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Hysteretic Systems and Step-Convergent Semistability

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Abstract

Hysteresis is usually characterized as a memory-dependent relationship between inputs and outputs. While various operator models have been proposed, it is often convenient for engineering applications to approximate hysteretic behavior by means of finite-dimensional differential models. In the present paper we show that step-convergent semistable systems (that is, semistable systems with convergent step response) give rise to multiple-valued maps under quasi-static operation. By providing a connection between semistability and hysteresis, our goal is to provide a class of differential models for representing hysteretic behavior.

1 Introduction

Hysteresis is a ubiquitous phenomenon in engineering applications. In structural mechanics, hysteresis arises from inelastic stress-strain due to thermal behavior, plasticity, and phase transitions, while in electromagnetic applications hysteresis is often found in magnetic materials [1, 6, 14, 16, 19]. More commonly, hysteresis is associated with mechanical couplings which often entail a deadzone-backlash interconnection. Hysteresis nonlinearities are of intense interest in control applications where they can give rise to limit cycles oscillations [10–12, 15, 17].

In some applications, feedback controllers are deliberately designed to be hysteretic as exemplified by a typical thermostat [18]. These hysteretic maps usually have unique forward and backward paths, unlike "natural" systems which can follow an arbitrary number of forward and backward paths.

Hysteresis is usually characterized as a memory-dependent relationship between inputs and outputs [13, 19]. To formalize this behavior, various operator models have been developed including the Duhem operator, Ishlinskii operator, Preisach model, and Krasnosel'skiĭ-Pokrovskiĭ hysteron. The mathematical foundation of these and other models is reviewed in [13], where relevant references are given.

For engineering applications, it is often convenient to approximate hysteretic behavior by means of finite-dimensional differential models. In structural mechanics, this approach is discussed in [5, 16, 20]. A closely related idea mentioned in [10], p. 6, is to approximate multiple-valued nonlinearities by single-valued nonlinearities in high-gain feedback loops. The same idea appears in [11], pp. 74-75, where a nonlinear feedback loop gives rise to hysteretic behavior. Both of these examples involve a deadzone nonlinearity and a semistable transfer function. To understand how these ingredients give rise to hysteretic behavior, we first review the concept of semistability.

The concept of semistability lies between Lyapunov stability and asymptotic stability. In the linear case, an asymptotically stable system has only open left half plane poles, while the poles of a Lyapunov stable system may lie in the open left half plane or on the imaginary axis so long as the latter are semisimple. On the other hand, the poles of a semistable system may lie in the open left half plane or at the origin, and the latter are semisimple. The class of linear semistable systems was identified in [8] and applied to damped rigid body motion (that is, mechanical systems with semidefinite stiffness) in [2]. Because of the pole at the origin, semistable systems have convergent free response to non-unique equilibria.

Recently, the concept of semistability was extended to nonlinear systems in [4], which provided Lyapunov tests. In this context, an equilibrium is semistable if it is Lyapunov stable and all trajectories beginning in a neighborhood of the equilibrium converge to Lyapunov stable equilibria. A system is semistable if all of its equilibria are semistable. The Lyapunov test for semistability given in [4] assumes semidefinite Lyapunov function and its derivative and requires that the largest invariant set of the zero set of the Lyapunov derivative consist only of equilibria.

Returning to the examples in [10, 11] mentioned above, these feedback models are nonlinear semistable systems with the additional property that the step response is convergent. In the present paper we show that step-convergent semistable systems give rise to multiple-valued maps under quasi-static operation, that is, hysteresis. By providing a connection between semistability and hysteresis, our goal is to provide a class of differential models for representing hysteretic behavior.

2 Motivating Example

To begin, we consider the system shown in Figure 1(a).We excite the system with sinusoidal position input $r(t)$, and plot the Lissajous figure of mass position $q(t)$ versus input $r(t)$ as shown in Figure 2(a). Next we replace $r(t)$ with a sequence of successively slower sinusoids and obtain Figure 2.Figure 2(b)shows that the response of the system in quasi-static (very low frequency) operation is modeled by a *single-valued* map corresponding to the DC gain of the system. If we excite the system with a triangle wave

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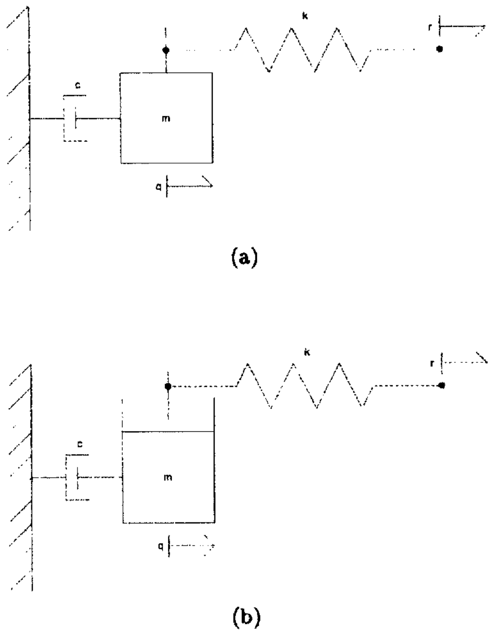

Figure 1: (a) Mass/Dashpot/Spring, (b) Mass/Dashpot/Spring with Deadzone

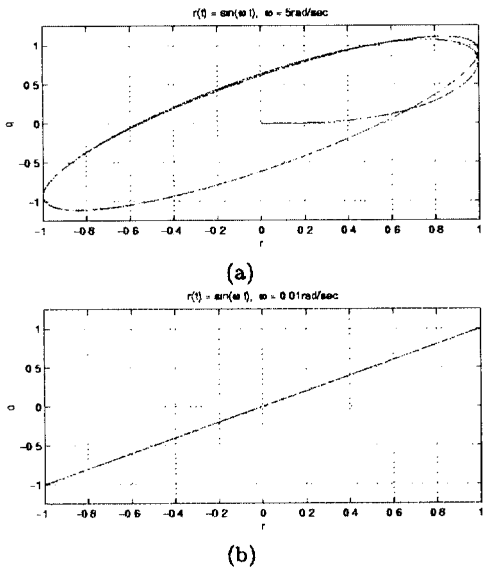

Figure 2: Response of System Shown in Figure 1(a).

input, we observe the same limiting behavior.

Next we introduce a gap where the spring attaches to the mass as shown in Figure 1(b).We again apply a sequence of successively slower sinusoidal inputs $r(t)$ as shown in Figure 3(a) and (c). In this case, the map obtained in quasi-static operation is *multi-valued.* As before, we also excite the system with a triangle wave input and observe the same limiting map as shown in Figures 3(b) and (d).This suggests that the quasi-static map is independent of the detailed behavior of the input sequence. The multi-valued behavior shown in Figure 3is characteristic of hysteretic systems. Note that $r(t)$ is a continuous, piecewise monotonic function and that the map only depends on the values of the local maxima and local minima of $r(t)$.

The key difference between the system responses shown in Figure 1(a) and Figure 1(b)is that, for constant $r(t)$, the system in Figure 1(a)has a unique equilibrium, whereas the system in Figure 1(b)has multiple equilibria. In addition, the former is asymptotically stable whereas the latter is semistable. Intuitively speaking, the trajectories of the semistable system rapidly converge to a member of the equilibrium set during quasi-static operation. In the next section we review the notion of semistability.

3 Step-Convergent Semistable Systems and Hysteresis

To investigate the relationship between hysteresis and semistability, we introduce the following framework. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be continuous, and let

$$\dot{x}(t) = f(x(t), u(t)). \tag{3.1}$$

We now review the following definitions for (3.1).

Consider (3.1) with constant $u(t) = u$ and equilibrium $\bar{x}$. The equilibrium $\bar{x}$ is *semistable* if it is Lyapunov stable and if there exists $\delta > 0$ such that $\|x(0) - \bar{x}\| < \delta$ implies $\lim_{t\to\infty} x(t)$ exists and is a Lyapunov stable equilibrium. The system (3.1) is *semistable* if all of its equilibria are semistable and $\lim_{t\to\infty} x(t)$ exists for all $x(0) \in \mathbb{R}^n$. Finally, (3.1) is *step-convergent semistable* if it is semistable for all constant $u \in \mathbb{R}$.

The linear system $\dot{x} = Ax$ is semistable if all eigenvalues of A lie in the open left half plane, except possibly for a semisimple eigenvalue at the origin. The following result concerns the system $\dot{x} = Ax + Bu$ with constant u.

Proposition 1 *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$. Then the linear system*

$$\dot{x} = Ax + Bu \tag{3.2}$$

is step-convergent semistable if and only if the system $\dot{x} = Ax$ is semistable and $AA^{\#}B = B$. In this case,

$$\lim_{t\to\infty} x(t) = (I - AA^{\#})x(0), \tag{3.3}$$

where $A^{\#}$ denotes the group inverse of A. ***Proof*** *Using equation (17), p. 176 of [7], the solution to (3.2) is given by*

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A\tau} Bu\, d\tau \\ &= e^{At}x(0) + \left[A^{\#}e^{At} + \left(I - AA^{\#}\right)t\right]Bu. \end{aligned} \tag{3.4}$$

For $\dot{x} = Ax$ semistable,

$$\lim_{t\to\infty} e^{At} = I - AA^{\#}. \tag{3.5}$$

Therefore the $\lim_{t\to\infty} x(t)$ exists if and only if $AA^{\#}B = B$, and is given by (3.3). □

The following test for semistability of (3.1) with constant u is given as Theorem 3.3 of [4].

Theorem 1 *Let $u \in \mathbb{R}$ be constant. Suppose every solution of (3.1) is bounded and there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x) \geq 0$ and $\dot{V} \leq 0$ for all $x \in \mathbb{R}^n$. If every point in the largest invariant subset $\mathcal{M}$ of $\dot{V}^{-1}(0)$ is a Lyapunov stable equilibrium point, then (3.1) is semistable.* □

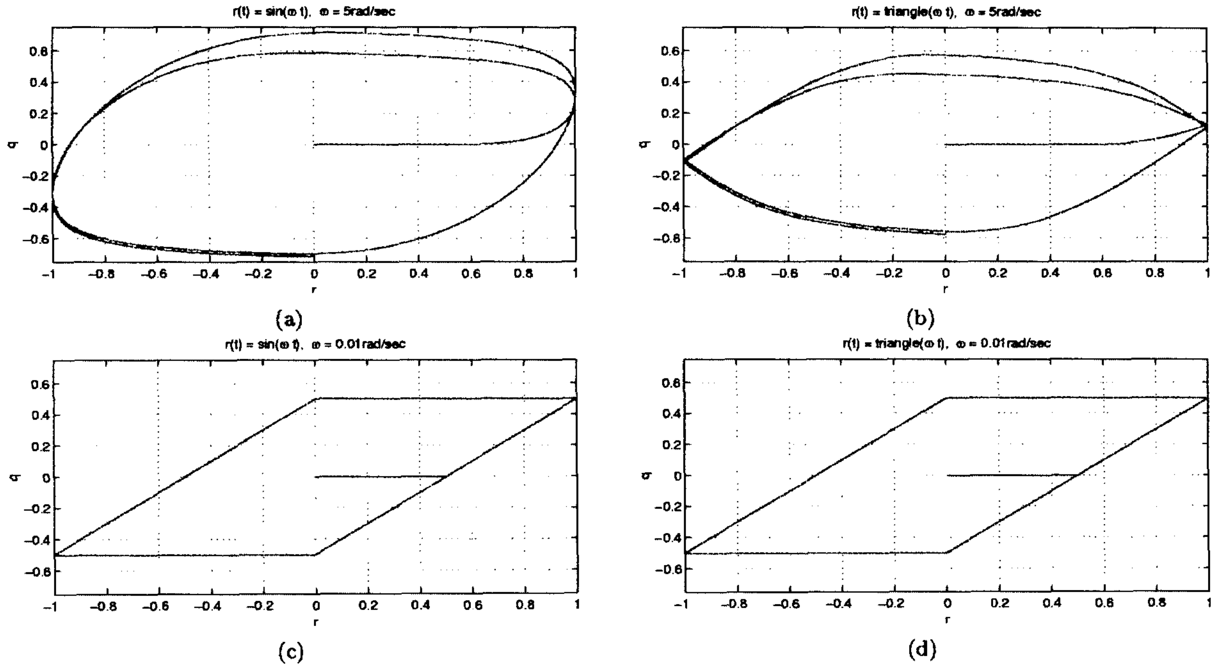

Figure 3: Response of System Shown in Figure 1(b)

To clarify the notion of step-convergent semistability, consider the SISO linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{3.6}$$

where $\dot{x} = Ax$ is semistable. Corresponding to (3.6) is the transfer function

$$\hat{y}(s) = \frac{1}{s} G_0(s)\hat{u}(s), \tag{3.7}$$

where $G_0(s)$ is asymptotically stable. Letting $\hat{u}(s) = \frac{1}{s}$ it follows that $\hat{y}(s)$ has a repeated pole at the origin and thus $\lim_{t\to\infty} y(t)$ does not exist. Hence the linear system (3.6) cannot be step-convergent semistable. This observation is not surprising since semistable systems have infinite gain at low frequency. Therefore if the system (3.1) is step-convergent semistable and not asymptotically stable, then it is not linear.

4 Hysteretic Models

In this section we illustrate several examples of step-convergent semistable systems that are not asymptotically stable. As noted above, such systems are necessarily nonlinear. To examine the quasi-static response we excite each system with a sequence of successively slower input signals. For each example we present the hysteretic map obtained.

4.1 Mass/Dashpot/Spring with Deadzone

Consider the mass-dashpot system driven through a spring with a deadzone as shown in Figure 1(b).The input is the free end of the spring ($u(t) = r(t)$) and the output is the position of the mass ($y(t) = q(t)$). The states of the system are $x(t) = [q(t)\ \dot{q}(t)]^{\mathrm{T}}$. The system obeys the differential equation

$$m\ddot{q}(t) + c\dot{q}(t) + kd_w\left(q(t) - r(t)\right) = 0, \tag{4.1}$$

where the deadzone function $d_w(z)$ with width $w > 0$ is defined as

$$d_w(z) \triangleq \begin{cases} z - \frac{w}{2} & \text{if} \quad z > \frac{w}{2} \\ z + \frac{w}{2} & \text{if} \quad z < -\frac{w}{2} \\ 0 & \text{else.} \end{cases} \tag{4.2}$$

To show that (4.1) is semistable for each constant r, consider the Lyapunov candidate

$$V(q,\dot{q}) = \frac{1}{2}\left(\frac{k}{m} d_w(q-r)^2 + \dot{q}^2\right), \tag{4.3}$$

where r is constant. Differentiating, we find

$$\dot{V}(q,\dot{q}) = -\frac{c}{m}\dot{q}^2. \tag{4.4}$$

Now we examine the set $\dot{q} = 0$. We see that the only trajectories that remain in $\dot{V} = \dot{q} = 0$ are those for which $d_w(q-r) = 0$, or for which $-\frac{w}{2} < q - r < \frac{w}{2}$. Hence, the set $\dot{q} = 0, q \in \left(r - \frac{w}{2}, r + \frac{w}{2}\right)$ contains all the equilibria (a continuum of them) and they are Lyapunov stable. Also note that since $\dot{q} = 0$ in this set, all trajectories converge. Since all trajectories converge to semistable equilibria, we conclude (4.1) is step-convergent semistable.

Figure 3shows the hysteretic maps derived under both sinusoidal and triangle inputs for $w = c = 1$, $k = 10$, and $m = 0.1$. Note that both responses have the same limit as $\dot{r}(t) \to 0$.

4.2 Single Integrator With Deadzone

Consider the single integrator system

$$m\dot{x}(t) + kd_w\left(x(t) - r(t)\right) = 0 \tag{4.5}$$

where $y(t) = r(t)$ and $x(t) = y(t)$. Using the Lyapunov function $V = \frac{1}{2}d_w(x-r)^2$ we see (4.5) is semistable. Similar to above, we apply both sinusoidal and triangle inputs, and compute the hysteretic map as $\dot{r}(t) \to 0$ for $m = 0.1$, and $k = w = 1$. Results are similar to those presented in Figure 3.

4.3 Mass/Spring with Square-Root Dashpot

Here we examine the finite-time stable system of the mass driven by a spring on a square-root dashpot, See Figure 1(a).The equation of motion for the system is

$$m\ddot{q}(t) + c\ \mathrm{sign}\dot{q}(t)\sqrt{|\dot{q}(t)|} + kd_w\left(q(t) - r(t)\right) = 0, \quad (4.6)$$

with $y(t) = r(t)$, $y(t) = q(t)$, and $x(t) = [q(t)\ \dot{q}(t)]^{\mathrm{T}}$. Since $\mathrm{sign}x\sqrt{|x|}$ is continuous, existence of solutions is guaranteed. See [3] for a discussion of uniqueness and finite time convergence. Results are similar to those presented in Figure 3.

4.4 Mass/Spring with Stiction

We now consider a mass sliding on a surface with stiction driven by a spring as in Figure 4. We model this system

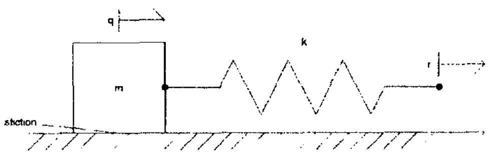

Figure 4: Mass/Spring with Stiction

as:

$$m\ddot{q}(t) + F_{\mathrm{f}} + k\left(q(t) - r(t)\right) = 0, \quad (4.7)$$

where the friction force F_{f} is given by [9]

$$F_{\mathrm{f}} = \left(1 - \kappa(t)\right)F_{\mathrm{s}} + \kappa(t)F_{d}, \quad (4.8)$$

$$F_{\mathrm{s}} = \mathrm{sat}_{\alpha_0+\alpha_1}\left(k_{\mathrm{s}}\eta(t) + d_{\mathrm{s}}\dot{q}(t)\right), \quad (4.9)$$

$$F_{d} = \alpha_2\dot{q}(t) + \alpha_0\frac{2}{\pi}\arctan\left(\alpha_3\dot{q}(t)\right), \quad (4.10)$$

$$\dot{\kappa}(t) = \tau\left(-\kappa(t) + 1 - e^{-\left(\frac{\dot{q}(t)}{\dot{q}(0)}\right)^2}\right), \quad (4.11)$$

$$\dot{\eta}(t) = \left(1 - \kappa(t)\right)\dot{q}(t) - \mu\kappa(t)\eta(t), \quad (4.12)$$

where

$$\mathrm{sat}_s x = \begin{cases} s & \text{if} \quad x \geq s \\ x & \text{if} \quad |x| < s \\ -s & \text{if} \quad x \leq -s. \end{cases} \quad (4.13)$$

In this example $u(t) = r(t)$, $y(t) = q(t)$, and $x(t) = [q(t)\ \dot{q}(t)\ \kappa(t)\ \eta(t)]^{\mathrm{T}}$. In Figure 5 we present the system response for both sinusoidal and triangle inputs with $m = 0.1$, $k = \tau = \alpha_3 = k_s = d_s = 100$, $\alpha_0 = \alpha_2 = \dot{q}(0) = 1$, $\alpha_1 = 1.5$, and $\mu = 1000$. Note that both responses have the same limit as $\dot{r}(t) \to 0$.

4.5 Mass/Dashpot with Bouc-Wen Restoring Force

Next we consider the Bouc-Wen restoring force model [5, 16, 20, 21]. The system consists of a mass, dashpot, and a nonlinear restoring force, modeled by

$$m\ddot{q}(t) + c\dot{q}(t) + kq(t) + z(t) = F(t) \quad (4.14)$$

$$\dot{z}(t) = \kappa\dot{q}(t) - \gamma\dot{q}(t)\left|z(t)\right|^n - \beta\left|\dot{q}(t)\right|\left|z(t)\right|^{n-1}z(t) \quad (4.15)$$

where $q(t), z(t) \in \mathbb{R}$. The parameters $m, c, \kappa, \beta, n \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}$ determine the system mass, damping, and the shape of the hysteresis map. In this example, $u(t) = r(t)$, $y(t) = q(t)$, and $x(t) = [q(t)\ \dot{q}(t)\ z(t)]^{\mathrm{T}}$. In the simulation, we take $m = c = k = \kappa = n = \gamma = 1$, and $\beta = 2$. In Figure 6 we present simulation results for both sinusoidal and triangle inputs. As can be seen, both responses have the same limit as $\dot{F}(t) \to 0$.

5 Conclusion

In this paper we introduced a new class of finite-dimensional differential models for representing hysteresis. The class consists of step-convergent semistable systems which are not asymptotically stable. Under quasi-static conditions, these systems yield multi-valued response maps. This property was demonstrated with several physical examples.

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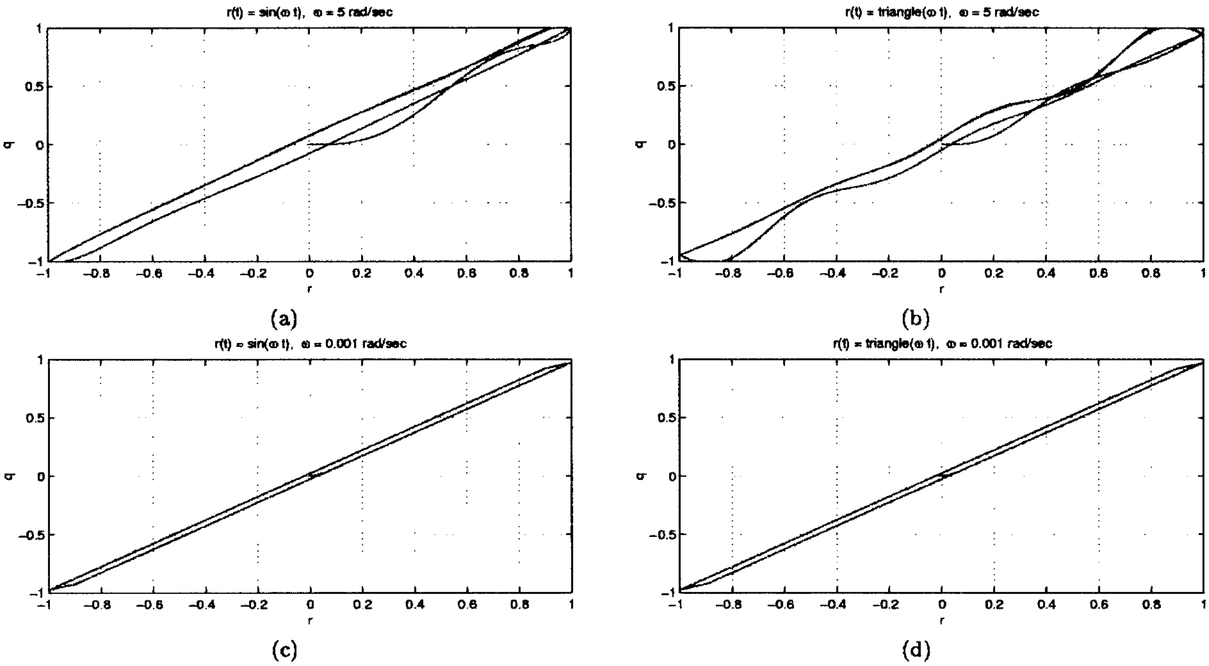

Figure 5: Mass/Spring with Stiction Response for both Sinusoidal and Triangle Inputs

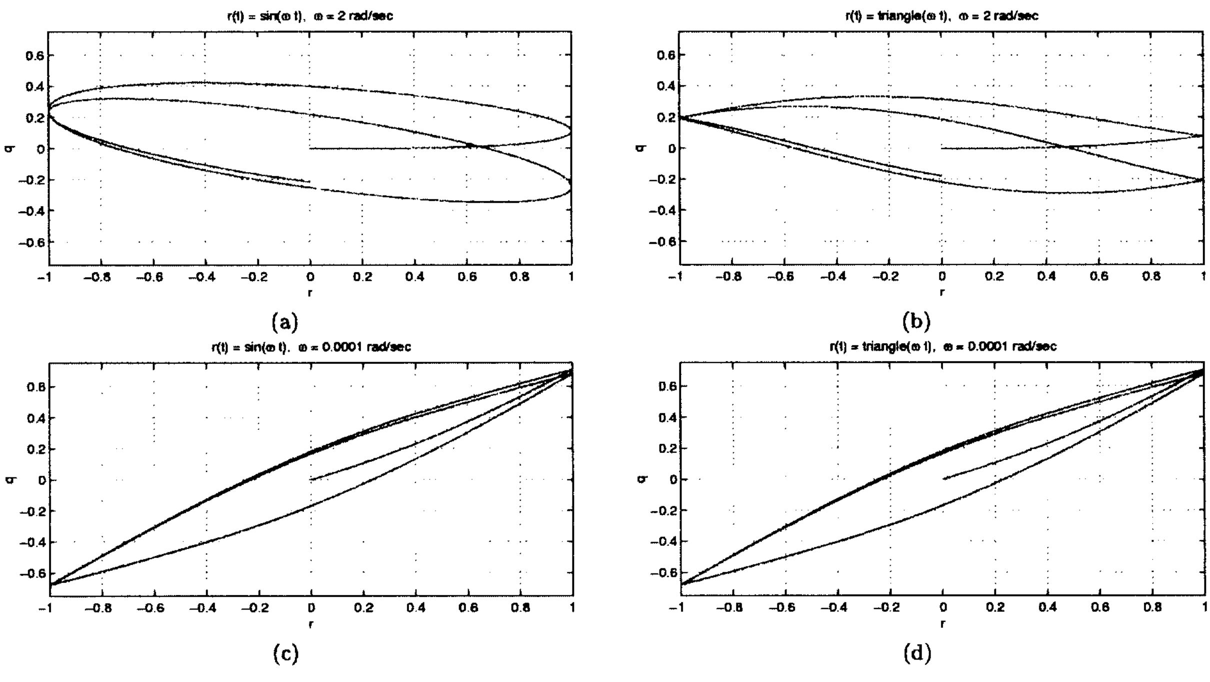

Figure 6: Mass/Dashpot with Bouc-Wen Restoring Force Hysteresis Map for Sinusoidal and Trapezoidal Inputs

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