mathcal{G}_2} & \text{if } \{v_{n_{i-1}},v_{n_i}\}\in E_2 \text{ and } \{v_{n_{i-1}},v_{n_i}\}\notin E_1\\ \in\{A_{\mathcal{G}_1},A_{\mathcal{G}_2}\} & \text{otherwise}\end{cases}$$

Proof. We prove this result by induction on the length l of the walk. First, assume that $l=1$. It follows from the definition of the adjacency matrix that $e_{n_1}^T A_{\mathcal{G}_1}e_{n_0}>0$ if and only if $\{v_{n_0},v_{n_1}\}\in E_1$ and $e_{n_1}^T A_{\mathcal{G}_2}e_{n_0}>0$ if and only if $\{v_{n_0},v_{n_1}\}\in E_2$. Therefore, there exists a walk of length 1 between v_{n_0} and v_{n_1} if and only if $e_{n_1}^T\Theta_1 e_{n_0}>0$.

Now, for induction, assume that the result holds for walks of length $l-1\geq 1$.

Next, we prove that the result holds for walks of length l. Note that there exists a walk $(v_{n_0},v_{n_1},\ldots,v_{n_l})$ of length l between v_{n_0} and v_{n_l} if and only if there exists a vertex $v_{n_1}\in V$ such that there exists a walk $(v_{n_1},v_{n_2},\ldots,v_{n_l})$ of length $l-1$ between v_{n_1} and v_{n_l}, and a walk (v_{n_0},v_{n_1}) of length 1 between v_{n_0} and v_1.

Since the result holds for walks of length of $l-1$, it follows that there exists a walk $(v_{n_1},v_{n_2},\ldots,v_{n_l})$ of length $l-1$ between v_{n_1} and v_{n_l} and only if

$$e_{n_l}^T\Theta_l\Theta_{l-1}\cdots\Theta_2 e_{n_1}>0$$

Furthermore, there exists a walk of length 1 between v_{n_0} and v_{n_1} and only if $e_{n_1}^T\Theta_1 e_{n_0}>0$.

Therefore, there exists a walk $(v_{n_0},v_{n_1},\ldots,v_{n_l})$ of length l between v_{n_0} and v_{n_l} if and only if there exists a vertex $v_{n_1}\in V$ such that

$$(e_{n_l}^T\Theta_l\Theta_{l-1}\cdots\Theta_2 e_{n_1})(e_{n_1}^T\Theta_1 e_{n_0})>0$$

Furthermore, note that

$$e_{n_l}^T\Theta_l\Theta_{l-1}\cdots\Theta_1 e_{n_0}=\sum_{k=1}^{N}(e_{n_l}^T\Theta_l\cdots\Theta_2 e_k)(e_k^T\Theta_1 e_{n_0})$$

Thus, $e_{n_l}^T\Theta_l\cdots\Theta_1 e_{n_0}>0$ if and only if there exists a $k\in\{1,\ldots,N\}$ such that $(e_{n_l}^T\Theta_l\cdots\Theta_2 e_k)(e_k^T\Theta_1 e_{n_0})>0$.

Therefore, there exists a walk $(v_{n_0},v_{n_2},\ldots,v_{n_l})$ of length l between v_{n_0} and v_{n_l} if and only if (2.1) is satisfied. □

The final result of this section is the contrapositive of the necessary condition of Lemma 2.5.

COROLLARY 2.1. *Consider the graph $\mathcal{G}_{12}=(V,E_1\cup E_2)$, and let $v_{n_0},v_{n_l}\in V$ be distinct. For all $i=1,\ldots,l$, let $E_{[i]}\in\{E_1,E_2\}$. Assume that there does not exist a walk $(v_{n_0},v_{n_1},\ldots,v_{n_l})$ of length l between v_{n_0} and v_{n_l} such that, for all $i=1,\ldots,l$, $\{v_{n_{i-1}},v_{n_i}\}\in E_{[i]}$. Then*

$$e_{n_l}^T\Theta_l\Theta_{l-1}\cdots\Theta_1 e_{n_0}=0$$

where, for $i=1,\ldots,l$,

$$\Theta_i\triangleq\begin{cases}A_{\mathcal{G}_1} & \text{if } E_{[i]}=E_1\\ A_{\mathcal{G}_2} & \text{if } E_{[i]}=E_2\end{cases}$$

3 Dynamics and Stability of Lumped-Parameter Structures

In this section, we present the dynamics of lumped-parameter structures. Specifically, we consider mass-spring-dashpot structures in which every pair of masses may or may not be connected by a spring, a dashpot, or a spring and dashpot in parallel. In addition, each mass may or may not be connected to a fixed wall by a spring, a dashpot, or a spring and dashpot in parallel. The N masses of the structure are denoted by $m_1,\ldots,m_N$. For all distinct $i,j=1,\ldots,N,c_{i,j}$ is the damping coefficient of the dashpot connecting the ith and jth masses, and $k_{i,j}$ is the stiffness of the spring connecting the ith and jth masses. If $c_{i,j}=0$ or $k_{i,j}=0$, then the ith and jth masses are not connected by a dashpot or a spring, respectively. For all $i=1,\ldots,N,c_i$ and k_i are the damping coefficient and spring stiffness, respectively, of the dashpot and spring connecting the ith mass to the wall. If $c_i=0$ or $k_i=0$, then the ith mass is not connected to the wall by a dashpot or a spring, respectively. For all $i=1,\ldots,N,q_i(t)$ is the position of the mass m_i, relative to an equilibria position, and $u_i(t)$ is the force acting on the mass m_i.

We consider lumped-parameter structures that consist of physical masses, springs, and dashpots, that is, the system must have positive masses, non-negative damping coefficients, and non-negative spring stiffnesses. Specifically, for all $i=1,\ldots,N$, $m_i>0$ and $c_i,k_i\geq 0$ and, for all distinct i, $j=1,\ldots,N,c_{i,j}\geq 0$, and $k_{i,j}\geq 0$.

For $N=3$, Figure 3 shows a 3-mass structure with all possible spring and dashpot connections. Figure 4 shows the possible spring and dashpot connections for the nth mass in an N-mass lumped-parameter structure.

The dynamics of the N-mass lumped-parameter structure are given by

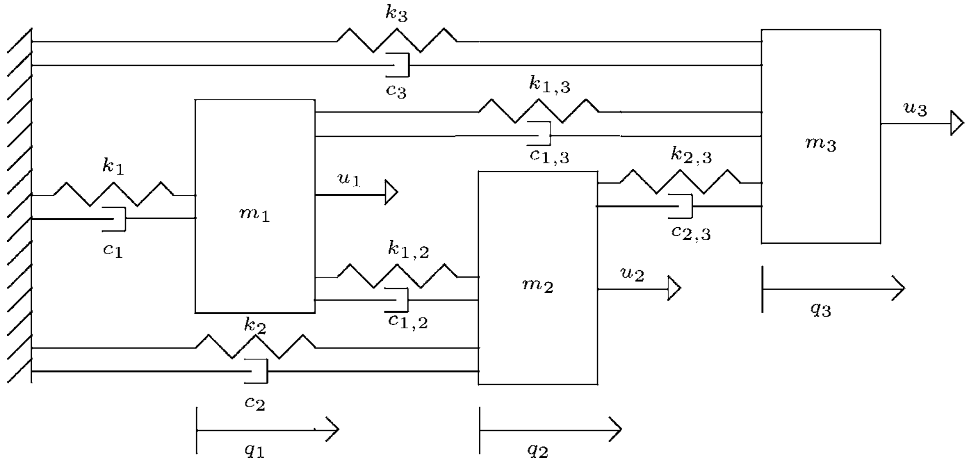

Fig. 3 3-mass structure with all possible spring and dashpot connections

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = u(t), \tag{3.1}$$

where

$$M \triangleq \mathrm{diag}(m_1, \ldots, m_N) \tag{3.2}$$

$$C \triangleq C_w + L_C \tag{3.3}$$

$$K \triangleq K_w + L_K \tag{3.4}$$

$$q(t) \triangleq [q_1(t) \cdots q_N(t)]^T \tag{3.5}$$

$$u(t) \triangleq [u_1(t) \cdots u_N(t)]^T \tag{3.6}$$

and

$$C_w \triangleq \mathrm{diag}(c_1, \ldots, c_N), \quad K_w \triangleq \mathrm{diag}(k_1, \ldots, k_N) \tag{3.7}$$

$$L_C \triangleq \begin{bmatrix} \sum_{j=2}^{N} c_{1,j} & -c_{1,2} & -c_{1,3} & \cdots & -c_{1,N} \\ -c_{2,1} & \sum_{j=1,j\neq 2}^{N} c_{2,j} & -c_{2,3} & \cdots & -c_{2,N} \\ -c_{3,1} & -c_{3,2} & \sum_{j=1,j\neq 3}^{N} c_{3,j} & & -c_{3,N} \\ \vdots & \vdots & & \ddots & \vdots \\ -c_{N,1} & -c_{N,2} & -c_{N,3} & \cdots & \sum_{j=1}^{N-1} c_{N,j} \end{bmatrix} \tag{3.8}$$

$$L_K \triangleq \begin{bmatrix} \sum_{j=2}^{N} k_{1,j} & -k_{1,2} & -k_{1,3} & \cdots & -k_{1,N} \\ -k_{2,1} & \sum_{j=1,j\neq 2}^{N} k_{2,j} & -k_{2,3} & \cdots & -k_{2,N} \\ -k_{3,1} & -k_{3,2} & \sum_{j=1,j\neq 3}^{N} k_{3,j} & & -k_{3,N} \\ \vdots & \vdots & & \ddots & \vdots \\ -k_{N,1} & -k_{N,2} & -k_{N,3} & \cdots & \sum_{j=1}^{N-1} k_{N,j} \end{bmatrix} \tag{3.9}$$

Note that the damping matrix C has a diagonal component C_w, which represents the dashpots connecting masses $m_1, \ldots, m_N$ to the wall, and a nondiagonal component L_C, which results from the dashpot interconnections among the N masses. Similarly, the stiffness matrix K has a diagonal component K_w and a nondiagonal component L_K. This distinction will be important in the following section when we associate several graphs with the lumped-parameter structure.

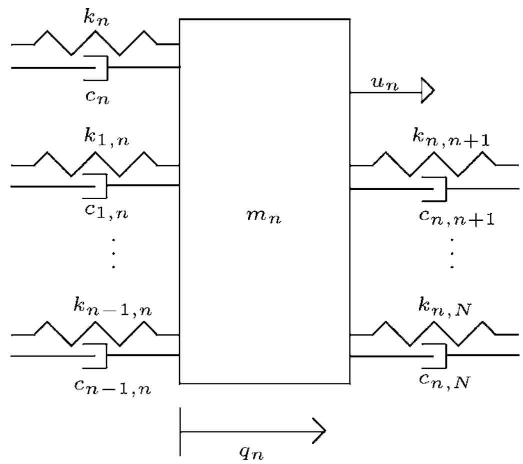

Fig. 4 Spring and dashpot connections to the *n*th mass of an *N*-mass structure

Next, we review the stability properties of lumped-parameter structures. The lumped-parameter structure (3.1)–(3.9) can be written as the first-order linear state-space system

$$\dot{x} = Ax + Bu \tag{3.10}$$

$$q = C_p x \tag{3.11}$$

where

$$A \triangleq \begin{bmatrix} 0 & I_N \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \quad B \triangleq \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \tag{3.12}$$

$$C_p \triangleq \begin{bmatrix} I_N & 0 \end{bmatrix} \tag{3.13}$$

and $x \triangleq [q_1 \cdots q_N \quad \dot{q}_1 \cdots \dot{q}_N]^T$. First, we characterize the eigenvalues of A in terms of the mass, stiffness, and damping matrices. This result can be found in [29, p. 203].

LEMMA 3.1. *Consider the lumped-parameter system* (3.10)–(3.13) *and let* $s \in \mathbb{C}$. *Then* $det(sI-A)=0$ *if and only if* $det(s^2M+sC+K)=0$.

The following stability result is a direct consequence of [30, Theorem 1].

LEMMA 3.2. *Consider the lumped-parameter system* (3.10)–(3.13). *Then the following statements are valid:*

(i) *A is Lyapunov stable if and only if $K+C$ is positive definite.*
(ii) *A is asymptotically stable if and only if (KM^{-1},C) is controllable and K is positive definite.*

4 Positive Zeros of Lumped-Parameter Structures

In this section, we analyze the zeros of lumped-parameter structures using the graph-theoretic tools presented in Sec. 2. To aid in our analysis, we associate three weighted graphs with the lumped-parameter structure (3.1)–(3.9). The masses are the vertices of the graphs, and the edges of the graphs are the dashpots, the springs, or the springs and dashpots. Furthermore, the weights associated with each edge are functions of the damping coefficient and the spring stiffness. Define the vertex set $V_M \triangleq \{m_1,\ldots,m_N\}$ and the edge sets

$$E_C \triangleq \{\{m_i,m_j\}: c_{i,j} > 0, \quad \text{where } i,j=1,\ldots,N \quad \text{and} \quad i \neq j\}$$

$$E_K \triangleq \{\{m_i,m_j\}: k_{i,j} > 0, \quad \text{where } i,j=1,\ldots,N \quad \text{and} \quad i \neq j\}$$

Define the weighted graphs $\mathcal{G}_C \triangleq (V_M,E_C)$, where, for all $\{m_i,m_j\} \in E_C$, the weight $c_{i,j}$ is assigned to the edge $\{m_i,m_j\}$, and $\mathcal{G}_K \triangleq (V_M,E_K)$, where, for all $\{m_i,m_j\} \in E_K$, the weight $k_{i,j}$ is assigned to the edge $\{m_i,m_j\}$.

By examining (3.8) and (3.9), it can be seen that L_C and L_K are the Laplacian matrices associated with $\mathcal{G}_C$ and $\mathcal{G}_K$, respectively. Then Lemma 2.1 implies that L_C and L_K are positive-semidefinite M-matrices. Since C_w and K_w are diagonal positive-semidefinite matrices, we conclude that the damping matrix $C=C_w+L_C$ and the stiffness matrix $K=K_w+L_K$ are positive semidefinite. We have thus proven the known fact that lumped-parameter structures of the form (3.1)–(3.9) have positive-semidefinite damping and stiffness matrices.

A notion of structural connectedness is needed to analyze the zeros of lumped-parameter structures. Roughly speaking, a lumped-parameter structure is structurally connected if it is a single structure rather than two or more disjoint structures. To formalize this idea, define the edge set $E_{CK} \triangleq E_C \cup E_K$.

DEFINITION 4.1. *The lumped-parameter structure* (3.1)–(3.9) *is structurally connected if the graph $\mathcal{G}_{CK} \triangleq (V_M,E_{CK})$ is connected.*

Definition 4.1 intuitively implies that (3.1)–(3.9) is structurally connected if and only if the force-to-motion transfer functions between every pair of masses is nonzero. Next, we characterize structural connectedness in terms of the damping and stiffness matrices.

LEMMA 4.1. *The lumped-parameter structure* (3.1)–(3.9) *is structurally connected if and only if $K+C$ is irreducible.*

Proof. Define the weighted graph $\mathcal{G}_{CK} \triangleq (V_M,E_{CK})$, where, for all $\{m_i,m_j\} \in E_{CK}$, the weight $k_{i,j}+c_{i,j}$ is assigned to the edge $\{m_i,m_j\}$. By examining (3.8) and (3.9), it follows that L_K+L_C is the Laplacian matrix associated with $\mathcal{G}_{CK}$. Lemma 2.2 implies that L_K+L_C is irreducible if and only if $\mathcal{G}_{CK}$ is connected. Since K_w+C_w is diagonal, it follows that $K+C=L_K+L_C+K_w+C_w$ is irreducible if and only if L_K+L_C is irreducible. Therefore, $K+C$ is irreducible if and only if $\mathcal{G}_{CK}$ is connected. □

We now present our main result on the zeros of lumped-parameter structures.

THEOREM 4.1. *Assume that the system* (3.10)–(3.13) *is structurally connected. Then, for all $i,j=1,\ldots,N$, the transfer function from $u_j(t)$ to $q_i(t)$ has no positive zeros. If, in addition, the graph $\mathcal{G}_K$ is connected and K is positive definite, then, for all $i,j=1,\ldots,N$, the transfer function from $u_j(t)$ to $q_i(t)$ has no nonnegative zeros.*

Proof. The transfer function from the force input u_j applied to mass m_j to the position q_i of mass m_i is

$$G_{i,j}(s) \triangleq e_i^T C_p(sI-A)^{-1}Be_j$$

where, for $i=1,\ldots,N, e_i$ is the ith column of I_N. Let $z>0$. For all $i,j=1,\ldots,N$,

$$G_{i,j}(z) = e_i^T C_p(zI-A)^{-1}Be_j = \begin{bmatrix} e_i^T & 0 \end{bmatrix} \times \begin{bmatrix} zI & -I \\ M^{-1}K & zI+M^{-1}C \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ M^{-1}e_j \end{bmatrix} \tag{4.1}$$

Since $z>0$, M is positive definite, and C and K are positive semidefinite, it follows that zI and $(1/z)M^{-1}$ (Mz^2+Cz+K) are nonsingular. Hence, Proposition 2.8.7 of [29] implies that

$$\begin{bmatrix} zI & -I \\ M^{-1}K & zI+M^{-1}C \end{bmatrix}^{-1} = \begin{bmatrix} \sharp & (zI)^{-1}(zI+M^{-1}C+M^{-1}K(zI)^{-1})^{-1} \\ \sharp & \sharp \end{bmatrix} = \begin{bmatrix} \sharp & (z^2I+M^{-1}Cz+M^{-1}K)^{-1} \\ \sharp & \sharp \end{bmatrix} \tag{4.2}$$

where $\sharp$ denotes an inconsequential entry. Combining (4.1) and (4.2) yields

$$\begin{aligned} G_{i,j}(z) &= \begin{bmatrix} e_i^T & 0 \end{bmatrix} \begin{bmatrix} \sharp & (z^2I+M^{-1}Cz+M^{-1}K)^{-1} \\ \sharp & \sharp \end{bmatrix} \begin{bmatrix} 0 \\ M^{-1}e_j \end{bmatrix} \\ &= e_i^T(Mz^2+Cz+K)^{-1}e_j \\ &= e_i^T[(Mz^2+C_wz+K_w)+(L_Cz+L_K)]^{-1}e_j \end{aligned} \tag{4.3}$$

Next, it follows from (3.8) and (3.9) that L_Cz+L_K is the Laplacian matrix of the weighted graph $\mathcal{G}_{CK} \triangleq (V_M,E_{CK})$, where, for all $\{m_i,m_j\} \in E$, the weight $c_{i,j}z+k_{i,j}$ is associated with the edge $\{m_i,m_j\}$. Furthermore, $Mz^2+C_wz+K_w$ is a diagonal positive-definite matrix. Since $\mathcal{G}_{CK}$ is connected and $Mz^2+C_wz+K_w$ is a diagonal positive-definite matrix, Lemma 2.4 implies that $(Mz^2+C_wz+K_w)+(L_Cz+L_K)$ is an irreducible, nonsingular M-matrix and every entry of $(Mz^2+Cz+K)^{-1}=[(Mz^2+C_wz+K_w)+(L_Cz+L_K)]^{-1}$ is positive. Therefore, for all $i,j=1,\ldots,N$, it follows from (4.3) that $G_{i,j}(z)>0$, and thus z is not a zero of $G_{i,j}(s)$.

Now consider the case $z=0$. It follows from (4.1) that

$$G_{i,j}(0) = \begin{bmatrix} e_i^T & 0 \end{bmatrix} \begin{bmatrix} 0 & -I \\ M^{-1}K & M^{-1}C \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ M^{-1}e_j \end{bmatrix} \tag{4.4}$$

Since $M>0$ and $K>0$ it follows that $M^{-1}K$ is nonsingular. Hence, Fact 2.15.2 of [29] implies that

$$\begin{bmatrix} 0 & -I \\ M^{-1}K & M^{-1}C \end{bmatrix}^{-1} = \begin{bmatrix} \sharp & K^{-1}M \\ \sharp & \sharp \end{bmatrix} \tag{4.5}$$

Combining (4.4) and (4.5) yields

$$G_{i,j}(0) = e_i^TK^{-1}e_j = e_i^T(K_w+L_K)^{-1}e_j$$

Since $\mathcal{G}_K$ is connected, it follows from Lemma 2.1 and Lemma 2.2 that L_K is an irreducible, singular, M-matrix. Since K_w is diagonal, it follows that $K=K_w+L_K$ is an irreducible M-matrix. Furthermore, since K is positive definite, it follows that K is an irreducible, nonsingular, M-matrix. It then follows from Lemma 2.3 that every entry of K^{-1} is positive. Therefore, $G_{i,j}(0)=e_i^TK^{-1}e_j$ is positive and $z=0$ is not a zero of $G_{i,j}(s)$. □

COROLLARY 4.1. *Assume that the system* (3.10)–(3.13) *is structurally connected. Then, for all $i,j=1,\ldots,N$, the transfer functions from $u_j(t)$ to $\dot{q}_i(t)$ and from $u_j(t)$ to $\ddot{q}_i(t)$ have no positive zeros.*

5 Complex Nonminimum Phase Zeros of Lumped-Parameter Structures

Theorem 4.1 and Corollary 4.1 guarantee that every force-to-motion transfer function of a structurally connected lumped-parameter structure has no positive zeros. However, these results do not guarantee that every force-to-motion transfer function is minimum phase; the force-to-motion transfer functions can have complex zeros in the open right half plane. In fact, for $N=3$, there exists lumped-parameter structures (3.1)–(3.9) that are structurally connected and have a nonminimum phase force-to-motion transfer function. Specifically, consider the 3-mass lumped-parameter structure in Fig. 3, where $m_1=m_2=m_3=1$ kg, $k_1=k_3=0$ kg/s^2, $k_2=k_{1,2}=k_{1,3}=k_{2,3}=5$ kg/s^2, $c_1=c_2=c_3=c_{1,2}=c_{2,3}=0$ kg/s, and $c_{1,3}=5$ kg/s. This system is structurally connected since the graph $\mathcal{G}_{CK}$ is connected. Furthermore, the transfer function from $u_1(t)$ to $q_3(t)$, given by

$$G_{3,1}(s) \triangleq \frac{5s^3+5s^2+75s+100}{s^6+10s^5+35s^4+200s^3+325s^2+250s+375}$$

is nonminimum phase. The zeros of $G_{3,1}(s)$ are approximately $0.150\pm j3.92$ and -1.30. In fact, for all $N\geq 3$, there exists a lumped-parameter structure (3.1)–(3.9) that is structurally connected and has a nonminimum phase force-to-motion transfer function.

6 Initial Undershoot in Lumped-Parameter Structures

Initial undershoot describes the qualitative behavior of the step response of a transfer function. A system has initial undershoot if the step response initially moves in the direction that is opposite to its asymptotic value. We now define initial undershoot and state a result classifying the existence of initial undershoot. The definition and result are given in [31–33].

DEFINITION 6.1. *Let $H(s)$ be a single-input single-output asymptotically stable transfer function with relative degree $r>0$. Let $y(t)$ be the step response of $H(s)$. Assume that $H(0)\neq 0$. Then the step response of $H(s)$ has initial undershoot if $y^{(r)}(0)\lim_{t\to\infty}y(t)<0$.*

LEMMA 6.1. *Let $H(s)$ be a single-input single-output asymptotically stable transfer function with relative degree $r>0$. Assume that $H(0)\neq 0$. Then the step response of $H(s)$ has initial undershoot if and only if $H(s)$ has an odd number of positive zeros.*

The main result of this section addresses the existence of initial undershoot in a force-to-position transfer function of an asymptotically stable lumped-parameter structure.

THEOREM 6.1. *Assume that the system* (3.10)–(3.13) *is structurally connected. Furthermore, assume that A is asymptotically stable and the graph $\mathcal{G}_K$ is connected. Then, for all $i,j=1,\ldots,N$, the step response of the transfer function from $u_j(t)$ to $q_i(t)$ does not exhibit initial undershoot.*

Proof. Let $G_{i,j}(s)$ be the transfer function from the force input on mass m_j to the position of mass m_i. Since A is asymptotically stable, Lemma 3.2 implies $K>0$. It follows from Theorem 4.1 that $G_{i,j}(s)$ has no non-negative zeros. Therefore, $G_{i,j}(0)\neq 0$ and $G_{i,j}(s)$ has no positive zeros. Since A is asymptotically stable, $G_{i,j}(s)$ is asymptotically stable. Since $G_{i,j}(s)$ is an asymptotically stable transfer function, $G_{i,j}(0)\neq 0$, and $G_{i,j}(s)$ has no positive zeros, it follows from Lemma 6.1 that $G_{i,j}(s)$ does not exhibit initial undershoot. □

7 Example: 3-Mass Lumped-Parameter Structure

Consider the structurally connected 3-mass structure shown in Fig. 3 whose dynamics are given by (3.1)–(3.9), where $N=3$. For this example, the masses are $m_1=m_2=m_3=5$ kg; the spring stiffnesses are $k_1=1$ kg/s^2, $k_2=2$ kg/s^2, $k_3=3$ kg/s^2, $k_{1,2}=12$ kg/s^2, $k_{1,3}=13$ kg/s^2, and $k_{2,3}=23$ kg/s^2; and the damping coefficients are $c_1=10$ kg/s, $c_2=20$ kg/s, $c_3=30$ kg/s, $c_{1,2}=120$ kg/s, $c_{1,3}=130$ kg/s, and $c_{2,3}=23$ kg/s.

The transfer functions from u_1 to q_1, from u_1 to q_2, and from u_1 to q_3 are

$$G_{1,1}(s) \triangleq \frac{0.2s^4+30.4s^3+734.24s^2+146.24s+7.312}{s^6+204s^5+10328.4s^4+39813.6s^3+11428.68s^2+1132.56s+37.752}$$

$$G_{2,1}(s) \triangleq \frac{4.8s^3+614.08s^2+122.72s+6.136}{s^6+204s^5+10328.4s^4+39813.6s^3+11428.68s^2+1132.56s+37.752}$$

$$G_{3,1}(s) \triangleq \frac{5.2s^3+606.12s^2+121.12s+6.056}{s^6+204s^5+10328.4s^4+39813.6s^3+11428.68s^2+1132.56s+37.752}$$

respectively. The zeros of $G_{1,1}(s)$ are approximately -121.9, -29.86, -0.1001, and -0.1003. The zeros of $G_{2,1}(s)$ are approximately -127.7, -0.1000, and -0.1001. The zeros of $G_{3,1}(s)$ are approximately -116.4, -0.1000, and -0.1001. Therefore, $G_{1,1}(s)$, $G_{2,1}(s)$, and $G_{3,1}(s)$ have no non-negative zeros as guaranteed by Theorem 4.1. Furthermore, Theorem 6.1 implies that the step responses of $G_{1,1}(s)$, $G_{2,1}(s)$, and $G_{3,1}(s)$ do not have initial undershoot. Figure 5 verifies that the step responses do not have initial undershoot.

8 Relative Degree of Lumped-Parameter Structures

In this section, we analyze the relative degree of the lumped-parameter structure (3.10)–(3.13). If we assume that (3.10)–(3.13) is structurally connected, then the ith and jth masses are connected by means of at least one sequence of springs and dashpots. To calculate the relative degree of the transfer function from $u_j(t)$ to $q_i(t)$, we consider all sequences of springs and dashpots that connect m_j to m_i. Equivalently, we consider all walks on the graph $\mathcal{G}_{CK}=(V_M,E_{CK})$ from m_j to m_i. For all $i,j=1,\ldots,N$ such that $i\neq j$, define

$$\Omega_{i,j} \triangleq \{\omega : \omega \text{ is a walk on } \mathcal{G}_{CK} \text{ from } m_j \text{ to } m_i\}$$

and, for all $i=j=1,\ldots,N$, define $\Omega_{i,j}\triangleq\emptyset$.

Let $\omega=(m_{n_0},m_{n_1},\ldots,m_{n_l})$ be a walk of length l on $\mathcal{G}_{CK}$ from m_{n_0} to m_{n_l}. Now, let $n_C(\omega)$ denote the number of edges in ω that are dashpots only, let $n_K(\omega)$ denote the number of edges in ω that are springs only, and let $n_{CK}(\omega)$ denote the number of edges in ω that are springs and dashpots, that is,

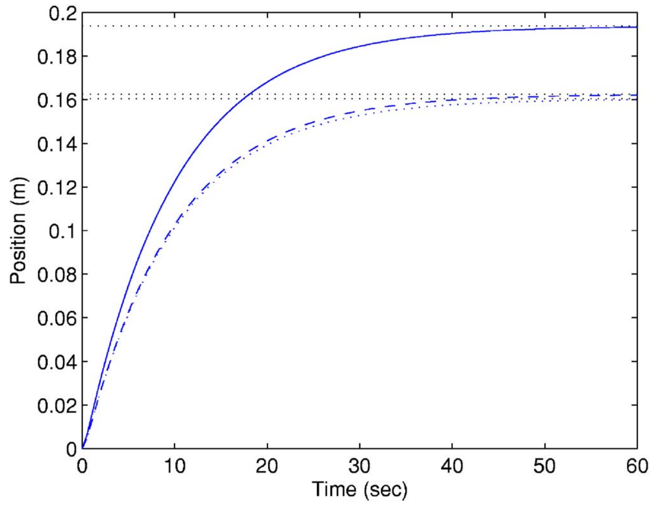

Fig. 5 The step responses of $G_{1,1}(s)$ (solid), $G_{2,1}(s)$ (dashed), and $G_{3,1}(s)$ (dotted) do not display an initial undershoot

$$n_C(\omega)=\sum_{i=1}^{l}\alpha_i \quad n_K(\omega)=\sum_{i=1}^{l}\beta_i \quad n_{CK}(\omega)=\sum_{i=1}^{l}\gamma_i$$

where, for $i=1,\ldots,l$,

$$\alpha_i=\begin{cases}1, & \text{if } \{m_{n_{i-1}},m_{n_i}\}\in E_C \text{ and } \{m_{n_{i-1}},m_{n_i}\}\notin E_K\\ 0, & \text{otherwise}\end{cases}$$

$$\beta_i=\begin{cases}1, & \text{if } \{m_{n_{i-1}},m_{n_i}\}\in E_K \text{ and } \{m_{n_{i-1}},m_{n_i}\}\notin E_C\\ 0, & \text{otherwise}\end{cases}$$

$$\gamma_i=\begin{cases}1, & \text{if } \{m_{n_{i-1}},m_{n_i}\}\in E_K\cap E_C\\ 0, & \text{otherwise}\end{cases}$$

The next result provides an expression for the relative degree of the transfer function from $u_j(t)$ to $q_i(t)$ and characterizes the sign of the first nonzero Markov parameter (often called the high-frequency gain).

THEOREM 8.1. *Assume that the system* (3.10)–(3.13) *is structurally connected. Then, for all $i,j=1,\ldots,N$, the relative degree of the transfer function from $u_j(t)$ to $q_i(t)$ is*

$$r_{i,j}\triangleq\min_{\omega\in\Omega_{i,j}}[2n_K(\omega)+n_C(\omega)+n_{CK}(\omega)]+2 \tag{8.1}$$

Furthermore, for all $i,j=1,\ldots,N$, the first nonzero Markov parameter $H_{r_{i,j}}\triangleq e_i^T C_p A^{r_{i,j}-1}Be_j$ of the transfer function from $u_j(t)$ to $q_i(t)$ is positive.

Proof. Let i and j be positive integers between 1 and N, and let $\omega\in\Omega_{i,j}$ be the minimizer in (8.1) so that $r_{i,j}=2n_K(\omega)+n_C(\omega)+n_{CK}(\omega)+2$. For all $n=1,\ldots,N$,

$$H_n\triangleq e_i^T C_p A^{n-1}Be_j=[e_i^T\ \ 0]\begin{bmatrix}0 & I\\ -M^{-1}K & -M^{-1}C\end{bmatrix}^{n-1}\begin{bmatrix}0\\ M^{-1}e_j\end{bmatrix} \tag{8.2}$$

To prove this result, it suffices to show that $H_0, H_1,\ldots,H_{r_{i,j}-1}=0$ and $H_{r_{i,j}}>0$.

Performing the matrix multiplications in (8.2) implies

$$H_n=[e_i^T\ \ 0]\begin{bmatrix}\sharp & \Gamma_n\\ \sharp & \sharp\end{bmatrix}\begin{bmatrix}0\\ M^{-1}e_j\end{bmatrix}=e_i^T\Gamma_n M^{-1}e_j \tag{8.3}$$

where $\sharp$ denotes an inconsequential entry, $\Gamma_1\triangleq 0$, $\Gamma_2\triangleq I$, and, for all $n=3,\ldots N, \Gamma_n\triangleq -M^{-1}C\Gamma_{n-1}-M^{-1}K\Gamma_{n-2}$. By manipulating the terms of (8.3), it follows that

$$H_n=e_i^T M^{(-1/2)}\bar{\Gamma}_n M^{(-1/2)}e_j=\frac{1}{\sqrt{m_i m_j}}e_i^T\bar{\Gamma}_n e_j$$

where $\bar{\Gamma}_1\triangleq 0$, $\bar{\Gamma}_2\triangleq I$, and, for all $n=3,\ldots,N$,

$$\bar{\Gamma}_n\triangleq P\bar{\Gamma}_{n-1}+Q\bar{\Gamma}_{n-2} \tag{8.4}$$

where $P\triangleq -M^{-1/2}CM^{-1/2}$ and $Q\triangleq -M^{-1/2}KM^{-1/2}$.

Since $\Gamma_1=0$, it follows that $H_1=0$. Since $\Gamma_2=I$, it follows that $H_2>0$ if and only if $i=j$, which is equivalent to $H_2>0$ if and only if $\Omega_{i,j}=\varnothing$. Thus, $H_2>0$ if and only if $r_{i,j}=2$.

Now, consider the case in which $\Omega_{i,j}\neq\varnothing$ and thus $r_{i,j}>2$. Note that P and Q can each be expressed as the sum of a weighted adjacency matrix and a diagonal negative-semidefinite matrix, that is,

$$P=A_{\mathcal{G}_C}+Dc \tag{8.5}$$

$$Q=A_{\mathcal{G}_K}+DK \tag{8.6}$$

where

$$D_C\triangleq=-\operatorname{diag}\left(\frac{c_1}{m_1}+\sum_{j=2}^{N}\frac{c_{1,j}}{m_1},\frac{c_2}{m_2}+\sum_{j=1,j\neq 2}^{N}\frac{c_{2,j}}{m_2},\ldots,\frac{c_N}{m_N}+\sum_{j=1,j\neq N}^{N}\frac{c_{N,j}}{m_N}\right)$$

$$D_K\triangleq=-\operatorname{diag}\left(\frac{k_1}{m_1}+\sum_{j=2}^{N}\frac{k_{1,j}}{m_1},\frac{k_2}{m_2}+\sum_{j=1,j\neq 2}^{N}\frac{k_{2,j}}{m_2},\ldots,\frac{k_N}{m_N}+\sum_{j=1,j\neq N}^{N}\frac{k_{N,j}}{m_N}\right)$$

are diagonal negative semidefinite, $A_{\mathcal{G}_C}$ is the weighted adjacency matrix associated with $\mathcal{G}_C$, where, for all $\{m_p,m_q\}\in E_C$, the weight $c_{p,q}/\sqrt{m_p m_q}>0$ is assigned to the edge $\{m_p,m_q\}$, and $A_{\mathcal{G}_K}$ is the weighted adjacency matrix associated with $\mathcal{G}_K$, where, for all $\{m_p,m_q\}\in E_K$, the weight $k_{p,q}/\sqrt{m_p m_q}>0$ is assigned to the edge $\{m_p,m_q\}$.

It follows from the recursion (8.4) that, for all $n=3,\ldots,N$,

$$\bar{\Gamma}_n={\sum}' P^{p_1}Q^{q_1}\cdots P^{p_{n-3}}Q^{q_{n-3}}P^{p_{n-2}} \tag{8.7}$$

where Σ' denotes the sum over all distinct products such that $p_1,\ldots,p_{n-2},q_1,\ldots,q_{n-3}\in\{0,1\}$ and $\Sigma_{j=1}^{n-2}p_j+2\Sigma_{j=1}^{n-3}q_j+2=n$. Combining (8.5)–(8.7) and performing the multiplications yields

$$\bar{\Gamma}_n={\sum}'(A_{\mathcal{G}_C}+D_C)^{p_1}(A_{\mathcal{G}_K}+D_K)^{q_1}\cdots(A_{\mathcal{G}_C}+D_C)^{p_{n-3}}(A_{\mathcal{G}_K}+D_K)^{q_{n-3}}(A_{\mathcal{G}_C}+D_C)^{p_{n-2}}={\sum}' A_{\mathcal{G}_C}^{p_1}A_{\mathcal{G}_K}^{q_1}\cdots A_{\mathcal{G}_C}^{p_{n-3}}A_{\mathcal{G}_K}^{q_{n-3}}A_{\mathcal{G}_C}^{p_{n-2}}+\Lambda_n \tag{8.8}$$

where, for all $n=3,\ldots,N$,

$$\Lambda_n={\sum}' D_C^{p_1}A_{\mathcal{G}_K}^{q_1}\cdots A_{\mathcal{G}_C}^{p_{n-3}}A_{\mathcal{G}_K}^{q_{n-3}}A_{\mathcal{G}_C}^{p_{n-2}}+{\sum}' D_C^{p_1}D_K^{q_1}\cdots A_{\mathcal{G}_C}^{p_{n-3}}A_{\mathcal{G}_K}^{q_{n-3}}A_{\mathcal{G}_C}^{p_{n-2}}+\cdots+{\sum}' A_{\mathcal{G}_C}^{p_1}D_K^{q_1}\cdots A_{\mathcal{G}_C}^{p_{n-3}}A_{\mathcal{G}_K}^{q_{n-3}}A_{\mathcal{G}_C}^{p_{n-2}}+\cdots+{\sum}' D_C^{p_1}D_K^{q_1}\cdots D_C^{p_{n-3}}D_K^{q_{n-3}}D_C^{p_{n-2}}$$

Since ω is a minimizer of (8.1), there does not exist a walk $\bar{\omega}\in\Omega_{i,j}$ such that $2n_K(\bar{\omega})+n_C(\bar{\omega})+n_{CK}(\bar{\omega})<2n_K(\omega)+n_C(\omega)+n_{CK}(\omega)$. Thus, Corollary 2.1 implies that, for all $n=3,\ldots,r_{i,j}-1$,

$$e_i^T A_{\mathcal{G}_C}^{p_1} A_{\mathcal{G}_K}^{q_1} \cdots A_{\mathcal{G}_C}^{p_{n-3}} A_{\mathcal{G}_K}^{q_{n-3}} A_{\mathcal{G}_C}^{p_{n-2}} e_j = 0 \tag{8.9}$$

where $p_1,\ldots,p_{n-2},q_1,\ldots,q_{n-3} \in \{0,1\}$ satisfy $\Sigma_{j=1}^{n-2} p_j + 2\Sigma_{j=1}^{n-3} q_j + 2 = n$. By combining (8.8) and (8.9), it follows that, for all $n = 3,\ldots,r_{i,j}-1$,

$$e_i^T \bar{\Gamma}_n e_j = e_i^T \Lambda_n e_j$$

Next, note that every term of $e_i^T \Lambda_n e_j$ of the form $e_i^T D_C^{p_1} D_K^{q_1} \cdots D_C^{p_{n-3}} D_K^{q_{n-3}} D_C^{p_{n-2}} e_j$ is zero because D_C and D_K are diagonal and $i \neq j$. The remaining terms of $e_i^T \Lambda_n e^j$ have the form of (8.9) where a negative semidefinite matrix D_C or D_K may appear between the matrices $A_{\mathcal{G}_C}$ and $A_{\mathcal{G}_K}$. Therefore, Lemma A.1 and (8.9) imply that for all $n = 3,\ldots,r_{i,j}-1$, $e_i^T \Lambda_n e_j = 0$. Thus, for all $n = 3,\ldots,r_{i,j}-1$, $e_i^T \bar{\Gamma}_n e_j = 0$, which implies that $H_n = 0$.

Now, it suffices to show that $H_{r_{i,j}} > 0$. Again, note that every term of $e_i^T \Lambda_n e_j$ of the form $e_i^T D_C^{p_1} D_K^{q_1} \cdots D_C^{p_{n-3}} D_K^{q_{n-3}} D_C^{p_{n-2}} e_j$ is zero because D_C and D_K are diagonal and $i \neq j$. The remaining terms of $e_i^T \Lambda_n e_j$ have the form of (8.9) where a negative semidefinite matrix D_C or D_K may appear between the matrices $A_{\mathcal{G}_C}$ and $A_{\mathcal{G}_K}$. Therefore, Lemma A.1 and (8.9) imply that $e_i^T \Lambda_{r_{i,j}} e_j = 0$. Therefore,

$$e_i^T \bar{\Gamma}_{r_{i,j}} e_j = \sum{}' \, e_i^T A_{\mathcal{G}_C}^{p_1} A_{\mathcal{G}_K}^{q_1} \cdots A_{\mathcal{G}_C}^{p_{n-3}} A_{\mathcal{G}_K}^{q_{n-3}} A_{\mathcal{G}_C}^{p_{n-2}} e_j \tag{8.10}$$

Furthermore, note that each product $e_i^T A_{\mathcal{G}_C}^{p_1} A_{\mathcal{G}_K}^{q_1} \cdots A_{\mathcal{G}_C}^{p_{n-3}} A_{\mathcal{G}_K}^{q_{n-3}} A_{\mathcal{G}_C}^{p_{n-2}} e_j$ is non-negative.

Since there exists a walk ω such that $r_{i,j} = 2n_K(\omega) + n_C(\omega) + n_{CK}(\omega) + 2$, it follows from Lemma 2.5 that there exists $p_1,\ldots,p_{r_{i,j}-2},q_1,\ldots,q_{r_{i,j}-3} \in \{0,1\}$ such that $\Sigma_{j=1}^{r_{i,j}-2} p_j + 2\Sigma_{j=1}^{r_{i,j}-3} q_j = r_{i,j}-2$ and

$$e_i^T A_{\mathcal{G}_C}^{p_1} A_{\mathcal{G}_K}^{q_1} \cdots A_{\mathcal{G}_C}^{p_{n-3}} A_{\mathcal{G}_K}^{q_{n-3}} A_{\mathcal{G}_C}^{p_{n-2}} e_j > 0$$

Therefore, at least one term in the summation (8.10) is positive. Thus, $e_i^T \bar{\Gamma}_{r_{i,j}} e_j > 0$, which implies that $H_{r_{i,j}} > 0$. □

Notice that the formula for relative degree provided by Theorem 8.1 does not depend on the specific values of the masses, spring constants, or damping coefficients. In fact, the relative degree depends only on the placement of the springs and dashpots.

9 Zeros and Relative Degree of Serially Connected Lumped-Parameter Structures

In this section, we consider the special case of a serially connected structure in which adjacent masses are connected by springs and dashpots, but nonadjacent masses are not connected to each other. This structure is shown in Fig. 6. The dynamics of the N-mass lumped-parameter structure shown in Fig. 6 are given by

$$M\ddot{q}(t) + C\dot{q}(t) + K_q(t) = u(t) \tag{9.1}$$

where

$$M \triangleq \operatorname{diag}(m_1,\ldots,m_N) \tag{9.2}$$

$$q(t) \triangleq [q_1(t) \ \ \cdots \ \ q_N(t)]^T \tag{9.3}$$

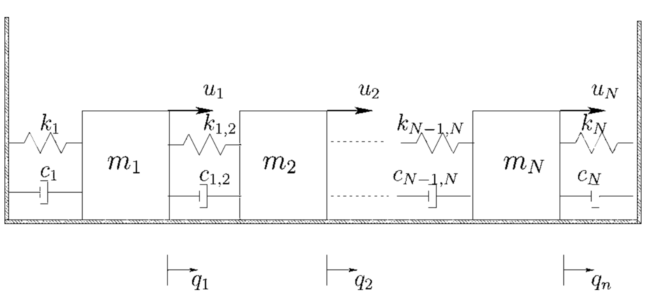

Fig. 6 Serially connected lumped-parameter structure

$$u(t) \triangleq [u_1(t) \ \ \cdots \ \ u_N(t)]^T \tag{9.4}$$

and

$$K \triangleq \begin{bmatrix} k_1 + k_{1,2} & -k_{1,2} & 0 & \cdots & 0 & 0 \\ -k_{1,2} & k_{1,2}+k_{2,3} & -k_{2,3} & \cdots & 0 & 0 \\ 0 & -k_{2,3} & k_{2,3}+k_{3,4} & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -k_{N-1,N} & k_{N-1,N}+k_N \end{bmatrix} \tag{9.5}$$

$$C \triangleq \begin{bmatrix} c_1 + c_{1,2} & -c_{1,2} & 0 & \cdots & 0 & 0 \\ -c_{1,2} & c_{1,2}+c_{2,3} & -c_{2,3} & \cdots & 0 & 0 \\ 0 & -c_{2,3} & c_{2,3}+c_{3,4} & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -c_{N-1,N} & c_{N-1,N}+c_N \end{bmatrix} \tag{9.6}$$

We assume that all spring constants and damping coefficients appearing in (9.5) and (9.6) are positive. It follows from [29, Fact 8.7.35] that C and K are positive definite. The lumped-parameter structure (9.1)–(9.6) can be written as the first-order linear state-space system (3.10)–(3.13). Since C and K are positive definite, it follows from Lemma 3.2 that A is asymptotically stable.

For all $i,j = 1,\ldots,N$, the transfer function from the force input on mass m_j to the position of mass m_i is

$$G_{i,j}(s) \triangleq e_i^T C_{\mathrm{p}} (sI - A)^{-1} B e_j$$

Now, we present our main result on the zeros of serially connected lumped-parameter structures.

THEOREM 9.1. *For all $i,j = 1,\ldots,N$, $G_{i,j}(s)$ has no closed-right-half-plane zeros.*

Proof. It follows from Theorem 4.1 that $G_{i,j}(s)$ has no non-negative zeros. Therefore, it suffices to consider complex non-minimum phase zeros. Let $z \in \mathbb{C}$ such that $\operatorname{Re} z \geq 0$ and $\operatorname{Im} z \neq 0$. For all $i,j = 1,\ldots,N$,

$$\begin{aligned} G_{i,j}(z) &= e_i^T C_{\mathrm{p}} (zI - A)^{-1} B e_j \\ &= [e_i^T \ \ 0] \begin{bmatrix} zI & -I \\ M^{-1}K & zI + M^{-1}C \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ M^{-1} e_j \end{bmatrix} \end{aligned} \tag{9.7}$$

Since $z \neq 0$, M is positive definite, and C and K are positive definite, it follows that zI and $(1/z)M^{-1}(Mz^2 + Cz + K)$ are invertible. Hence, Proposition 2.8.7 of [29] implies that

$$\begin{bmatrix} zI & -I \\ M^{-1}K & zI + M^{-1}C \end{bmatrix}^{-1} = \begin{bmatrix} \sharp & (z^2 I + M^{-1}Cz + M^{-1}K)^{-1} \\ \sharp & \sharp \end{bmatrix} \tag{9.8}$$

where $\sharp$ denotes an inconsequential entry. Combining (9.7) and (9.8) yields

$$G_{i,j}(z) = e_i^T (Mz^2 + Cz + K)^{-1} e_j.$$

Next, we show that the leading (and trailing) principal subdeterminants of $Mz^2 + Cz + K$ are nonzero. Let ν be an integer between 1 and N, and let $\bar{M}$, $\bar{C}$, and $\bar{K}$ be the $\nu \times \nu$ leading (or trailing) principal submatrices of M, C, and K, respectively, so that $\bar{M}z^2 + \bar{C}z + \bar{K}$ is a leading (or trailing) principal submatrix of $Mz^2 + Cz + K$. Note that $\bar{M}$, $\bar{C}$, and $\bar{K}$ have the same form as M, C, and K. More precisely, $\bar{M}$, $\bar{C}$, and $\bar{K}$ are the mass, damping, and stiffness matrices of a serially connected lumped-parameter structure with ν masses. Therefore, $\bar{M}$, $\bar{C}$, and $\bar{K}$ are positive definite. Lemma 3.2 implies that

$$\bar{A} \triangleq \begin{bmatrix} 0 & I \\ -\bar{M}^{-1}\bar{K} & -\bar{M}\bar{C} \end{bmatrix}$$

is asymptotically stable, and thus $\det(zI-\bar{A}) \neq 0$. Then, Lemma 3.1 implies that $\det(\bar{M}z^2+\bar{C}z+\bar{K}) \neq 0$. Therefore, the leading (and trailing) principal subdeterminants of Mz^2+Cz+K are nonzero. Furthermore, Mz^2+Cz+K is a tridiagonal matrix all of whose superdiagonal and subdiagonal entries are nonzero. It thus follows from Lemma B.1 with $\mathcal{A}=Mz^2+Cz+K$ that every entry of $(Mz^2+Cz+K)^{-1}$ is nonzero. Hence, $G_{i,j}(z)=e_i^T(Mz^2+Cz+K)^{-1}e_j \neq 0$, and thus, z is not a zero of $G_{i,j}(s)$. □

The next result, which follows immediately from Theorem 8.1, provides a simple formula for the relative degree of $G_{i,j}(s)$ for serially connected structures.

THEOREM 9.2. *For all*, $i,j=1,\ldots,N$, *the relative degree of* $G_{i,j}(s)$ *is*

$$r_{i,j} \triangleq |i-j|+2$$

Furthermore, for all $i,j=1,\ldots,N$, *the first nonzero Markov parameter* $H_{r_{i,j}} \triangleq e_i^T C_p A^{r_{i,j}-1} B e_j$ *of* $G_{i,j}(s)$ *is positive.*

10 Conclusions

This paper showed that every SISO force-to-motion transfer function of a lumped-parameter structure has no positive (real open-right-half-plane) zeros. In addition, every SISO force-to-position transfer function of a spring-connected lumped-parameter structure has no nonnegative (real closed-right-half-plane) zeros. As a consequence, the step response of every asymptotically stable SISO force-to-position transfer function does not exhibit initial undershoot. In addition, we obtained a formula for the relative degree of the SISO force-to-motion transfer functions. This formula depends on the placement of springs and dashpots, but does not depend on the specific values of the spring constants and damping coefficients. Finally, we showed that every SISO force-to-position transfer function of a serially connected lumped-parameter structure is minimum phase.

Appendix A

LEMMA A.1. *Let* $X,\bar{X} \in \mathbb{R}^{N\times N}$ *be non-negative. Let* $p_1,\ldots,p_l$ *and* $\bar{p}_1,\ldots,\bar{p}_l$ *all be non-negative integers, and let* $D_1,\ldots D_l \in \mathbb{R}^{N\times N}$ *and* $\bar{D}_1,\ldots,\bar{D}_l \in \mathbb{R}^{N\times N}$ *all be diagonal positive or negative semidefinite. Assume that*

$$e_j^T X^{p_1}\bar{X}^{\bar{p}_1}X^{p_2}\bar{X}^{\bar{p}_2}\cdots X^{p_l}\bar{X}^{\bar{p}_l}e_i = 0 \qquad \text{(A1)}$$

Then

$$e_j^T D_1 X^{p_1}\bar{D}_1\bar{X}^{\bar{p}_1}D_2X^{p_2}\bar{D}_2\bar{X}^{\bar{p}_2}\cdots D_lX^{p_l}\bar{D}_l\bar{X}^{p_l}e_i = 0 \qquad \text{(A2)}$$

Proof. It follows from (A1) that $\operatorname{tr}(X^{p_1}\bar{X}^{\bar{p}_1}X^{p_2}\bar{X}^{\bar{p}_2}\cdots X^{p_l}\bar{X}^{\bar{p}_l}e_ie_j^T)=0$. Since X and $\bar{X}$ are non-negative, it follows that $X^{p_1}\bar{X}^{\bar{p}_1}X^{p_2}\bar{X}^{\bar{p}_2}\cdots X^{p_l}\bar{X}^{\bar{p}_l}e_ie_j^T$ has all zero entries along the diagonal. Since $D_1$is diagonal positive (or negative) semidefinite, it follows that $D_1X^{p_1}\bar{X}^{\bar{p}_1}X^{p_2}\bar{X}^{\bar{p}_2}\cdots X^{p_l}\bar{X}^{\bar{p}_l}e_ie_j^T$ has all zero entries along the diagonal and is non-negative (or nonpositive). Thus $\operatorname{tr}(D_1X^{p_1}\bar{X}^{\bar{p}_1}X^{p_2}\bar{X}^{\bar{p}_2}\cdots X^{p_l}\bar{X}^{\bar{p}_l}e_ie_j^T)=0$, which implies $\operatorname{tr}(\bar{X}^{\bar{p}_1}X^{p_2}\bar{X}^{\bar{p}_2}\cdots X^{p_l}\bar{X}^{\bar{p}_l}e_ie_j^TD_1X^{p_1})=0$. Since $\bar{X}^{\bar{p}_1}X^{p_2}\bar{X}^{\bar{p}_2}\cdots X^{p_l}\bar{X}^{\bar{p}_l}e_ie_j^TD_1X^{p_1}$ is non-negative (or nonpositive), it has all zero entries along the diagonal. Since D_1 is diagonal positive (or negative) semidefinite, it follows that $\bar{D}_1\bar{X}^{\bar{p}_1}X^{p_2}\bar{X}^{\bar{p}_2}\cdots X^{p_l}\bar{X}^{\bar{p}_l}e_ie_j^TD_1X^{p_1}$ has all zero entries along the diagonal and is non-negative (or non-negative).

Continuing with this analysis implies that $\bar{X}^{\bar{p}_l}e_ie_j^TD_1X^{p_1}\bar{D}_1\bar{X}^{\bar{p}_1}D_2X^{p_2}\bar{D}_2\bar{X}^{\bar{p}_2}\cdots D_lX^{p_l}$ has all zero entries along the diagonal. Then since $\bar{D}_l$ is diagonal, it follows that $\bar{D}_l\bar{X}^{\bar{p}_l}e_ie_j^TD_1X^{p_1}\bar{D}_1\bar{X}^{\bar{p}_1}D_2X^{p_2}\bar{D}_2\bar{X}^{\bar{p}_2}\cdots D_lX^{p_l}$ has all zero entries along the diagonal. Therefore, $\operatorname{tr}(\bar{D}_l\bar{X}^{\bar{p}_l}e_ie_j^TD_1X^{p_1}\bar{D}_1\bar{X}^{\bar{p}_1}D_2X^{p_2}\bar{D}_2\bar{X}^{\bar{p}_2}\cdots D_lX^{p_l})=0$, which implies (A2). □

Appendix B

LEMMA B.1. *Let* $\mathcal{A} \in \mathbb{C}^{n\times n}$ *be tridiagonal, and assume that every entry of the superdiagonal and subdiagonal of* $\mathcal{A}$ *is nonzero. Furthermore, assume that every leading principal subdeterminant and every trailing principal subdeterminant of* $\mathcal{A}$ *is nonzero. Then every entry of* $\mathcal{A}^{-1}$ *is nonzero.*

Proof. Let

$$\mathcal{A} = \begin{bmatrix} a_1 & b_1 & & & & \\ c_1 & a_2 & b_2 & & & \\ & c_2 & a_3 & b_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & c_{n-2} & a_{n-1} & b_{n-1} \\ & & & & c_{n-1} & a_n \end{bmatrix}$$

For $i,j=1,\ldots,n$, let $\mathcal{A}_{(i,j)}$ denote the (i,j) entry of $\mathcal{A}$. Since $(\mathcal{A}^{-1})_{(i,j)}=(1/\det\mathcal{A})(-1)^{i+j}\det\mathcal{A}_{[j,i]}$, where $\mathcal{A}_{[j,i]}$ is the cofactor of $\mathcal{A}_{(i,j)}$, by assigning $\mathcal{A}$ is nonsingular, it suffices to show that the determinant of every cofactor is nonzero.

Assume that $i \leq j$, and write

$$\mathcal{A}_{[i,j]} = \left[\begin{array}{ccccccccccc|ccccc} a_1 & b_1 & & & & & & & & & & & & & \\ c_1 & \ddots & \ddots & & & & & & & & & & & & \\ & \ddots & a_{1-2} & b_{i-2} & & & & & & & & & & & \\ & & c_{i-2} & a_{i-1} & b_{i-1} & & & & & & & & & & \\ \hline & & & & c_i & a_{i+1} & b_{i+1} & & & & & & & & \\ & & & & & \ddots & \ddots & \ddots & & & & & & & \\ & & & & & & c_{j-3} & a_{j-2} & b_{j-2} & & & & & & \\ & & & & & & & c_{j-2} & a_{j-1} & & & & & & \\ & & & & & & & & c_{j-1} & & b_j & & & & \\ & & & & & & & & & & a_{j+1} & \ddots & & & \\ & & & & & & & & & & c_{j+1} & \ddots & b_{n-2} & & \\ & & & & & & & & & & & \ddots & a_{n-1} & b_{n-1} & \\ & & & & & & & & & & & & c_{n-1} & a_n & \end{array}\right]$$

where the horizontal line shows where the ith row has been removed, and the vertical line shows where the jth column has been removed. Therefore,

$$\det \mathcal{A}_{[i,j]} = \det \begin{bmatrix} \Omega_{11} & \Omega_{12} & 0 \\ 0 & \Omega_{22} & \Omega_{23} \\ 0 & 0 & \Omega_{33} \end{bmatrix} = (\det \Omega_{11})(\det \Omega_{22})(\det \Omega_{33}) \tag{B1}$$

where

$$\Omega_{11} \triangleq \begin{bmatrix} a_1 & b_1 & & \\ c_1 & \ddots & \ddots & \\ & \ddots & a_{1-2} & b_{i-2} \\ & & c_{i-2} & a_{i-1} \end{bmatrix} \in \mathbb{C}^{(i-1)\times(i-1)}$$

$$\Omega_{12} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ b_{i-1} & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{(i-1)\times(j-i)}$$

$$\Omega_{22} \triangleq \begin{bmatrix} c_i & a_{i+1} & b_{i+1} & & & \\ & \ddots & \ddots & \ddots & & \\ & & c_{j-3} & a_{j-2} & b_{j-2} \\ & & & c_{j-2} & a_{j-1} \\ & & & & c_{j-1} \end{bmatrix} \in \mathbb{C}^{(j-i)\times(j-i)}$$

$$\Omega_{23} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ b_j & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{(j-i)\times(n-j)}$$

$$\Omega_{33} \triangleq \begin{bmatrix} a_{j+1} & b_{j+1} & & \\ c_{j+1} & \ddots & \ddots & \\ & \ddots & a_{n-1} & b_{n-1} \\ & & c_{n-1} & a_n \end{bmatrix} \in \mathbb{C}^{(n-j)\times(n-j)}$$

Since Ω_{11} is a leading principal submatrix of A and Ω_{33} is a trailing principal submatrix of A, it follows that $\det \Omega_{11} \neq 0$ and $\det \Omega_{33} \neq 0$. Furthermore, $\det \Omega_{22} \neq 0$ since Ω_{22} is an upper-triangular matrix with nonzero diagonal entries $c_i, \ldots, c_{j-1}$. Since $\det \Omega_{11} \neq 0$, $\det \Omega_{22} \neq 0$, and $\det \Omega_{33} \neq 0$, it follows from (B1) that $\det \mathcal{A}_{[i,j]} \neq 0$.

Next, assume that $i > j$. Then

$$\det \mathcal{A}_{[i,j]} = \det \begin{bmatrix} \bar{\Omega}_{11} & 0 & 0 \\ \bar{\Omega}_{21} & \bar{\Omega}_{22} & 0 \\ 0 & \bar{\Omega}_{32} & \bar{\Omega}_{33} \end{bmatrix} = (\det \bar{\Omega}_{11})(\det \bar{\Omega}_{22})(\det \bar{\Omega}_{33})$$

where $\bar{\Omega}_{11}$ is a leading principal submatrix of $\mathcal{A}$, $\bar{\Omega}_{33}$ is a trailing principal submatrix of $\mathcal{A}$, and $\bar{\Omega}_{22}$ is a lower-triangular matrix with nonzero diagonal entries. Thus, for $i > j$, $\det \mathcal{A}_{[i,j]} \neq 0$. Therefore, for $i, j = 1, \ldots, n$, $\det \mathcal{A}_{[i,j]} \neq 0$, and every entry of $\mathcal{A}^{-1}$ is nonzero. □

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