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H_2 OPTIMAL TUNING OF PASSIVE ISOLATORS AND ABSORBERS

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ABSTRACT

Isolators and absorbers have been studied extensively as a means of passively suppressing vibration in mechanical systems. This paper derives the optimal tuning parameters for these passive suppression schemes with an H_2 performance criterion. The H_2 norm of a system is briefly reviewed, and its selection as a performance criterion is motivated. The optimal tuning scheme in the case of the absorber is then compared to the classical work of Den Hartog [1] and Snowdon [2]. The comparison shows little improvement in the H_2 cost over the classical scheme, which suggests that the approximately optimal H_∞ analysis of the absorber respects the optimal H_2 cost. However, the reverse is generally not true, as in the case with increasing absorber masses.

INTRODUCTION

Passive isolators and absorbers provide one of the principal means for suppressing undesirable vibrations. Techniques for tuning isolators and absorbers have thus been extensively studied in the vibration control literature [2–5]. These studies have generally focused on the choice of design parameters to achieve a suitable shape of the transmissibility for a given disturbance spectrum. For example, the classical tuning of the Den Hartog absorber given in [1,2] provides damping values that approximately minimize the peak transmissibility. From a modern systems approach, this design is approximately optimal from the point of view of the H_∞ performance criterion [6].

In contrast to these techniques, semi-active and active control approaches to vibration suppression are often based upon precise optimality criteria, in particular, a quadratic H_2 performance index with the associated LQR and LQG control designs [7–13]. Quadratic optimality is useful for minimizing mean-square response levels in the presence of broadband (white) or narrow-band (colored) stochastic noise disturbances [14].

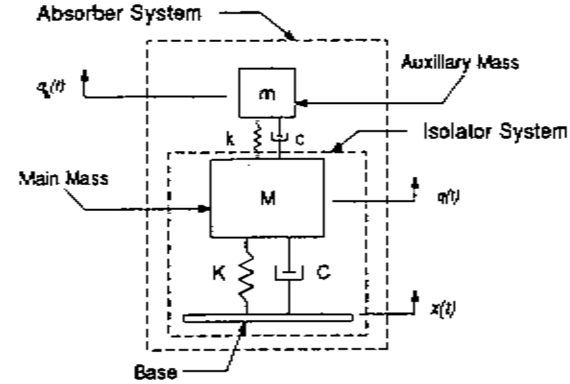

FIGURE 1: ISOLATOR/ABSORBER SYSTEM

The goal of the present paper is to revisit the problem of passive isolator and absorber design from the point of view of H_2 optimality. For our analysis we consider the standard spring/dashpot isolator and one-degree-of-freedom mass/spring/dashpot absorber in the presence of white or impulsive shock disturbances. In both cases we optimize over the available design parameters. In the case of the isolator, optimization is performed with respect to the main spring and dashpot, while for the absorber, optimization is performed with respect to the auxiliary mass, spring, and dashpot. The resulting optimal tuning parameters for the absorber are then compared with the classical absorber tuning parameters given in [1, 2].

H_2 OPTIMAL PERFORMANCE CRITERION

The physical interpretation of the H_2 norm in the frequency and time domains motivates its use as a performance criterion for quantifying vibration suppression. Referring to Figure 1, it is desirable to suppress the displacement $q(t)$ of the main mass in the presence of base displacement $x(t)$. Because of reciprocity, the transfer function from base displacement to main mass displacement, that is, the transmissibility, is

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equivalent to the transfer function from main mass force to base force [15].

Both the isolator and absorber are considered as vibration suppression schemes. The isolator consists of the spring/dashpot interconnection between the main mass and the base, while the addition of an auxiliary mass/spring/dashpot to the main mass of the isolator constitutes the absorber system as shown in Figure 1.

The classical results in [1, 2] derive absorber tuning parameters by minimizing the peak frequency response. The derivation of these system parameters can be viewed as an approximately optimal H_∞ design of the transmissibility.

An alternative approach is to formulate this classical problem from an H_2 optimality standpoint. The H_2 norm of a transfer function has three interpretations that have physical significance in quantifying vibration suppression. A brief discussion of these interpretations is given below.

Impulse Response Interpretation

Consider the linear time-invariant system

$$\dot{x} = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{1}$$

$$y(t) = Cx(t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $q(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, and $C \in \mathbb{R}^{1\times n}$. Now assume that A is asymptotically stable and consider the performance criterion given by the quadratic functional

$$J(x_0, u) \triangleq \int_0^\infty y^{\mathrm{T}}(t)y(t)\mathrm{d}t. \tag{3}$$

If $x_0 = 0$ and $u(t) = \delta(t)$, where $\delta(t)$ is the unit impulse at $t = 0$, the impulse response $y(t) = H(t)$ is given by

$$H(t) = Ce^{At}B, \tag{4}$$

and the cost functional (3) is the L_2 norm of the impulse response function $H(\cdot)$ defined by

$$\|H(\cdot)\|_2^2 \triangleq \int_0^\infty \|H(t)\|^2\mathrm{d}t. \tag{5}$$

The L_2 norm of the impulse response can be evaluated in terms of the controllability and observability Gramians. When A is asymptotically stable the L_2 norm of $H(\cdot)$ becomes

$$\begin{aligned}\|H(\cdot)\|_2^2 &= \int_0^\infty Ce^{At}BB^{\mathrm{T}}e^{A^{\mathrm{T}}t}C^{\mathrm{T}}\mathrm{d}t \qquad (6)\\ &= \int_0^\infty B^{\mathrm{T}}e^{A^{\mathrm{T}}t}C^{\mathrm{T}}Ce^{At}B\mathrm{d}t,\end{aligned}$$

and can be further written as

$$\|H(\cdot)\|_2^2 = CQC^{\mathrm{T}} = B^{\mathrm{T}}PB, \tag{7}$$

where $Q, P \in \mathbb{R}^{n\times n}$, defined by

$$Q = \int_0^\infty e^{At}BB^{\mathrm{T}}e^{A^{\mathrm{T}}t}\mathrm{d}t \tag{8}$$

$$P = \int_0^\infty e^{A^{\mathrm{T}}t}C^{\mathrm{T}}Ce^{At}\mathrm{d}t, \tag{9}$$

satisfy the Lyapunov equations

$$0 = AQ + QA^{\mathrm{T}} + BB^{\mathrm{T}} \tag{10}$$

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C. \tag{11}$$

Symbolic evaluation of the H_2 norm, which is based on the closed form expression from the solution of (10) and (11), can be obtained by using the method given in Jury and Dewey [16]. This algorithm has been implemented using *Mathematica* and is used in later sections.

Stochastic Interpretation

Alternatively, consider the case in which $u(t)$ is a zero-mean normalized Gaussian white noise disturbance with covariance

$$\mathbb{E}[u(t)u^{\mathrm{T}}(\tau)] = \delta(t - \tau). \tag{12}$$

Now consider the performance to be the mean-square response of the system from equilibrium in the presence of this disturbance as given by

$$J_s(x_0, u(t)) \triangleq \lim_{t\to\infty} \mathbb{E}[y^{\mathrm{T}}(t)y(t)]. \tag{13}$$

Defining the state covariance by

$$Q(t) \triangleq \mathbb{E}[x(t)x^{\mathrm{T}}(t)], \tag{14}$$

yields

$$J_s(x_0, u(t)) = \lim_{t\to\infty} CQ(t)C^{\mathrm{T}}, \tag{15}$$

where $Q(t)$ solves the Lyapunov differential equation

$$\dot{Q}(t) = AQ(t) + Q(t)A^{\mathrm{T}} + BB^{\mathrm{T}}, \tag{16}$$

$$Q(0) = \mathbb{E}[x_0x_0^{\mathrm{T}}] = 0.$$

If A is asymptotically stable then the steady state covariance $Q \triangleq \lim_{t\to\infty} Q(t)$ exists and satisfies (10). Thus $J_s(x_0, u(t))$ is given by

$$J_s(x_0, u(t)) = CQC^{\mathrm{T}}. \tag{17}$$

Power Spectral Density Interpretation

Again, consider a linear time-invariant system with transfer function $G(\jmath\omega)$ and a zero-mean normalized white Gaussian disturbance. The power spectral density $S_q(\omega)$ of the response is then given by

$$S_q(\omega) = |G(\jmath\omega)|^2, \tag{18}$$

and the total mean-square power of the response is

$$\|G(\cdot)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\jmath\omega)|^2 d\omega. \quad (19)$$

It follows from Parseval's Theorem that

$$\int_0^{\infty} H(t)H^T(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\jmath\omega)G^*(\jmath\omega)d\omega, \quad (20)$$

which relates the L_2 norm of the impulse response function in the time domain to the H_2 norm of its transform.

This expression can be interpreted as the average power of the response in the presence of a white noise input.

H_2 Vibration Cost

Consider the transmissibility of the isolator/absorber system as

$$T(s) = \frac{q(s)}{x(s)}, \quad (21)$$

where $x(s)$ and $q(s)$ are the Laplace transforms of the displacements in Figure 1. Since the H_2 norm of the impulse response function quantifies vibration levels we define the vibration cost as

$$J_{\text{vibration}} = \|T(\cdot)\|_2^2. \quad (22)$$

The three interpretations of the H_2 norm in the previous sections give $J_{\text{vibration}}$ the following physical meanings:

1. Noting that $T(s) = \frac{sq(s)}{sx(s)}$, $J_{\text{vibration}}$ is the total kinetic energy per unit mass of the main mass due to an impulsive velocity at the base.
2. $J_{\text{vibration}}$ is the mean-square steady state kinetic energy per unit mass of the main mass in the presence of a white velocity disturbance at the base.
3. $J_{\text{vibration}}$ is the average power output of the main mass in the presence of a white noise disturbance at the base.

The cost $J_{\text{vibration}}$ can also include colored disturbances by augmenting the transmissibility with an appropriate filter. However, for simplicity we consider only broadband disturbances.

H_2 OPTIMAL ISOLATOR TUNING

Consider the isolator modeled as

$$M\ddot{q} = K(x-q) + C(\dot{x}-\dot{q}). \quad (23)$$

By taking the Laplace transform, the transmissibility is given by

$$T(s) = \frac{q(s)}{x(s)} = \frac{Cs+K}{Ms^2+Cs+K} \quad (24)$$

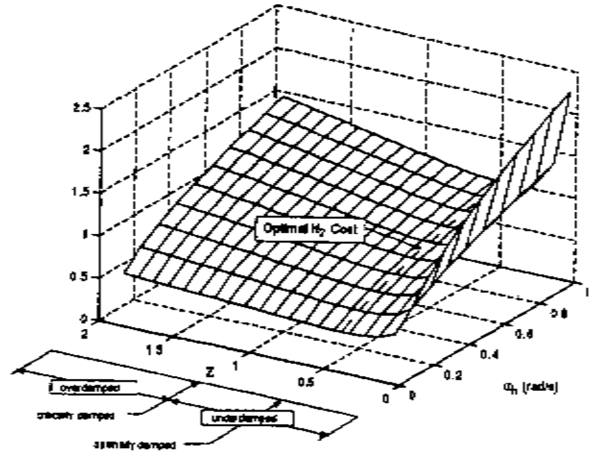

FIGURE 2: ISOLATOR H_2 COST

and the H_2 norm of this transmissibility is

$$J_{\text{vibration}} = \|T(\cdot)\|_2^2 \quad (25)$$
$$= \int_{-\infty}^{\infty}\left|\frac{C\jmath\omega+K}{K-C\jmath\omega-M\omega^2}\right|^2 d\omega = \sqrt{\frac{K}{2C}+\frac{C}{2M}}.$$

To minimize $J_{\text{vibration}}$ with respect to C we set $\frac{\partial}{\partial C}J_{\text{vibration}} = 0$, which yields

$$C_{\text{optimal}} = \sqrt{KM}. \quad (26)$$

With the damping ratio defined by $\Lambda \triangleq \frac{C}{2\sqrt{KM}}$ the optimal damping ratio is

$$\Lambda_{\text{optimal}} = \frac{1}{2}. \quad (27)$$

Substituting Λ_{optimal} back into the performance criterion (26) yields a minimum H_2 norm of

$$J_{\text{vibration}} = \|T(\cdot)\|_2^2 = \omega_n, \quad (28)$$

where ω_n is the natural frequency of the isolator. Figure 2 illustrates the minimum of the H_2 cost along the line $\Lambda = \frac{1}{2}$. Additionally, with $\Lambda = \Lambda_{\text{optimal}}$ the input frequency yielding the largest vibration amplitude occurs at

$$\omega_{\max} = \sqrt{\sqrt{3}-1}\,\omega_n \cong .8556\,\omega_n \quad (29)$$

with maximum transmissibility

$$T_{\max} = \sqrt{\frac{2}{3}\sqrt{3}+1} \cong 1.4679 = 3.334 \text{ dB}. \quad (30)$$

The transmissibility and impulse response of the isolator for several damping ratios, including the Λ_{optimal}, are shown in Figures 3 and 4.

H_2 OPTIMAL ABSORBER TUNING

In order to further suppress vibration in the isolator system, an absorber comprised of an auxiliary mass/spring/dashpot is commonly added to the main mass. In this section we outline the optimal tuning parameters for the absorber.

The dynamics of the isolator/absorber system are given by

$$m\ddot{q}_a(t) = -k(q_a - q) - c(\dot{q}_a - \dot{q}), \quad (31)$$

$$M\ddot{q} = -K(q - x) - C(q - x) + k(q_a - q) + c(\dot{q}_a - \dot{q}), \quad (32)$$

By eliminating q_a from these equations and taking the Laplace transform, the transmissibility is given by

$$T(s) = \frac{Cms^3 + (Km + Cc)s^2 + (Ck + Kc)s + Kk}{Mms^4 + (c(m+M) + Cm)s^3 + (k(m+M) + Km + Cc)s^2 + (Ck + Kc)s + Kk}, \quad (33)$$

As in [1, 2] we consider the case of undamped main mass $C = 0$ in the following analysis. In this case, the transmissibility in (33) becomes

$$T(s) = \frac{Kms^2 + Kcs + Kk}{Mms^4 + c(m+M)s^3 + (k(m+M) + Km)s^2 + Kcs + Kk}, \quad (34)$$

and the H_2 norm is given by

$$J_{\text{vibration}} = \frac{c^2Km + k^2m^2 - kKm^2 + K^2m^2 + c^2KM + 2k^2mM - 2kKmM + k^2M^2}{2cKm^2}. \quad (35)$$

This quantity is now minimized over the absorber design parameters. First, inspection of $J_{\text{vibration}}$ shows that as m becomes large $J_{\text{vibration}}$ asymptotically decreases and thus there does not exist a minimum with respect to m. The auxiliary mass should generally be chosen as large as design constraints will allow to best minimize this cost functional. Next, by leaving $J_{\text{vibration}}$ in terms of the auxiliary mass m and minimizing over c and k, the optimal parameters become

$$c_{\text{optimal}} = \sqrt{\frac{k^2m^2 - kKm^2 + K^2m^2 + 2k^2mM - 2kKmM + k^2M^2}{K(m+M)}} \quad (36)$$

$$k_{\text{optimal}} = \frac{m(m+2M)}{2(m+M)^2}K. \quad (37)$$

For these values the optimal damping ratio is

$$\zeta_{\text{optimal}} = \frac{1}{2}\sqrt{(\frac{1}{\eta} - \eta)^2 + \mu}. \quad (38)$$

Furthermore, the natural frequency ratio, locked natural frequency, absorber natural frequency, and mass ratio are given by

$$\eta = \frac{\omega_a}{\omega_{\text{locked}}}, \quad \omega_{\text{locked}} = \sqrt{\frac{K}{M+m}}, \quad (39)$$

$$\omega_a = \sqrt{\frac{k}{m}}, \quad \mu = \frac{m}{M+m}.$$

Additionally, since k_{optimal} is independent of the auxiliary damping c, (37) can be written as

$$k_{\text{optimal}} = \frac{1}{2}\mu(2-\mu)K. \quad (40)$$

Now substituting k_{optimal} into (36) yields the damping ratio

$$\zeta_{\text{optimal}} = \sqrt{\frac{\mu(4-\mu)}{8(2-\mu)}}. \quad (41)$$

Equations (40) and (41) constitute the simultaneous optimization of c and k.

Den Hartog [1] gave the tuning parameter for the auxiliary spring as

$$k_{\text{DenHartog}} = \mu(1-\mu)K, \quad (42)$$

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while Snowdon [2] later developed the approximately optimal H_∞ tuning parameter for ζ as

$$\zeta_{\text{Snowdon}} = \sqrt{\frac{3\mu}{8(1-\mu)}}. \quad (43)$$

For clarity, note that the mass ratio $\mu = \frac{m}{M+m}$ defined in (40) and used in expressions (38)-(46) differs from the classical definition $\mu_{\text{classical}} = \frac{M}{m+M}$. The transmissibility and impulse response for these values are shown in Figures 5 and 6. It becomes evident through examination of these plots that the approximately optimal H_∞ analysis and the optimal

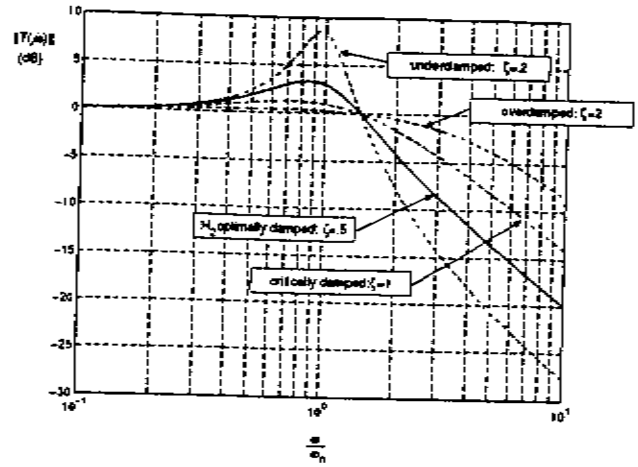

FIGURE 3: ISOLATOR TRANSMISSIBILITY

H_2 analysis yield tuning parameters that are almost identical. Table 1 lists the H_2 costs for the normalized H_2 optimally tuned isolator, the normalized H_2 optimally tuned absorber, and the Den Hartog/Snowdon normalized tuned absorber.

Table 1: Vibration Performance

Tuned System	H_2 Cost
H_2 Optimal Isolator	ω_n
H_2 Optimal Absorber	$\frac{1}{2}\sqrt{\frac{(4-\mu)}{\mu}}\ \omega_n$
Den Hartog/Snowdon Absorber	$\frac{5}{2\sqrt{6\mu}}\omega_n$

By examining this table, comparisons of the H_2 cost can be made. The H_2 optimal isolator always does better than the Den Hartog/Snowdon absorber, and the fraction of improvement is given by

$$1 - \frac{2\sqrt{6\mu}}{5}. \tag{44}$$

It can also be seen that the H_2 optimal isolator will do better than the H_2 optimal absorber when $\mu < \frac{4}{5}$ by a fraction improvement of

$$1 - 2\sqrt{\frac{\mu}{4-\mu}}. \tag{45}$$

Finally, the H_2 optimal absorber does better than the Den Hartog/Snowdon absorber with a fraction improvement

$$1 - \frac{\sqrt{6}}{5}\sqrt{4-\mu}. \tag{46}$$

Typically these improvements are on the order of 3%–5%. Figure 7 shows these H_2 costs normalized by ω_n.

CONCLUSIONS

Optimal tuning parameters for an undamped main mass absorber from an H_2 optimal perspective were derived. It was

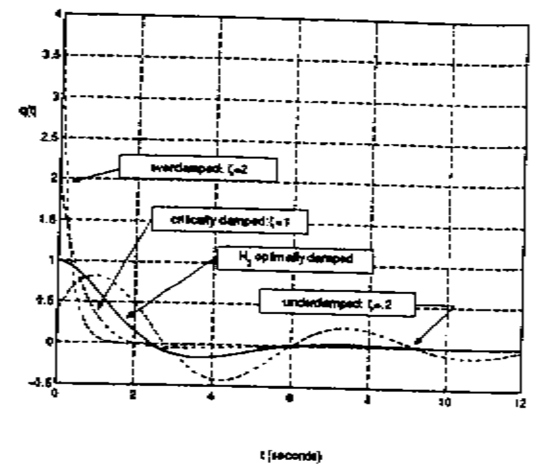

FIGURE 4: ISOLATOR NORMALIZED IMPULSE RESPONSE

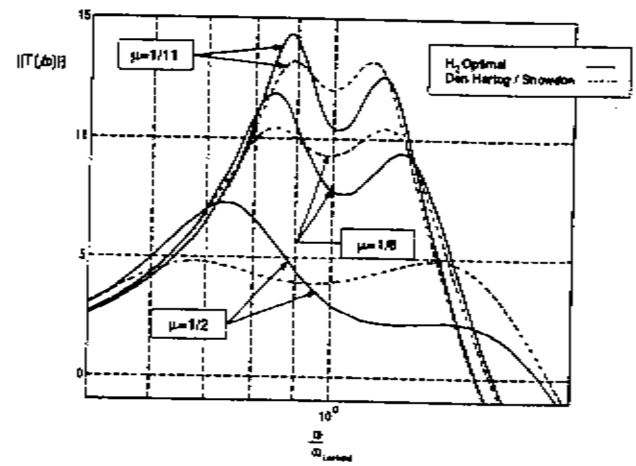

FIGURE 5: ABSORBER TRANSMISSIBILITY COMPARISON FOR VARIOUS VALUES OF THE MASS RATIO μ

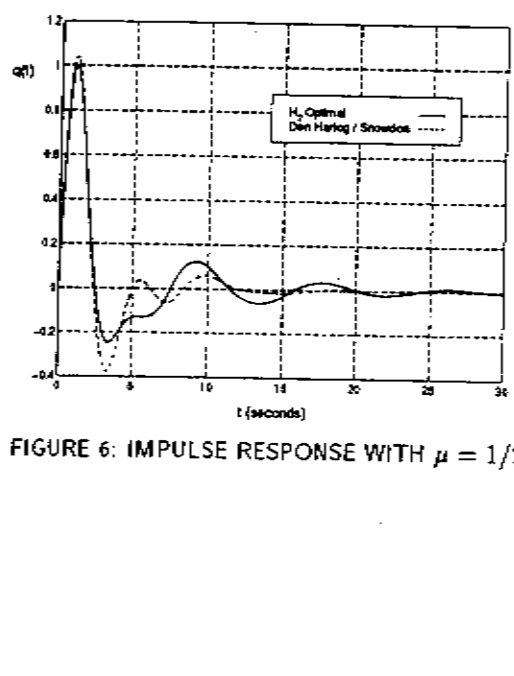

FIGURE 6: IMPULSE RESPONSE WITH $\mu = 1/2$

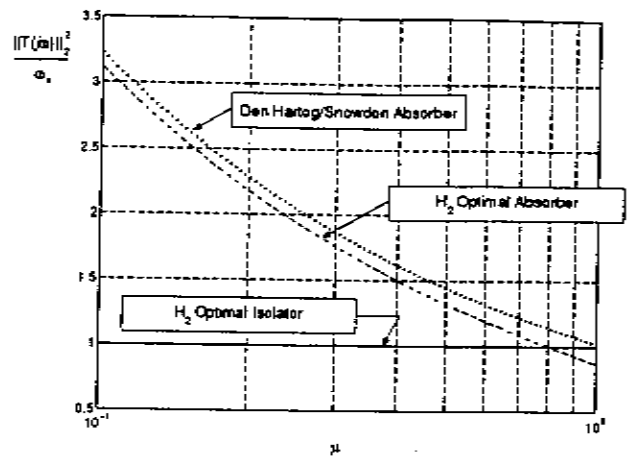

FIGURE 7: VIBRATION COST COMPARISON

shown in Section 2 that the H_2 norm entails several interpretations for quantifying the effects of vibration. Alternatively, Den Hartog and Snowdon approximately optimize the system from an H_∞ (harmonic amplification) perspective. The comparison of the H_2 optimal scheme to the classical one yields little improvement in the H_2 cost as seen in Figure 7. Additionally, the H_∞ norm of the H_2 optimal system becomes increasingly large for larger mass ratios μ. The case of equal main mass and auxiliary mass yields a 3 dB degradation in H_∞ performance.

An outcome of particular interest is the improvement of $J_{\text{vibration}}$ when the damping ratio at the main mass can be chosen as $\zeta_{\text{optimal}} = \frac{1}{2}$. This large improvement shows the authority that the main mass damping has over suppressing vibration in the system.

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