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On the Zeros, Initial Undershoot, and Relative Degree of Lumped-Mass Structures

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Abstract— **This paper considers lumped-mass structures where each mass in the structure has a single degree of freedom. Specifically, we analyze the zeros and relative degree of the single-input single-output (SISO) transfer function from the force applied to an arbitrary mass to the position, velocity, or acceleration of another mass.**

1. Introduction

One of the main impediments to achievable performance in linear time-invariant control systems is the presence of nonminimum phase zeros [1, 2]. The role of nonminimum phase zeros in limiting both achievable performance and robust stability suggests the importance of understanding the mechanisms that give rise to such zeros in flexible structures. This issue is discussed in [3], where it is shown that nonminimum phase zeros arise in noncolocated transfer functions for beam models when multiple mechanisms are involved for energy transfer, for example, bending and torsion. Furthermore, it is shown in [4] that nonminimum phase zeros arise in noncolocated transfer functions for beam models when the dynamics are dispersive, as occurs in bending.

Graph theory can provide a systematic framework for analyzing structures and dynamical systems [5, 6]. In particular, [5] uses graph theory to derive expressions for the component forces at the end of individual structural members. In [6], the dynamic equations of motion for a class of rigid body systems are derived using graph-theoretic tools.

In the present paper, we use graph-theoretic results to examine the zeros and relative degree of lumped-mass structures. We show that every single-input single-output (SISO) force-to-motion transfer function of a lumped-mass structure has no positive (real open-right-half-plane) zeros. Furthermore, every SISO force-to-position transfer function for a spring-connected lumped-mass structure has no nonnegative (real closed-right-half-plane) zeros. As a consequence of this result, the step response of every asymptotically stable SISO force-to-position transfer function for a spring-connected lumped-mass structure does not exhibit initial undershoot. We also derive a formula for the relative degree of every SISO force-to-motion transfer function.

2. Graph Theory Preliminaries

There is a natural relationship between lumped-mass structures and graphs. The masses of a lumped-mass structure represent the vertices of a graph, while the springs and dashpots connecting the masses represent the edges of the graph. For example, the 4-mass structure in Figure 1 is represented by the 4-vertex graph in Figure 1. Furthermore, the stiffnesses and damping coefficients can determine the weights associated with the edges. For example, the stiffnesses of the springs in Figure 1 are the weights associated with the edges in Figure 1. In this section, we present definitions and basic results that are useful for analyzing the zeros of lumped-mass structures.

Let $V = \{v_1, v_2, \ldots, v_N\}$. The N elements of V are *vertices*, and V is the *vertex set*. Define $\mathcal{E} \triangleq \{\{v_{n_1}, v_{n_2}\} : v_{n_1}, v_{n_2} \in V, v_{n_1} \neq v_{n_2}\}$, and let $E \subseteq \mathcal{E}$. The elements of E are *edges*, and E is the *edge set*. Since the elements of E are sets and thus are unordered, the edges do not have directions. Thus all graphs considered in this paper are undirected graphs. Furthermore, we do not consider multiple lines since the elements of E are distinct, and we do not consider loops since, for all $i = 1, \ldots, N$, $\{v_i, v_i\} \notin E$.

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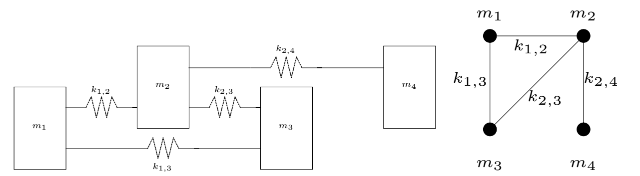

Fig. 1. A 4-mass structure with spring interconnections and the 4-vertex graph representing the 4-mass structure.

Definition 2.1. *$\mathcal{G} = (V, E)$ is a* graph*. If, in addition, for all $\{v_i, v_j\} \in E$, a weight $w_{i,j} > 0$ is assigned to the edge $\{v_i, v_j\}$, then $\mathcal{G}$ is a* weighted graph*.*

Definition 2.2. *Let $\mathcal{G} = (V, E)$ be a graph, and let $v_{n_0}, v_{n_l} \in V$ be distinct. A* walk of length l *from v_{n_0} to v_{n_l} is the $(l+1)$-tuple $(v_{n_0}, v_{n_1}, \ldots, v_{n_l}) \in V \times \cdots \times V$ such that, for all $i = 1, 2, \ldots, l$, $\{v_{n_{i-1}}, v_{n_i}\} \in E$.*

Definition 2.3. *The graph $\mathcal{G} = (V, E)$ is* connected *if, for all distinct $\alpha, \beta \in V$, there exists a walk between α and β.*

The *weighted adjacency matrix* $A_\mathcal{G} \in \mathbb{R}^{N \times N}$ associated with the weighted graph $\mathcal{G} = (V, E)$ is defined as

$$A_\mathcal{G} \triangleq \begin{bmatrix} 0 & w_{1,2} & w_{1,2} & \ldots & w_{1,N} \\ w_{2,1} & 0 & w_{2,3} & \ldots & w_{2,N} \\ w_{3,1} & w_{3,2} & 0 & & w_{3,N} \\ \vdots & \vdots & & \ddots & \vdots \\ w_{N,1} & w_{N,2} & w_{N,3} & \ldots & 0 \end{bmatrix}, \tag{2.1}$$

where, for all $\{v_i, v_j\} \in E$, $w_{i,j} = w_{j,i} > 0$ is the weight assigned to the edge $\{v_i, v_j\}$ and, for all $\{v_i, v_j\} \notin E$, $w_{i,j} = 0$.

The *Laplacian matrix* $L_\mathcal{G} \in \mathbb{R}^{N \times N}$ associated with the weighted graph $\mathcal{G} = (V, E)$ is defined as

$$L_\mathcal{G} \triangleq D_\mathcal{G} - A_\mathcal{G}, \tag{2.2}$$

where $D_\mathcal{G} \triangleq \operatorname{diag}\left(\sum_{i=2}^{N} w_{1,i}, \sum_{i=1, i\neq 2}^{N} w_{2,i}, \ldots, \sum_{i=1}^{N-1} w_{N,i}\right)$.

Next, we present two results concerning the Laplacian matrix. These results can be found in [7, p. 144] and [8, Theorem 3.16], respectively. In this paper, a matrix is positive semidefinite if it is symmetric with all nonnegative eigenvalues. Furthermore, a matrix is positive definite if it is symmetric with all positive eigenvalues.

Lemma 2.1. *The Laplacian matrix $L_\mathcal{G}$ is a singular, positive-semidefinite M-matrix.*

Lemma 2.2. *The graph $\mathcal{G} = (V, E)$ is connected if and only if its Laplacian matrix $L_\mathcal{G}$ is irreducible.*

The following result, which concerns nonsingular M-matrices, is given by [9, Theorem 2.7].

Lemma 2.3. *Let $A_M \in \mathbb{R}^{N\times N}$ be an irreducible Z-matrix. Then A_M is a nonsingular M-matrix if and only if every entry of A_M^{-1} is positive.*

The next result of this section, which follows immediately from lemmas 2.1-2.3, is used to analyze the zeros of lumped-mass structures.

Lemma 2.4. *Assume the graph $\mathcal{G} = (V, E)$ is connected. Then the Laplacian matrix $L_\mathcal{G}$ is an irreducible, singular, positive-semidefinite M-matrix. Furthermore, let $D \in \mathbb{R}^{N\times N}$ be positive definite and diagonal. Then $D + L_\mathcal{G}$ is an irreducible, nonsingular M-matrix, and thus every entry of $(D + L_\mathcal{G})^{-1}$ is positive.*

Now, we present results regarding the weighted adjacency matrices of two different graphs having the same vertex set; these results are used to analyze the relative degree of the transfer functions for lumped-mass structures. Let $E_1 \subseteq \mathcal{E}$ and $E_2 \subseteq \mathcal{E}$, and consider the weighted graphs $\mathcal{G}_1 = (V, E_1)$ and $\mathcal{G}_2 = (V, E_2)$. Let $A_{\mathcal{G}_1} \in \mathbb{R}^{N\times N}$ and $A_{\mathcal{G}_2} \in \mathbb{R}^{N\times N}$ be the weighted adjacency matrices associated with the weighted graph $\mathcal{G}_1 = (V, E_1)$ and $\mathcal{G}_2 = (V, E_2)$, respectively.

Lemma 2.5. *Consider the graph $\mathcal{G}_{12} = (V, E_1 \cup E_2)$, and let $v_{n_0}, v_{n_l} \in V$ be distinct. Then there exists a walk $(v_{n_0}, v_{n_1}, \ldots, v_{n_l})$ of length l on the graph $\mathcal{G}_{12}$ between v_{n_0} and v_{n_l} if and only if*

$$e_{n_l}^{\mathrm{T}} \Theta_l \Theta_{l-1} \cdots \Theta_1 e_{n_0} > 0, \tag{2.3}$$

where, for all $i = 1, \ldots, l$, $\Theta_i \in \mathbb{R}^{N\times N}$ satisfies

$$\Theta_i \begin{cases} = A_{\mathcal{G}_1}, & \text{if } \{v_{n_{i-1}}, v_{n_i}\} \in E_1 \\ & \text{and } \{v_{n_{i-1}}, v_{n_i}\} \notin E_2, \\ = A_{\mathcal{G}_2}, & \text{if } \{v_{n_{i-1}}, v_{n_i}\} \in E_2 \\ & \text{and } \{v_{n_{i-1}}, v_{n_i}\} \notin E_1, \\ \in \{A_{\mathcal{G}_1}, A_{\mathcal{G}_2}\}, & \text{otherwise}. \end{cases} \tag{2.4}$$

Proof. We prove this result by induction on the length l of the walk. First, assume that $l = 1$. It follows from the definition of the adjacency matrix that $e_{n_1}^{\mathrm{T}} A_{\mathcal{G}_1} e_{n_0} > 0$ if and only if $\{v_{n_0}, v_{n_1}\} \in E_1$ and $e_{n_1}^{\mathrm{T}} A_{\mathcal{G}_2} e_{n_0} > 0$ if and only if $\{v_{n_0}, v_{n_1}\} \in E_2$. Therefore, there exists a walk of length 1 between v_{n_0} and v_{n_1} if and only if $e_{n_1}^{\mathrm{T}} \Theta_1 e_{n_0} > 0$.

Now, for induction, assume that the result holds for walks of length $l - 1 \geq 1$.

Next, we prove that the result holds for walks of length l. Note that there exists a walk $(v_{n_0}, v_{n_1}, \ldots, v_{n_l})$ of length l between v_{n_0} and v_{n_l} if and only if there exists a vertex $v_{n_1} \in V$ such that there exists a walk $(v_{n_1}, v_{n_2}, \ldots, v_{n_l})$ of length $l - 1$ between v_{n_1} and v_{n_l}, and a walk (v_{n_0}, v_{n_1}) of length 1 between v_{n_0} and v_1.

Since the result holds for walks of length of $l-1$, it follows that there exists a walk $(v_{n_1}, v_{n_2}, \ldots, v_{n_l})$ of length $l-1$ between v_{n_1} and v_{n_l} if and only if $e_{n_l}^{\mathrm{T}} \Theta_l \Theta_{l-1} \cdots \Theta_2 e_{n_1} > 0$. Furthermore, there exists a walk of length 1 between v_{n_0} and v_{n_1} if and only if $e_{n_1}^{\mathrm{T}} \Theta_1 e_{n_0} > 0$.

Therefore, there exists a walk $(v_{n_0}, v_{n_1}, \ldots, v_{n_l})$ of length l between v_{n_0} and v_{n_l} if and only if there exists a vertex $v_{n_1} \in V$ such that $\left(e_{n_l}^{\mathrm{T}} \Theta_l \Theta_{l-1} \cdots \Theta_2 e_{n_1}\right)\left(e_{n_1}^{\mathrm{T}} \Theta_1 e_{n_0}\right) > 0$. Furthermore, note that $e_{n_l}^{\mathrm{T}} \Theta_l \Theta_{l-1} \cdots \Theta_1 e_{n_0} = \sum_{k=1}^{N} \left(e_{n_l}^{\mathrm{T}} \Theta_l \cdots \Theta_2 e_k\right)\left(e_k^{\mathrm{T}} \Theta_1 e_{n_0}\right)$. Thus, $e_{n_l}^{\mathrm{T}} \Theta_l \cdots \Theta_1 e_{n_0} > 0$ if and only if there exists an $k \in \{1, \ldots, N\}$ such that $\left(e_{n_l}^{\mathrm{T}} \Theta_l \cdots \Theta_2 e_k\right)\left(e_k^{\mathrm{T}} \Theta_1 e_{n_0}\right) > 0$.

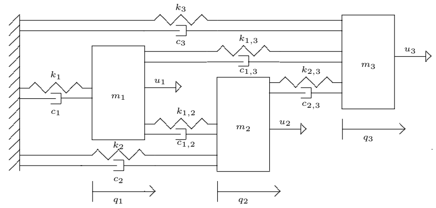

Fig. 2. 3-mass structure with all possible spring and dashpot connections.

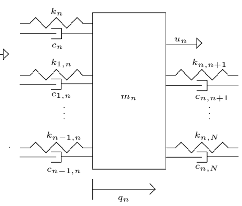

Fig. 3. Spring and dashpot connections to the nth mass.

Therefore, there exists a walk $(v_{n_0}, v_{n_2}, \ldots, v_{n_l})$ of length l between v_{n_0} and v_{n_l} if and only if (2.3) is satisfied. □

The final result of this section is the contrapositive of the necessary condition of Lemma 2.5.

Corollary 2.1. *Consider the graph $\mathcal{G}_{12} = (V, E_1 \cup E_2)$, and let $v_{n_0}, v_{n_l} \in V$ be distinct. For all $i = 1, \ldots, l$, let $\bar{E}_i = E_1$ or $\bar{E}_i = E_2$. Assume that there does not exist a walk $(v_{n_0}, v_{n_1}, \ldots, v_{n_l})$ of length l between v_{n_0} and v_{n_l} such that, for all $i = 1, \ldots, l$, $\{v_{n_{i-1}}, v_{n_i}\} \in \bar{E}_i$. Then*

$$e_{n_l}^{\mathrm{T}} \Theta_l \Theta_{l-1} \cdots \Theta_1 e_{n_0} = 0, \tag{2.5}$$

where, for $i = 1, \ldots, l$, $\Theta_i \triangleq \begin{cases} A_{\mathcal{G}_1}, & \text{if } \bar{E}_i = E_1, \\ A_{\mathcal{G}_2}, & \text{if } \bar{E}_i = E_2. \end{cases}$

3. The Dynamics and Stability Properties of Lumped-Mass Structures

In this section, we present the dynamics of lumped-mass structures. Specifically, we consider mass-spring-dashpot structures in which every pair of masses may or may not be connected by a spring, a dashpot, or a spring and dashpot in parallel. In addition, each mass may or may not be connected to a fixed wall by a spring, a dashpot, or a spring and dashpot in parallel. The N masses of the structure are denoted by $m_1, \ldots, m_N$. For all distinct $i, j = 1, \ldots, N$, $c_{i,j}$ is the damping coefficient of the dashpot connecting the ith and jth masses, and $k_{i,j}$ is the stiffness of the spring connecting the ith and jth masses. If $c_{i,j} = 0$ or $k_{i,j} = 0$, then the ith and jth masses are not connected by a dashpot or a spring, respectively. For all $i = 1, \ldots, N$, c_i and k_i are the damping coefficient and spring stiffness, respectively, of the dashpot and spring connecting the ith mass to the wall. If $c_i = 0$ or $k_i = 0$, then the ith mass is not connected to the wall by a dashpot or a spring, respectively. For all $i = 1, \ldots, N$, $q_i(t)$ is the position of the mass m_i relative to an equilibria position, and $u_i(t)$ is the force acting on the mass m_i.

We consider lumped-mass structures that consist of physical masses, springs, and dampers, that is, the system must have positive masses, nonnegative damping coefficients, and nonnegative spring stiffnesses. Specifically, for all $i = 1, \ldots, N$, $m_i > 0$ and $c_i, k_i \geq 0$ and, for all $i, j = 1, \ldots, N$ such that $i \neq j$, $c_{i,j} \geq 0$ and $k_{i,j} \geq 0$.

For $N = 3$, Figure 2 shows a 3-mass structure with all possible spring and dashpot connections. Figure 3 shows the possible spring and dashpot connections for the nth mass in an N-mass lumped-mass structure.

The dynamics of the N-mass lumped-mass structure are given by

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = u(t), \tag{3.1}$$

where $M \triangleq \operatorname{diag}(m_1, \ldots, m_N)$, $C \triangleq C_\mathrm{w} + L_C$, $K \triangleq K_\mathrm{w} + L_K$, $q(t) \triangleq \begin{bmatrix} q_1(t) & \cdots & q_N(t) \end{bmatrix}^\mathrm{T}$, $u(t) \triangleq \begin{bmatrix} u_1(t) & \cdots & u_N(t) \end{bmatrix}^\mathrm{T}$, and

$$C_\mathrm{w} \triangleq \operatorname{diag}(c_1, \ldots, c_N), \quad K_\mathrm{w} \triangleq \operatorname{diag}(k_1, \ldots, k_N), \tag{3.2}$$

$$L_C \triangleq \begin{bmatrix} \sum_{j=2}^N c_{1,j} & -c_{1,2} & \cdots & -c_{1,N} \\ -c_{2,1} & \sum_{j=1, j\neq 2}^N c_{2,j} & \cdots & -c_{2,N} \\ \vdots & & \ddots & \vdots \\ -c_{N,1} & -c_{N,2} & \cdots & \sum_{j=1}^{N-1} c_{N,j} \end{bmatrix}, \tag{3.3}$$

$$L_K \triangleq \begin{bmatrix} \sum_{j=2}^N k_{1,j} & -k_{1,2} & \cdots & -k_{1,N} \\ -k_{2,1} & \sum_{j=1, j\neq 2}^N k_{2,j} & \cdots & -k_{2,N} \\ \vdots & & \ddots & \vdots \\ -k_{N,1} & -k_{N,2} & \cdots & \sum_{j=1}^{N-1} k_{N,j} \end{bmatrix}. \tag{3.4}$$

Note that the damping matrix C has a diagonal component C_w, which represents the dashpots connecting masses $m_1, \ldots, m_N$ to the wall, and a non-diagonal component L_C, which results from the dashpot interconnections among the N masses. Similarly, the stiffness matrix K has a diagonal component K_w and a non-diagonal component L_K. This distinction will be important in the following section when we associate several graphs with the lumped-mass structure.

Next, we review the stability properties of lumped-mass structures. The lumped-mass structure (3.1)-(3.4) can be written as the first-order linear state-space system

$$\dot{x} = Ax + Bu, \quad q = C_\mathrm{p} x, \tag{3.5}$$

where

$$A \triangleq \begin{bmatrix} 0 & I_N \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}, \tag{3.6}$$

$$C_\mathrm{p} \triangleq \begin{bmatrix} I_N & 0 \end{bmatrix}, \tag{3.7}$$

and $x \triangleq \begin{bmatrix} q_1 & \cdots & q_N & \dot{q}_1 & \cdots & \dot{q}_N \end{bmatrix}$. First, we characterize the eigenvalues of A in terms of the mass, stiffness, and damping matrices. This result can be found in [10, p. 203]

Lemma 3.1. *Consider the lumped-mass system (3.5)-(3.7) and let $s \in \mathbb{C}$. Then* $\det(sI - A) = 0$ *if and only if* $\det(s^2 M + sC + K) = 0$.

The following stability result is from [11, Theorem 1].

Lemma 3.2. *Consider the lumped-mass system (3.5)-(3.7). Then the following statements are valid.*

(i) A is Lyapunov stable if and only if $K+C$ is positive definite.

(ii) A is asymptotically stable if and only if $\left(KM^{-1}, C\right)$ is controllable and K is positive definite.

4. The Positive Zeros of Lumped-Mass Structures

In this section, we analyze the zeros of lumped-mass structures using the graph-theoretic tools presented in Section 2. To aid in our analysis, we associate three weighted graphs with the lumped-mass structure (3.1)-(3.4). The masses are the vertices of the graphs, and the edges of the graphs are the dashpots, the springs, or the springs and dashpots. Furthermore, the weights associated with each edge are functions of the damping coefficient and the spring stiffness. Define the vertex set $V_M \triangleq \{m_1, \ldots, m_N\}$ and the edge sets

$$E_C \triangleq \{\{m_i, m_j\} : c_{i,j} > 0, \text{ where } i, j = 1, \ldots, N \text{ and } i \neq j\},$$

$$E_K \triangleq \{\{m_i, m_j\} : k_{i,j} > 0, \text{ where } i, j = 1, \ldots, N \text{ and } i \neq j\}.$$

Define the weighted graphs $\mathcal{G}_C \triangleq (V_M, E_C)$, where, for all $\{m_i, m_j\} \in E_C$, the weight $c_{i,j}$ is assigned to the edge $\{m_i, m_j\}$, and $\mathcal{G}_K \triangleq (V_M, E_K)$, where, for all $\{m_i, m_j\} \in E_K$, the weight $k_{i,j}$ is assigned to the edge $\{m_i, m_j\}$.

By examining (3.3) and (3.4), it can be seen that L_C and L_K are the Laplacian matrices associated with $\mathcal{G}_C$ and $\mathcal{G}_K$, respectively. Then Lemma 2.1 implies that L_C and L_K are positive-semidefinite M-matrices. Since C_w and K_w are diagonal positive-semidefinite matrices, we conclude that the damping matrix $C = C_\mathrm{w} + L_C$ and the stiffness matrix $K = K_\mathrm{w} + L_K$ are positive semidefinite. We have thus proven the known fact that lumped-mass structures of the form (3.1)-(3.4) have positive-semidefinite damping and stiffness matrices.

A notion of structural connectedness is needed to analyze the zeros of lumped-mass structures. Roughly speaking, a lumped-mass structure is structurally connected if it is a single structure rather than two or more disjoint structures. To formalize this idea, define the edge set $E_{CK} \triangleq E_C \cup E_K$.

Definition 4.1. *The lumped-mass structure (3.1)-(3.4) is* structurally connected *if the graph $\mathcal{G}_{CK} \triangleq (V_M, E_{CK})$ is connected.*

Definition 4.1 intuitively implies that (3.1)-(3.4) is structurally connected if and only if the force-to-motion transfer functions between every pair of masses is nonzero. Next, we characterize structural connectedness in terms of the damping and stiffness matrices.

Lemma 4.1. *The lumped-mass structure (3.1)-(3.4) is structurally connected if and only if $K + C$ is irreducible.*

Proof. Define the weighted graph $\mathcal{G}_{CK} \triangleq (V_M, E_{CK})$, where, for all $\{m_i, m_j\} \in E_{CK}$, the weight $k_{i,j} + c_{i,j}$ is assigned to the edge $\{m_i, m_j\}$. By examining (3.3) and (3.4), it follows that $L_K + L_C$ is the Laplacian matrix associated with $\mathcal{G}_{CK}$. Lemma 2.2 implies that $L_K + L_C$ is irreducible if and only if $\mathcal{G}_{CK}$ is connected. Since $K_\mathrm{w} + C_\mathrm{w}$ is diagonal, it follows that $K + C = L_K + L_C + K_\mathrm{w} + C_\mathrm{w}$ is irreducible if and only if $L_K + L_C$ is irreducible. Therefore, $K + C$ is irreducible if and only if $\mathcal{G}_{CK}$ is connected. □

We now present our main result on the zeros of lumped-mass structures.

Theorem 4.1. *Assume that the system (3.5)-(3.7) is structurally connected. Then, for all $i, j = 1, \ldots, N$, the transfer function from $u_j(t)$ to $q_i(t)$ has no positive zeros. If, in addition, the graph $\mathcal{G}_K$ is connected and K is positive definite, then, for all $i, j = 1, \ldots, N$, the transfer function from $u_j(t)$ to $q_i(t)$ has no nonnegative zeros.*

Proof. The transfer function from the force input u_j applied to mass m_j to the position q_i of mass m_i is $G_{i,j}(s) \triangleq e_i^\mathrm{T} C_\mathrm{p} (sI - A)^{-1} B e_j$, where, for $i = 1, \ldots, N$, e_i is the ith column of I_N. Let $z > 0$. For all $i, j = 1, \ldots, N$,

$$\begin{aligned} G_{i,j}(z) &= e_i^\mathrm{T} C_\mathrm{p} (zI - A)^{-1} B e_j \\ &= \begin{bmatrix} e_i^\mathrm{T} & 0 \end{bmatrix} (sI - A)^{-1} \begin{bmatrix} 0 \\ M^{-1} e_j \end{bmatrix}. \end{aligned} \tag{4.1}$$

Since $z > 0$, M is positive definite, and C and K are positive semidefinite, it follows that zI and $\frac{1}{z}M^{-1}\left(Mz^2 + Cz + K\right)$ are nonsingular. Hence, Proposition 2.8.7 of [10] implies that

$$(sI - A)^{-1} = \begin{bmatrix} zI & -I \\ M^{-1}K & zI + M^{-1}C \end{bmatrix}^{-1} = \begin{bmatrix} \sharp & (z^2 I + M^{-1}Cz + M^{-1}K)^{-1} \\ \sharp & \sharp \end{bmatrix}, \quad (4.2)$$

where $\sharp$ denotes an inconsequential entry. Combining (4.1) and (4.2) yields

$$G_{i,j}(z) = e_i^{\mathrm{T}}(Mz^2 + Cz + K)^{-1}e_j = e_i^{\mathrm{T}}[(Mz^2 + C_{\mathrm{w}}z + K_{\mathrm{w}}) + (L_C z + L_K)]^{-1}e_j. \quad (4.3)$$

Next, it follows from (3.3) and (3.4) that $L_C z + L_K$ is the Laplacian matrix of the weighted graph $\mathcal{G}_{CK} \triangleq (V_M, E_{CK})$, where, for all $\{m_i, m_j\} \in E$, the weight $c_{i,j}z + k_{i,j}$ is associated with the edge $\{m_i, m_j\}$. Furthermore, $Mz^2 + C_{\mathrm{w}}z + K_{\mathrm{w}}$ is a diagonal positive-definite matrix. Since $\mathcal{G}_{CK}$ is connected and $Mz^2 + C_{\mathrm{w}}z + K_{\mathrm{w}}$ is a diagonal positive-definite matrix, Lemma 2.4 implies that $(Mz^2 + C_{\mathrm{w}}z + K_{\mathrm{w}}) + (L_C z + L_K)$ is an irreducible, nonsingular M-matrix and every entry of $(Mz^2 + Cz + K)^{-1} = [(Mz^2 + C_{\mathrm{w}}z + K_{\mathrm{w}}) + (L_C z + L_K)]^{-1}$ is positive. Therefore, for all $i, j = 1, \ldots, N$, it follows from (4.3) that $G_{i,j}(z) > 0$, and thus z is not a zero of $G_{i,j}(s)$.

Now consider the case $z = 0$. It follows from (4.1) that

$$G_{i,j}(0) = \begin{bmatrix} e_i^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} 0 & -I \\ M^{-1}K & M^{-1}C \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ M^{-1}e_j \end{bmatrix}. \quad (4.4)$$

Since $M > 0$ and $K > 0$ it follows that $M^{-1}K$ is nonsingular. Hence, Fact 2.15.2 of [10] implies that

$$\begin{bmatrix} 0 & -I \\ M^{-1}K & M^{-1}C \end{bmatrix}^{-1} = \begin{bmatrix} \sharp & K^{-1}M \\ \sharp & \sharp \end{bmatrix}. \quad (4.5)$$

Combining (4.4) and (4.5) yields $G_{i,j}(0) = e_i^{\mathrm{T}}K^{-1}e_j = e_i^{\mathrm{T}}(K_{\mathrm{w}} + L_K)^{-1}e_j$. Since $\mathcal{G}_K$ is connected, it follows from Lemma 2.1 and Lemma 2.2 that L_K is an irreducible, singular, M-matrix. Since K_{w} is diagonal, it follows that $K = K_{\mathrm{w}} + L_K$ is an irreducible M-matrix. Furthermore, since K is positive definite, it follows that K is an irreducible, nonsingular, M-matrix. It then follows from Lemma 2.3 that every entry of K^{-1} is positive. Therefore, $G_{i,j}(0) = e_i^{\mathrm{T}}K^{-1}e_j$ is positive and $z = 0$ is not a zero of $G_{i,j}(s)$. □

Corollary 4.1. *Assume that the system (3.5)-(3.7) is structurally connected. Then, for all $i, j = 1, \ldots, N$, the transfer functions from $u_j(t)$ to $\dot{q}_i(t)$ and from $u_j(t)$ to $\ddot{q}_i(t)$ have no positive zeros.*

5. The Complex Nonminimum Phase Zeros of Lumped-Mass Structures

Theorem 4.1 and Corollary 4.1 guarantee that every force-to-motion transfer function of a structurally connected lumped-mass structure has no positive zeros. However, these results do not guarantee that every force-to-motion transfer function is minimum phase; the force-to-motion transfer functions can have complex zeros in the open right half plane. In fact, for $N = 3$, there exists a lumped-mass structure (3.1)-(3.4) that is structurally connected and has a nonminimum phase force-to-motion transfer function. Specifically, consider the 3-mass lumped-mass structure in Figure 2, where $m_1 = m_2 = m_3 = 1$ kg, $k_1 = k_3 = 0$ kg/s^2, $k_2 = k_{1,2} = k_{1,3} = k_{2,3} = 5$ kg/s^2, $c_1 = c_2 = c_3 = c_{1,2} = c_{2,3} = 0$ kg/s, and $c_{1,3} = 5$ kg/s. This system is structurally connected since the graph $\mathcal{G}_{CK}$ is connected. Furthermore, the transfer function from $u_1(t)$ to $q_3(t)$, given by

$$G_{1,3}(s) \triangleq \frac{5s^3 + 5s^2 + 75s + 100}{s^6 + 10s^5 + 35s^4 + 200s^3 + 325s^2 + 250s + 375},$$

is nonminimum phase. The zeros of $G_{1,3}(s)$ are approximately $0.150 \pm \jmath 3.92$ and -1.30. In fact, for all $N \geq 3$, there exists a lumped-mass structure (3.1)-(3.4) that is structurally connected and has a nonminimum phase force-to-motion transfer function.

6. Initial Undershoot in Lumped-Mass Structures

Initial undershoot describes the qualitative behavior of the step response of a transfer function. A system has initial undershoot if the step response initially moves in the direction that is opposite its asymptotic value. We now define initial undershoot and state a result classifying the existence of initial undershoot. The definition and result are given in [12–14].

Definition 6.1. *Let $H(s)$ be a single-input single-output asymptotically stable transfer function with relative degree $r > 0$. Let $y(t)$ be the step response of $H(s)$. Assume that $H(0) \neq 0$. Then the step response of $H(s)$ has* initial undershoot *if $y^{(r)}(0)\lim_{t\to\infty} y(t) < 0$.*

Lemma 6.1. *Let $H(s)$ be a single-input single-output asymptotically stable transfer function with relative degree $r > 0$. Assume that $H(0) \neq 0$. Then the step response of $H(s)$ has initial undershoot if and only if $H(s)$ has an odd number of positive zeros.*

The main result of this section addresses the existence of initial undershoot in a force-to-position transfer function of an asymptotically stable lumped-mass structure.

Theorem 6.1. *Assume that the system (3.5)-(3.7) is structurally connected. Furthermore, assume that A is asymptotically stable and the graph $\mathcal{G}_K$ is connected. Then, for all $i, j = 1, \ldots, N$, the step response of the transfer function from $u_j(t)$ to $q_i(t)$ does not exhibit initial undershoot.*

Proof. Let $G_{i,j}(s)$ be the transfer function from the force input on mass m_j to the position of mass m_i. Since A is asymptotically stable, Lemma 3.2 implies $K > 0$. It follows from Theorem 4.1 that $G_{i,j}(s)$ has no nonnegative zeros. Therefore, $G_{i,j}(0) \neq 0$ and $G_{i,j}(s)$ has no positive zeros. Since A is asymptotically stable, $G_{i,j}(s)$ is asymptotically stable. Since $G_{i,j}(s)$ is an asymptotically stable transfer function, $G_{i,j}(0) \neq 0$, and $G_{i,j}(s)$ has no positive zeros, it follows from Lemma 6.1 that $G_{i,j}(s)$ does not exhibit initial undershoot. □

7. Example: 3-Mass Lumped-Mass Structure

Consider the structurally connected 3-mass structure shown in Figure 2 whose dynamics are given by (3.1)-(3.4), where $N = 3$. For this example, the masses are $m_1 = m_2 = m_3 = 5$ kg; the spring stiffnesses are $k_1 = 1$ kg/s^2, $k_2 = 2$ kg/s^2, $k_3 = 3$ kg/s^2, $k_{1,2} = 12$ kg/s^2, $k_{1,3} = 13$ kg/s^2, and $k_{2,3} = 23$ kg/s^2; and the damping coefficients are $c_1 = 10$ kg/s, $c_2 = 20$ kg/s, $c_3 = 30$ kg/s, $c_{1,2} = 120$ kg/s, $c_{1,3} = 130$ kg/s, and $c_{2,3} = 230$ kg/s.

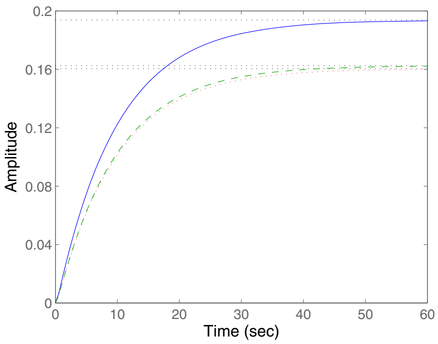

Fig. 4. The step responses of $G_{1,1}(s)$ (solid), $G_{2,1}(s)$ (dashed), and $G_{3,1}(s)$ (dotted) do not display initial undershoot.

The transfer functions from u_1 to q_1, from u_1 to q_2, and from u_1 to q_3 are

$$G_{1,1}(s) = \frac{0.2s^4 + 30.4s^3 + 734.24s^2 + 146.24s + 7.312}{p(s)},$$
$$G_{2,1}(s) = \frac{4.8s^3 + 614.08s^2 + 122.72s + 6.136}{p(s)},$$
$$G_{3,1}(s) = \frac{5.2s^3 + 606.12s^2 + 121.12s + 6.056}{p(s)},$$

respectively, where $p(s) \triangleq s^6 + 204s^5 + 10328.4s^4 + 39813.6s^3 + 11428.68s^2 + 1132.56s + 37.752$. The zeros of $G_{1,1}(s)$ are approximately -121.9, -29.86, -0.1001, and -0.1003. The zeros of $G_{2,1}(s)$ are approximately -127.7, -0.1000, and -0.1001. The zeros of $G_{3,1}(s)$ are approximately -116.4, -0.1000, and -0.1001. Therefore, $G_{1,1}(s)$, $G_{2,1}(s)$, and $G_{3,1}(s)$ have no nonnegative zeros as guaranteed by Theorem 4.1. Furthermore, Theorem 6.1 implies that the step responses of $G_{1,1}(s)$, $G_{2,1}(s)$, and $G_{3,1}(s)$ do not have initial undershoot. Figure 4 verifies that the step responses do not have initial undershoot.

8. Relative Degree of Lumped-Mass Structures

In this section, we analyze the relative degree of the lumped-mass structure (3.5)-(3.7). If we assume that (3.5)-(3.7) is structurally connected, then the ith and jth masses are connected by means of a sequence of springs and dashpots. To calculate the relative degree of the transfer function from $u_j(t)$ to $q_i(t)$, we consider all sequences of springs and dashpots that connect m_j to m_i. More precisely, we consider all walks on the graph $\mathcal{G}_{CK} = (V_M, E_{CK})$ from m_j to m_i. For all $i, j = 1, \ldots, N$ such that $i \neq j$, define

$$\Omega_{i,j} \triangleq \{\omega : \omega \text{ is a walk on } \mathcal{G}_{CK} \text{ from } m_j \text{ to } m_i\}, \quad (8.1)$$

and, for all $i = j = 1, \ldots, N$, define $\Omega_{i,j} \triangleq \emptyset$.

Let $\omega = (m_{n_0}, m_{n_1}, \ldots, m_{n_l})$ be a walk of length l on $\mathcal{G}_{CK}$ from from m_{n_0} to m_{n_l}. Now, count the number of edges $n_C(\omega)$ in the walk ω that are dashpots only, the number of edges $n_K(\omega)$ in the walk ω that are springs only, and the number of edges $n_{CK}(\omega)$ in the walk ω that are springs and dashpots, that is,

$$n_C(\omega) = \sum_{i=1}^{l} \alpha_i, \quad n_K(\omega) = \sum_{i=1}^{l} \beta_i, \quad n_{CK}(\omega) = \sum_{i=1}^{l} \gamma_i,$$

where, for $i = 1, \ldots, l$,

$$\alpha_i = \begin{cases} 1, & \text{if } \{m_{n_{i-1}}, m_{n_i}\} \in E_C \text{ and } \{m_{n_{i-1}}, m_{n_i}\} \notin E_K, \\ 0, & \text{otherwise}, \end{cases}$$
$$\beta_i = \begin{cases} 1, & \text{if } \{m_{n_{i-1}}, m_{n_i}\} \in E_K \text{ and } \{m_{n_{i-1}}, m_{n_i}\} \notin E_C, \\ 0, & \text{otherwise}, \end{cases}$$
$$\gamma_i = \begin{cases} 1, & \text{if } \{m_{n_{i-1}}, m_{n_i}\} \in E_K \cap E_C, \\ 0, & \text{otherwise}. \end{cases}$$

The next result provides an expression for the relative degree of the transfer function from $u_j(t)$ to $q_i(t)$ and characterizes the sign of the first non-zero Markov parameter (often called the high-frequency gain).

Theorem 8.1. *Assume that the system (3.5)-(3.7) is structurally connected. Then, for all $i, j = 1, \ldots, N$, the relative degree of the transfer function from $u_j(t)$ to $q_i(t)$ is*

$$r_{i,j} \triangleq \min_{\omega \in \Omega_{i,j}} 2n_K(\omega) + n_C(\omega) + n_{CK}(\omega) + 2. \quad (8.2)$$

Furthermore, for all $i, j = 1, \ldots, N$, the first non-zero Markov parameter of the transfer function from $u_j(t)$ to $q_i(t)$ is

$$H_{r_{i,j}} \triangleq e_i^{\mathrm{T}} C_{\mathrm{p}} A^{r_{i,j}-1} B e_j > 0. \quad (8.3)$$

Proof. Let i and j be positive integers between 1 and N, and let $\omega \in \Omega_{i,j}$ be the minimizer in (8.2) so that $r_{i,j} = 2n_K(\omega) + n_C(\omega) + n_{CK}(\omega) + 2$. For all $n = 1, \ldots, N$,

$$H_n \triangleq e_i^{\mathrm{T}} C_{\mathrm{p}} A^{n-1} B e_j = \begin{bmatrix} e_i^{\mathrm{T}} & 0 \end{bmatrix} A^{n-1} \begin{bmatrix} 0 \\ M^{-1} e_j \end{bmatrix}. \quad (8.4)$$

To prove (8.2) and (8.3), it suffices to show that $H_0, H_1, \ldots, H_{r_{i,j}-1} = 0$ and $H_{r_{i,j}} > 0$.Performing the matrix multiplications in (8.4) implies

$$H_n = \begin{bmatrix} e_i^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \sharp & \Gamma_n \\ \sharp & \sharp \end{bmatrix} \begin{bmatrix} 0 \\ M^{-1} e_j \end{bmatrix} = e_i^{\mathrm{T}} \Gamma_n M^{-1} e_j, \quad (8.5)$$

where $\sharp$ denotes an inconsequential entry, $\Gamma_1 \triangleq 0$, $\Gamma_2 \triangleq I$, and, for all $n = 3, \ldots, N$, $\Gamma_n \triangleq -M^{-1} C \Gamma_{n-1} - M^{-1} K \Gamma_{n-2}$. By manipulating the terms of (8.5), it follows that

$$H_n = e_i^{\mathrm{T}} M^{-\frac{1}{2}} \bar{\Gamma}_n M^{-\frac{1}{2}} e_j = \frac{1}{\sqrt{m_i m_j}} e_i^{\mathrm{T}} \bar{\Gamma}_n e_j, \quad (8.6)$$

where $\bar{\Gamma}_1 \triangleq 0$, $\bar{\Gamma}_2 \triangleq I$, and, for all $n = 3, \ldots, N$,

$$\bar{\Gamma}_n \triangleq P \bar{\Gamma}_{n-1} + Q \bar{\Gamma}_{n-2}, \quad (8.7)$$

where $P \triangleq -M^{-\frac{1}{2}} C M^{-\frac{1}{2}}$ and $Q \triangleq -M^{-\frac{1}{2}} K M^{-\frac{1}{2}}$.

Since $\Gamma_1 = 0$, it follows that $H_1 = 0$. Since $\Gamma_2 = I$, it follows that $H_2 > 0$ if and only if $i = j$, which is equivalent to $H_2 > 0$ if only if $\Omega_{i,j} = \emptyset$. Thus, $H_2 > 0$ if and only if $r_{i,j} = 2$.

Now, consider the case where $\Omega_{i,j} \neq \emptyset$ and thus $r_{i,j} > 2$. Note that P and Q can each be expressed as the sum of a weighted adjacency matrix and a diagonal negative semidefinite matrix, that is,

$$P = A_{\mathcal{G}_C} + D_C, \quad Q = A_{\mathcal{G}_K} + D_K, \quad (8.8)$$

where $D_C \triangleq -\mathrm{diag}\left(\frac{c_1}{m_1} + \sum_{j=2}^{N} \frac{c_{1,j}}{m_1}, \frac{c_2}{m_2} + \sum_{j=1, j\neq 2}^{N} \frac{c_{2,j}}{m_2}, \ldots, \frac{c_N}{m_N} + \sum_{j=1, j\neq N}^{N} \frac{c_{N,j}}{m_N}\right)$ and $D_K \triangleq -\mathrm{diag}\left(\frac{k_1}{m_1} + \sum_{j=2}^{N} \frac{k_{1,j}}{m_1}, \frac{k_2}{m_2} + \sum_{j=1, j\neq 2}^{N} \frac{k_{2,j}}{m_2}, \ldots, \frac{k_N}{m_N} + \sum_{j=1, j\neq N}^{N} \frac{k_{N,j}}{m_N}\right)$ are diagonal negative semidefinite, $A_{\mathcal{G}_C}$ is the weighted adjacency matrix associated with $\mathcal{G}_C$, where, for all $\{m_p, m_q\} \in E_C$, the weight $\frac{c_{p,q}}{\sqrt{m_p m_q}} > 0$ is assigned to the edge $\{m_p, m_q\}$, $A_{\mathcal{G}_K}$ is the weighted adjacency matrix associated with $\mathcal{G}_K$, where, for all $\{m_p, m_q\} \in E_K$, the weight $\frac{k_{p,q}}{\sqrt{m_p m_q}} > 0$ is assigned to the edge $\{m_p, m_q\}$.

It follows from the regression (8.7) that, for all $n = 3, \ldots, N$,

$$\bar{\Gamma}_n = \sum{}' P^{p_1} Q^{q_1} \cdots P^{p_{n-3}} Q^{q_{n-3}} P^{p_{n-2}}, \tag{8.9}$$

where $\sum'$ denotes the sum over all distinct products such that $p_1, \ldots, p_{n-2}, q_1, \ldots, q_{n-3} \in \{0, 1\}$ and $\sum_{j=1}^{n-2} p_j + 2\sum_{j=1}^{n-3} q_j + 2 = n$. Combining (8.8)-(8.9) and performing the multiplications yields

$$\begin{aligned}\bar{\Gamma}_n &= \sum{}'(A_{\mathcal{G}_C} + D_C)^{p_1}(A_{\mathcal{G}_K} + D_K)^{q_1} \cdots (A_{\mathcal{G}_C} + D_C)^{p_{n-3}} \\ &\quad \times (A_{\mathcal{G}_K} + D_K)^{q_{n-3}}(A_{\mathcal{G}_C} + D_C)^{p_{n-2}} \\ &= \Lambda_n + \sum{}' A_{\mathcal{G}_C}^{p_1} A_{\mathcal{G}_K}^{q_1} \cdots A_{\mathcal{G}_C}^{p_{n-3}} A_{\mathcal{G}_K}^{q_{n-3}} A_{\mathcal{G}_C}^{p_{n-2}}, \end{aligned} \tag{8.10}$$

where, for all $n = 3, \ldots, N$,

$$\begin{aligned}\Lambda_n &= \sum{}' D_C^{p_1} A_{\mathcal{G}_K}^{q_1} \cdots A_{\mathcal{G}_C}^{p_{n-3}} A_{\mathcal{G}_K}^{q_{n-3}} A_{\mathcal{G}_C}^{p_{n-2}} \\ &\quad + \sum{}' D_C^{p_1} D_K^{q_1} \cdots A_{\mathcal{G}_C}^{p_{n-3}} A_{\mathcal{G}_K}^{q_{n-3}} A_{\mathcal{G}_C}^{p_{n-2}} \\ &\quad + \cdots + \sum{}' A_{\mathcal{G}_C}^{p_1} D_K^{q_1} \cdots A_{\mathcal{G}_C}^{p_{n-3}} A_{\mathcal{G}_K}^{q_{n-3}} A_{\mathcal{G}_C}^{p_{n-2}} \\ &\quad + \cdots + \sum{}' D_C^{p_1} D_K^{q_1} \cdots D_C^{p_{n-3}} D_K^{q_{n-3}} D_C^{p_{n-2}}. \end{aligned} \tag{8.11}$$

Since ω is the minimizer of (8.2), then there does not exist a walk $\bar{\omega} \in \Omega_{i,j}$ such that $2n_K(\bar{\omega}) + n_C(\bar{\omega}) + n_{CK}(\bar{\omega}) < 2n_K(\omega) + n_C(\omega) + n_{CK}(\omega)$. Thus, Corollary 2.1 implies that, for all $n = 3, \ldots, r_{i,j} - 1$,

$$e_i^{\mathrm{T}} A_{\mathcal{G}_C}^{p_1} A_{\mathcal{G}_K}^{q_1} \cdots A_{\mathcal{G}_C}^{p_{n-3}} A_{\mathcal{G}_K}^{q_{n-3}} A_{\mathcal{G}_C}^{p_{n-2}} e_j = 0, \tag{8.12}$$

where $p_1, \ldots, p_{n-2}, q_1, \ldots, q_{n-3} \in \{0, 1\}$ satisfy $\sum_{j=1}^{n-2} p_j + 2\sum_{j=1}^{n-3} q_j + 2 = n$. By combining (8.10) and (8.12), it follows that, for all $n = 3, \ldots, r_{i,j} - 1$, $e_i^{\mathrm{T}} \bar{\Gamma}_n e_j = e_i^{\mathrm{T}} \Lambda_n e_j$.

Next, note that every term of $e_i^{\mathrm{T}} \Lambda_n e_j$ of the form $e_i^{\mathrm{T}} D_C^{p_1} D_K^{q_1} \cdots D_C^{p_{n-3}} D_K^{q_{n-3}} D_C^{p_{n-2}} e_j$ is zero because D_C and D_K are diagonal and $i \neq j$. The remaining terms of $e_i^{\mathrm{T}} \Lambda_n e_j$ have the form of (8.12) where a negative semidefinite matrix D_C or D_K may appear between the matrices $A_{\mathcal{G}_C}$ and $A_{\mathcal{G}_K}$. Therefore, Lemma A.1 and (8.12) imply that for all $n = 3, \ldots, r_{i,j} - 1$, $e_i^{\mathrm{T}} \Lambda_n e_j = 0$. Thus, for all $n = 3, \ldots, r_{i,j} - 1$, $e_i^{\mathrm{T}} \bar{\Gamma}_n e_j = 0$, which implies that $H_n = 0$.

Now, it suffice to show that $H_{r_{i,j}} > 0$. Again, note that every term of $e_i^{\mathrm{T}} \Lambda_n e_j$ of the form $e_i^{\mathrm{T}} D_C^{p_1} D_K^{q_1} \cdots D_C^{p_{n-3}} D_K^{q_{n-3}} D_C^{p_{n-2}} e_j$ is zero because D_C and D_K are diagonal and $i \neq j$. The remaining terms of $e_i^{\mathrm{T}} \Lambda_n e_j$ have the form of (8.12) where a negative semidefinite matrix D_C or D_K may appear between the matrices $A_{\mathcal{G}_C}$ and $A_{\mathcal{G}_K}$. Therefore, Lemma A.1 and (8.12) imply that $e_i^{\mathrm{T}} \Lambda_{r_{i,j}} e_j = 0$. Therefore,

$$e_i^{\mathrm{T}} \bar{\Gamma}_{r_{i,j}} e_j = \sum{}' e_i^{\mathrm{T}} A_{\mathcal{G}_C}^{p_1} A_{\mathcal{G}_K}^{q_1} \cdots A_{\mathcal{G}_C}^{p_{n-3}} A_{\mathcal{G}_K}^{q_{n-3}} A_{\mathcal{G}_C}^{p_{n-2}} e_j. \tag{8.13}$$

Furthermore, note that each product $e_i^{\mathrm{T}} A_{\mathcal{G}_C}^{p_1} A_{\mathcal{G}_K}^{q_1} \cdots A_{\mathcal{G}_C}^{p_{n-3}} A_{\mathcal{G}_K}^{q_{n-3}} A_{\mathcal{G}_C}^{p_{n-2}} e_j$ is nonnegative.

Since there exists a walk ω such that $r_{i,j} = 2n_K(\omega) + n_C(\omega) + n_{CK}(\omega) + 2$, it follows from Lemma 2.5 that there exist $p_1, \ldots, p_{r_{i,j}-2}, q_1, \ldots, q_{r_{i,j}-3} \in \{0, 1\}$ such that $\sum_{j=1}^{r_{i,j}-2} p_j + 2\sum_{j=1}^{r_{i,j}-3} q_j = r_{i,j} - 2$ and $e_i^{\mathrm{T}} A_{\mathcal{G}_C}^{p_1} A_{\mathcal{G}_K}^{q_1} \cdots A_{\mathcal{G}_C}^{p_{n-3}} A_{\mathcal{G}_K}^{q_{n-3}} A_{\mathcal{G}_C}^{p_{n-2}} e_j > 0$. Therefore, at least one term in the summation (8.13) is positive. Thus, $e_i^{\mathrm{T}} \bar{\Gamma}_{r_{i,j}} e_j > 0$, which implies that $H_{r_{i,j}} > 0$. $\square$

Notice, that the formula for relative degree provided by Theorem 8.1 does not depend on the specific values of the masses, spring constants, or damping coefficient. In fact, the relative degree depends only on the placement of the springs and dashpots.

Appendix A: Product of Nonnegative Matrices

In this appendix, we provide a result on the product of nonnegative matrices.

Lemma A.1. *Let $X, \bar{X} \in \mathbb{R}^{N \times N}$ be nonnegative. Let $p_1, \ldots, p_l$ and $\bar{p}_1, \ldots, \bar{p}_l$ be nonnegative integers, and let $D_1, \ldots, D_l \in \mathbb{R}^{N \times N}$ and $\bar{D}_1, \ldots, \bar{D}_l \in \mathbb{R}^{N \times N}$ be diagonal positive or negative semidefinite. Assume that $e_j^{\mathrm{T}} X^{p_1} \bar{X}^{\bar{p}_1} X^{p_2} \bar{X}^{\bar{p}_2} \cdots X^{p_l} \bar{X}^{\bar{p}_l} e_i = 0$. Then $e_j^{\mathrm{T}} D_1 X^{p_1} \bar{D}_1 \bar{X}^{\bar{p}_1} D_2 X^{p_2} \bar{D}_2 \bar{X}^{\bar{p}_2} \cdots D_l X^{p_l} \bar{D}_l \bar{X}^{\bar{p}_l} e_i = 0$.*

Proof. It follows from that $\mathrm{tr}\left(X^{p_1} \bar{X}^{\bar{p}_1} X^{p_2} \bar{X}^{\bar{p}_2} \cdots X^{p_l} \bar{X}^{\bar{p}_l} e_i e_j^{\mathrm{T}}\right) = 0$. Since X and $\bar{X}$ are nonnegative, it follows that $X^{p_1} \bar{X}^{\bar{p}_1} X^{p_2} \bar{X}^{\bar{p}_2} \cdots X^{p_l} \bar{X}^{\bar{p}_l} e_i e_j^{\mathrm{T}}$ has all zero entries along the diagonal. Since D_1 is diagonal positive (or negative) semidefinite, it follows that $D_1 X^{p_1} \bar{X}^{\bar{p}_1} X^{p_2} \bar{X}^{\bar{p}_2} \cdots X^{p_l} \bar{X}^{\bar{p}_l} e_i e_j^{\mathrm{T}}$ has all zero entries along the diagonal and is nonnegative (or nonpositive). Thus $\mathrm{tr}\left(D_1 X^{p_1} \bar{X}^{\bar{p}_1} X^{p_2} \bar{X}^{\bar{p}_2} \cdots X^{p_l} \bar{X}^{\bar{p}_l} e_i e_j^{\mathrm{T}}\right) = 0$, which implies $\mathrm{tr}\left(\bar{X}^{\bar{p}_1} X^{p_2} \bar{X}^{\bar{p}_2} \cdots X^{p_l} \bar{X}^{\bar{p}_l} e_i e_j^{\mathrm{T}} D_1 X^{p_1}\right) = 0$. Since $\bar{X}^{\bar{p}_1} X^{p_2} \bar{X}^{\bar{p}_2} \cdots X^{p_l} \bar{X}^{\bar{p}_l} e_i e_j^{\mathrm{T}} D_1 X^{p_1}$ is nonnegative (or nonpositive), it has all zero entries along the diagonal. Since D_1 is diagonal positive (or negative) semidefinite, it follows that $\bar{D}_1 \bar{X}^{\bar{p}_1} X^{p_2} \bar{X}^{\bar{p}_2} \cdots X^{p_l} \bar{X}^{\bar{p}_l} e_i e_j^{\mathrm{T}} D_1 X^{p_1}$ has all zero entries along the diagonal and is nonnegative (or nonnegative).

Continuing with this analysis will yield $\bar{X}^{\bar{p}_l} e_i e_j^{\mathrm{T}} D_1 X^{p_1} \bar{D}_1 \bar{X}^{\bar{p}_1} D_2 X^{p_2} \bar{D}_2 \bar{X}^{\bar{p}_2} \cdots D_l X^{p_l}$ has all zero entries along the diagonal. Then since $\bar{D}_l$ is diagonal, it follows that $\bar{D}_l \bar{X}^{\bar{p}_l} e_i e_j^{\mathrm{T}} \; D_1 X^{p_1} \bar{D}_1 \bar{X}^{\bar{p}_1} D_2 X^{p_2} \bar{D}_2 \bar{X}^{\bar{p}_2} \cdots D_l X^{p_l}$ has all zero entries along the diagonal. Therefore, $\mathrm{tr}\left(\bar{D}_l \bar{X}^{\bar{p}_l} e_i e_j^{\mathrm{T}} D_1 X^{p_1} \bar{D}_1 \bar{X}^{\bar{p}_1} D_2 X^{p_2} \bar{D}_2 \bar{X}^{\bar{p}_2} \cdots D_l X^{p_l}\right) = 0$. $\square$

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