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Subspace Identification for Nonlinear Systems That are Linear in Unmeasured States[1]

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Abstract

In this paper we apply subspace methods to the identification of a class of multi-input multi-output discrete-time nonlinear time-varying systems. Specifically, we study the identification of systems that are nonlinear in measured data and linear in unmeasured states. We present numerical simulations to demonstrate the efficacy of the method.

1 Introduction

System identification is the process of estimating system dynamics using measured data. These identified models can then be used for control and estimator design, model validation, system analysis, and output prediction. Linear system identification has been well studied [1, 3, 8–11, 13–15], and nonlinear system identification has received increasing attention [5, 7, 16–18, 20].

Subspace methods have been applied to linear systems with great success [1, 10, 15, 19]. Three of the most developed and widely used algorithms, CVA, N4SID, and MOESP have been implemented in MATLAB packages [2, 6, 12]. These methods are computationally simple and naturally applied to MIMO systems. Recently, consistency of one subspace approach under modest conditions has been shown [4].

Nonlinear and time-varying subspace identification methods have been devised in [17, 18, 20]. In this paper we study systems that are nonlinear in measured data and linear in unmeasured states, see Figure 1. Our

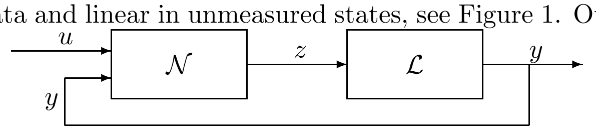

Figure 1: Block diagram of a system linear in unmeasured states: $\mathcal{N}$ is a nonlinear function of current and past data, $\mathcal{L}$ is a linear dynamic system.

method consists of three steps. First, we approximate the nonlinearities in the system as finite sums of known basis functions with unknown coefficients. Then we estimate the state sequence. Finally we use the state sequence to estimate the unknown system coefficients. Our method allows for the incorporation of prior knowledge through the selection of the basis functions.

This paper is organized as follows. In Section 2 we define the problem and the notation to be used throughout the paper. In Section 3 we derive the main results of the paper in the zero noise case. In Section 4 we estimate the system order and the state sequence in the presence of noise. In Section 5 we use the state estimate to calculate the system coefficients. In Section 6 we summarize the identification procedure. Finally, in Section 7 we demonstrate the method with numerical examples.

2 Nonlinear Subspace Identification

Here we study systems of the form

$$x(k+1) = Ax(k) + w(k) + F(k, u(k), \dots, u(k-b), y(k), \dots, y(k-b)), \quad (2.1)$$

$$y(k) = Cx(k) + Ew(k) + v(k) + G(k, u(k), \dots, u(k-b), y(k-1), \dots, y(k-b)) \quad (2.2)$$

This structure models multi-input, multi-output systems that depend linearly on the unmeasured states, but nonlinearly on the measured quantities $k \in \mathbb{Z}_+$, u, and y. For convenience we rewrite (2.1), (2.2) as

$$x(k+1) = Ax(k) + w(k) + F(k, u(k-b:k), y(k-b:k)), \quad (2.3)$$

$$y(k) = Cx(k) + Ew(k) + v(k) + G(k, u(k-b:k), y(k-b:k-1)), \quad (2.4)$$

where we use the convention

$$a(i:j) \triangleq \begin{bmatrix} a(i) & \cdots & a(j) \end{bmatrix}, \qquad i \le j, \quad (2.5)$$

and b is the number of delays to consider in the model, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $F : \mathbb{R} \times \mathbb{R}^{m\times b+1} \times \mathbb{R}^{p\times b+1} \to \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^p$, $C \in \mathbb{R}^{p\times n}$, $G : \mathbb{R} \times \mathbb{R}^{m\times b+1} \times \mathbb{R}^{p\times b} \to \mathbb{R}^p$. In addition $w(k) \in \mathbb{R}^n$ and $v(k) \in \mathbb{R}^p$ represent state and output noise, respectively. The presence of E in (2.4) is used to model correlated state and measurement noise.

The model (2.3), (2.4) includes classical model structures as special cases. For example, to capture a Hammerstein system, in which the nonlinearities are functions of the input, we write (2.3) and (2.4) as

$$x(k+1) = Ax(k) + F(u(k)) + w(k) \quad (2.6)$$

$$y(k) = Cx(k) + G(u(k)) + Ew(k) + v(k). \quad (2.7)$$

For a nonlinear-feedback system, in which the input is a nonlinear function of the output , we write (2.3) and (2.4) as

$$x(k+1) = Ax(k) + Bu(k) + F(y(k)) + w(k) \quad (2.8)$$

$$y(k) = Cx(k) + Du(k) + Ew(k) + v(k). \quad (2.9)$$

We assume F and G can be exactly represented as linear combinations of a finite set of known basis functions $f_i : \mathbb{R} \times \mathbb{R}^{m\times b+1} \times \mathbb{R}^{p\times b+1} \to \mathbb{R}$, $g_i : \mathbb{R} \times \mathbb{R}^{m\times b+1} \times \mathbb{R}^{p\times b} \to \mathbb{R}$, and $h_i : \mathbb{R} \times \mathbb{R}^{m\times b+1} \times \mathbb{R}^{p\times b} \to \mathbb{R}$ with unknown matrix coefficients $B_1 \in \mathbb{R}^{n\times r}$, $D_1 \in \mathbb{R}^{p\times s}$, $B_2 \in \mathbb{R}^{n\times t}$, and $D_2 \in \mathbb{R}^{p\times t}$. Using this notation we can write

$$F(k, u(k-b:k), y(k-b:k)) = B_1 f(k) + B_2 h(k), \quad (2.10)$$

$$G(k, u(k-b:k), y(k-b:k-1)) = D_1 g(k) + D_2 h(k), \quad (2.11)$$

where $f(k) \in \mathbb{R}^r$, $g(k) \in \mathbb{R}^s$, and $h(k) \in \mathbb{R}^t$ are given by

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$$f(k) \triangleq \begin{bmatrix} f_1(k, u(k-b:k), y(k-b:k)) \\ \vdots \\ f_r(k, u(k-b:k), y(k-b:k)) \end{bmatrix}, \quad (2.12)$$

$$g(k) \triangleq \begin{bmatrix} g_1(k, u(k-b:k), y(k-b:k-1)) \\ \vdots \\ g_s(k, u(k-b:k), y(k-b:k-1)) \end{bmatrix}, \quad (2.13)$$

$$h(k) \triangleq \begin{bmatrix} h_1(k, u(k-b:k), y(k-b:k-1)) \\ \vdots \\ h_t(k, u(k-b:k), y(k-b:k-1)) \end{bmatrix}. \quad (2.14)$$

Note that g and h are functions of delayed outputs only, while f is a function of both delayed and current outputs. If the functions F and G cannot be exactly represented as a finite sum of basis functions, the expansions (2.10) and (2.11) can be regarded as approximations. The basis functions in f, g, and h are sorted according to whether they appear in the expansion of F, G, or both. Specifically, $f(k)$ is a list of the basis functions that appear only in the expansion of F; $g(k)$ is a list of the basis functions that appear only in the expansion of G; and $h(k)$ is a list of the basis functions that appear in the expansions of both F and G. Without this convention, $Z_q(k)$ defined below would not have full row rank.

With the notation (2.10), (2.11) we can rewrite (2.3) and (2.4) as

$$x(k+1) = Ax(k) + B_1 f(k) + B_2 h(k) + w(k), \quad (2.15)$$

$$y(k) = Cx(k) + D_1 g(k) + D_2 h(k) + Ew(k) + v(k). \quad (2.16)$$

Using (2.15) and (2.16), we construct the block-matrix equation

$$Y_q(k) = \Gamma_q x(k:k+\ell-2q) + \Lambda_q Z_q(k) + \Upsilon_q N_q(k), \quad (2.17)$$

where q is a user-defined window length denoting the number of block rows in (2.17), and ℓ is a second user-defined window length which denotes the number of columns $-2q+1$ in (2.17). ℓ is often used to denote the number of measurements of u and y available. For notational convenience, we define $\sigma \triangleq r+s+t$. Next, we define the block-Hankel matrix $Y_q(k) \in \mathbb{R}^{pq \times \ell-2q+1}$ as

$$Y_q(k) \triangleq \begin{bmatrix} y(k:k+\ell-2q) \\ \vdots \\ y(k+q-1:k+\ell-q-1) \end{bmatrix},$$

the extended observability matrix $\Gamma \in \mathbb{R}^{pq \times n}$ as

$$\Gamma_q \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix},$$

the block-Toeplitz matrix $\Lambda_q \in \mathbb{R}^{pq \times q\sigma}$ as

$$\Lambda_q \triangleq \left[\begin{matrix} 0 & D_1 & D_2 \\ CB_1 & 0 & CB_2 \\ CAB_1 & 0 & CAB_2 \\ \vdots & \vdots & \vdots \\ CA^{q-2}B_1 & 0 & CA^{q-2}B_2 \end{matrix}\right. \left.\begin{matrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & D_1 & D_2 & \cdots & 0 \\ CB_1 & 0 & CB_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ CA^{q-3}B_1 & 0 & CA^{q-3}B_2 & \cdots & D_2 \end{matrix}\right],$$

the block-Hankel matrix $Z_q(k) \in \mathbb{R}^{q\sigma \times \ell-2q+1}$ as

$$Z_q(k) \triangleq \begin{bmatrix} z(k:k+\ell-2q) \\ \vdots \\ z(k+q-1:k+\ell-q-1) \end{bmatrix},$$

the column vector $z(k) \in \mathbb{R}^{\sigma}$ as

$$z(k) \triangleq \begin{bmatrix} f(k) \\ g(k) \\ h(k) \end{bmatrix},$$

the block-Toeplitz matrix $\Upsilon_q \in \mathbb{R}^{pq \times q(n+p)}$ as

$$\Upsilon_q \triangleq \begin{bmatrix} E & I & 0 & 0 & 0 & 0 & \cdots & 0 \\ C & 0 & E & I & 0 & 0 & \cdots & 0 \\ CA & 0 & C & 0 & E & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ CA^{q-2} & 0 & CA^{q-3} & 0 & CA^{q-4} & 0 & \cdots & I \end{bmatrix},$$

and the block-Hankel matrix $N_q(k) \in \mathbb{R}^{q(n+p) \times \ell-2q+1}$ as

$$N_q(k) \triangleq \begin{bmatrix} w(k:k+\ell-2q) \\ v(k:k+\ell-2q) \\ \vdots \\ w(k+q-1:k+\ell-q-1) \\ v(k+q-1:k+\ell-q-1) \end{bmatrix}.$$

Finally, we define the data matrix $\Delta_q(k) \in \mathbb{R}^{q(p+\sigma) \times \ell-2q+1}$ as

$$\Delta_q(k) \triangleq \begin{bmatrix} Y_q(k) \\ Z_q(k) \end{bmatrix}. \quad (2.24)$$

3 State Reconstruction

In this section, we give conditions under which the state sequence can be exactly reconstructed from noise free data. For $V \in \mathbb{R}^{n \times m}$ let $\mathcal{R}(V)$ denote the range (column space) of V. Then $\mathcal{R}(V^{\mathrm{T}})$ is the row space of V. Let $V^{\mathrm{L}} \triangleq \left(V^{\mathrm{T}}V\right)^{-1}V^{\mathrm{T}}$ and $V^{\mathrm{R}} \triangleq V^{\mathrm{T}}\left(VV^{\mathrm{T}}\right)^{-1}$ denote left and right inverses of V, respectively. We also define the projection $\Pi_V \triangleq V^{\mathrm{R}}V = V^{\mathrm{T}}\left(VV^{\mathrm{T}}\right)^{-1}V$ and $\Pi_V^{\perp} \triangleq I - \Pi_V$. Note that $V\Pi_V = V$ and $V\Pi_V^{\perp} = 0$.

Theorem 1 *Assume the following conditions are satisfied:*

i) rank $\Gamma_q = n$

ii) $w(k)$ and $v(k)$ are zero for all $k \in \mathbb{Z}_+$.

iii) rank $\begin{bmatrix} Z_q(k) \\ Z_q(k+q) \end{bmatrix} = 2q\sigma$ *for all $k \in \mathbb{Z}_+$.*

iv) $\mathrm{rank}(x(k:k+\ell-2q)\Pi^{\perp}_{Z_q(k)}) = \mathrm{rank}\ x(k:k+\ell-2q)$ *for all $k \in \mathbb{Z}_+$.*

v) rank $x(k:k+\ell-2q)=n$ *for all* $k\in\mathbb{Z}_+$.
Then

$$q \geq n/p, \tag{3.1}$$

$$\ell \geq 2q(\sigma+1)+n-1, \tag{3.2}$$

$$n = \operatorname{rank}\begin{bmatrix}\Delta_q(k)\\ \Delta_q(k+q)\end{bmatrix} - 2q\sigma \text{ for all } k\in\mathbb{Z}_+, \tag{3.3}$$

$$\mathcal{R}(x(k+q:k+\ell-q)^{\mathrm{T}}) = \mathcal{R}(\Delta_q(k)^{\mathrm{T}})\cap\mathcal{R}(\Delta_q(k+q)^{\mathrm{T}}) \quad \text{for all } k\in\mathbb{Z}_+. \tag{3.4}$$

Proof (3.1) follows directly from *i*). To prove (3.4) we first show that (3.4) holds with "=" replaced with "⊆". Then we show that the left and right hand sides of (3.4) have the same dimension. Let $k\in\mathbb{Z}_+$.

First note that *ii*) implies that (2.17) has the form

$$Y_q(k) = \Gamma_q x(k:k+\ell-2q) + \Lambda_q Z_q(k) \tag{3.5}$$

and

$$Y_q(k+q) = \Gamma_q x(k+q:k+\ell-q) + \Lambda_q Z_q(k+q). \tag{3.6}$$

By *i*) we can solve (3.5) and (3.6) for the state matrices

$$x(k:k+\ell-2q) = \begin{bmatrix}\Gamma_q^{\mathrm{L}} & -\Gamma_q^{\mathrm{L}}\Lambda_q\end{bmatrix}\Delta_q(k), \tag{3.7}$$

$$x(k+q:k+\ell-q) = \begin{bmatrix}\Gamma_q^{\mathrm{L}} & -\Gamma_q^{\mathrm{L}}\Lambda_q\end{bmatrix}\Delta_q(k+q). \tag{3.8}$$

Using (2.15) and (3.7) the state matrix $x(k+q:k+\ell-q)$ can also be written as

$$\begin{aligned} x(k+q:k+\ell-q) &= A^q x(k:k+\ell-2q) + \Psi_q Z_q(k)\\ &= A^q\begin{bmatrix}\Gamma_q^{\mathrm{L}} & -\Gamma_q^{\mathrm{L}}\Lambda_q\end{bmatrix}\Delta_q(k) + \Psi_q Z_q(k)\\ &= \begin{bmatrix}A^q\Gamma_q^{\mathrm{L}} & \Psi_q - A^q\Gamma_q^{\mathrm{L}}\Lambda_q\end{bmatrix}\Delta_q(k), \end{aligned} \tag{3.9}$$

where $\Psi_q\in\mathbb{R}^{n\times q\sigma}$ is given by

$$\Psi_q \triangleq \begin{bmatrix} A^{q-1}\begin{bmatrix}B_1 & 0 & B_2\end{bmatrix} & A^{q-2}\begin{bmatrix}B_1 & 0 & B_2\end{bmatrix} & \cdots & \begin{bmatrix}B_1 & 0 & B_2\end{bmatrix}\end{bmatrix}. \tag{3.10}$$

(3.8) and (3.9) imply

$$\mathcal{R}(x(k+q:k+\ell-q)^{\mathrm{T}}) \subseteq \mathcal{R}(\Delta_q(k)^{\mathrm{T}})\cap\mathcal{R}(\Delta_q(k+q)^{\mathrm{T}}). \tag{3.11}$$

Next we show that the left and right hand sides of (3.4) have the same dimension. By *iii*), $\Pi^{\perp}_{Z_q(k)}$ exists. We multiply (3.5) on the right by $\Pi^{\perp}_{Z_q(k)}$ and use $Z_q(k)\Pi^{\perp}_{Z_q(k)}=0$ to obtain

$$Y_q(k)\Pi^{\perp}_{Z_q(k)} = \Gamma_q x(k:k+\ell-2q)\Pi^{\perp}_{Z_q(k)}. \tag{3.12}$$

Using *i*), *iv*), and *v*) yields

$$\begin{aligned}\operatorname{rank}(Y_q(k)\Pi^{\perp}_{Z_q(k)}) &= \operatorname{rank}(\Gamma_q x(k:k+\ell-2q)\Pi^{\perp}_{Z_q(k)})\\ &= \operatorname{rank}(x(k:k+\ell-2q)\Pi^{\perp}_{Z_q(k)})\\ &= \operatorname{rank}(x(k:k+\ell-2q))\\ &= n.\end{aligned} \tag{3.13}$$

Using *iii*) we can calculate

$$\begin{aligned}\operatorname{rank}(\Delta_q(k)) &= \operatorname{rank}\begin{bmatrix}Y_q(k)\\ Z_q(k)\end{bmatrix} = \operatorname{rank}\begin{bmatrix}Y_q(k)\Pi^{\perp}_{Z_q(k)}\\ Z_q(k)\end{bmatrix}\\ &= \operatorname{rank}(Z_q(k)) + \operatorname{rank}(Y_q(k)\Pi^{\perp}_{Z_q(k)}) = q\sigma+n.\end{aligned} \tag{3.14}$$

Since (3.14) hold at time $k+q$

$$\operatorname{rank}(\Delta_q(k+q)) = q\sigma+n. \tag{3.15}$$

Similarly,

$$\operatorname{rank}(\Delta_{2q}(k)) = 2q\sigma+n, \tag{3.16}$$

Writing $Z_{2q}(k)=\begin{bmatrix}Z_q(k)\\ Z_q(k+q)\end{bmatrix}$ and $Y_{2q}(k)=\begin{bmatrix}Y_q(k)\\ Y_q(k+q)\end{bmatrix}$ we see

$$\operatorname{rank}\begin{bmatrix}\Delta_q(k)\\ \Delta_q(k+q)\end{bmatrix} = \operatorname{rank}(\Delta_{2q}(k)) = 2q\sigma+n, \tag{3.17}$$

proving (3.3). Since $\Delta_{2q}(k)\in\mathbb{R}^{2q(p+\sigma)\times\ell-2q+1}$ it follows that (3.2). We can now calculate the dimension of the right hand side of (3.4),

$$\begin{aligned}\dim(\mathcal{R}(\Delta_q(k)^{\mathrm{T}})\cap\mathcal{R}(\Delta_q(k+q)^{\mathrm{T}})) &= \operatorname{rank}(\Delta_q(k))\\ &+ \operatorname{rank}(\Delta_q(k+q)) - \operatorname{rank}\begin{bmatrix}\Delta_q(k)\\ \Delta_q(k+q)\end{bmatrix} = n.\end{aligned} \tag{3.18}$$

(3.4) follows from *v*). □

Now we calculate the intersection of the row spaces of $\Delta_q(k)$ and $\Delta_q(k+q)$ and thus a representation of the state matrix $x(k+q:k+\ell-q)$.

Proposition 2 *Assume i) - v) of Theorem 1. Let* M, L_{11}, L_{22}, $L_{11}-L_{12}L_{22}^{-1}L_{21}$, *and* $L_{22}-L_{21}L_{11}^{-1}L_{12}$ *be nonsingular, where* $M\in\mathbb{R}^{\ell-2q+1\times\ell-2q+1}$, $L_{11}\in\mathbb{R}^{q(p+\sigma)\times q(p+\sigma)}$, $L_{12}\in\mathbb{R}^{q(p+\sigma)\times q(p+\sigma)}$, $L_{21}\in\mathbb{R}^{q(p+\sigma)\times q(p+\sigma)}$, *and* $L_{22}\in\mathbb{R}^{q(p+\sigma)\times q(p+\sigma)}$. *Consider the singular value decomposition*

$$L\begin{bmatrix}\Delta_q(k)\\ \Delta_q(k+q)\end{bmatrix}M = \begin{bmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{bmatrix}\begin{bmatrix}S_{11} & 0\\ 0 & 0\end{bmatrix}V^{\mathrm{T}}, \tag{3.19}$$

where $L=\begin{bmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{bmatrix}\in\mathbb{R}^{2q(p+\sigma)\times 2q(p+\sigma)}$, $S_{11}\in\mathbb{R}^{2q\sigma+n\times 2q\sigma+n}$, $U_{11}\in\mathbb{R}^{q(p+\sigma)\times 2q\sigma+n}$, $U_{12}\in\mathbb{R}^{q(p+\sigma)\times 2pq-n}$, $U_{21}\in\mathbb{R}^{q(p+\sigma)\times 2q\sigma+n}$, $U_{22}\in\mathbb{R}^{q(p+\sigma)\times 2pq-n}$, *and* $V\in\mathbb{R}^{\ell-2q+1\times\ell-2q+1}$. *Also let* $U_r\in\mathbb{R}^{2pq-n\times n}$, $S_r\in\mathbb{R}^{n\times n}$, *and* $V_r\in\mathbb{R}^{2q\sigma+n\times n}$ *be defined through the singular value decomposition*

$$\begin{aligned}&\left(U_{12}^{\mathrm{T}}L_{11}+U_{22}^{\mathrm{T}}L_{21}\right)\Big(\left(L_{11}-L_{12}L_{22}^{-1}L_{21}\right)^{-1}U_{11}\\ &\qquad -L_{11}^{-1}L_{21}\left(L_{22}-L_{21}L_{11}^{-1}L_{12}\right)^{-1}U_{21}\Big)S_{11}\\ &\quad = \begin{bmatrix}U_r & U_s\end{bmatrix}\begin{bmatrix}S_r & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_r^{\mathrm{T}}\\ V_s^{\mathrm{T}}\end{bmatrix} = U_rS_rV_r^{\mathrm{T}}.\end{aligned} \tag{3.20}$$

Then there exists nonsingular $T\in\mathbb{R}^{n\times n}$ *such that*

$$\begin{aligned}Tx(k+q:k+\ell-q) &= U_r^{\mathrm{T}}\left(U_{12}^{\mathrm{T}}L_{11}+U_{22}^{\mathrm{T}}L_{21}\right)\Delta_q(k)\\ &= \begin{bmatrix}S_rV_r^{\mathrm{T}} & 0\end{bmatrix}V^{\mathrm{T}}M^{-1}\\ &= -U_r^{\mathrm{T}}\left(U_{12}^{\mathrm{T}}L_{12}+U_{22}^{\mathrm{T}}L_{22}\right)\Delta_q(k+q).\end{aligned} \tag{3.21}$$

Proof Rewriting the singular value decomposition (3.19) as

$$\begin{bmatrix}U_{11}^{\mathrm{T}} & U_{21}^{\mathrm{T}}\\ U_{12}^{\mathrm{T}} & U_{22}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{bmatrix}\begin{bmatrix}\Delta_q(k)\\ \Delta_q(k+q)\end{bmatrix}M = \begin{bmatrix}S_{11} & 0\\ 0 & 0\end{bmatrix}V^{\mathrm{T}}, \tag{3.22}$$

$$\begin{bmatrix}U_{11}^{\mathrm{T}}\left(L_{11}\Delta_q(k)+L_{12}\Delta_q(k+q)\right)\\ U_{12}^{\mathrm{T}}\left(L_{11}\Delta_q(k)+L_{12}\Delta_q(k+q)\right)\\ +U_{21}^{\mathrm{T}}\left(L_{21}\Delta_q(k)+L_{22}\Delta_q(k+q)\right)\\ +U_{22}^{\mathrm{T}}\left(L_{21}\Delta_q(k)+L_{22}\Delta_q(k+q)\right)\end{bmatrix} = \begin{bmatrix}S_{11} & 0\\ 0 & 0\end{bmatrix}V^{\mathrm{T}}M^{-1}, \tag{3.23}$$

we see that

$$\left(U_{12}^{\mathrm{T}}L_{11}+U_{22}^{\mathrm{T}}L_{21}\right)\Delta_q(k) = -\left(U_{12}^{\mathrm{T}}L_{12}+U_{22}^{\mathrm{T}}L_{22}\right)\Delta_q(k+q). \tag{3.24}$$

(3.24) has $2pq-n$ rows, where $2pq-n\geq n$ by Theorem 1. In Theorem 1 we proved that only n of these rows are linearly independent. We now select n linear combinations of these rows to find a minimal set of vectors to span the intersection of the row spaces of $\Delta_q(k)$ and $\Delta_q(k+q)$ and thus find a representation of the state matrix $x(k+q:k+\ell-q)$. Rewriting (3.19) we have

$$\begin{bmatrix}\Delta_q(k)\\ \Delta_q(k+q)\end{bmatrix} = L^{-1}\begin{bmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{bmatrix}\begin{bmatrix}S_{11} & 0\\ 0 & 0\end{bmatrix}V^{\mathrm{T}}M^{-1}$$
$$= \begin{bmatrix}\left(L_{11}-L_{12}L_{22}^{-1}L_{21}\right)^{-1} & -L_{11}^{-1}L_{21}\left(L_{22}-L_{21}L_{11}^{-1}L_{12}\right)^{-1}\\ -L_{22}^{-1}L_{21}\left(L_{11}-L_{12}L_{22}^{-1}L_{21}\right)^{-1} & \left(L_{22}-L_{21}L_{11}^{-1}L_{12}\right)^{-1}\end{bmatrix}\begin{bmatrix}U_{11}S_{11} & 0\\ U_{21}S_{11} & 0\end{bmatrix}V^{\mathrm{T}}M^{-1}$$
$$= \begin{bmatrix}\left(L_{11}-L_{12}L_{22}^{-1}L_{21}\right)^{-1}U_{11}S_{11} - L_{11}^{-1}L_{21}\left(L_{22}-L_{21}L_{11}^{-1}L_{12}\right)^{-1}U_{21}S_{11} & 0\\ -L_{22}^{-1}L_{21}\left(L_{11}-L_{12}L_{22}^{-1}L_{21}\right)^{-1}U_{11}S_{11} + \left(L_{22}-L_{21}L_{11}^{-1}L_{12}\right)^{-1}U_{21}S_{11} & 0\end{bmatrix} \times V^{\mathrm{T}}M^{-1}. \quad (3.26)$$

Then

$$\left(U_{12}^{\mathrm{T}}L_{11}+U_{22}^{\mathrm{T}}L_{21}\right)\Delta_q(k) = \left(U_{12}^{\mathrm{T}}L_{11}+U_{22}^{\mathrm{T}}L_{21}\right) \times \begin{bmatrix}\left(L_{11}-L_{12}L_{22}^{-1}L_{21}\right)^{-1}U_{11}S_{11} - L_{11}^{-1}L_{21}\left(L_{22}-L_{21}L_{11}^{-1}L_{12}\right)^{-1}U_{21}S_{11} & 0\end{bmatrix} \times V^{\mathrm{T}}M^{-1}$$
$$= \begin{bmatrix}\left(U_{12}^{\mathrm{T}}L_{11}+U_{22}^{\mathrm{T}}L_{21}\right)\left(\left(L_{11}-L_{12}L_{22}^{-1}L_{21}\right)^{-1}U_{11} - L_{11}^{-1}L_{21}\left(L_{22}-L_{21}L_{11}^{-1}L_{12}\right)^{-1}U_{21}\right)S_{11} & 0\end{bmatrix} \times V^{\mathrm{T}}M^{-1}$$
$$= \begin{bmatrix}U_rS_rV_r^{\mathrm{T}} & 0\end{bmatrix}V^{\mathrm{T}}M^{-1} = U_r\begin{bmatrix}S_rV_r^{\mathrm{T}} & 0\end{bmatrix}V^{\mathrm{T}}M^{-1}. \quad (3.27)$$

Thus

$$U_r^{\mathrm{T}}\left(U_{12}^{\mathrm{T}}L_{11}+U_{22}^{\mathrm{T}}L_{21}\right)\Delta_q(k) = \begin{bmatrix}S_rV_r^{\mathrm{T}} & 0\end{bmatrix}V^{\mathrm{T}}M^{-1} = -U_r^{\mathrm{T}}\left(U_{12}^{\mathrm{T}}L_{12}+U_{22}^{\mathrm{T}}L_{22}\right)\Delta_q(k+q) \quad (3.28)$$

is a basis for the row space of $\left(U_{12}^{\mathrm{T}}L_{11}+U_{22}^{\mathrm{T}}L_{21}\right)\Delta_q(k) = -\left(U_{12}^{\mathrm{T}}L_{12}+U_{22}^{\mathrm{T}}L_{22}\right)\Delta_q(k+q)$, and a realization of the state matrix $x(k+q:k+\ell-q)$. □

4 Noise Effects

Since $w(k)$ and $v(k)$ are present in real systems, our goal is to apply the above results without *ii*). Referring to (3.3) and (3.21), we define the following approximations

$$\hat{n} \triangleq \operatorname{rank}\left(L\begin{bmatrix}\Delta_q(k)\\ \Delta_q(k+q)\end{bmatrix}M\right) - 2q\sigma, \quad (4.1)$$

$$\hat{x}(k+q:k+\ell-q) \triangleq U_r^{\mathrm{T}}\left(U_{12}^{\mathrm{T}}L_{11}+U_{22}^{\mathrm{T}}L_{21}\right)\Delta_q(k) = -U_r^{\mathrm{T}}\left(U_{12}^{\mathrm{T}}L_{12}+U_{22}^{\mathrm{T}}L_{22}\right)\Delta_q(k+q), \quad (4.2)$$

for n and $x(k+q:k+\ell-q)$, where we have arbitrarily set $T=I$ in (3.21) to obtain (4.2). T can be used to select the basis for the realization of the state sequence.

The problem of rank determination is central to estimating the order of the unknown system (4.1). The presence of $w(k)$ and $v(k)$ will generally add rank to $L\begin{bmatrix}\Delta_q(k)\\ \Delta_q(k+q)\end{bmatrix}M$. For computational purposes, we use the following technique to estimate the rank of $L\begin{bmatrix}\Delta_q(k)\\ \Delta_q(k+q)\end{bmatrix}M$. Define the singular value decomposition

$$L\begin{bmatrix}\Delta_q(k)\\ \Delta_q(k+q)\end{bmatrix}M \triangleq USV^{\mathrm{T}}. \quad (4.3)$$

Let $s(k)$ be defined as

$$s(k) \triangleq \begin{cases}\frac{S(k,k)}{S(k+1,k+1)}, & 2q\sigma \le k < \min(2q(p+\sigma), \ell-2q+1) \text{ and } S(k+1,k+1)\neq 0\\ 0 & \text{else.}\end{cases}$$

Then

$$\operatorname{numrank}\left(L\begin{bmatrix}\Delta_q(k)\\ \Delta_q(k+q)\end{bmatrix}M\right) \triangleq k^\star, \quad (4.5)$$

where $k^\star$ is given by $\quad s(k^\star) = \max s(k). \quad (4.6)$

In (4.1) we know that $L\begin{bmatrix}\Delta_q(k)\\ \Delta_q(k+q)\end{bmatrix}M$ has rank $\geq 2q\sigma$, so we define $s(k)$ only for $k \geq 2q\sigma$.

5 Coefficient Estimation

Now that we have an estimate $\hat{x}(k+q:k+\ell-q) \in \mathbb{R}^{n\times \ell-2q+1}$ of the state sequence $x(k+q:k+\ell-q)$ we proceed to estimate A, B_1, B_2, C, D_1, and D_2 as well as the covariance matrices of the noise sequences $Q \triangleq \mathcal{E}[ww^{\mathrm{T}}]$ and $R \triangleq \mathcal{E}[vv^{\mathrm{T}}]$, and the correlation matrix E. Consider the cost function

$$J(\hat{A},\hat{B}_1,\hat{B}_2,\hat{C},\hat{D}_1,\hat{D}_2) \triangleq \left\|\begin{bmatrix}\hat{x}(k+q+1:k+\ell-q)\\ y(k+q:k+\ell-q-1)\end{bmatrix} - \begin{bmatrix}\hat{A} & \hat{B}_1 & 0 & \hat{B}_2\\ \hat{C} & 0 & \hat{D}_1 & \hat{D}_2\end{bmatrix}\begin{bmatrix}\hat{x}(k+q:k+\ell-q-1)\\ z(k+q:k+\ell-q-1)\end{bmatrix}\right\|_{\mathrm{F}} = J_1(\hat{A},\hat{B}_1,\hat{B}_2) + J_2(\hat{C},\hat{D}_1,\hat{D}_2)$$

where

$$J_1(\hat{A},\hat{B}_1,\hat{B}_2) \triangleq \left\|\hat{x}(k+q+1:k+\ell-q) - \begin{bmatrix}\hat{A} & \hat{B}_1 & \hat{B}_2\end{bmatrix}R_1(k+q:k+\ell-q-1)\right\|_{\mathrm{F}},$$
$$J_2(\hat{C},\hat{D}_1,\hat{D}_2) \triangleq \left\|y(k+q:k+\ell-q-1) - \begin{bmatrix}\hat{C} & \hat{D}_1 & \hat{D}_2\end{bmatrix}R_2(k+q:k+\ell-q-1)\right\|_{\mathrm{F}},$$

and $\|\cdot\|_{\mathrm{F}}$ is the Frobenius matrix norm and

$$R_1(k) \triangleq \begin{bmatrix}\hat{x}(k)\\ f(k)\\ h(k)\end{bmatrix}, \qquad R_2(k) \triangleq \begin{bmatrix}\hat{x}(k)\\ g(k)\\ h(k)\end{bmatrix}.$$

Proposition 3 *The matrices*

$$\begin{bmatrix}\hat{A} & \hat{B}_1 & \hat{B}_2\end{bmatrix} \triangleq \hat{x}(k+q+1:k+\ell-q) \times R_1(k+q:k+\ell-q-1)^{\mathrm{R}}, \quad (5.2)$$
$$\begin{bmatrix}\hat{C} & \hat{D}_1 & \hat{D}_2\end{bmatrix} \triangleq y(k+q:k+\ell-q-1) \times R_2(k+q:k+\ell-q-1)^{\mathrm{R}}, \quad (5.3)$$

minimize the cost functions J_1 *and* J_2.

Proposition 4 E *and the noise covariance matrices can be estimated as*

$$\hat{Q} \triangleq \Sigma_{11}, \quad (5.4)$$
$$\hat{E} \triangleq \Sigma_{21}\Sigma_{11}^{-1}, \quad (5.5)$$
$$\hat{R} \triangleq \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{21}^{\mathrm{T}}, \quad (5.6)$$

where

$$\Sigma_{11} \triangleq \hat{x}(k+q+1:k+\ell-q) \times \Pi^{\perp}_{R_1(k+q:k+\ell-q-1)}\hat{x}(k+q+1:k+\ell-q)^{\mathrm{T}}, \quad (5.7)$$
$$\Sigma_{21} \triangleq y(k+q:k+\ell-q-1) \times \left(I - \Pi_{R_1(k+q:k+\ell-q-1)} - \Pi_{R_2(k+q:k+\ell-q-1)} + \Pi_{R_1(k+q:k+\ell-q-1)}\Pi_{R_2(k+q:k+\ell-q-1)}\right) \times \hat{x}(k+q+1:k+\ell-q)^{\mathrm{T}}, \quad (5.8)$$
$$\Sigma_{22} \triangleq y(k+q:k+\ell-q-1) \times \Pi^{\perp}_{R_2(k+q:k+\ell-q-1)}y(k+q:k+\ell-q-1)^{\mathrm{T}}. \quad (5.9)$$

6 The Algorithm

Here we list the steps in the nonlinear subspace identification algorithm.

1. Collect input-output data. Choose the window lengths q and ℓ. ℓ must be less than or equal to the number of data pairs available.

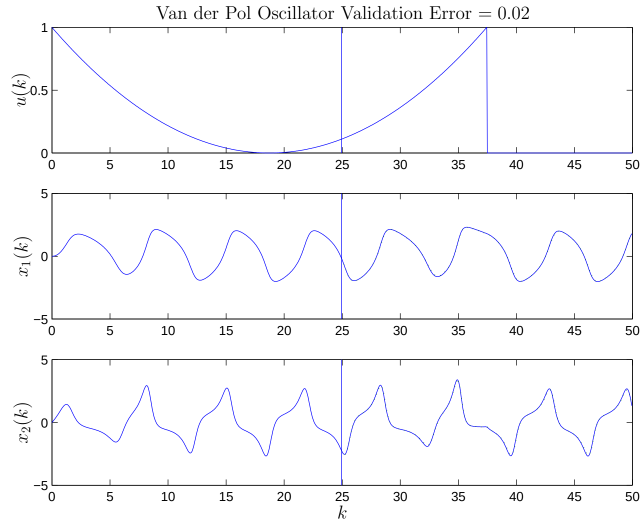

Figure 2: Time Traces of the Van der Pol Oscillator with 4 basis functions, vertical line indicates end of ID data set and beginning of validation data set, solid line indicates signal from true continuous time system, dashed line indicates signal from estimated discrete time system

2. Select the time frame b and basis functions f_i, g_i, and h_i to model the system.
3. Construct $\Delta_q(k)$ and $\Delta_q(k+q)$ as in (2.24).
4. Select weighting matrices L and M.
5. Calculate the primary singular value decomposition of $L\begin{bmatrix}\Delta_q(k)\\ \Delta_q(k+q)\end{bmatrix}M$ in (3.19) and obtain $\hat{n}$ in (4.1) as well as U_{11}, U_{12}, U_{21}, U_{22}, and S_{11}.
6. Calculate the second singular value decomposition in (3.20) and obtain U_r.
7. Estimate the state matrix in (4.2).
8. Estimate the system matrices $\hat{A}$, $\hat{B}_1$, $\hat{B}_2$, $\hat{C}$, $\hat{D}_1$, and $\hat{D}_2$ in (5.2) and (5.3). Estimate $\hat{E}$ and the noise covariance matrices $\hat{Q}$ and $\hat{R}$ in (5.4), (5.5), and (5.6).

7 Examples

Here we apply the nonlinear subspace identification algorithm to several nonlinear systems. We define the validation error as

$$e \triangleq \frac{\|y(1:\ell)-\hat{y}(1:\ell)\|_{\mathrm{F}}}{\|y(1:\ell)\|_{\mathrm{F}}}. \tag{7.1}$$

7.1 Van der Pol Oscillator

Here we consider the system

$$\dot{x}_1 = x_2 \tag{7.2}$$

$$\dot{x}_2 = -\omega^2 x_1 + \epsilon\omega\left(1-\mu^2 x_1^2\right)x_2 + u \tag{7.3}$$

with $\omega=\epsilon=\mu=1$. We excite this continuous time system with a zero-order-held sequence of $\ell=1000$ inputs with time interval $\tau=0.05s$, and measure ℓ outputs x_1 and x_2 with sampling rate $1/\tau=20$Hz. We choose all polynomials up to third order in x_1, x_2, and u as our basis functions. We then estimate a discrete time system (2.15), (2.16). To validate, we continue the input sequence and measure outputs of both the true continuous time system and the estimated discrete time system and compare the results. See Figure 2 and Figure 3.

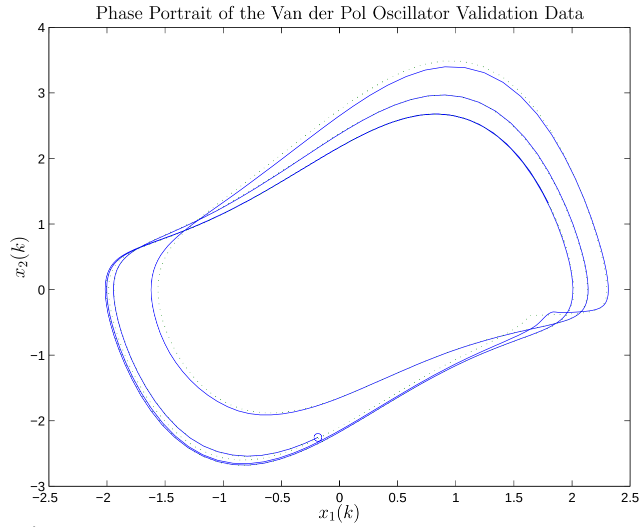

Figure 3: x_1 vs. x_2 Validation Signals for the Van der Pol Oscillator with 20 basis functions, solid line indicates signal from true continuous time system, dashed line indicates signal from estimated discrete time system

7.2 Planar Articulated Spacecraft

We model the planar motion of two flexible bodies linked by a hinge. For simplicity, we model only the first flexure mode of each link. The equations of motion are

$$\ddot{\theta}_1 = \frac{\sin\theta\left(\dot{\theta}_1^2\cos\theta-\lambda_2\dot{\theta}_2^2\right)}{-\lambda_1\lambda_2+\cos^2\theta}+\sqrt{2}\delta_1\ddot{\zeta}_1+T, \tag{7.4}$$

$$\ddot{\theta}_2 = -\frac{\sin\theta\left(\dot{\theta}_2^2\cos\theta-\lambda_1\dot{\theta}_1^2\right)}{-\lambda_1\lambda_2+\cos^2\theta}+\sqrt{2}\delta_2\ddot{\zeta}_2-T, \tag{7.5}$$

$$0 = \ddot{\zeta}_1+2c_1\omega_1\dot{\zeta}_1+\omega_1^2\zeta_1+\sqrt{2}\delta_1\ddot{\theta}_1, \tag{7.6}$$

$$0 = \ddot{\zeta}_2+2c_2\omega_2\dot{\zeta}_2+\omega_2^2\zeta_2+\sqrt{2}\delta_2\ddot{\theta}_2, \tag{7.7}$$

where $\lambda_1=J_1/\eta$, $\lambda_2=J_2/\eta$, $\eta=d_1d_2m$, $J_1=I_1+md_1^2$, $J_2=I_2+md_2^2$, $m=\frac{m_1m_2}{m_1+m_2}$ is the reduced mass, d_1 and d_2 are the distances from the hinge point to the center of mass of each body, m_1 and m_2 are the masses of the two bodies, $\theta=\theta_1-\theta_2$ is the angle between the two bodies, θ_1 and θ_2 are the angular positions of the two bodies with respect to an inertial frame, ζ_1 and ζ_2 are the fundamental flexible modes of each body, ω_1 and ω_2 are the modal frequencies, δ_1 and δ_2 are the coupling coefficients, and T is the control torque applied between the two bodies. Since $\lambda_1\lambda_2>1$ these equations do not have a singularity. We measure $\ell=2000$ data points with sampling rate $1/\tau=20$Hz. We take measurements of θ_1, θ_2, $\dot{\theta}_1$, and $\dot{\theta}_2$, and set

$$z(k)=h(k)=\begin{bmatrix} u(k-1)\\ \dot{\theta}_1^2(k-1)\sin(\theta_1(k-1)-\theta_2(k-1))\\ \dot{\theta}_2^2(k-1)\sin(\theta_1(k-1)-\theta_2(k-1))\\ \dot{\theta}_1^2(k-1)\sin 2(\theta_1(k-1)-\theta_2(k-1))\\ \dot{\theta}_2^2(k-1)\sin 2(\theta_1(k-1)-\theta_2(k-1))\end{bmatrix}, \tag{7.8}$$

a function of delayed data, with the nonlinearities periodic in θ. We obtain a fourth order discrete time model and plot the identification and validation data in Figure 4.

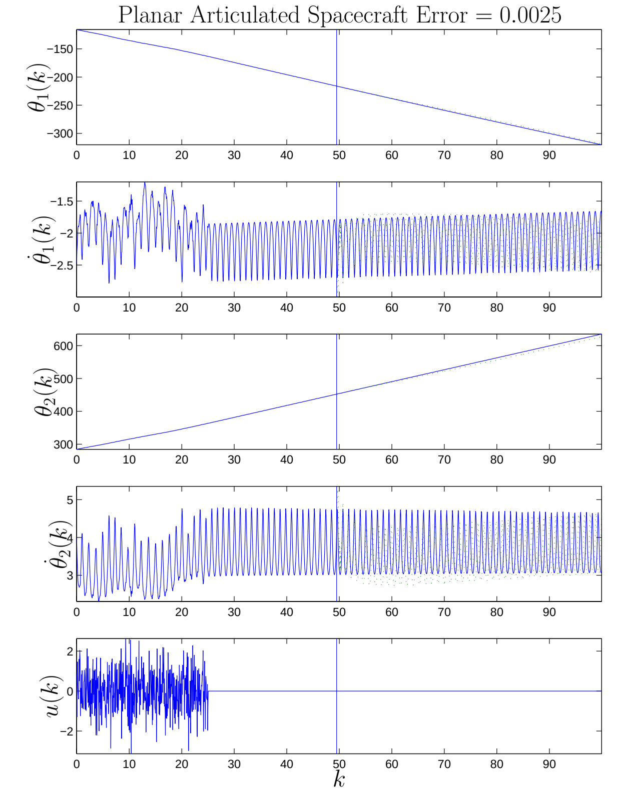

Figure 4: Time Trace of Measured Variables and Estimates, vertical line indicates end of ID data set and beginning of validation data set, solid line indicates signal from true continuous time system, dashed line indicates signal from estimated discrete time system

8 Conclusion

We presented a subspace-based identification method for identifying nonlinear time-varying systems that are nonlinear in measured data and linear in unmeasured states. We applied the algorithm to two numerical examples. Future work will focus on experimental applications, extending the class of identifiable systems, and methods for basis function selection.

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