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Adaptive Harmonic Steady State Control for Disturbance Rejection

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I. INTRODUCTION

The use of feedback control for disturbance rejection is of fundamental importance in a broad range of applications, and the development of effective algorithms is an ongoing area of research. For well-modeled plants with broadband disturbance, classical LQG theory can be applied with weighting filters introduced to shape controller effort in accordance with the disturbance spectrum and performance objectives [1]. On the other hand, if the disturbance is tonal or multi-tonal with known spectrum, then a model of the exogenous signal can be embedded in the controller to produce high gain feedback at frequencies that comprise the spectrum of the disturbance [2].

An alternative approach, which is applicable in the case of tonal or multi-tonal disturbances with known spectrum, allows the system to reach harmonic steady state and uses measurements of the steady state response amplitude and phase to determine the required control signal. This technique was developed independently within two research communities. For helicopter vibration reduction, this technique is known as *higher harmonic control* [3, 4]. The same technique was developed independently for active rotor balancing, in this case known as *convergent control* [5]. We refer to this algorithm as HSS (harmonic steady state) control. Connections between higher harmonic control and internal model control are discussed in [6].

When the plant and the disturbance are not well modeled, then the problem can be significantly more challenging. Within the active noise control literature, a host of adaptive algorithms have been developed inspired by digital signal processing techniques. These techniques are based on LMS updating of FIR filters [7]. Another approach involves continuously adjusting the frequency, magnitude and phase of the control input to cancel the disturbance [8]. Alternative techniques, which require limited modeling of the plant dynamics and disturbance spectrum, have also been developed; see, for example, [9].

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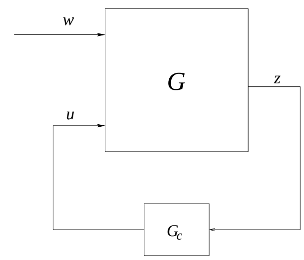

Fig. 1. Harmonic Steady State Control Architecture

Implementation of HSS control requires knowledge of the frequency response of the transfer function between the control input and the measurements at the disturbance frequency. In practice, this information is obtained through modeling or off-line identification. When this information is uncertain or when the plant is subject to change, instability can occur. Consequently, the robustness of HSS control is analyzed in [5] for both additive and multiplicative model uncertainty.

Adaptive extensions of HSS control that remove the need to independently model the control-to-measurement frequency response have been considered. Specifically, in [10, 11] the least squares procedure was proposed for estimating this transfer function. Analysis of convergence of the estimates and performance of HSS with simultaneous estimation were not discussed in [10, 11]. In the present paper we develop a unified framework for analyzing the properties and performance of adaptive harmonic steady state control, thus extending and including most of the previous literature on harmonic steady state control.

II. HARMONIC PERFORMANCE ANALYSIS

Assume for convenience that the disturbance $w \in \mathbb{R}^d$ acting on the plant is a single harmonic, with constant amplitude and phase. The HSS control algorithm waits until the output $z \in \mathbb{R}^p$ reaches harmonic steady state, and then measures the amplitude and phase of the output. With this information, the control input $u \in \mathbb{R}^m$ is determined to minimize the effect of disturbance. As shown in Fig. 1, the HSS control algorithm is a feedback controller, and thus can potentially destabilize the plant, although not in the usual LTI sense. As indicated in Fig. 1, we assume that the disturbance signal is unmeasured and thus is not available for feedback.

The inputs u, w and the output z are related by

$$z = \begin{bmatrix} G_{zw} & G_{zu} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}, \tag{2.1}$$

where $G_{zw}(s)$ and $G_{zu}(s)$ are multi-input multi-output continuous-time transfer functions, and, for all $i = 1,\ldots,p$, $j = 1,\ldots,d$, and $l = 1,\ldots,m$, $G_{z_i w_j}(s)$ and $G_{z_i u_l}(s)$ are the SISO entries of $G_{zw}(s)$ and $G_{zu}(s)$, respectively. In HSS control, the update of the control input u is not performed continuously but rather at specified times t_k. The control input is harmonic with the same spectrum as the disturbance, and the amplitude and phase of the control input are updated at t_k. The time interval $t_{k+1} - t_k$ between two successive updates need not be constant but must be sufficiently large to allow the output z to reach harmonic steady state. At steady state, the amplitude and phase of the output z are completely determined by the amplitude and phase of the disturbance and control input. Assuming that the disturbance w is harmonic with frequency ω_1 and the output z has reached harmonic steady state within the time interval $[t_k, t_{k+1}]$, $w(t)$, $z(t)$, and $u(t)$ have components

$$\begin{aligned} w(t) &= \left[\hat{w}_1 \mathrm{Re}(e^{\jmath(\omega_1 t+\phi_1)}) \ \cdots \ \hat{w}_d \mathrm{Re}(e^{\jmath(\omega_1 t+\phi_d)})\right]^{\mathrm{T}}, \\ z(t) &= \left[\hat{z}_{1k} \mathrm{Re}(e^{\jmath(\omega_1 t+\theta_{1k})}) \ \cdots \ \hat{z}_{pk} \mathrm{Re}(e^{\jmath(\omega_1 t+\theta_{pk})})\right]^{\mathrm{T}}, \\ u(t) &= \left[\hat{u}_{1k} \mathrm{Re}(e^{\jmath(\omega_1 t+\psi_{1k})}) \ \cdots \ \hat{u}_{mk} \mathrm{Re}(e^{\jmath(\omega_1 t+\psi_{mk})})\right]^{\mathrm{T}}, \end{aligned} \tag{2.2}$$

where $\hat{w}_i \in \mathbb{R}$, $\hat{z}_{ik} \in \mathbb{R}$, $\hat{u}_{ik} \in \mathbb{R}$ are the amplitudes, and $\phi_i \in \mathbb{R}$, $\theta_{ik} \in \mathbb{R}$, $\psi_{ik} \in \mathbb{R}$ are the phase angles, of the ith component of $w(t)$, $z(t)$, and $u(t)$, respectively. Note that the amplitude $\hat{w}_i$ and phase ϕ_i of the ith component of $w(t)$ are independent of the time interval, and, furthermore, ϕ_i is determined by the choice of t_0.

Next, define $w_{\mathrm{s}_i} \in \mathbb{R}$, $w_{\mathrm{c}_i} \in \mathbb{R}$, $z_{\mathrm{s}_{i_k}} \in \mathbb{R}$, $z_{\mathrm{c}_{i_k}} \in \mathbb{R}$, $u_{\mathrm{s}_{i_k}} \in \mathbb{R}$, and $u_{\mathrm{c}_{i_k}} \in \mathbb{R}$ by

$$\begin{aligned} w_{\mathrm{s}_i} &\triangleq -\hat{w}_i \sin(\phi_i), \ w_{\mathrm{c}_i} \triangleq \hat{w}_i \cos(\phi_i), \\ z_{\mathrm{s}_{i_k}} &\triangleq -\hat{z}_{ik} \sin(\theta_{ik}), \ z_{\mathrm{c}_{i_k}} \triangleq \hat{z}_{ik} \cos(\theta_{ik}), \\ u_{\mathrm{s}_{i_k}} &\triangleq -\hat{u}_{ik} \sin(\psi_{ik}), \ u_{\mathrm{c}_{i_k}} \triangleq \hat{u}_{ik} \cos(\psi_{ik}). \end{aligned} \tag{2.3}$$

Note that for all $i = 1,\ldots,d$, w_{s_i} and w_{c_i} are constants determined by the choice of t_0, and that $u_{\mathrm{s}_{i_k}}$ and $u_{\mathrm{c}_{i_k}}$ are determined by the control law. Define $w \in \mathbb{R}^{2d}$, $z_k \in \mathbb{R}^{2p}$, and $u_k \in \mathbb{R}^{2m}$ by

$$\begin{aligned} &w \triangleq [w_{\mathrm{s}_1} w_{\mathrm{c}_1} \cdots w_{\mathrm{s}_d} w_{\mathrm{c}_d}]^{\mathrm{T}}, z_k \triangleq \left[z_{\mathrm{s}_{1_k}} z_{\mathrm{c}_{1_k}} \cdots z_{\mathrm{s}_{p_k}} z_{\mathrm{c}_{p_k}}\right]^{\mathrm{T}}, \\ &u_k \triangleq \left[u_{\mathrm{s}_{1_k}} u_{\mathrm{c}_{1_k}} \cdots u_{\mathrm{s}_{m_k}} u_{\mathrm{c}_{m_k}}\right]^{\mathrm{T}}. \end{aligned} \tag{2.4}$$

It follows from (2.1)-(2.4) that the system dynamics in terms of w, z_k, and u_k are given by

$$z_k = T u_k + W w, \tag{2.5}$$

where for all $i = 1,\ldots,p$, $j = 1,\ldots,m$, and $l = 1,\ldots,d$, the entries $T_{ij} \in \mathbb{R}^{2\times 2}$ and $W_{il} \in \mathbb{R}^{2\times 2}$ of $T \in \mathbb{R}^{2p\times 2m}$ and $W \in \mathbb{R}^{2p\times 2d}$, respectively, are defined by

$$\begin{aligned} T_{ij} &\triangleq \begin{bmatrix} \mathrm{Re}(G_{z_i u_j}(\jmath\omega_1)) & -\mathrm{Im}(G_{z_i u_j}(\jmath\omega_1)) \\ \mathrm{Im}(G_{z_i u_j}(\jmath\omega_1)) & \mathrm{Re}(G_{z_i u_j}(\jmath\omega_1)) \end{bmatrix}, \\ W_{il} &\triangleq \begin{bmatrix} \mathrm{Re}(G_{z_i w_l}(\jmath\omega_1)) & -\mathrm{Im}(G_{z_i w_l}(\jmath\omega_1)) \\ \mathrm{Im}(G_{z_i w_l}(\jmath\omega_1)) & \mathrm{Re}(G_{z_i w_l}(\jmath\omega_1)) \end{bmatrix}. \end{aligned} \tag{2.6}$$

Replacing k by $k+1$ in (2.5), and subtracting the resulting equation from (2.5) yields the disturbance-free update model

$$z_{k+1} = z_k + T(u_{k+1} - u_k). \tag{2.7}$$

When the disturbance $w(t)$ is a sum of sinusoids of multiple frequencies, the above analysis carries through with minor modifications.

III. The HSS Algorithm

Consider the cost function

$$J(z_k, u_k) \triangleq z_k^{\mathrm{T}} Q z_k + 2 z_k^{\mathrm{T}} S u_k + u_k^{\mathrm{T}} R u_k, \tag{3.1}$$

where $Q \in \mathbb{R}^{2p\times 2p}$, $S \in \mathbb{R}^{2p\times 2m}$, and $R \in \mathbb{R}^{2m\times 2m}$ are weighting matrices such that R is positive definite and $\begin{bmatrix} Q & S \\ S^{\mathrm{T}} & R \end{bmatrix}$ is positive semidefinite. Substituting z_k from (2.5) into (3.1) yields

$$\begin{aligned} J(z_k, u_k) = u_k^{\mathrm{T}} \left(T^{\mathrm{T}} Q T + T^{\mathrm{T}} S + S^{\mathrm{T}} T + R\right) u_k + \\ 2 u_k^{\mathrm{T}} (T^{\mathrm{T}} Q + S^{\mathrm{T}}) W w + w^{\mathrm{T}} W^{\mathrm{T}} Q W w. \end{aligned} \tag{3.2}$$

Since (3.2) involves only u_k and w we define

$$\mathcal{J}(w, u_k) \triangleq J(z_k, u_k), \tag{3.3}$$

and write $\mathcal{J}(w, u_k)$ as

$$\mathcal{J}(w, u_k) = \left[(Ww)^{\mathrm{T}} \ u_k^{\mathrm{T}}\right] \begin{bmatrix} Q & QT + S \\ T^{\mathrm{T}} Q + S^{\mathrm{T}} & D \end{bmatrix} \begin{bmatrix} Ww \\ u_k \end{bmatrix}, \tag{3.4}$$

where the positive-semidefinite matrix D is defined by

$$D \triangleq T^{\mathrm{T}} Q T + T^{\mathrm{T}} S + S^{\mathrm{T}} T + R. \tag{3.5}$$

To determine u_k that minimizes $\mathcal{J}(w, u_k)$, we set

$$\frac{\partial \mathcal{J}(w, u_k)}{\partial u_k} = 2 D u_k + 2(T^{\mathrm{T}} Q + S^{\mathrm{T}}) W w = 0. \tag{3.6}$$

Assuming D is positive definite, the optimal control law is given by

$$u_{\mathrm{opt}} \triangleq u_{k,\mathrm{opt}} = -D^{-1}(T^{\mathrm{T}} Q + S^{\mathrm{T}}) W w \tag{3.7}$$

and the minimum cost is

$$\mathcal{J}(w, u_{\mathrm{opt}}) = w^{\mathrm{T}} W^{\mathrm{T}} \left[\mathbb{Q} - (QT + S) D^{-1} (T^{\mathrm{T}} Q + S^{\mathrm{T}})\right] W w. \tag{3.8}$$

Since u_{opt} depends on w whose measurement is not available, we derive an equivalent control law that can be used for all $k \geqslant 1$.

Setting $k = 0$ in (2.5), yields

$$W w = z_0 - T u_0 \tag{3.9}$$

and hence substituting (3.9) into (2.5) yields

$$z_k = z_0 + T(u_k - u_0). \tag{3.10}$$

From (3.9) the optimal control law $u_{\rm opt}$ in (3.7) can be expressed as

$$u_{\rm opt} = -D^{-1}(T^{\rm T}Q + S^{\rm T})(z_0 - Tu_0). \qquad (3.11)$$

IV. Convergence Analysis of HSS Algorithm

Note that $u_{\rm opt}$ given by (3.11) is independent of k, and hence remains constant for all $k \geqslant 1$. Substituting (3.11) in (3.10) and (3.1), the optimal value of z_k for all $k \geqslant 1$ is given by

$$z_{\rm opt} \triangleq z_{k,\rm opt} = \left[I - TD^{-1}(T^{\rm T}Q + S^{\rm T})\right](z_0 - Tu_0), \qquad (4.1)$$

and thus

$$J(z_{\rm opt}, u_{\rm opt}) = (z_0 - Tu_0)^{\rm T}[Q - (QT + S)D^{-1} (T^{\rm T}Q + S^{\rm T})](z_0 - Tu_0). \qquad (4.2)$$

Using (3.10), the optimal control law can be expressed recursively as

$$u_{k+1,\rm opt} = -D^{-1}(T^{\rm T}Q + S^{\rm T})(z_{k,\rm opt} - Tu_{k,\rm opt}). \quad (4.3)$$

The state-space representation of the system dynamics with the optimal control law is

$$\begin{bmatrix} z_{k+1,\rm opt} \\ u_{k+1,\rm opt} \end{bmatrix} = A \begin{bmatrix} z_{k,\rm opt} \\ u_{k,\rm opt} \end{bmatrix}, \qquad (4.4)$$

where $A \in \mathbb{R}^{2(p+m)\times 2(p+m)}$ is defined by

$$A \triangleq \begin{bmatrix} I_{2p} - TM & -(I_{2p} - TM)T \\ -M & MT \end{bmatrix} \qquad (4.5)$$

and $M \in \mathbb{R}^{2m\times 2p}$ is defined by $M \triangleq D^{-1}(T^{\rm T}Q + S^{\rm T})$. Note that $A^2 = A$ and hence A is an idempotent matrix, and its eigenvalues are either 0 or 1. In fact, A can be factored as

$$A = \begin{bmatrix} I_{2p} & T \\ 0 & I_{2m} \end{bmatrix} \begin{bmatrix} I_{2p} & 0 \\ -M & 0_{2m} \end{bmatrix} \begin{bmatrix} I_{2p} & -T \\ 0 & I_{2m} \end{bmatrix}, \qquad (4.6)$$

which implies that

$$\operatorname{spec}(A) = \operatorname{spec}(I_{2p}) \cup \operatorname{spec}(0_{2m}). \qquad (4.7)$$

Since A is idempotent, with the initial conditions z_0 and u_0, (4.4) implies that for all $k = 1, 2, \ldots,$

$$\begin{bmatrix} z_{\rm opt} \\ u_{\rm opt} \end{bmatrix} = A^k \begin{bmatrix} z_0 \\ u_0 \end{bmatrix} = A \begin{bmatrix} z_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} (I - TM)(z_0 - Tu_0) \\ -M(z_0 - Tu_0) \end{bmatrix}. \quad (4.8)$$

Consequently, the optimum values of $u_{\rm opt}$ in (3.11) and $z_{\rm opt}$ in (4.1) are attained after the first update.

Note that if $S = 0$, then $z_{\rm opt}$ in (4.1) can be expressed as

$$z_{\rm opt} = Q^{-1}(Q^{-1} + TR^{-1}T^{\rm T})^{-1}(z_0 - Tu_0). \qquad (4.9)$$

Hence,

$$\|z_{\rm opt}\| \leqslant \frac{\sigma_{\max}(R)}{\sigma_{\min}(Q)\sigma_l(TT^{\rm T})}\|z_0 - Tu_0\|, \qquad (4.10)$$

where $l \triangleq \min(2p, 2m)$.

Note that, if $\frac{\sigma_{\max}(R)}{\sigma_{\min}(Q)}$ is large (minimum energy control), it follows from (4.10) that $\|z_{\rm opt}\|$ may be large, indicating poor performance. Alternatively, if $\frac{\sigma_{\max}(R)}{\sigma_{\min}(Q)}$ is small (cheap control), then (4.10) implies that the $\|z_{\rm opt}\|$ is small and hence the performance is good.

V. Robustness of HSS Control

Implementation of HSS requires knowledge of T. An erroneous model of T can result in degraded performance and possible instability. If an estimate $\hat{T}$ of T is given, the control law defined in (4.3) becomes

$$\hat{u}_{k+1} = -\hat{M}(z_k - \hat{T}\hat{u}_k), \qquad (5.1)$$

where

$$\hat{M} \triangleq \hat{D}^{-1}(\hat{T}^{\rm T}Q + S^{\rm T}) \in \mathbb{R}^{2m\times 2p} \qquad (5.2)$$

assuming that

$$\hat{D} \triangleq \hat{T}^{\rm T}Q\hat{T} + S^{\rm T}\hat{T} + \hat{T}^{\rm T}S + R \qquad (5.3)$$

is positive definite. The state-space representation of the system-dynamics with (5.1) is

$$\begin{bmatrix} z_{k+1} \\ \hat{u}_{k+1} \end{bmatrix} = \hat{A} \begin{bmatrix} z_k \\ \hat{u}_k \end{bmatrix}, \qquad (5.4)$$

where $\hat{A}$ is defined by

$$\hat{A} \triangleq \begin{bmatrix} (I_{2p} - T\hat{M}) & T(\hat{M}\hat{T} - I_{2p}) \\ -\hat{M} & \hat{M}\hat{T} \end{bmatrix} \qquad (5.5)$$

and $\Delta T \triangleq \hat{T} - T$.

It is useful to factor $\hat{A}$ as

$$\hat{A} = \begin{bmatrix} I_{2p} & T \\ 0 & I_{2m} \end{bmatrix} \begin{bmatrix} I_{2p} & 0 \\ -\hat{M} & \hat{M}\Delta T \end{bmatrix} \begin{bmatrix} I_{2p} & -T \\ 0 & I_{2m} \end{bmatrix}, \qquad (5.6)$$

which shows that

$$\operatorname{spec}(\hat{A}) = \operatorname{spec}(I_{2p}) \cup \operatorname{spec}(\hat{M}\Delta T). \qquad (5.7)$$

Hence the HSS algorithm is stable if and only if

$$\operatorname{sprad}(\hat{M}\Delta T) < 1. \qquad (5.8)$$

Assuming $S = 0$, it follows that

$$\operatorname{sprad}(\hat{M}\Delta T) = \operatorname{sprad}\left((\hat{T}^{\rm T}Q\hat{T} + R)^{-1}\hat{T}^{\rm T}Q\Delta T\right) \leqslant \frac{(\sigma_{\max}(T) + \sigma_{\max}(\Delta T))\sigma_{\max}(Q)\sigma_{\max}(\Delta T)}{\sigma_{\min}(R)}. \qquad (5.9)$$

Therefore, it can be shown that, if

$$\sigma_{\max}(\Delta T) < \frac{-\sigma_{\max}(T)}{2} + \frac{1}{2}\sqrt{\sigma_{\max}(T)^2 + 4\frac{\sigma_{\min}(R)}{\sigma_{\max}(Q)}}, \qquad (5.10)$$

then $\operatorname{sprad}(\hat{M}\Delta T) < 1$.

If $\frac{\sigma_{\min}(R)}{\sigma_{\max}(Q)}$ is large (minimum energy control), then according to (5.10), HSS control possesses a high degree of robustness and a sufficient condition for the stability of HSS algorithm is approximately given by

$$\sigma_{\max}(\Delta T) < \sqrt{\frac{\sigma_{\min}(R)}{\sigma_{\max}(Q)}}. \qquad (5.11)$$

However, if $\frac{\sigma_{\min}(R)}{\sigma_{\max}(Q)}$ is small (cheap control), then (5.10) implies that robustness is compromised.

From (5.6), it follows that

$$\hat{A}^k = \begin{bmatrix} I_{2p} & T \\ 0 & I_{2m} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\sum_{i=1}^{k-1} (\hat{M}\Delta T)^i \hat{M} & (\hat{M}\Delta T)^k \end{bmatrix} \begin{bmatrix} I_{2p} & -T \\ 0 & I_{2m} \end{bmatrix}. \quad (5.12)$$

Now assume that HSS control is stable, that is, (5.8) is satisfied. In this case, $\lim_{k\to\infty}(\hat{M}\Delta T)^k = 0$, and let $\Gamma \triangleq \lim_{k\to\infty} \sum_{i=1}^{k-1} (\hat{M}\Delta T)^i = (I - \hat{M}\Delta T)^{-1}$. Hence, the limiting values of z_k and $\hat{u}_k$ are given by

$$\lim_{k\to\infty} \begin{bmatrix} z_k \\ \hat{u}_k \end{bmatrix} = \lim_{k\to\infty} \hat{A}^k \begin{bmatrix} z_0 \\ \hat{u}_0 \end{bmatrix} = \begin{bmatrix} (I - T\Gamma\hat{M})(z_o - T\hat{u}_o) \\ -\Gamma\hat{M}(z_o - T\hat{u}_o) \end{bmatrix}. \quad (5.13)$$

Now, consider the case where the estimate $\hat{T}$ of T involves a multiplicative error, that is, $\hat{T} = T(I + \Delta T_{\rm mul})$, where $\Delta T_{\rm mul} \in \mathbb{R}^{2m\times 2m}$. Then $\hat{T}$ can be expressed equivalently by $\hat{T} = T + \Delta T$, where $\Delta T = T\Delta T_{\rm mul}$. A sufficient condition for stability of HSS control is ${\rm sprad}(\hat{M}T\Delta T_{\rm mul}) < 1$. Following a procedure similar to the one discussed in (5.9) and (5.10) for additive uncertainty, it can be shown that the HSS algorithm is stable if

$$\sigma_{\max}(\Delta T_{\rm mul}) < -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\sigma_{\min}(R)}{\sigma_{\max}^2(T)\sigma_{\max}(Q)}}, \quad (5.14)$$

VI. Adaptive HSS Control Algorithm

Here we discuss online identification of the matrix T, which will then be used as the basis for an adaptive extension of HSS control discussed in the previous sections. Define $\Delta z_k \in \mathbb{R}^{2p}$ and $\Delta u_k \in \mathbb{R}^{2m}$ by

$$\Delta z_k \triangleq z_k - z_{k-1}, \quad \Delta u_k \triangleq u_k - u_{k-1}, \quad (6.1)$$

and $\Delta Z_k \in \mathbb{R}^{2p\times k}$ and $\Delta U_k \in \mathbb{R}^{2m\times k}$ by

$$\Delta Z_k \triangleq [\Delta z_1 \ \cdots \ \Delta z_k], \quad \Delta U_k \triangleq [\Delta u_1 \ \cdots \ \Delta u_k]. \quad (6.2)$$

The system dynamics equation (2.7) can be represented by

$$\Delta z_k = T\Delta u_k. \quad (6.3)$$

Hence, it follows from (6.1) that

$$\Delta Z_k = T\Delta U_k. \quad (6.4)$$

Assuming $\Delta U_k \Delta U_k^{\rm T}$ is nonsingular, we define

$$P_k \triangleq (\Delta U_k \Delta U_k^{\rm T})^{-1}, \quad (6.5)$$

and it follows from (6.4) that the least squares estimate $\hat{T}_{{\rm LS}_k}$ of T is given by $\hat{T}_{{\rm LS}_k} = \Delta Z_k \Delta U_k^{\rm T} P_k$.

The recursive least squares method is used to iteratively update $\hat{T}_{{\rm LS}_k}$ based on the past and current values of Δz_k and Δu_k. Since $\Delta u_k \Delta u_k^{\rm T}$ is positive semidefinite for all k and

$$\Delta U_k \Delta U_k^{\rm T} = \sum_{i=1}^{k} \Delta u_i \Delta u_i^{\rm T}, \quad (6.6)$$

it follows that if $\Delta U_{k_0} \Delta U_{k_0}^{\rm T}$ is nonsingular, then $\Delta U_k \Delta U_k^{\rm T}$ is nonsingular for all $k > k_0$. Hence the recursive procedure for determining $T_{{\rm LS}_k}$ for all $k > k_0$ is given by

$$K_{k+1} = (1 + \Delta u_{k+1}^{\rm T} P_k \Delta u_{k+1})^{-1} \Delta u_{k+1}^{\rm T} P_k, \quad (6.7)$$

$$\hat{T}_{{\rm LS}_{k+1}} = \hat{T}_{{\rm LS}_k} + \varepsilon_{k+1} K_{k+1}, \quad (6.8)$$

$$P_{k+1} = P_k (I - \Delta u_{k+1} K_{k+1}), \quad (6.9)$$

where

$$\varepsilon_{k+1} \triangleq \Delta z_{k+1} - \hat{T}_{{\rm LS}_k} \Delta u_{k+1}. \quad (6.10)$$

Note that

$$\operatorname{rank}(\Delta U_k \Delta U_k^{\rm T}) = \operatorname{rank}(\Delta U_k) \leqslant \min(k, 2m). \quad (6.11)$$

Since $\Delta U_k \Delta U_k^{\rm T}$ is $2m \times 2m$, it follows from (6.11) that $\Delta U_k \Delta U_k^{\rm T}$ is singular for all $k < 2m$. Hence the recursive procedure (6.7)-(6.9) cannot be used for $k < 2m$.

A suboptimal way to determine an estimate $\hat{T}_k$ of T is to replace P_k in (6.7)-(6.9) by $\hat{P}_k$, that is,

$$K_{k+1} = (1 + \Delta u_{k+1}^{\rm T} \hat{P}_k \Delta u_{k+1})^{-1} \Delta u_{k+1}^{\rm T} \hat{P}_k, \quad (6.12)$$

$$\hat{T}_{k+1} = \hat{T}_k + \varepsilon_{k+1} K_{k+1}, \quad (6.13)$$

$$\hat{P}_{k+1} = \hat{P}_k (I - \Delta u_{k+1} K_{k+1}), \quad (6.14)$$

where $\hat{P}_0$ is positive definite but otherwise arbitrary and ε_k is defined by (6.10) with $\hat{T}_{{\rm LS}_k}$ replaced by $\hat{T}_k$. It follows from (6.12)-(6.14) that $\hat{P}_k$ is positive definite for all $k \geqslant 0$ and is given by

$$\hat{P}_k = (\hat{P}_0^{-1} + \Delta U_k \Delta U_k^{\rm T})^{-1}. \quad (6.15)$$

Furthermore, $\hat{T}_k$ is given by

$$\hat{T}_k = (\hat{T}_0 \hat{P}_0^{-1} + \Delta Z_k \Delta U_k^{\rm T})(\hat{P}_0^{-1} + \Delta U_k \Delta U_k^{\rm T})^{-1}. \quad (6.16)$$

Since $\hat{P}_0$ is positive definite, the inverse in (6.16) always exists, and hence the recursive procedure can be used for all $k \geqslant 0$. The updated estimate $\hat{T}_k$ is used at each control update step to calculate the control law u_{k+1}, which is given by

$$u_{k+1} = -\hat{M}_k (z_k - \hat{T}_k u_k), \quad (6.17)$$

where $\hat{M}_k$ is defined by

$$\hat{M}_k \triangleq (\hat{T}_k^{\rm T} Q \hat{T}_k + S^{\rm T} \hat{T}_k + \hat{T}_k^{\rm T} S + R)^{-1} (\hat{T}_k^{\rm T} Q + S^{\rm T}). \quad (6.18)$$

VII. Convergence Analysis of the Estimate $\hat{T}_k$

Define $\Delta T_k \in \mathbb{R}^{2p\times 2m}$ by

$$\Delta T_k \triangleq \hat{T}_k - T, \quad (7.1)$$

where $\hat{T}_k$ is updated using (6.12)-(6.14). Next, define the function $V(\Delta T, \hat{P})$ by

$$V(\Delta T, \hat{P}) \triangleq \Delta T \hat{P}^{-1} \Delta T^{\rm T} \quad (7.2)$$

and ΔV_k by

$$\Delta V_k = V(\Delta T_{k+1}, \hat{P}_{k+1}) - V(\Delta T_k, \hat{P}_k), \quad (7.3)$$

where $\hat{P}_k$ is updated using (6.12)-(6.14) and $\hat{P}_0$ is the positive definite matrix used to initialize (6.14). Subtracting T from both sides of (6.13) yields

$$\Delta T_{k+1} = \Delta T_k + \varepsilon_{k+1} K_{k+1}. \tag{7.4}$$

Using (6.3) and (7.1), ε_{k+1} can be expressed as

$$\varepsilon_{k+1} = -\Delta T_k \Delta u_{k+1}. \tag{7.5}$$

Substituting (6.12)-(6.14) and (7.5) into (7.3) yields

$$\Delta V_k = -\frac{\varepsilon_{k+1}\varepsilon_{k+1}^{\mathrm{T}}}{1+\Delta u_{k+1}^{\mathrm{T}}\hat{P}_k\Delta u_{k+1}}, \tag{7.6}$$

and hence $V(\Delta T_k, \hat{P}_k)$ is non-increasing. Since $V(\Delta T_k, \hat{P}_k) \geqslant 0$, it follows that $\lim_{k\to\infty} V(\Delta T_k, \hat{P}_k)$ exists and is nonnegative. Hence

$$\lim_{k\to\infty} -\frac{\varepsilon_{k+1}\varepsilon_{k+1}^{\mathrm{T}}}{1+\Delta u_{k+1}^{\mathrm{T}}\hat{P}_k\Delta u_{k+1}} = \lim_{k\to\infty} \Delta V_k = 0. \tag{7.7}$$

Next we show that Δu_k is bounded. Substituting (6.4) into (6.16) yields

$$\hat{T}_k = (\hat{T}_0\hat{P}_0^{-1} + T\Delta U_k \Delta U_k^{\mathrm{T}})(\hat{P}_0^{-1} + \Delta U_k \Delta U_k^{\mathrm{T}})^{-1}. \tag{7.8}$$

Post-multiplying (7.8) by $\hat{P}_0^{-1} + \Delta U_k \Delta U_k^{\mathrm{T}}$ yields

$$\Delta T_k = \Delta T_0 \hat{P}_0^{-1}(\hat{P}_0^{-1} + \Delta U_k \Delta U_k^{\mathrm{T}})^{-1}. \tag{7.9}$$

If $\Delta U_{k_0}\Delta U_{k_0}^{\mathrm{T}}$ is nonsingular, then, for all $k \geqslant k_0$, (7.9) implies that

$$\sigma_{\max}(\Delta T_k) \leqslant \frac{\sigma_{\max}(\Delta T_0)}{\sigma_{\min}(\hat{P}_0)\sigma_{\min}(\Delta U_{k_0}\Delta U_{k_0}^{\mathrm{T}})}. \tag{7.10}$$

Hence, if $\hat{P}_0$ is chosen to be sufficiently large, then $\sigma_{\max}(\Delta T_k)$ can be made sufficiently small and even made to satisfy condition (5.10) and in that case it follows from (5.9) that, for all $k \geqslant k_0$,

$$\sigma_{\max}(\hat{M}_k \Delta T_k) < 1. \tag{7.11}$$

The state space representation of the system dynamics with the control law (6.17) is

$$\begin{bmatrix} z_{k+1} \\ u_{k+1} \end{bmatrix} = \hat{A}_k \begin{bmatrix} z_k \\ u_k \end{bmatrix}, \tag{7.12}$$

where $\hat{A}_k$ is defined by (5.5) with $\hat{T}$ and $\hat{M}$ replaced by $\hat{T}_k$ and $\hat{M}_k$ respectively. Hence for all $k \geqslant k_0$

$$\begin{bmatrix} z_k \\ u_k \end{bmatrix} = \prod_{i=k_0}^{k} \hat{A}_i \begin{bmatrix} z_{k_0} \\ u_{k_0} \end{bmatrix}. \tag{7.13}$$

Note that $\prod_{i=k_0}^{k} \hat{A}_i$ can be factored as

$$\prod_{i=k_0}^{k}\hat{A}_i = \begin{bmatrix} I_{2p} & T \\ 0 & I_{2m} \end{bmatrix} \begin{bmatrix} I & 0 \\ \hat{A}_{21} & \prod_{i=k_0}^{k}(\hat{M}_i\Delta T_i) \end{bmatrix} \begin{bmatrix} I_{2p} & -T \\ 0 & I_{2m} \end{bmatrix}, \tag{7.14}$$

where $\hat{A}_{21} = -\hat{M}_{k_0}\left(I + \sum_{i=k_0}^{k-1}(\prod_{j=0}^{i}\Delta T_j \hat{M}_{j+1})\right)$.

From (7.11) it follows that

$$\sigma_{\max}\left(\prod_{i=k_0}^{k}(\hat{M}_i\Delta T_i)\right) \leqslant \prod_{i=k_0}^{k}\left(\sigma_{\max}(\hat{M}_i\Delta T_i)\right) < 1 \tag{7.15}$$

and hence it follows from (7.13)-(7.15) that z_k and u_k are bounded. Hence, for all $k = 0, 1, \ldots$, let $\gamma > 0$ satisfy $\|u_k\| < \gamma$ and thus

$$\|\Delta u_k\| = \|u_{k+1} - u_k\| < 2\gamma. \tag{7.16}$$

From (6.5) and (6.6) it follows that $\hat{P}_{k+1} \leqslant \hat{P}_k$, which implies that

$$\hat{P}_k \leqslant \hat{P}_0. \tag{7.17}$$

Hence, it follows from (7.17) and (7.16) that

$$1 + \Delta u_k^{\mathrm{T}}\hat{P}_k\Delta u_k \leqslant 1 + 4\gamma^2\lambda_{\max}(\hat{P}_0) \tag{7.18}$$

and thus (7.8) implies that

$$\lim_{k\to\infty} \varepsilon_k = 0. \tag{7.19}$$

Taking the limit as $k \to \infty$ of (6.13) yields

$$\lim_{k\to\infty}\left(\hat{T}_{k+1} - \hat{T}_k\right) = \lim_{k\to\infty} \frac{\varepsilon_{k+1}\Delta u_{k+1}^{\mathrm{T}}\hat{P}_k}{1+\Delta u_{k+1}^{\mathrm{T}}\hat{P}_k\Delta u_{k+1}}. \tag{7.20}$$

From (7.16)-(7.20) it follows that,

$$\lim_{k\to\infty}\left(\hat{T}_{k+1} - \hat{T}_k\right) = 0. \tag{7.21}$$

Thus, $\{\hat{T}_k\}$ is a Cauchy sequence, and hence $\hat{T}_k$ converges. However, there is no guarantee that $\hat{T}_k$ will converge to T. In fact, it can be shown that there are certain choices of $\hat{P}_0$ and $\hat{T}_0$ such that $\hat{T}_k$ will not converge to T. Conditions on $\hat{T}_0$ and $\hat{P}_0$ that guarantee convergence of $\hat{T}_k$ to T are not available. However, the steady state performance of the adaptive HSS control depends on the steady-state value of the estimate $\hat{T}_k$. Note that using (6.16) the RLS estimate $\hat{T}_k$ can be expressed as

$$\hat{T}_k = T + \Delta T_0 \hat{P}_0^{-1}\hat{P}_k. \tag{7.22}$$

Consequently, if $\hat{P}_k \to 0$ as $k \to \infty$, then $\hat{T}_k \to T$ as $k \to \infty$. It follows from the theory of RLS [12] that, if Δu_k is persistently exciting, then $\hat{P}_k \to 0$ as $k \to \infty$. However, since u_k is given by the adaptive control law (6.17), Δu_k may not be persistently exciting.

Next, define $\tilde{\delta}_k \in \mathbb{R}^{2m}$ by

$$\tilde{\delta}_k \triangleq \begin{bmatrix} 0_{1\times i-1} & \delta_{k,i} & 0_{1\times 2m-i} \end{bmatrix}^{\mathrm{T}} \tag{7.23}$$

and $\delta_{k,i} \in \mathbb{R}$ by $\delta_{k,i} \triangleq \delta\,\mathrm{sign}(\Delta u_{k,i})$, where $\Delta u_{k,i}$ is the ith component of Δu_k, $\delta > 0$, and i is the remainder when k is divided by $2m$. Next, define $\Delta\tilde{u}_k$ by $\Delta\tilde{u}_k \triangleq \Delta u_k + \tilde{\delta}_k$. It can be shown that if δ is sufficiently large, then $\Delta\tilde{u}_k$ is persistently exciting. The modified control law is given by

$$u_{k+1} = u_k + \Delta\tilde{u}_k. \tag{7.24}$$

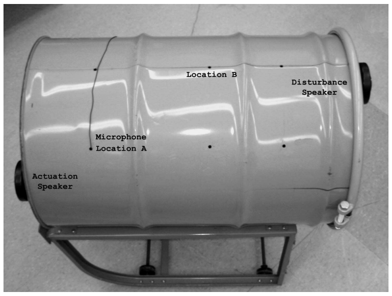

Fig. 2. Top view of the acoustic drum with two end-mounted speakers and one internal microphone. The microphone is initially at Location A and is moved to Location B while the HSS control algorithm is operational.

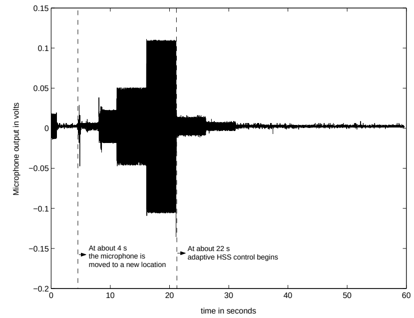

Fig. 3. Disturbance rejection using fixed-model and adaptive HSS control. The microphone is moved at $t \approx 4$ s and the output diverges. Adaptive HSS control begins at $t \approx 22$ s, and convergence is achieved.

VIII. Experimental Example: Noise Cancellation in an Acoustic Drum

The acoustic drum (see Fig. 2) has two end-mounted speakers, and up to six microphones suspended inside the drum through holes drilled along the top. Though the equations of motion of the acoustic drum is not similar to that of the duct [1], the input-output response is linear, and hence HSS control can be used to reject a disturbance with a known harmonic spectrum.

A constant-amplitude, single-tone disturbance signal $w(t)$ with frequency 10 Hz ($\omega = 20\pi$ rad/s) is produced by the disturbance speaker. The actuation speaker produces the control $u(t)$ for cancelling the disturbance. The control objective is to reduce the output $z(t)$ measured by microphone 1 at Location A.

To estimate T, a sinusoidal input with frequency 10 Hz, amplitude $\hat{u}_1$, and phase angle ψ_1 is applied to the system, in N separate trials. The amplitude and phase of the output, $\hat{z}_1$ and θ_1, are used to determine an initial estimate $T_{\rm LS}$ of T using the batch least squares procedure. A dSPACE 1003 system is used to determine the vector z_k from measurements of $z(t)$ and the update u_k is computed by a Simulink implementation of the HSS control algorithm.

Fig. 3 shows the performance of HSS control with $T = T_{\rm LS}$. Next, we consider the case in which T is uncertain. At $t \approx 4$ s the microphone is moved from its original location (Location A) to a new location (Location B), resulting in a change in the system dynamics. Since conventional HSS control is unaware of this change, the modified system is unstable and the output diverges. At $t \approx 22$ s, adaptive HSS control begins, and stability is recovered, providing disturbance rejection at the new location.

IX. Conclusion

In this paper we developed adaptive harmonic steady state control for disturbance rejection. HSS control extends higher harmonic control and convergent control developed for helicopter vibration reduction and rotor imbalance control. HSS control is applicable to stable systems with tonal or multi-tonal disturbances. The adaptive HSS algorithm is easy to implement and robust in the sense that no modeling information is required aside from the number of disturbance harmonics.

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