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THERMODYNAMIC MODELLING OF INTERCONNECTED SYSTEMS, PART I: CONSERVATIVE COUPLING

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In this paper we extend results of Wyatt, Siebert and Tan by deriving an energy flow model in terms of thermodynamic energy rather than stored energy as in the standard Statistical Energy Analysis (SEA) approach to energy flow modelling. This modified energy flow model shows that energy flows according to thermodynamic energy and that this property holds for an arbitrary number of non-identical subsystems independently of the strength of the coupling. These results are compared with SEA energy flow predictions by means of illustrative examples. In particular, it is shown that for multiple coupled oscillators, SEA energy flow predictions based upon blocked energy may be erroneous, while predictions based upon thermodynamic energy are correct. In particular, the thermodynamic energy flow model correctly predicts zero net energy flow in the case of equal temperature subsystems.

1. INTRODUCTION

Although classical dynamics provides models for multi-degree-of-freedom vibrational systems, the inherent uncertainties and high dimensionality of many practical problems have led researchers to develop stochastic energy flow techniques. In this regard Statistical Energy Analysis, known as SEA, has been extensively developed and successfully applied to practical problems in vibrations and acoustics [1–11]. A brief readable account is given in Chapter 6 of reference [10].

From a system-theoretic point of view, however, the theoretical foundation of SEA remains unsatisfying, since the precise mathematical assumptions of the theory have not been completely specified. In addition, SEA itself has inherent limitations, such as the requirement of either weak subsystem coupling or identical subsystems [3].

Our motivation for examining SEA is twofold. First, we believe that thermodynamic modelling of large scale interconnected systems is an efficient approach for dealing with both uncertainty and dimensionality [12, 13], especially in the high frequency range [14–18]. In this regard, we assert that thermodynamic concepts need not be limited to the realm of statistical mechanics of molecular systems, but rather are applicable to low-dimensional, low frequency, deterministic systems. This point of view has been put forth in reference [19] and further emphasized in reference [20].

Our second motivation for this work is the long range goal of developing robust feedback controllers for large scale uncertain systems. In this regard SEA has already inspired some robust feedback control techniques [21–24].

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Prior work on the foundations of SEA has focused on compartmental modelling [25, 26]. Such models were used in an SEA context in reference [6], and re-examined in references [27, 28] by exploiting M-matrix properties [29].

The starting point for the present paper is the insightful work of Wyatt, Siebert and Tan [30]. This paper, which was motivated by SEA and the ideas of reference [19], proposes a significant departure from the usual SEA formulation. In essence, the contribution of reference [30] was not to define subsystem energy in terms of "blocked" or isolated stored (potential plus kinetic) modal energy. Rather, they propose defining subsystem energy as the ratio of the external input power to the damping coefficient. We call this ratio the *thermodynamic energy*. Thus, the energy (and hence "temperature") of a system is not fundamentally its stored energy content, but rather its ability to shed heat. Since the thermodynamic energy of a second order system is equivalent to its uncoupled mechanical energy (see section 6), this observation is consistent with the discussion in reference [31, see page 131].

The present paper, thus, has three goals. First, we re-state and, for completeness, re-prove the results obtained in reference [30]. In this regard we reformulate the multiple lossless coupled electrical network of reference [30] as the feedback interconnection of a positive real transfer function and a strictly positive real transfer function. This reformulation allows us to use state space and transfer function methods to render these results accessible to the linear systems and control community, while paving the way for application to mechanical systems (*a là* SEA) in later sections.

Next, we go beyond the results of reference [30] by deriving an energy flow model involving pairwise subsystem energy flow (Theorem 3.2). If the lossless coupling is purely imaginary—for example, a stiffness coupling—and the disturbances are mutually uncorrelated, then we prove that energy always flows from higher energy ("hotter") subsystems to lower energy ("colder") subsystems. Similar results are obtained for time domain models with white noise disturbances (section 4). It is important to note that these results hold for an arbitrary number of non-identical subsystems under strong (but purely imaginary) coupling.

Finally, we apply these results to coupled mechanical subsystems to contrast our results with those of SEA theory. Specifically, we show that for three coupled oscillators under strong coupling, energy flows according to thermodynamic energy and *not* according to isolated ("blocked") stored subsystem energy. In fact, we show that for a system with equal temperatures (equipartition of energy) there is no net energy flow, although SEA erroneously predicts such flow. These results demonstrate that thermodynamic modelling is also effective for low-dimensional, low frequency, deterministic systems.

By using thermodynamic modelling techniques developed in this paper, we further examine connections with SEA in another paper [32].

2. DEFINITIONS AND ASSUMPTIONS

In this paper we consider r scalar subsystems $z_1(s), \ldots, z_r(s)$ interconnected by a linear time-invariant lossless coupling $L(s)$. An electrical representation of this interconnection involving scalar impedances $z_i(s)$ is given in Figure 1, which is adapted from reference [30]. Each subsystem $z_i(s)$ is assumed to be a strictly positive real and thus asymptotically stable scalar transfer function. The disturbances $w_i(t)$ are zero-mean, wide-sense stationary random processes with power spectral density matrix $S_{ww}(\omega)$, where $w \triangleq [w_1 \cdots w_r]^{\mathrm{T}}$. Note that, for $i, j = 1, \ldots, r$,

$$S_{ww}(\omega)_{(i,i)} = S_{w_i w_i}(\omega), \qquad S_{ww}(\omega)_{(i,j)} = S_{ww}(\omega)_{(j,i)} = S_{w_i w_j}(\omega) = S_{w_j w_i}(\omega), \tag{1}$$

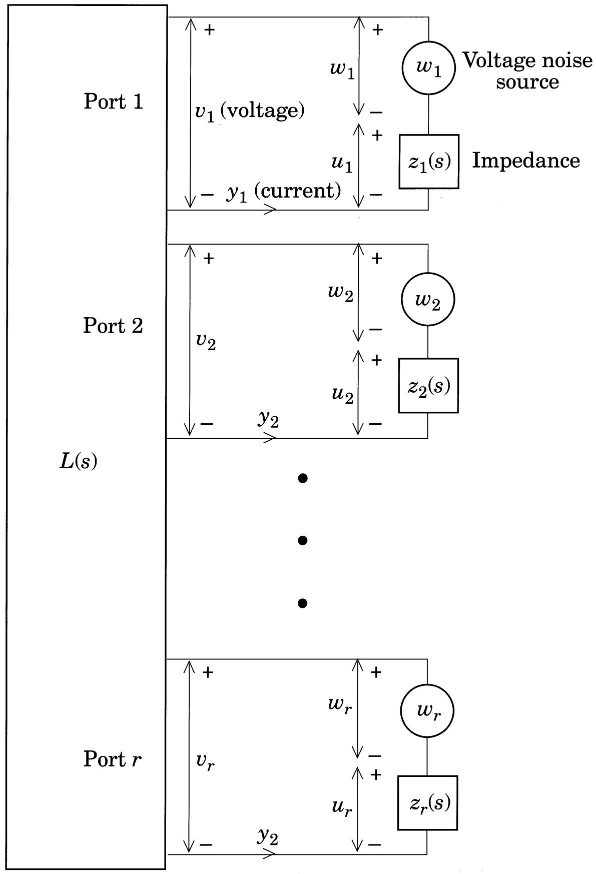

Figure 1. An electrical representation of coupled impedance subsystems.

where $S_{w_iw_i}(\omega)$ and $S_{w_iw_j}(\omega)$ are the power spectral density of $w_i(t)$ and the cross-power spectral density of $w_i(t)$ and $w_j(t)$, respectively (a list of notation is given in Appendix F). If $w_i(t)$ and $w_j(t)$ are uncorrelated, then $S_{w_iw_j}(\omega) = 0$. However, unlike reference [30], we allow $S_{w_iw_j}(\omega) \neq 0$. Since $z_i(s)$ is strictly positive real, it follows that

$$c_i(\omega) \triangleq \mathrm{Re}\,[z_i(\mathrm{j}\omega)] > 0, \qquad i = 1, \ldots, r, \quad \omega \in \mathscr{R}, \tag{2}$$

where $c_i(\omega)$ is the frequency dependent resistance or damping of the ith subsystem. For convenience, define the $r \times r$ diagonal transfer function

$$Z(s) \triangleq \mathrm{diag}\,(z_1(s), z_2(s), \ldots, z_r(s)), \tag{3}$$

and the frequency dependent resistance or damping matrix

$$C_d(\omega) \triangleq \mathrm{diag}\,(c_1(\omega), c_2(\omega), \ldots, c_r(\omega)) = \mathrm{Re}\,[Z(\mathrm{j}\omega)]. \tag{4}$$

The lossless r-port impedance coupling $L(s)$ is an $r \times r$ skew-Hermitian transfer function; that is,

$$L(\mathrm{j}\omega) = -L(\mathrm{j}\omega)^*, \qquad \omega \in \mathscr{R}. \tag{5}$$

This condition implies that $\mathrm{Re}\,[L(\mathrm{j}\omega)]$ is skew-symmetric and that $\mathrm{Im}\,[L(\mathrm{j}\omega)]$ is symmetric.

For later use we recast Figure 1 in a slightly different form involving $Z^{-1}(s)$, which is also strictly positive real. By defining the r-dimensional vectors

$$u \triangleq [u_1 \cdots u_r]^{\mathrm{T}}, \qquad y \triangleq [y_1 \cdots y_r]^{\mathrm{T}}, \qquad v \triangleq [v_1 \cdots v_r]^{\mathrm{T}},$$

it can be seen that Figure 2 is equivalent to Figure 1. Figure 2 will be useful in applying our results to mechanical systems for which v denotes force and y denotes velocity. Since $Z^{-1}(s)$ is a strictly positive real transfer function and $L(s)$ is a positive real transfer function, closed loop stability is guaranteed [33].

3. ENERGY FLOW ANALYSIS IN THE FREQUENCY DOMAIN

In this section we analyze energy flow among the coupled systems $z_i(s)$ shown in Figure 1. We use E to denote energy variables and P for power variables. Throughout this section we consider the situation in which the system is in steady state and all random processes are stationary.

At first, we consider the power associated with the coupling matrix $L(s)$ only. From Figure 2 it follows that

$$y = Z^{-1}(s)u = Z^{-1}(s)(w - v) = Z^{-1}(s)(w - L(s)y). \tag{6}$$

Since $Z(s)$ is square and invertible, it follows from equation (6) that

$$y = (I + Z^{-1}(s)L(s))^{-1}Z^{-1}(s)w = (L(s) + Z(s))^{-1}w. \tag{7}$$

On the other hand,

$$v = L(s)y = L(s)(L(s) + Z(s))^{-1}w. \tag{8}$$

Remark 3.1. Since $L(\mathrm{j}\omega) + L^*(\mathrm{j}\omega) = 0$ and $Z^{-1}(\mathrm{j}\omega) + Z^{-*}(\mathrm{j}\omega) > 0$, it follows from Lemma 2.1 in reference [33] that $L(\mathrm{j}\omega) + Z(\mathrm{j}\omega)$ is invertible for all $\omega \in \mathscr{R}$.

Since $v(t)$, $y(t)$ and $w(t)$ are stationary random processes, it follows from equations (7) and (8) that the cross-spectral density matrix $S_{vy}(\omega)$ is given by

$$S_{vy}(\omega) = L(\mathrm{j}\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}S_{ww}(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}. \tag{9}$$

Thus, the cross-spectral density $S_{v_i y_i}$ of $v_i(t)$ and $y_i(t)$ is given by $S_{v_i y_i}(\omega) = S_{vy}(\omega)_{(i,i)}$.

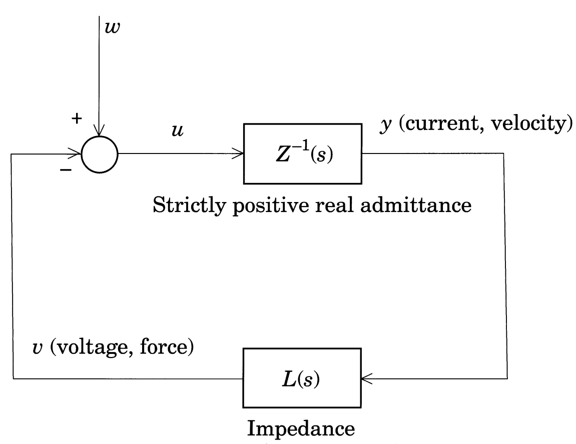

Figure 2. A feedback representation of coupled electrical or mechanical subsystems.

Now we let P^c denote the steady state average coupling energy flow matrix from v through the coupling $L(s)$ to y,

$$P^c \triangleq -\mathscr{E}[v(t)y^{\mathrm{T}}(t)], \tag{10}$$

where the minus sign denotes the fact that P^c is the energy flow exiting from $L(s)$. Thus the steady state average energy flow P_i^c exiting from the ith port of $L(s)$ is given by

$$P_i^c \triangleq P_{(i,i)}^c = -\mathscr{E}[v_i(t)y_i(t)]. \tag{11}$$

Furthermore, standard identities yield [10]

$$\begin{aligned} P^c = -R_{vy}(0) &= \frac{-1}{2\pi}\int_{-\infty}^{\infty} S_{vy}(\omega)\,\mathrm{d}\omega \\ &= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega t}\mathscr{E}[v(\tau)y^{\mathrm{T}}(\tau+t)]\,\mathrm{d}t\,\mathrm{d}\omega \\ &= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\cos\omega t - \mathrm{j}\sin\omega t)\mathscr{E}[v(\tau)y^{\mathrm{T}}(\tau+t)]\,\mathrm{d}t\,\mathrm{d}\omega \\ &= \frac{-1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Re}\,[S_{vy}(\omega)]\,\mathrm{d}\omega, \end{aligned} \tag{12}$$

where R_{vy} is the cross-correlation matrix of $v(t)$ and $y(t)$.

Next we define the average coupling energy flow matrix per unit bandwidth by

$$E^c(\omega) \triangleq \frac{-1}{2\pi}\,\mathrm{Re}\,[S_{vy}(\omega)], \tag{13}$$

so that the average energy flow per unit bandwidth exiting from the ith port of $L(s)$ is given by $E_i^c(\omega) = E^c(\omega)_{(i,i)}$. Thus

$$P^c = \int_{-\infty}^{\infty} E^c(\omega)\,\mathrm{d}\omega, \qquad P_i^c = \int_{-\infty}^{\infty} E_i^c(\omega)\,\mathrm{d}\omega. \tag{14}$$

The following result, which appears as equation (3.1) in reference [30], can be interpreted as the statement of conservation of energy at the lossless coupling $L(s)$.

Lemma 3.1. The system shown in Figure 2 satisfies

$$\sum_{i=1}^{r} E_i^c(\omega) = 0, \qquad \omega \in \mathscr{R}. \tag{15}$$

Proof. It follows from equations (9) and (13) that

$$\begin{aligned} \sum_{i=1}^{r} E_i^c(\omega) &= \frac{-1}{2\pi}\,\mathrm{tr}\,[\mathrm{Re}\,[S_{vy}(\omega)]] = \frac{-1}{4\pi}\,\mathrm{tr}\,[S_{vy}(\omega) + S_{vy}^*(\omega)] \\ &= \frac{-1}{4\pi}\,\mathrm{tr}\,[L(\mathrm{j}\omega)(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-1}S_{ww}(\omega)(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-*} \\ &\quad + (L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-1}S_{ww}(\omega)(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega)^{-*}L(\mathrm{j}\omega)^*] \\ &= \frac{-1}{4\pi}\,\mathrm{tr}\,[(L(\mathrm{j}\omega)+L(\mathrm{j}\omega)^*)(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-1}S_{ww}(\omega)(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-*}] \\ &= 0. \end{aligned}$$

□

Next we consider energy flow through the subsystems $z_i(s)$. From Figure 2 it follows that

$$u = Z(s)y = Z(s)(L(s) + Z(s))^{-1}w. \tag{16}$$

By using equations (7) and (16), the cross-spectral density matrix $S_{uy}(\omega)$ can be obtained as

$$S_{uy}(\omega) = Z(\mathrm{j}\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}S_{ww}(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}, \tag{17}$$

and the cross-spectral density $S_{u_iy_i}(\omega)$ of $u_i(t)$ and $y_i(t)$ is given by $S_{u_iy_i}(\omega) = S_{uy}(\omega)_{(i,i)}$. By denoting P^d as the steady state average energy dissipation rate matrix from u through the subsystems $z_i(s)$ to y and applying standard identities, we obtain

$$P^d \triangleq -\mathscr{E}[u(t)y^{\mathrm{T}}(t)] = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Re}\,[S_{uy}(\omega)]\,\mathrm{d}\omega, \tag{18}$$

where the minus sign denotes the energy flow exiting from $Z^{-1}(s)$; that is, dissipated by the subsystems. Thus the steady state average energy flow through the ith subsystem P_i^d is given by

$$P_i^d \triangleq P_{(i,i)}^d = -\mathscr{E}[u_i(t)y_i(t)]. \tag{19}$$

Then, in a manner similar to $E^c(\omega)$, we define the average energy dissipation rate matrix per unit bandwidth by

$$E^d(\omega) \triangleq \frac{-1}{2\pi}\,\mathrm{Re}\,[S_{uy}(\omega)], \tag{20}$$

so that the average rate of energy flow through the ith subsystem per unit bandwidth is given by $E_i^d(\omega) = E^d(\omega)_{(i,i)}$. Thus

$$P^d = \int_{-\infty}^{\infty} E^d(\omega)\,\mathrm{d}\omega, \qquad P_i^d = \int_{-\infty}^{\infty} E_i^d(\omega)\,\mathrm{d}\omega. \tag{21}$$

The energy flow through each subsystem is the sum of the rate of energy storage and the rate of energy dissipation. In steady state, since the rate of energy storage is zero, it follows that the energy flow through each subsystem is equal to the rate of energy dissipation.

Finally, we consider the external power generated by the disturbances. From Figure 2, the cross-spectral density matrix $S_{wy}(\omega)$ can be obtained as

$$S_{wy}(\omega) = S_{ww}(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}, \tag{22}$$

so that the cross-spectral density $S_{w_iy_i}(\omega)$ of $w_i(t)$ and $y_i(t)$ is given by $S_{w_iy_i}(\omega) = S_{wy}(\omega)_{(i,i)}$. Letting P^e denote the steady state average external power matrix from external disturbances w to y and applying standard identities, we obtain

$$P^e \triangleq \mathscr{E}[w(t)y^{\mathrm{T}}(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Re}\,[S_{wy}(\omega)]\,\mathrm{d}\omega, \tag{23}$$

and the steady state average external power generated at the ith subsystem P_i^e is given by

$$P_i^e \triangleq P_{(i,i)}^e = \mathscr{E}[w_i(t)y_i(t)]. \tag{24}$$

Next we define the average external power matrix per unit bandwidth $E^e(\omega)$ as

$$E^e(\omega) \triangleq \frac{1}{2\pi}\,\mathrm{Re}\,[S_{wy}(\omega)], \tag{25}$$

so that the average power generated at the ith subsystem per unit bandwidth is given by $E_i^e(\omega) = E^e(\omega)_{(i,i)}$. Thus,

$$P^e = \int_{-\infty}^{\infty} E^e(\omega)\,\mathrm{d}\omega, \qquad P_i^e = \int_{-\infty}^{\infty} E_i^e(\omega)\,\mathrm{d}\omega. \tag{26}$$

The energy flow per unit bandwidth quantities $E_i^c(\omega)$, $E_i^d(\omega)$ and $E_i^e(\omega)$ satisfy the following relations.

Lemma 3.2. The system shown in Figure 2 satisfies

$$E^c(\omega) + E^d(\omega) + E^e(\omega) = 0, \qquad \omega \in \mathscr{R}. \tag{27}$$

Proof. From equations (13), (20) and (25), it follows that

$$\begin{aligned} E^c(\omega)+E^d(\omega) + E^e(\omega) &= S_{vy}(\omega) + S_{uy}(\omega) + S_{wy}(\omega) \\ &= \frac{-1}{2\pi}\,\mathrm{Re}\,[(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}S_{ww}(\omega)(L(\mathrm{j}\omega) \\ &\quad + Z(\mathrm{j}\omega))^{-*} - S_{ww}(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}] \\ &= \frac{-1}{2\pi}\,\mathrm{Re}\,[S_{ww}(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*} - S_{ww}(\omega)(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-*}] \\ &= 0. \end{aligned}$$

□

Corollary 3.1. The system shown in Figure 2 satisfies

$$E_i^c(\omega) + E_i^d(\omega) + E_i^e(\omega) = 0, \qquad i = 1, \ldots, r, \quad \omega \in \mathscr{R}, \tag{28}$$

and

$$\sum_{i=1}^{r} [E_i^d(\omega) + E_i^e(\omega)] = 0, \qquad \omega \in \mathscr{R}. \tag{29}$$

Proof. Equation (28) corresponds to the (i, i) element of equation (27), while equations (15) and (28) yield equation (29). □

Lemma 3.1 and Corollary 3.1 describe the properties of the average energy flows per unit bandwidth among the coupled subsystems. That is, equation (28) shows that at each subsystem the effect of external power generated by the disturbances is to change the rate of energy dissipation and energy flow through the coupling, but, from equation (15), the total energy flow through all of the ports is zero. Thus, as shown by equation (29), the total external power is used to change the total rate of subsystem energy dissipation in the steady state. Furthermore, from the following results we can make the same arguments for the energy flows.

Corollary 3.2. The system in Figure 2 satisfies

$$\sum_{i=1}^{r} P_i^c = 0, \qquad P^c + P^d + P^e = 0, \qquad P_i^c + P_i^d + P_i^e = 0, \qquad i = 1, \ldots, r, \tag{30–32}$$

and

$$\sum_{i=1}^{r} (P_i^d + P_i^e) = 0. \tag{33}$$

Proof. These results are obtained by integrating equations (15), (27), (28) and (29) over $(-\infty, \infty)$. □

Now we decompose $S_{ww}(\omega)$ into its incoherent (diagonal) portion $\mathrm{Inc}\,[S_{ww}(\omega)]$ and its coherent (off-diagonal) portion $\mathrm{Coh}\,[S_{ww}(\omega)]$, so that $S_{ww}(\omega) = \mathrm{Inc}\,[S_{ww}(\omega)] + \mathrm{Coh}\,[S_{ww}(\omega)]$. As will be seen, this decomposition allows us to show how energy flow is affected by disturbance correlation. Hence $E^c(\omega)$, $E^d(\omega)$ and $E^e(\omega)$ can be correspondingly decomposed as

$$E^c(\omega) = E^c_{Inc}(\omega) + E^c_{Coh}(\omega), \qquad E^d(\omega) = E^d_{Inc}(\omega) + E^d_{Coh}(\omega),$$
$$E^e(\omega) = E^e_{Inc}(\omega) + E^e_{Coh}(\omega), \tag{34–36}$$

where

$$E^c_{Inc}(\omega) \triangleq \frac{-1}{2\pi}\,\mathrm{Re}\,[L(\mathrm{j}\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}\,\mathrm{Inc}\,[S_{ww}(\omega)](L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}], \tag{37}$$

$$E^c_{Coh}(\omega) \triangleq \frac{-1}{2\pi}\,\mathrm{Re}\,[L(\mathrm{j}\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}\,\mathrm{Coh}\,[S_{ww}(\omega)](L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}], \tag{38}$$

and $E^d_{Inc}(\omega)$, $E^d_{Coh}(\omega)$, $E^e_{Inc}(\omega)$ and $E^e_{Coh}(\omega)$ are similarly defined from equations (17), (20), (22) and (25). Note that $\mathrm{Inc}\,[S_{ww}(\omega)]$ and $\mathrm{Coh}\,[S_{ww}(\omega)]$ are diagonal and off-diagonal matrices, respectively, although the matrix $E^c_{Inc}(\omega)$ is not necessarily diagonal, while $E^c_{Coh}(\omega)$ may have non-zero diagonal elements.

Next we define the *thermodynamic energy matrix*, $E^{th}(\omega)$, as

$$E^{th}(\omega) \triangleq \tfrac{1}{2}\, C_d^{-1/2}(\omega) S_{ww}(\omega) C_d^{-1/2}(\omega). \tag{39}$$

Hence the *steady state thermodynamic cross energy* between the ith subsystem and the jth subsystem, $E^{th}_{ij}(\omega)$, and the *steady state thermodynamic energy* of the ith subsystem, $E^{th}_i(\omega)$, are given by

$$E^{th}_{ij}(\omega) \triangleq E^{th}_{(i,j)}(\omega) = \frac{S_{w_i w_j}(\omega)}{2\sqrt{c_i(\omega)c_j(\omega)}}, \qquad E^{th}_i(\omega) \triangleq E^{th}_{ii}(\omega) = \frac{S_{w_i w_i}(\omega)}{2c_i(\omega)}. \tag{40, 41}$$

Furthermore, since the diagonal portion $\{E^{th}(\omega)\}$ of $E^{th}(\omega)$ is given by

$$\{E^{th}(\omega)\} \triangleq \tfrac{1}{2}\, C_d^{-1/2}(\omega)\,\mathrm{Inc}\,[S_{ww}(\omega)] C_d^{-1/2}(\omega), \tag{42}$$

it follows that

$$\mathrm{Inc}\,[S_{ww}(\omega)] = 2C_d(\omega)\{E^{th}(\omega)\}. \tag{43}$$

Substituting equation (43) into equation (37) yields

$$E^c_{Inc}(\omega) = \frac{-1}{\pi}\,\mathrm{Re}\,[L(\mathrm{j}\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1} C_d(\omega)\{E^{th}(\omega)\}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}]. \tag{44}$$

Equation (44) shows that the thermodynamic energies act as driving forces for the average energy flows at each frequency. In a similar manner, we obtain

$$E^d_{Inc}(\omega) = \frac{-1}{\pi}\,\mathrm{Re}\,[Z(\mathrm{j}\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1} C_d(\omega)\{E^{th}(\omega)\}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}]. \tag{45}$$

If the disturbances $w_i(t)$ are mutually uncorrelated, that is, $\mathrm{Coh}\,[S_{ww}(\omega)] = 0$, then $E^{th}_i(\omega)$ plays the role of temperature in determining energy flow among subsystems, as shown by the following result.

Theorem 3.1. The system shown in Figure 2 satisfies

$$\sum_{i=1}^{r} [E^c_{Inc,i}(\omega)]/[E^{th}_i(\omega)] \geqslant 0, \qquad \omega \in \mathscr{R}, \tag{46}$$

where $E^c_{Inc,i}(\omega) \triangleq E^c_{Inc}(\omega)_{(i,i)}$.

Proof. See Appendix A.

The result (46) for the case $r = 2$ and uncorrelated disturbances was obtained in reference [30] and used to show that energy flows from the higher energy subsystem to the lower energy subsystem in analogy with the Second Law of Thermodynamics. We now derive a result that demonstrates this phenomenon for an arbitrary number of subsystems excited by possibly correlated disturbances. To do this, decompose $E^c_i(\omega)$ and $E^d_i(\omega)$ as

$$E^c_i(\omega) = E^c_{Inc,i}(\omega) + E^c_{Coh,i}(\omega), \qquad \omega \in \mathscr{R}, \tag{47}$$

and

$$E^d_i(\omega) = E^d_{Inc,i}(\omega) + E^d_{Coh,i}(\omega), \qquad \omega \in \mathscr{R}, \tag{48}$$

where $E^c_{Coh,i}(\omega)$, $E^d_{Inc,i}(\omega)$ and $E^d_{Coh,i}(\omega)$ are defined in a similar manner.

Theorem 3.2. Consider the system shown in Figure 2. For $i = 1, \ldots, r$, $E^c_{Inc,i}(\omega)$, $E^c_{Coh,i}(\omega)$ and $E^d_{Coh,i}(\omega)$ are given by

$$E^c_{Inc,i}(\omega) = \sum_{\substack{j=1 \\ j \neq i}}^{r} [\delta_{ij}(\omega) E^{th}_j(\omega) - \delta_{ji}(\omega) E^{th}_i(\omega)], \qquad \omega \in \mathscr{R}, \tag{49}$$

$$E^c_{Coh,i}(\omega) = \sum_{p=1}^{r} \left[\sum_{\substack{q=1 \\ q \neq p}}^{r} \delta_{ipq}(\omega) E^{th}_{pq}(\omega) - \sum_{\substack{q=1 \\ q \neq i}}^{r} \delta_{piq}(\omega) E^{th}_{iq}(\omega) \right], \qquad \omega \in \mathscr{R}, \tag{50}$$

$$E^d_{Inc,i}(\omega) = -\delta_{ii}(\omega) E^{th}_i(\omega) - \sum_{\substack{j=1 \\ j \neq i}}^{r} \delta_{ij}(\omega) E^{th}_j(\omega), \qquad \omega \in \mathscr{R}, \tag{51}$$

and

$$E^d_{Coh,i}(\omega) = -\sum_{p=1}^{r} \sum_{\substack{q=1 \\ q \neq p}}^{r} \delta_{ipq}(\omega) E^{th}_{pq}(\omega), \qquad \omega \in \mathscr{R}, \tag{52}$$

where, for $\omega \in \mathscr{R}$,

$$\delta_{ipq}(\omega) \triangleq \frac{1}{\pi} c_i(\omega) \sqrt{c_p(\omega) c_q(\omega)}\, \mathrm{Re}\, [(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}_{(i,p)} (L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}_{(q,i)}],$$
$$i, p, q = 1, \ldots, r, \tag{53}$$

and

$$\delta_{ij}(\omega) \triangleq \delta_{ijj}(\omega) = \frac{1}{\pi} c_i(\omega) c_j(\omega) |(Z(\mathrm{j}\omega) + L(\mathrm{j}\omega))^{-1}_{(i,j)}|^2. \tag{54}$$

Proof. See Appendix B.

From equation (54), it can be seen that $\delta_{ij}(\omega) \geqslant 0$, $i, j = 1, \ldots, r$, $\omega \in \mathscr{R}$. Thus it can be seen from equation (49) that incoherent energy flow among the subsystems is governed by the thermodynamic energy of each subsystem.

By introducing additional assumptions as follows below, we can guarantee that energy flows from higher energy subsystems to lower energy subsystems for couplings of arbitrary strength.

Corollary 3.3. Consider the coupled system in Figure 2. If either $r = 2$ or $\mathrm{Re}\,[L(\mathrm{j}\omega)] = 0$, $\omega \in \mathscr{R}$, then

$$\delta_{ij}(\omega) = \delta_{ji}(\omega), \qquad i, j = 1, \ldots, r, \quad \omega \in \mathscr{R}, \tag{55}$$

and thus

$$E^{c}_{Inc,i}(\omega) = \sum_{\substack{j=1 \\ j \neq i}}^{r} \delta_{ij}(\omega)[E^{th}_{j}(\omega) - E^{th}_{i}(\omega)], \qquad i = 1, \ldots, r, \quad \omega \in \mathscr{R}. \tag{56}$$

Proof. Since $L(\mathrm{j}\omega)$ is skew-Hermitian, it follows that if $r = 2$ then

$$|[(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}]_{(1,2)}|^2 = |[(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}]_{(2,1)}|^2.$$

Furthermore, if $\mathrm{Re}\,[L(\mathrm{j}\omega)] = 0$, $\omega \in \mathscr{R}$, then $L(\mathrm{j}\omega) = \mathrm{Im}\,[L(\mathrm{j}\omega)]$ is symmetric. Thus, in both cases, $(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}$ is symmetric, so that equation (55) is an immediate consequence of equation (54). Equation (56) follows directly from equation (49). □

If $\mathrm{Coh}\,[S_{ww}(\omega)] = 0$, then equation (56) can be interpreted thermodynamically as saying that energy flow $E^{c}_{i}(\omega)$ is proportional to energy differences and flows from the higher energy subsystems to the lower energy subsystems. Note that if $\mathrm{Re}\,[L(\mathrm{j}\omega)] = 0$, then equations (55) and (56) hold independently of the number of subsystems and the strength of the coupling. The condition $\mathrm{Re}\,[L(\mathrm{j}\omega)] = 0$ holds for stiffness couplings.

However, if $\delta_{ij}(\omega) \neq \delta_{ji}(\omega)$, then equation (49) allows reverse flow; that is, energy flow from a lower energy subsystem to a higher energy subsystem. Conditions for such a reverse flow are stated in the following result.

Proposition 3.1. Consider the coupled system shown in Figure 2 and assume that $\mathrm{Coh}\,[S_{ww}(\omega)] = 0$. Then,

$$\delta_{ij}(\omega)E^{th}_{j}(\omega) - \delta_{ji}(\omega)E^{th}_{i}(\omega) > 0, \tag{57}$$

if and only if

$$\frac{|(Z(\mathrm{j}\omega) + L(\mathrm{j}\omega)^{-1}_{(i,j)}|^2}{|(Z(\mathrm{j}\omega) + L(\mathrm{j}\omega))^{-1}_{(j,i)}|^2} > \frac{E^{th}_{i}(\omega)}{E^{th}_{j}(\omega)}. \tag{58}$$

Proof. This result follows immediately from equation (54). □

Proposition 3.1 says that even though $E^{th}_{i}(\omega) > E^{th}_{j}(\omega)$, energy flows from the jth (lower energy) subsystem to the ith (higher energy) subsystem if the inequality (58) does hold.

The following useful result concerns the energy flow coefficients $\delta_{ij}(\omega)$.

Corollary 3.4. The coupled system in Figure 2 satisfies

$$\sum_{\substack{j=1 \\ j \neq 1}}^{r} [\delta_{ij}(\omega) - \delta_{ji}(\omega)] = 0, \qquad i = 1, \ldots, r. \tag{59}$$

Proof. See Appendix C.

We now consider the relationship between the thermodynamic energy $E^{th}_{i}(\omega)$, the thermodynamic cross-energy $E^{th}_{ij}(\omega)$ and the root mean square (r.m.s.) velocity per unit bandwidth of the ith subsystem $y_i(\omega)$, $i = 1, \ldots, r$, defined by

$$y_i(\omega) \triangleq \sqrt{(1/2\pi)S_{y_i y_i}(\omega)}. \tag{60}$$

By using equation (7), $y_i(\omega)$ can be written as

$$y_i(\omega) = [(1/2\pi)[(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}S_{ww}(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}]_{(i,i)}]^{1/2}, \tag{61}$$

and the following result can be obtained.

Proposition 3.2. The r.m.s. velocity, $y_i(\omega)$ is given by

$$y_i(\omega) = \frac{1}{\sqrt{c_i(\omega)}}\left[\sum_{j=1}^{r} \delta_{ij}(\omega)E_j^{th}(\omega) + \sum_{p=1}^{r}\sum_{\substack{q=1\\ q\neq p}}^{r} \delta_{ipq}(\omega)E_{pq}^{th}(\omega)\right]^{1/2}, \qquad i = 1, \ldots, r, \quad (62)$$

where $\delta_{ipq}(\omega)$ and $\delta_{ij}(\omega)$ are defined by equations (53) and (54), respectively.

Proof. This result can be obtained in the same manner as equations (51) and (52). □

Note that the first term and second term in equation (62) result from Inc $[S_{ww}(\omega)]$ and Coh $[S_{ww}(\omega)]$, respectively. Since $E_i^d(\omega) = E_{Inc,i}^d(\omega) + E_{Coh,i}^d(\omega)$, by comparing equations (51) and (52) with equation (62), it follows that the mean square velocity per unit bandwidth of the ith subsystem $y_i^2(\omega)$ satisfies

$$c_i(\omega)y_i^2(\omega) = -E_i^d(\omega).$$

Finally, we consider the rate of energy dissipation $E_{Inc,i}^d(\omega)$ in equation (51), which shows that $E_{Inc,i}^d(\omega)$ depends on $E_j^{th}(\omega)$ for $j = 1, \ldots, r$. By rewriting equation (51), $E_{Inc,i}^d(\omega)$ can be expressed as a function of $E_i^{th}(\omega)$ only; that is,

$$E_{Inc,i}^d(\omega) = -\delta_i(\omega)E_i^{th}(\omega), \qquad i = 1, \ldots, r, \quad \omega \in \mathscr{R}, \quad (63)$$

where

$$\delta_i(\omega) \triangleq \sum_{j=1}^{r} \delta_{ij}(\omega)\frac{E_j^{th}(\omega)}{E_i^{th}(\omega)}. \quad (64)$$

Note that $\delta_i(\omega) \geqslant 0$, $i = 1, \ldots, r$, $\omega \in \mathscr{R}$.

Theorem 3.2 is illustrated in Figure 3 for the case $r = 3$ and Coh $[S_{ww}(\omega)] = 0$. In Figure 3 it is shown that there exist energy flows $\delta_{ji}(\omega)E_i^{th}(\omega)$ and $\delta_{ij}(\omega)E_j^{th}(\omega)$ between the ith

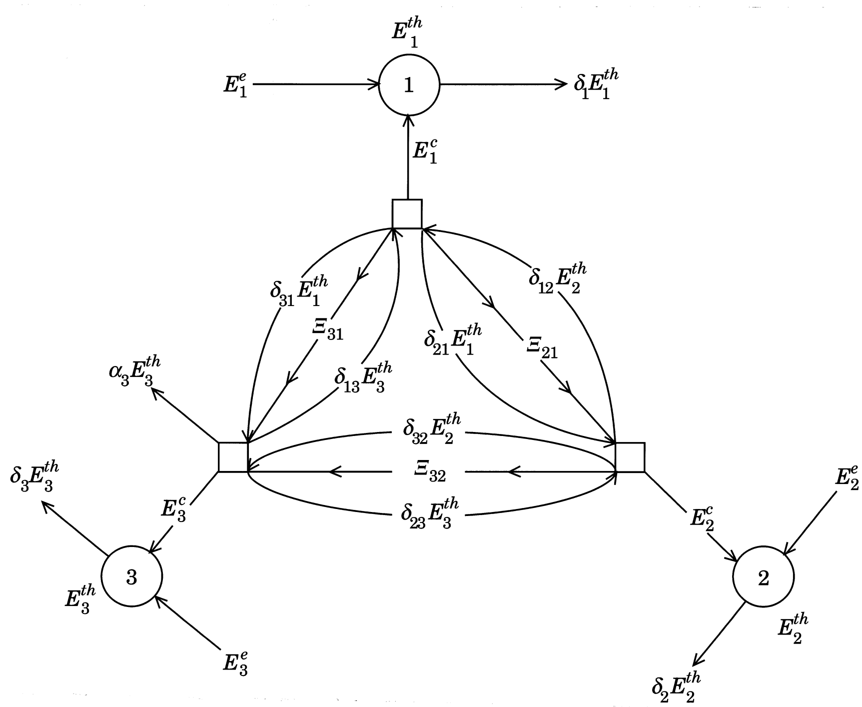

Figure 3. The subsystem energy flow diagram (frequency domain).

and jth subsystems, and that the difference between these two energy flows is the net energy flow $\Xi_{ij}(\omega) \triangleq \delta_{ij}(\omega)E_j^{th}(\omega) - \delta_{ji}(\omega)E_i^{th}(\omega)$. By equation (49), the average energy flow through the ith port per unit bandwidth $E_i^c(\omega)$ can thus be expressed as

$$\sum_{\substack{j=1 \\ j \neq i}}^{r} \Xi_{ij}(\omega).$$

As Corollary 3.3 shows, if $\delta_{ij}(\omega) = \delta_{ji}(\omega)$, $i, j = 1, \ldots, r$, then there is a net energy flow from higher energy subsystems to lower energy subsystems.

4. COMPARTMENTAL MODELLING AND TIME DOMAIN ANALYSIS

In this section we consider an alternative point of view involving compartmental modelling and time domain analysis. To do this we invoke two main assumptions. First, $w(t)$ is assumed to be a white noise vector, given by

$$w(t) = D\tilde{w}(t), \tag{65}$$

where $\tilde{w}(t) \triangleq [\tilde{w}_1(t) \cdots \tilde{w}_n(t)]^{\mathrm{T}}$ is a normalized white noise vector the intensity matrix of which is the $n \times n$ identity matrix and $D \in \mathscr{R}^{r \times n}$ is a constant matrix. Then, the intensity matrix S_{ww} of $w(t) \triangleq [w_1(t) \cdots w_r(t)]^{\mathrm{T}}$ is given by $S_{ww} = DD^{\mathrm{T}}$. Note that

$$S_{ww(i,i)} = S_{w_i w_i}, \qquad S_{ww(i,j)} = S_{ww(j,i)} = S_{w_i w_j} = S_{w_j w_i}. \tag{66}$$

Our second assumption is that the real part of $c_i = c_i(\mathrm{j}\omega)$ in equation (2) is constant. As before, let $C_d = \mathrm{diag}\,(c_1, c_2, \ldots, c_r)$. Under these assumptions, the cross-thermodynamic energy, E_{ij}^{th}, in equation (40) and the thermodynamic energy, E_i^{th}, in equation (41) become

$$E_{ij}^{th} = S_{w_i w_j}/2\sqrt{c_i c_j}, \qquad E_i^{th} = E_{ii}^{th} = S_{w_i w_i}/2c_i, \tag{67}$$

respectively.

Next we consider a realization of the feedback system in Figure 2. Let $Z^{-1}(s)$ and $L(s)$ have the realizations

$$\dot{x}_Z(t) = A_Z x_Z(t) + B_Z u(t), \qquad y(t) = C_Z x_Z(t), \tag{68, 69}$$

$$\dot{x}_L(t) = A_L x_L(t) + B_L y(t), \qquad v(t) = C_L x_L(t) + D_L y(t), \tag{70, 71}$$

respectively. Since $u = w - v$, the augmented system (68)–(71) is given by

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}\tilde{w}(t), \qquad x(0) = x_0, \tag{72}$$

where

$$x(t) \triangleq \begin{bmatrix} x_Z(t) \\ x_L(t) \end{bmatrix}, \qquad \tilde{A} \triangleq \begin{bmatrix} A_Z - B_Z D_L C_Z & -B_Z C_L \\ B_L C_Z & A_L \end{bmatrix}, \qquad \tilde{B} \triangleq \begin{bmatrix} B_Z D \\ 0 \end{bmatrix}.$$

Define C_1 and C_2 by

$$C_1 \triangleq [C_Z \quad 0], \qquad C_2 \triangleq [D_L C_Z \quad C_L], \tag{73}$$

so that $y(t) = C_1 x(t)$ and $v(t) = C_2 x(t)$. Then, the realizations for equations (7) for (8) are given by

$$(L(s) + Z(s))^{-1} D \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline C_1 & 0 \end{array}\right], \qquad L(s)(L(s) + Z(s))^{-1} D \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline C_2 & 0 \end{array}\right].$$

The following results provide expressions for P^c, P^e and P^d.

Theorem 4.1. Consider the system given by equation (72). Then P^c, P^e and P^d are given by

$$P^c = -C_2\tilde{Q}C_1^{\mathrm{T}}, \qquad P^e = \tfrac{1}{2}D\tilde{B}^{\mathrm{T}}C_1^{\mathrm{T}}, \qquad P^d = (C_2\tilde{Q} - \tfrac{1}{2}D\tilde{B}^{\mathrm{T}})C_1^{\mathrm{T}}, \tag{74–76}$$

where the steady state covariance $\tilde{Q} \triangleq \mathscr{E}[x(t)x^{\mathrm{T}}(t)]$ satisfies the algebraic Lyapunov equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{B}\tilde{B}^{\mathrm{T}}. \tag{77}$$

Proof. See Appendix D.

The following result provides an alternative characterization of P_i^d.

Corollary 4.1. Consider the system given by equation (72). Then, for $i = 1, \ldots, r$, P_i^d is given by

$$P_i^d = -(C_d C_1 \tilde{Q} C_1^{\mathrm{T}})_{(i,i)}, \tag{78}$$

where the steady state covariance $\tilde{Q}$ satisfies the algebraic Lyapunov equation (77).

Proof. See Appendix E.

As in the previous section, by decomposing S_{ww} into the incoherent portion Inc $[S_{ww}]$ and the coherent portion Coh $[S_{ww}]$, P_i^c, P_i^d and P_i^d can be decomposed as follows.

Proposition 4.1. Consider the system given by equation (72). Then, for $i = 1, \ldots, r$, P_i^c, P_i^d, and P_i^e are given by

$$P_i^c = P_{Inc,i}^c + P_{Coh,i}^c, \qquad P_i^d = P_{Inc,i}^d + P_{Coh,i}^d, \qquad P_i^e = P_{Inc,i}^e + P_{Coh,i}^e, \tag{79–81}$$

where

$$P_{Inc,i}^c \triangleq -(C_2\tilde{Q}_{Inc}C_1^{\mathrm{T}})_{(i,i)}, \qquad P_{Coh,i}^c \triangleq -(C_2\tilde{Q}_{Coh}C_1^{\mathrm{T}})_{(i,i)},$$
$$P_{Inc,i}^d \triangleq -(C_dC_1\tilde{Q}_{Inc}C_1^{\mathrm{T}})_{(i,i)}, \qquad P_{Coh,i}^d \triangleq -(C_dC_1\tilde{Q}_{Coh}C_1^{\mathrm{T}})_{(i,i)},$$
$$P_{Inc,i}^e \triangleq \tfrac{1}{2}(\mathrm{Inc}\,[S_{ww}]B^{\mathrm{T}}C_1^{\mathrm{T}})_{(i,i)}, \qquad P_{Coh,i}^e \triangleq \tfrac{1}{2}(\mathrm{Coh}[S_{ww}]B^{\mathrm{T}}C_1^{\mathrm{T}})_{(i,i)},$$

and $\tilde{Q}_{Inc}$ and $\tilde{Q}_{Coh}$ satisfy

$$0 = \tilde{A}\tilde{Q}_{Inc} + \tilde{Q}_{Inc}\tilde{A}^{\mathrm{T}} + B\,\mathrm{Inc}\,[S_{ww}]B^{\mathrm{T}}, \quad 0 = \tilde{A}\tilde{Q}_{Coh} + \tilde{Q}_{\mathrm{Coh}}\tilde{A}^{\mathrm{T}} + B\,\mathrm{Coh}\,[S_{ww}]B^{\mathrm{T}}. \tag{82, 83}$$

Proof. The above results follow from equations (74) and (75) by using $DD^{\mathrm{T}} = S_{ww} = \mathrm{Inc}\,[S_{ww}] + \mathrm{Coh}\,[S_{ww}]$. □

Proposition 4.2. Consider the system given by equation (72). Then, for $i = 1, \ldots, r$,

$$P_{Inc,i}^c + P_{Inc,i}^d + P_{Inc,i}^e = 0, \qquad P_{Coh,i}^c + P_{Coh,i}^d + P_{Coh,i}^e = 0. \tag{84, 85}$$

Proof. This is obvious from Corollary 3.2. □

Now we obtain a compartmental model for the coupled system. Compartmental models involve non-negative state variables that exchange and dissipate energy in accordance with conservation laws [25, 26, 28]. Since energy flow in the coupled system satisfies a conservation law, a compartmental model can be derived.

Theorem 4.2. Define

$$\sigma_{ij} \triangleq \int_{-\infty}^{\infty} \delta_{ij}(\omega)\,\mathrm{d}\omega, \qquad i \neq j, \quad i, j = 1, \ldots, r, \tag{86}$$

$$\sigma_i \triangleq \int_{-\infty}^{\infty} \delta_i(\omega)\,\mathrm{d}\omega, \qquad i = 1, \ldots, r, \tag{87}$$

$$\Pi_{ij} \triangleq \sigma_{ij}E_j^{th} - \sigma_{ji}E_i^{th}, \qquad i, j = 1, \ldots, r, \tag{88}$$

where $\delta_{ij}(\omega)$ and $\delta_i(\omega)$ are defined in equations (54) and (64), respectively. Then, for each $i = 1, \ldots, r$,

$$P^d_{Inc,i} = -\sigma_i E^{th}_i, \qquad P^c_{Inc,i} = \sum_{\substack{j=1 \\ j\neq 1}}^{r} \Pi_{ij}. \tag{89, 90}$$

Consequently,

$$-\sigma_i E^{th}_i + \sum_{\substack{j=1 \\ j\neq i}}^{r} \Pi_{ij} + P^e_{Inc,i} = 0, \qquad i = 1, \ldots, r. \tag{91}$$

Proof. From equations (14), (49) and (86), equation (90) is obtained as

$$P^c_{Inc,i} = \int_{-\infty}^{\infty} E^c_{Inc,i}(\omega)\,\mathrm{d}\omega = \int_{-\infty}^{\infty} \sum_{\substack{j=1 \\ j\neq i}}^{r} (\delta_{ij}(\omega)\,\mathrm{d}\omega E^{th}_j - \delta_{ji}(\omega)\,\mathrm{d}\omega\, E^{th}_i)$$

$$= \sum_{\substack{j=1 \\ j\neq i}}^{r} \left[\int_{-\infty}^{\infty} \delta_{ij}(\omega)\,\mathrm{d}\omega\, E^{th}_j - \int_{-\infty}^{\infty} \delta_{ji}(\omega)\,\mathrm{d}\omega\, E^{th}_i\right] = \sum_{\substack{j=1 \\ j\neq i}}^{r} [\sigma_{ij}E^{th}_j - \sigma_{ji}E^{th}_i]$$

$$= \sum_{\substack{j=1 \\ j\neq i}}^{r} \Pi_{ij}.$$

In a similar manner, equation (89) can be derived directly from equation (63). Finally, by using Proposition 4.2, equation (91) can be obtained. □

Obviously, $\sigma_i \geqslant 0$, $\sigma_{ij} \geqslant 0$ and $P^e_{Inc,i} \geqslant 0$.

Equations (91) and (88) represent a steady state compartmental model [6, 25, 26, 28]. These two equations can be expressed by the matrix equation

$$AE = P_e, \tag{92}$$

where the non-negative vectors E and P_e are defined by

$$E \triangleq [E^{th}_1 \cdots E^{th}_r]^{\mathrm{T}}, \qquad P_e \triangleq [P^e_{Inc,1} \cdots P^e_{Inc,r}]^{\mathrm{T}},$$

and the matrix A is given by

$$A_{(i,i)} \triangleq \sigma_i + \sum_{\substack{j=1 \\ j\neq i}}^{r} \sigma_{ji}, \quad A_{(i,j)} \triangleq -\sigma_{ij}, \qquad i, j = 1, \ldots, r.$$

Note that the matrix A is an M-matrix [29].

The following time domain results are analogous to Corollary 3.3, Corollary 3.4, Proposition 3.1 and Proposition 3.2.

Corollary 4.2. Consider that coupled system in Fig. 2. If either $r = 2$ or $\mathrm{Re}\,[L(\mathrm{j}\omega)] = 0$, then

$$\sigma_{ij} = \sigma_{ji}, \qquad i, j = 1, \ldots, r, \tag{93}$$

and

$$P^c_{Inc,i} = \sum_{\substack{j=1 \\ j\neq i}}^{r} \sigma_{ij}(E^{th}_j - E^{th}_i), \qquad i = 1, \ldots, r. \tag{94}$$

Proof. The result follows directly from Corollary 3.3. □

Corollary 4.3. The coupled system in Figure 2 satisfies

$$\sum_{\substack{j=1 \\ j\neq i}}^{r} (\sigma_{ij} - \sigma_{ji}) = 0, \qquad i = 1, \ldots, r. \tag{95}$$

Proof. This result follows immediately from Corollary 3.4. □

Proposition 4.3. Consider the coupled system in Figure 2 and assume that Coh $[S_{ww}] = 0$. Then

$$\sigma_{ij} E_j^{th} - \sigma_{ji} E_i^{th} > 0, \tag{96}$$

if and only if

$$\frac{\int_{-\infty}^{\infty} |(Z(\mathrm{j}\omega) + L(\mathrm{j}\omega))^{-1}_{(i,j)}|^2}{\int_{-\infty}^{\infty} |(Z(\mathrm{j}\omega) + L(\mathrm{j}\omega))^{-1}_{(j,i)}|^2} > \frac{E_i^{th}}{E_j^{th}}. \tag{97}$$

Proof. The result follows directly from Proposition 3.1 □

The steady state mean square velocity of the ith subsystem

$$\mathscr{E}[y_i^2(t)] = \int_{-\infty}^{\infty} y_i^2(\omega)\, \mathrm{d}\omega,$$

where $y_i(\omega)$ is defined by equation (60), is given by the following result.

Proposition 4.4. The coupled system in Figure 2 satisfies

$$\mathscr{E}[y_i^2(t)] = \frac{1}{c_i}\left(\sum_{j=1}^{r} \sigma_{ij} E_j^{th} + \sum_{p=1}^{r} \sum_{\substack{q=1 \\ q\neq p}}^{r} \sigma_{ipq} E_{pq}^{th} \right), \qquad i = 1, \ldots, r, \tag{98}$$

where $\sigma_{ipq} \triangleq \int_{-\infty}^{\infty} \delta_{ipq}(\omega)\, \mathrm{d}\omega$.

Proof. The result follows directly from Proposition 3.2. □

Although Theorem 4.2 provides expressions for σ_i and σ_{ij}, the integration is difficult, especially for $r \geqslant 3$. Next we introduce an algebraic expression for these coefficients.

Proposition 4.5. Consider the coupled system in Figure 2, and let σ_{ij} and σ_i, $i, j = 1, \ldots, r$, be defined by equations (86) and (87). Then σ_i and σ_{ij} are given by

$$\sigma_i = \sum_{j=1}^{r} \sigma_{ij} \frac{E_j^{th}}{E_i^{th}}, \qquad i = 1, \ldots, r, \tag{99}$$

and

$$\sigma_{ij} = c_i c_j \frac{1}{\pi} \int_{-\infty}^{\infty} |H_{ij}(\mathrm{j}\omega)|^2\, \mathrm{d}\omega = 2 c_i c_j (C_1 \tilde{Q}_j C_1^{\mathrm{T}})_{(i,i)}, \qquad i \neq j, \quad i, j = 1, \ldots, r, \tag{100}$$

where $H_{ij}(\mathrm{j}\omega) \triangleq e_i^{\mathrm{T}} C_1 (\mathrm{j}\omega I - \tilde{A})^{-1} B e_j$ and, for $j = 1, \ldots, r$, the $r \times r$ matrix $\tilde{Q}_j$ satisfies the Lyapunov equation

$$0 = \tilde{A}\tilde{Q}_j + \tilde{Q}_j \tilde{A}^{\mathrm{T}} + B e_j e_j^{\mathrm{T}} B^{\mathrm{T}}. \tag{101}$$

Proof. By applying Parseval's Theorem to the definition of $\delta_{ij}(\omega)$ in equation (54), we obtain

$$\begin{aligned}\sigma_{ij} &= \int_{-\infty}^{\infty} \delta_{ij}(\omega)\,\mathrm{d}\omega \\ &= \frac{1}{\pi} c_i c_j \int_{-\infty}^{\infty} [(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}]_{(i,j)}[(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}]_{(j,i)}\,\mathrm{d}\omega \\ &= \frac{1}{\pi} c_i c_j \int_{-\infty}^{\infty} [e_i^{\mathrm{T}}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}e_j][e_j^{\mathrm{T}}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}e_i]\,\mathrm{d}\omega \\ &= 2c_i c_j \frac{1}{2\pi} \int_{-\infty}^{\infty} [e_i^{\mathrm{T}} C_1(\mathrm{j}\omega I - \tilde{A})^{-1}Be_j][e_i^{\mathrm{T}} C_1(\mathrm{j}\omega I - \tilde{A})^{-1}Be_j]^*\,\mathrm{d}\omega \\ &= 2c_i c_j e_i^{\mathrm{T}} C_1 \tilde{Q}_j C_1^{\mathrm{T}} e_i = 2c_i c_j (C_1 \tilde{Q}_j C_1^{\mathrm{T}})_{(i,i)},\end{aligned}$$

where $\tilde{Q}_j$ satisfies equation (101). □

Equation (100) shows that σ_{ij} is given by H_2 norm of the transfer function $H_{ij}(s) = e_i^{\mathrm{T}} C_1(sI - \tilde{A})Be_j$. In section 7, we shall use closed form expressions for this integral, which are given in references [35, 36]. These expressions are based upon an explicit solution of the matrix $\tilde{Q}_j$ given by equation (101).

Finally, Theorem 4.2 is illustrated in Figure 4 for the case $r = 3$ and Coh $[S_{ww}] = 0$. This time domain representation of energy flow has the same interpretation as Figure 3.

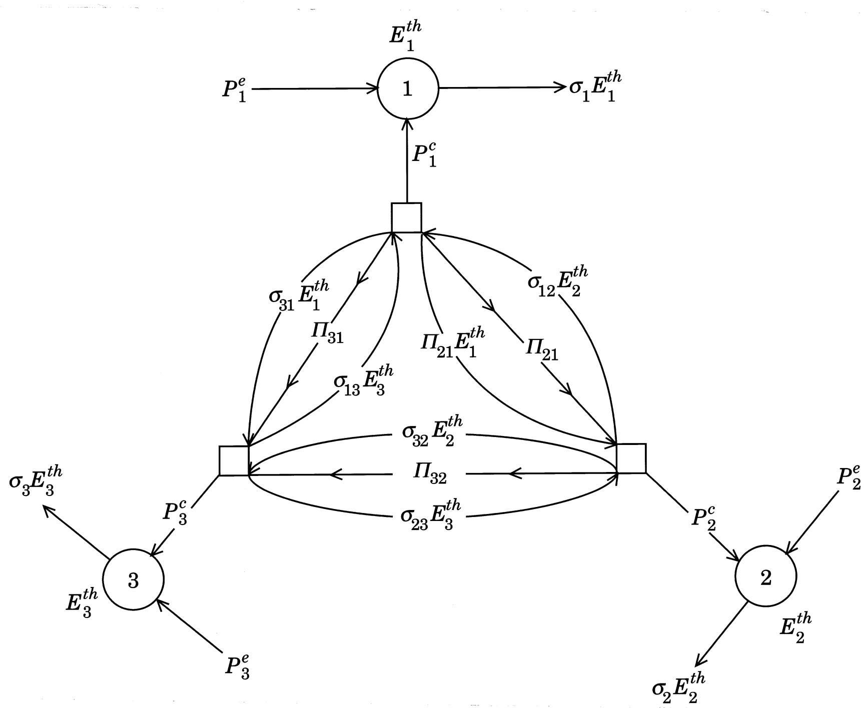

Figure 4. The subsystem energy flow diagram (time domain).

5. EQUIPARTITION OF ENERGY

In this section, we consider a condition relating to equipartition of energy concerning $E^c_{Inc,i}(\omega)$ and $P^c_{Inc,i}$. First we consider the frequency domain and use equation (59) to obtain the following result.

Theorem 5.1. Consider the coupled system in Figure 2 and assume that

$$E^{th}_i(\omega) = E^{th}_j(\omega), \qquad i, j = 1, \ldots, r, \quad \omega \in \mathcal{R}. \tag{102}$$

Then

$$E^c_{Inc,i}(\omega) = 0, \qquad i = 1, \ldots, r, \quad \omega \in \mathcal{R}. \tag{103}$$

Proof. From Corollary 3.4 and equation (102) it follows that

$$E^c_{Inc,i}(\omega) = \sum_{\substack{j=1 \\ j \neq i}}^{r} [\delta_{ij}(\omega) E^{th}_j(\omega) - \delta_{ji}(\omega) E^{th}_i(\omega)]$$

$$= \sum_{\substack{j=1 \\ j \neq i}}^{r} [\delta_{ij}(\omega) - \delta_{ji}(\omega)] E^{th}_i(\omega) = 0. \qquad \square$$

Theorem 5.1 says that there is no net energy flow among subsystems when every subsystem has the same thermodynamic energy; that is, when equipartition of energy holds.

In the time domain we obtain a similar result. By assuming white noise, the following result can be obtained.

Corollary 5.1. Consider the coupled system in Figure 2 and assume that $S_{w_i w_i}$ and c_i are constant for $i = 1, \ldots, r$. If

$$E^{th}_i = E^{th}_j, \qquad i, j = 1, \ldots, r, \tag{104}$$

then

$$P^c_{Inc,i} = 0, \qquad i = 1, \ldots, r. \tag{105}$$

Proof. From equation (14) and Theorem 5.1 it follows that

$$P^c_{Inc,i} = \int_{-\infty}^{\infty} E^c_{Inc,i}(\omega)\, \mathrm{d}\omega = 0. \qquad \square$$

Now we consider the converse of Theorem 5.1 and Corollary 5.1. For convenience, we define the $r \times r$ matrix $\mathscr{H}(\omega)$ by

$$\mathscr{H}(\omega)_{(i,j)} \triangleq \delta_{ij}(\omega), \qquad \mathscr{H}(\omega)_{(i,i)} \triangleq -\sum_{\substack{j=1 \\ j \neq i}}^{r} \delta_{ji}(\omega), \qquad i, j = 1, \ldots, r.$$

Note that since $e^{\mathrm{T}} \mathscr{H}(\omega) = 0$, $\omega \in \mathcal{R}$, where $e = [1 \cdots 1]^{\mathrm{T}}$, it follows that rank $\mathscr{H}(\omega) \leqslant r - 1$, $\omega \in \mathcal{R}$.

Theorem 5.2. Consider the coupled system in Figure 2 and assume that rank $\mathscr{H}(\omega) = r - 1$ for all $\omega \in \mathcal{R}$. If

$$E^c_{Inc,i}(\omega) = 0, \qquad i = 1, \ldots, r, \quad \omega \in \mathcal{R}, \tag{106}$$

then

$$E^{th}_i(\omega) = E^{th}_j(\omega), \qquad i, j = 1, \ldots, r, \quad \omega \in \mathcal{R}. \tag{107}$$

Proof. From equations (49) and (106), we obtain

$$\mathscr{H}(\omega) E(\omega) = 0, \tag{108}$$

where the vector $E(\omega)$ is defined by

$$E(\omega) \triangleq [E_1^{th}(\omega) \cdots E_r^{th}(\omega)]^{\mathrm{T}}.$$

From Corollary 3.4, $\mathscr{H}(\omega)e = 0$ and, by assumption, rank $\mathscr{H}(\omega) = r - 1$. Thus, it follows from equation (108) that $E(\omega) = \gamma(\omega)e$, where, for each $\omega \in \mathscr{R}$, $\gamma(\omega)$ is a non-zero real number. Thus $E_i^{th}(\omega) = \gamma(\omega)$, $i = 1, \ldots, r$, which proves equation (107). □

In the same manner as Thoerem 5.2, by defining

$$\mathscr{S}_{(i,j)} \triangleq \sigma_{ij}, \quad \mathscr{S}_{(i,i)} \triangleq -\sum_{\substack{j=1 \\ j \neq i}}^{r} \sigma_{ji}, \qquad i, j = 1, \ldots, r,$$

we obtain the following result.

Corollary 5.2. Consider the coupled system in Figure 2 and assume that rank $\mathscr{S} = r - 1$. If

$$P_{Inc,i}^{c} = 0, \qquad i = 1, \ldots, r, \tag{109}$$

then

$$E_i^{th} = E_j^{th}, \qquad i, j = 1, \ldots, r. \tag{110}$$

Proof. This result can be proved in the same manner as Theorem 5.2. □

Remark 5.1. Theorem 5.2 and Corollary 5.2 are closely related to Corollary 2.2 in reference [28].

Remark 5.2. Since $\mathscr{H}(\omega)e = e^{\mathrm{T}}\mathscr{H}(\omega) = 0$, rank $\mathscr{H}(\omega) = r - 1$ and $\mathscr{S}e = e^{\mathrm{T}}\mathscr{S} = 0$, rank $\mathscr{S} = r - 1$, it follows that $\mathscr{H}(\omega)$ and $\mathscr{S}$ are EP matrices (reference [34], p. 74).

6. RELATIONSHIP TO MECHANICAL ENERGY

In the previous sections, we considered energy flow from the point of view of thermodynamic energy. Now we introduce the mechanical energy generally used in the SEA approach and discuss the relationship between these two types of energy. Since uncorrelated white noise is considered in the SEA approach, we assume that the disturbance filter $D = I$, which implies that $S_{ww} = I$.

Consider the mass–damper–spring system in Figure 5. The thermodynamic energy E^{th} of this system can be obtained as

$$E^{th} = 1/2c. \tag{111}$$

On the other hand, mechanical energy is usually defined as the average stored energy at steady state. Then, the mechanical energy of this uncoupled system E^{u} can be obtained as

$$E^{u} \triangleq \tfrac{1}{2}\,(m\mathscr{E}[\dot{x}^2] + k\mathscr{E}[x^2]) = \tfrac{1}{2}\left(m\,\frac{1}{2cm} + k\,\frac{1}{2ck}\right) = \frac{1}{2c}\,.$$

Thus, we find

$$E^{u} = E^{th}; \tag{112}$$

that is, when white noise is assumed, the thermodynamic energy equals the mechanical energy of the uncoupled system.

Next, we consider the case in which the oscillator in Figure 5 is coupled with $r - 1$ similar oscillators by springs as shown in Figure 6 ($r = 2$) or Figure 7 ($r = 3$). In the SEA approach [3], the *blocked energy* of the ith oscillator is defined as (see p. 377 in reference [10])

$$E_i^{bl} \triangleq \tfrac{1}{2}\left(m_i\mathscr{E}[\dot{x}_i^2] + \left(k_i + \sum_{\substack{j=1 \\ j \neq i}}^{r} K_{ij}\right)\mathscr{E}[x_i^2]\right), \tag{113}$$

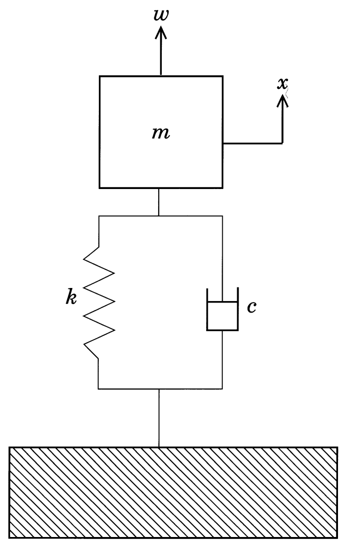

Figure 5. The single oscillator.

while the *coupled mechanical energy* of the ith oscillator E_i^{mec} is defined by

$$E_i^{mec} \triangleq \frac{1}{2}\left(m_i \mathscr{E}\left[\sum_{\substack{j=1\\ j\neq i}}^{r} (\dot{x}_i - \dot{x}_j)^2 \right] + \sum_{\substack{j=1\\ j\neq i}}^{r} K_{ij}\mathscr{E}[x_i - x_j]^2 \right).$$

Note that the blocked energy ignores the relative velocity and relative displacement which contribute to the coupled mechanical energy. Nevertheless, when the coupling stiffnesses K_{ij} are small, it follows that

$$E_i^{bl} \simeq E_i^u = E_i^{th} \triangleq 1/2c_i, \tag{114}$$

where E_i^{th} and E_i^u are the thermodynamic energy and the uncoupled mechanical energy of the ith subsystem, respectively. If the oscillators are coupled by springs only, it follows

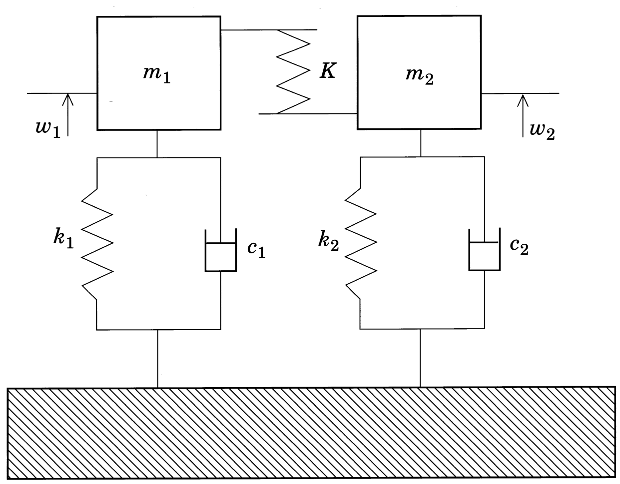

Figure 6. The two coupled oscillator system.

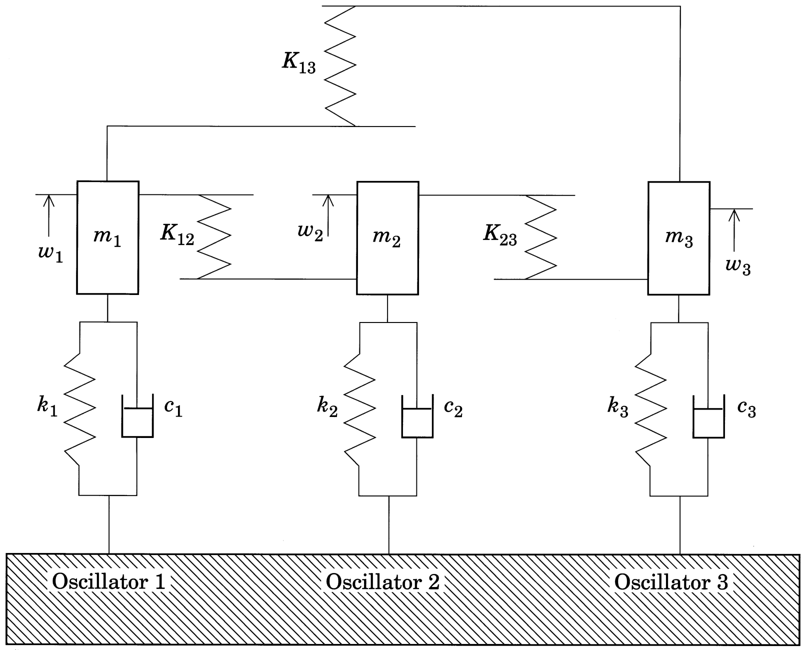

Figure 7. The three coupled oscillator system.

that $\mathrm{Re}\,[L(\mathrm{j}\omega)] = 0$. Thus Corollaries 3.3 and 4.2 guarantee that energy flows from higher energy subsystems to lower energy ones according to E_i^u. Note that this property does not depend on the number of subsystems or the strength of coupling.

On the other hand, if E_i^{bl} is considered, as in the usual SEA approach, energy would flow from higher energy subsystems to lower energy ones when the coupling is weak because E_i^{bl} is close to E_i^u. However, if the coupling becomes strong, such an energy flow can no longer be guaranteed. In fact, as shown in the next section, energy does not generally flow according to E_i^{bl}.

7. EXAMPLES

In this section we consider second order subsystems interconnected by either a stiffness coupling or a gyroscopic coupling. For convenience, we assume that $\mathrm{Coh}\,[S_{ww}(\omega)] = \mathrm{Coh}\,[S_{ww}] = 0$.

First we reconsider the state space model in equation (72). By assuming that each subsystem is a second order system, the state space vector x_Z in equation (68) is comprised of the position vector x_{pos} and the velocity vector x_{vel}. We now define an output matrix C_p so that $x_{pos} = C_p x_Z$ and assume that $L(s) = G_L + C_L/s$, where $G_L = -G_L^{\mathrm{T}}$ is the gyroscopic part of $L(s)$, and $C_L = C_L^{\mathrm{T}}$ is the stiffness part of $L(s)$. Then, from equations (8) and (69) it follows that

$$v = (G_L + C_L/s)y = G_L C_Z x_Z + C_L(C_Z x_Z/s) = (G_L C_Z + C_L C_p)x_Z. \tag{115}$$

Then by using $u = w - v$, it follows that

$$\begin{aligned} \dot{x}_Z &= A_Z x_Z + B_Z(w - G_L C_Z + C_L C_p)x_Z) \\ &= (A_Z - B_Z(G_L C_Z + C_L C_p))x_Z + B_Z w. \end{aligned} \tag{116}$$

Thus we can redefine x, $\tilde{A}$, $\tilde{B}$, C_1 and C_2 in equations (72) and (73) as

$$x \triangleq x_z, \qquad \tilde{A} \triangleq A_Z - B_Z(G_L C_Z + C_L C_p), \qquad \tilde{B} \triangleq B_Z D,$$
$$C_1 \triangleq C_Z, \qquad C_2 \triangleq G_L C_Z + C_L C_p.$$

When the coupling has only stiffnesses, as in this example, then $G_L = 0$ and $L(s)$ is given by C_L/s. Then, a minimal realization of the coupled system is obtained by defining

$$x \triangleq x_Z, \qquad \tilde{A} \triangleq A_Z - B_Z C_L C_p, \qquad \tilde{B} \triangleq B_Z D, \qquad C_1 \triangleq C_Z, \qquad C_2 \triangleq C_L C_p.$$

Now $L(s)$ has a zero real part so that the energy flows from higher energy subsystems to lower energy subsystems according to Corollaries 3.3 and 4.1.

On the other hand, if the coupling is purely gyroscopic, then $C_L = 0$ and $L(s) = G_L$. Then, a minimal realization of the coupled system is obtained by defining

$$x \triangleq x_Z, \qquad \tilde{A} \triangleq A_Z - B_Z G_L C_Z, \qquad \tilde{B} \triangleq B_Z D, \qquad C_1 \triangleq C_Z, \qquad C_2 \triangleq G_L C_Z.$$

If $r > 2$, then Corollaries 3.3 and 4.1 do not apply. Thus if the conditions in Propositions 3.1 and 4.3 hold, then reverse flow occurs.

First we consider oscillators coupled by springs as shown in Figures 6 and 7.

7.1. EXAMPLE 1

Consider the system consisting of two coupled oscillators, shown in Figure 6. From equation (14), power flow between the oscillators is given by

$$E_1^c(\omega) = \frac{K^2 \Delta_1 \Delta_2 \omega^2}{\pi \Gamma(\omega)} (E_2^{th}(\omega) - E_1^{th}(\omega)), \tag{117}$$

$$E_2^c(\omega) = \frac{K^2 \Delta_1 \Delta_2 \omega^2}{\pi \Gamma(\omega)} (E_1^{th}(\omega) - E_2^{th}(\omega)), \tag{118}$$

where

$$\Gamma(\omega) \triangleq [\Delta_1 \Delta_2 \omega^2 + \kappa^2 - (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)]^2 + \omega^2[\Delta_1(\omega^2 - \omega_2^2) + \Delta_2(\omega^2 - \omega_1^2)]^2, \tag{119}$$

and

$$\Delta_1 \triangleq c_1/m_1, \qquad \Delta_2 \triangleq c_2/m_2, \qquad \omega_1^2 \triangleq (K + k_1)/m_1, \qquad \omega_2^2 \triangleq (K + k_2)/m_2,$$
$$\kappa \triangleq K/\sqrt{m_1 m_2}. \tag{120}$$

From equations (117) and (118), we find that if $E_1^{th}(\omega) > E_2^{th}(\omega)$, then $E_1^c(\omega) < 0$ and $E_2^c(\omega) > 0$; that is, energy flows from oscillator 1 (higher thermodynamic energy) to oscillator 2 (lower thermodynamic energy) as guaranteed by Corollary 3.3. Furthermore, by using equation (54) we obtain

$$\delta_{12}(\omega) = \frac{K^2 \Delta_1 \Delta_2 \omega^2}{\pi \Gamma(\omega)} \geqslant 0. \tag{121}$$

On the other hand, the average blocked energy per unit bandwidth $E_i^{bl}(\omega)$, $i = 1, \ldots, r$, is defined by [3, 10]

$$E_i^{bl}(\omega) \triangleq \tfrac{1}{2} \left(m_i + \frac{k_i + \sum_{\substack{j=1 \\ j \neq i}}^{r} K_{ij}}{\omega^2} \right) y_i^2(\omega),$$

where $y_i(\omega)$ is defined by equation (60). Thus, by using equation (62) in Proposition 3.2 we obtain

$$E_1^{bl}(\omega) = \frac{\omega^2 + \omega_1^2}{2\Delta_1}[\delta_{11}(\omega)E_1^{th}(\omega) + \delta_{12}(\omega)E_2^{th}(\omega)], \tag{122}$$

$$E_2^{bl}(\omega) = \frac{\omega^2 + \omega_2^2}{2\Delta_2}[\delta_{12}(\omega)E_1^{th}(\omega) + \delta_{22}(\omega)E_2^{th}(\omega)], \tag{123}$$

where

$$\delta_{11}(\omega) = \frac{\Delta_1^2[\Delta_2^2\omega^2 + (\omega^2 - \omega_2^2)^2]}{\Gamma(\omega)}, \qquad \delta_{22}(\omega) = \frac{\Delta_2^2[\Delta_1^2\omega^2 + (\omega^2 - \omega_1^2)^2]}{\Gamma(\omega)}.$$

To examine energy flow, we assume for simplicity that the disturbances $w_1(t)$ and $w_2(t)$ have the same intensity s_{ww}; that is, $S_{ww} = \mathrm{diag}\,(s_{ww}, s_{ww})$. Then, from equations (122) and (123), we have

$$\tilde{E}^{bl}(\omega) \triangleq E_1^{bl}(\omega) - E_2^{bl}(\omega) = \frac{A_1\omega^6 + A_2\omega^4 + A_3\omega^2 + A_4}{4\pi}s_{ww}, \tag{124}$$

where

$$A_1 \triangleq 1/m_2 - 1/m_1, \qquad A_2 \triangleq \omega_2^2(m_2 - 2m_1)/m_1^2 - \omega_1^2(m_1 - 2m_2)/m_2^2,$$

$$A_3 \triangleq \omega_2^2/m_1 - \omega_1^2/m_2 + (\Delta_2/m_1)^2 - (\Delta_1/m_2)^2 + A_1(\kappa^2 + 2\omega_1^2\omega_2^2),$$

$$A_4 \triangleq \frac{(k_2 - k_1)[(k_1 + k_2)K + k_1k_2]}{m_1^2m_2^2}.$$

If $k_2 > k_1$, it follows from equation (124) that

$$\tilde{E}^{bl}(0) = \frac{k_2 - k_1}{4\pi[k_1k_2 + (k_1 + k_2)K]}s_{ww} > 0, \tag{125}$$

which implies that there exists a frequency range $(0, \omega_0)$ within which $E_1^{bl}(\omega) > E_2^{bl}(\omega)$. According to SEA, this inequality predicts net energy flow from subsystem 1 to subsystem 2. Now assume that $c_1 > c_2$, so that $E_2^{th} > E_1^{th}$. Then the energy flow determined by equations (117) and (118) satisfies $E_1^c(\omega) > 0$ and $E_2^c(\omega) < 0$. Since $E_1^c(\omega)$ is the energy flow per unit bandwidth entering subsystem 1, the net energy flow is from subsystem 2 to subsystem 1. This fact shows that the blocked energy flow incorrectly predicts reverse flow in the frequency range $(0, \omega_0)$.

Finally, we consider the total energy flow. With $r = 2$, equation (94) implies

$$P_1^c = -P_2^c = \sigma_{12}(E_2^{th} - E_1^{th}). \tag{126}$$

By using either equation (100) in Proposition 4.5 or the integral formulas in references [35, 36], we obtain

$$\sigma_{12} = \int_{-\infty}^{\infty}\delta_{12}(\omega)\,\mathrm{d}\omega = \frac{\kappa^2\Delta_1\Delta_2(\Delta_1 + \Delta_2)}{\Lambda}, \tag{127}$$

where

$$\Lambda \triangleq \kappa^2(\Delta_1 + \Delta_2)^2 + \Delta_1\Delta_2[(\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\Delta_1\omega_2^2 + \Delta_2\omega_1^2)]. \tag{128}$$

To obtain an energy flow relationship in terms of blocked energies, we again use references [35, 36] with equation (124) to obtain

$$E_2^{bl} - E_1^{bl} = \int_{-\infty}^{\infty}[E_2^{bl}(\omega) - E_1^{bl}(\omega)]\,\mathrm{d}\omega = \Upsilon(E_2^{th} - E_1^{th}), \tag{129}$$

where

$$\Upsilon \triangleq \frac{\Delta_1 \Delta_2}{\Delta}\,[(\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\Delta_1\omega_2^2 + \Delta_2\omega_1^2)]. \tag{130}$$

Now equations (126) and (129) imply

$$P_1^c = -P_2^c = \eta_{12}(E_2^{bl} - E_1^{bl}), \tag{131}$$

where

$$\eta_{12} \triangleq \frac{\sigma_{12}}{\Upsilon} = \frac{\kappa^2(\Delta_1 + \Delta_2)}{(\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\Delta_1\omega_2^2 + \Delta_2\omega_1^2)}.$$

This result was obtained in reference [3]. Since σ_{12} and η_{12} are non-negative, the total energy, integrated over ω, flows from the higher energy oscillator to the lower energy oscillator according to *both* the thermodynamic energy and the blocked energy although, as shown above, the energy flow prediction based upon the blocked energy may be incorrect in certain frequency ranges.

7.2. EXAMPLE 2

Next we analyze the three coupled oscillator system shown in Figure 7, where $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$ and the other parameters are changed as shown in subsequent figures. Furthermore, let the disturbances $w_i(t)$, $i = 1, 2, 3$, be white noise with unit intensity; that is, $D = I$. Although energy flows according to both E^{th} and E^{bl} in the weak coupling case shown in Figure 8, energy flows according to E^{th} but not according to E^{bl} when the coupling is strong, as shown in Figures 9 and 10. Note that in Figure 10, SEA erroneously predicts reverse flows between all pairs of subsystems. The additionally interesting fact is that, when the rate of energy dissipation and external parameter are balanced at one subsystem, then, as shown by subsystem 2 in Figure 9, this subsystem acts only as an energy conduit. By increasing the damping of this subsystem, subsystem 2 again serves as an energy sink, as shown in Figure 11. Note that the energy flow prediction based on the thermodynamic energy E_i^{th} is correct even if some of the subsystems are not directly excited and thus have zero thermodynamic energy. In such a situation, energy flows from the subsystems directly excited (non-zero thermodynamic energy subsystems) to the subsystems which are not excited directly (zero thermodynamic energy subsystems) according to equation (88). Additionally, there is no energy flow between zero thermodynamic energy subsystems 2 and 3. These features are shown in Figure 12. The next result, shown in Figure 13, illustrates that there is no energy flow if all of the subsystems have the same thermodynamic energy, even though there exist differences in the coupled mechanical energy among subsystems. In these last two cases SEA erroneously predicts energy flow among subsystems, although, in fact, there is none. As an interesting example we now consider the three coupled oscillator system shown in Figure 14. In Figure 14 only oscillator 1 is subject to a disturbance force, while only oscillator 2 is directly coupled with oscillator 1. As can be seen in Figure 15, there is no energy flow between the zero thermodynamic energy oscillators 2 and 3. Additionally, oscillator 3 receives energy flow indirectly from oscillator 1. This fact can be interpreted as follows. For the interconnected system which involves more than two subsystems, the absence of physical coupling between the ith subsystem and the jth subsystem does not necessarily imply that the coupling coefficient σ_{ij} is zero. For example, in Figure 14, although there is no physical coupling between oscillator 1 and oscillator 3, that is, $K_{13} = 0$, σ_{13} is not zero and thus energy flows from oscillator 1 to oscillator 3. Next, we consider the case of gyroscopic coupling.

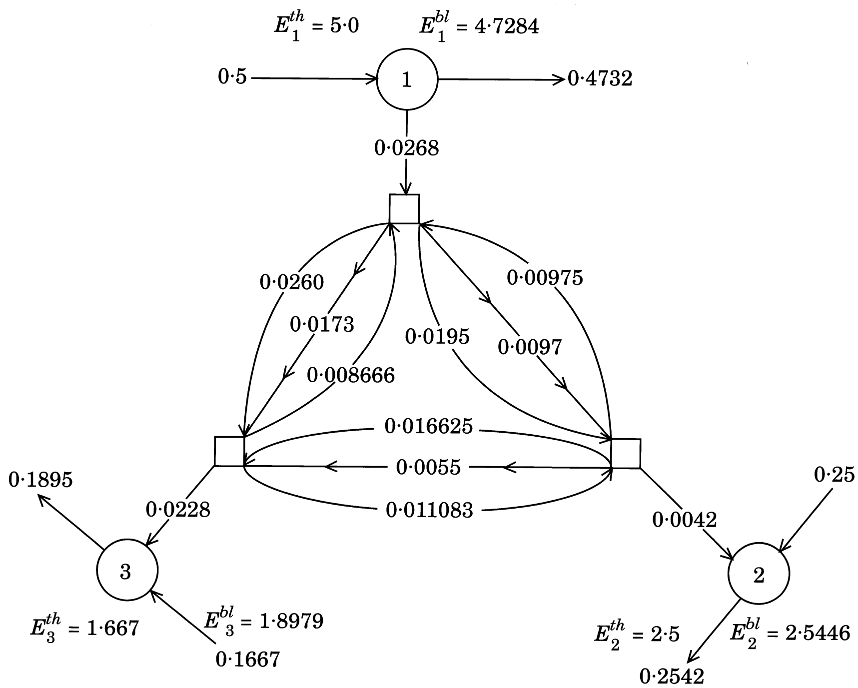

Figure 8. The weak coupling case: $E_1^{th} > E_2^{th} > E_3^{th}$, $E_1^{bl} > E_2^{bl} > E_3^{bl}$. $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}2$, $c_3 = 0{\cdot}3$, $K_{12} = 0{\cdot}05$, $K_{13} = 0{\cdot}07$, $K_{23} = 0{\cdot}1$.

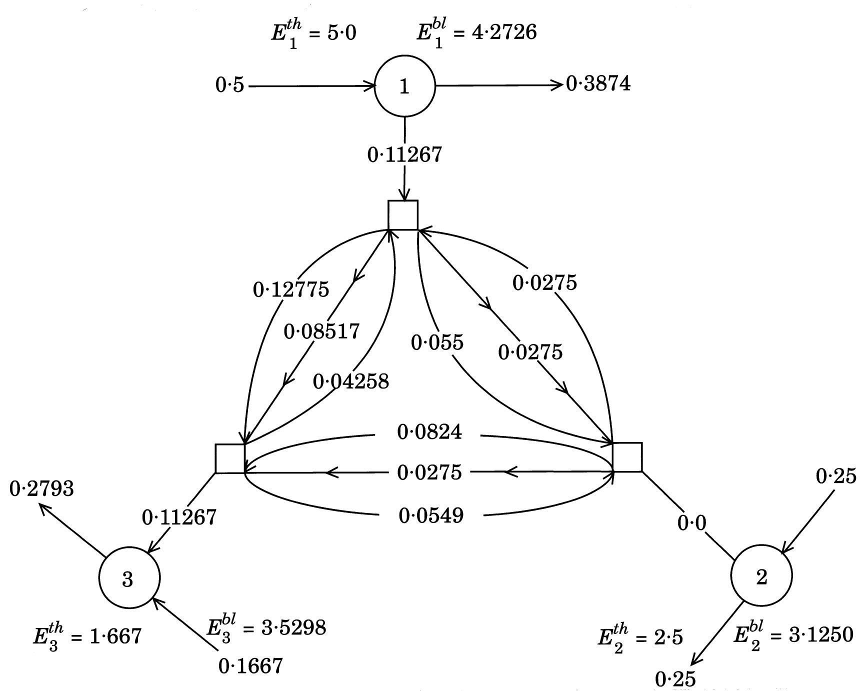

Figure 9. The strong coupling case: $E_1^{th} > E_2^{th} > E_3^{th}$, $E_1^{bl} > E_3^{bl} > E_2^{bl}$. $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}2$, $c_3 = 0{\cdot}3$, $K_{12} = 1{\cdot}0$, $K_{13} = 2{\cdot}0$, $K_{23} = 3{\cdot}0$.

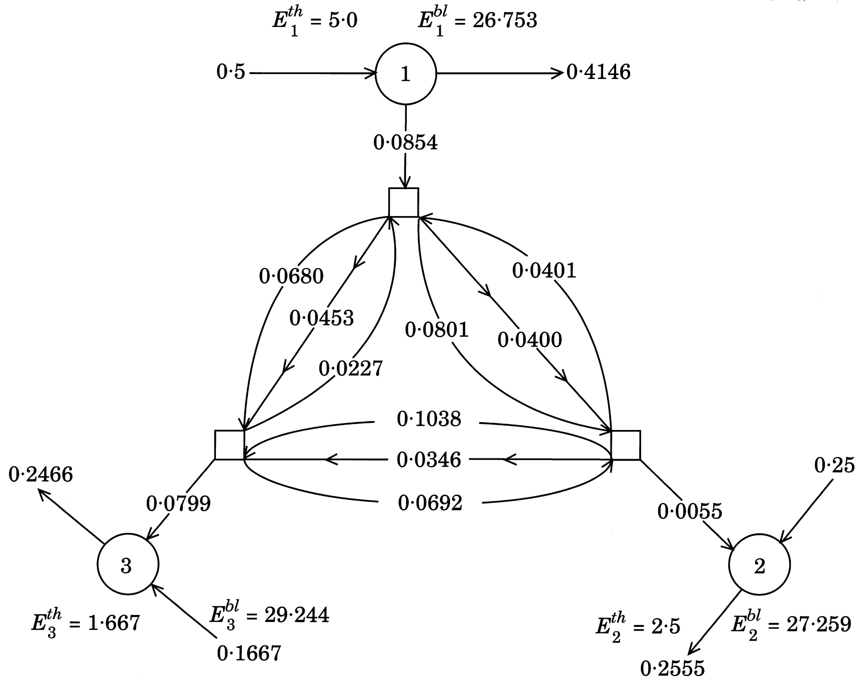

Figure 10. The very strong coupling case: $E_1^{th} > E_2^{th} > E_3^{th}$, $E_3^{bl} > E_2^{bl} > E_1^{bl}$. $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}2$, $c_3 = 0{\cdot}3$, $K_{12} = 50{\cdot}0$, $K_{13} = 60{\cdot}0$, $K_{23} = 70{\cdot}0$.

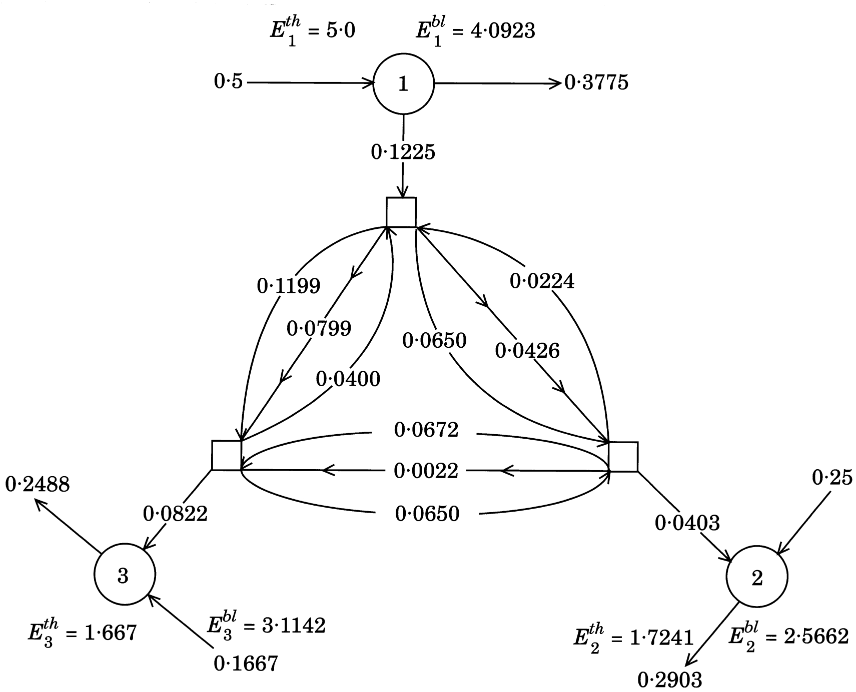

Figure 11. The strong coupling case with increased damping: $E_1^{th} > E_2^{th} > E_3^{th}$, $E_1^{bl} > E_3^{bl} > E_2^{bl}$. $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}29$, $c_3 = 0{\cdot}3$, $K_{12} = 1{\cdot}0$, $K_{13} = 2{\cdot}0$, $K_{23} = 3{\cdot}0$.

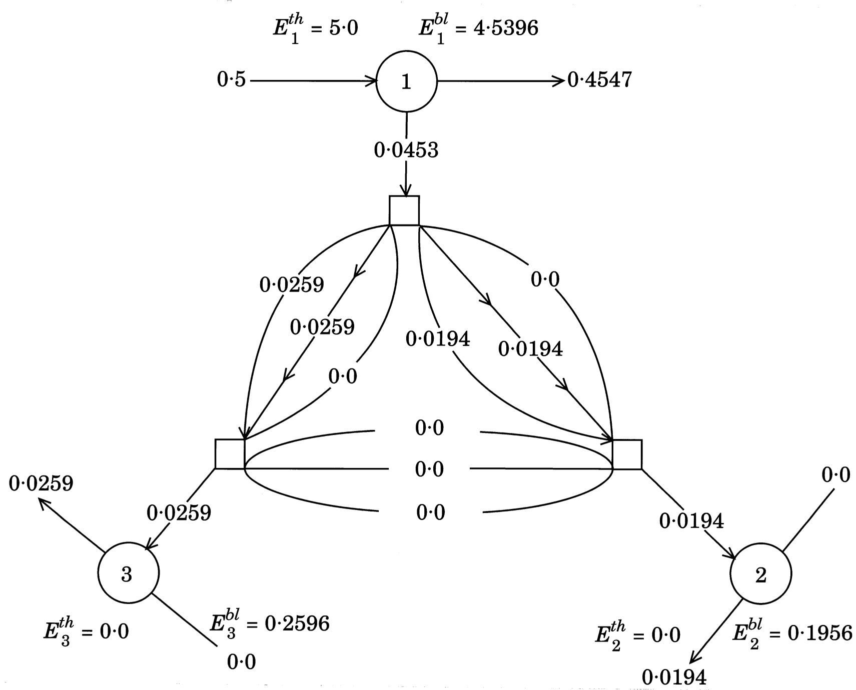

Figure 12. The indirectly excited subsystem case: $E_1^{th} > E_2^{th} = E_3^{th} = 0$, $E_1^{bl} > E_3^{bl} > E_2^{bl}$. $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}2$, $c_3 = 0{\cdot}3$, $K_{12} = 0{\cdot}05$, $K_{13} = 0{\cdot}07$, $K_{23} = 0{\cdot}1$.

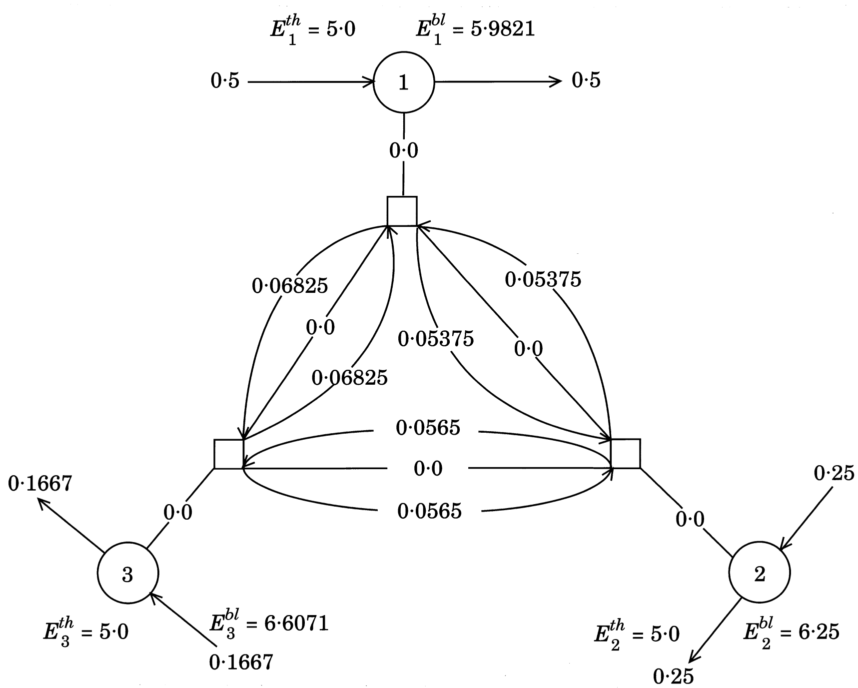

Figure 13. Equipartition of energy: $E_1^{th} = E_2^{th} = E_3^{th}$, $E_3^{bl} > E_2^{bl} > E_1^{bl}$. $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}1$, $c_3 = 0{\cdot}1$, $K_{12} = 1{\cdot}0$, $K_{13} = 2{\cdot}0$, $K_{23} = 3{\cdot}0$.

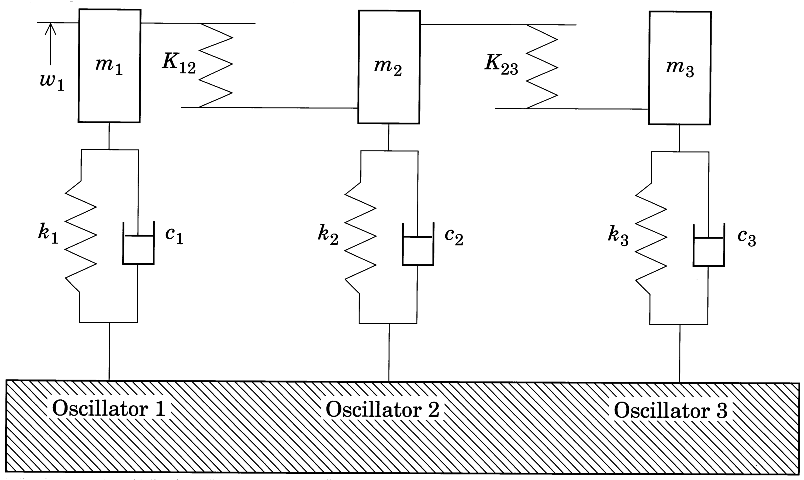

Figure 14. The three coupled oscillator system without K_{13}.

7.3. EXAMPLE 3

Consider the interconnected system composed of the three second order subsystems $z_i(s)$, $i = 1, 2, 3$, where

$$z_i(s) = (s^2 + c_i s + k_i)/s, \tag{132}$$

and the gyroscopic (skew-symmetric) coupling L, where

$$L = G_L = \begin{bmatrix} 0 & -a_1 & a_2 \\ a_1 & 0 & -a_3 \\ -a_2 & a_3 & 0 \end{bmatrix}. \tag{133}$$

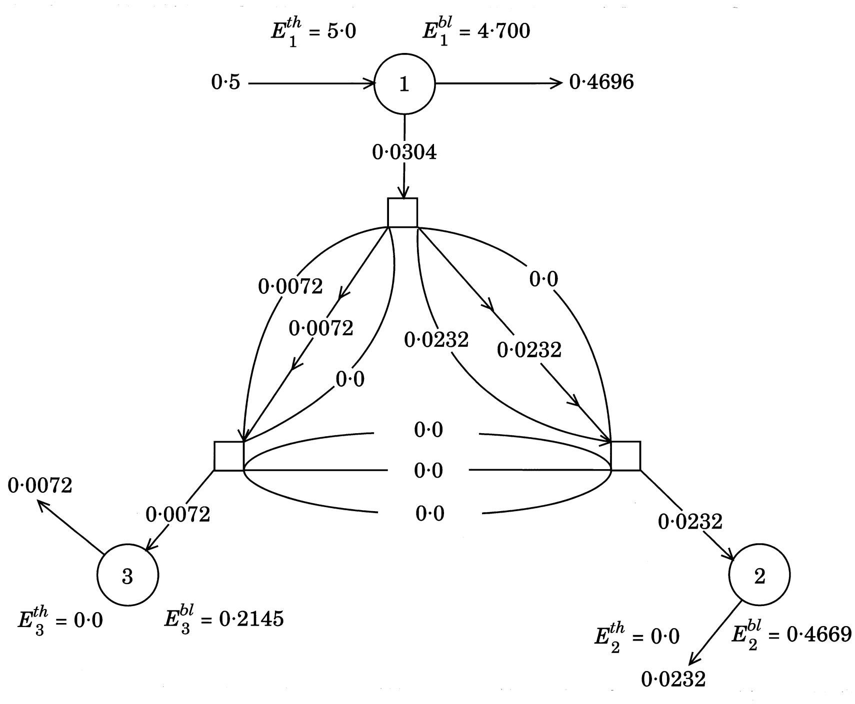

Figure 15. The energy flow in the coupled oscillator system of Figure 14. $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}2$, $c_3 = 0{\cdot}3$, $K_{12} = 0{\cdot}05$, $K_{23} = 0{\cdot}1$.

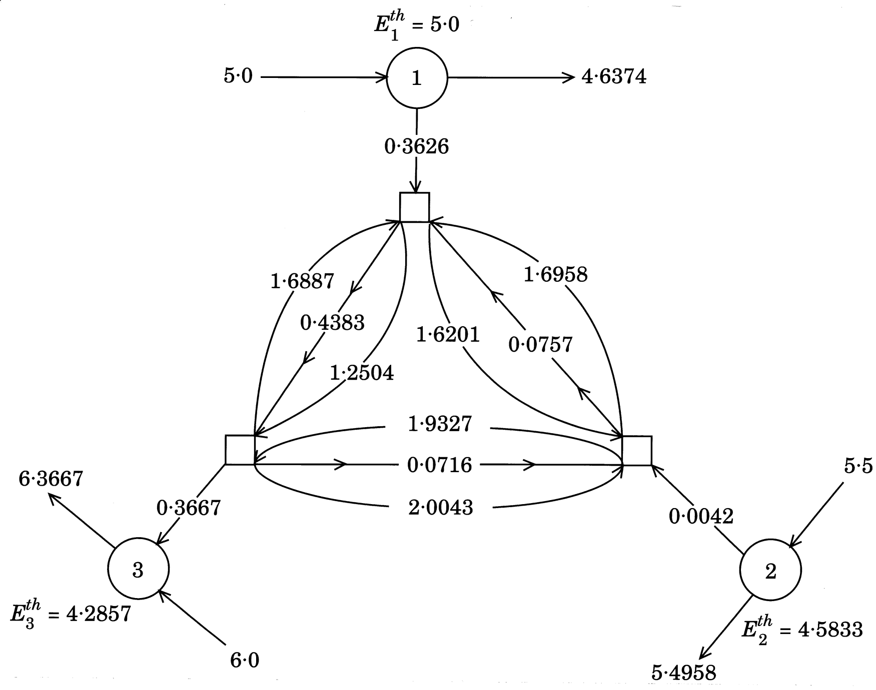

Figure 16. Reverse energy flow due to gyroscopic coupling.

Let $c_1 = 1$, $c_2 = 1{\cdot}2$, $c_3 = 1{\cdot}4$, $k_1 = 2$, $k_2 = 3$, $k_3 = 4$, $a_1 = 10$, $a_2 = 15$ and $a_3 = 20$. Furthermore, each disturbance force $w_i(t)$, $i = 1, 2, 3$, has the intensity $S_{w_1w_1} = 10$, $S_{w_2w_2} = 11$ and $S_{w_3w_3} = 12$, respectively. The result is shown in Figure 16, which indicates that reverse flow occurs between subsystems 1 and 2, and between subsystems 2 and 3; that is, the inequality (97) of Proposition 4.3 holds for $(i, j) = (1, 2)$ and (2, 3).

8. CONCLUSIONS

In this paper, we have extended the work of Wyatt, Siebert and Tan [30] on energy flow modelling of coupled subsystems. It has been shown that energy flow models based upon thermodynamic energy rather than stored energy can be used to predict energy flow from higher energy subsystems to lower energy subsystems. This model has been compared with the standard Statistical Energy Analysis (SEA) approach for mechanical systems. In contrast to the usual SEA formulation, which requires weak coupling or identical subsystems, the thermodynamic formulation holds for an arbitrary number of non-identical subsystems with arbitrarily strong coupling. This feature was demonstrated by means of a system involving three coupled oscillators.

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APPENDIX A: PROOF OF THEOREM 3.1

From equations (13) and (42), it follows that

$$\sum_{i=1}^{r} \frac{E^c_{Inc,i}(\omega)}{E^{th}_i(\omega)} = \frac{-1}{4\pi} \operatorname{tr}\left[\{E^{th}(\omega)\}^{-1}(S_{vy}(\omega) + S^*_{vy}(\omega))\right]$$

$$= \frac{-1}{4\pi} \operatorname{tr}[\{E^{th}(\omega)\}^{-1}[L(\mathrm{j}\omega)(L(\mathrm{j}\omega)$$

$$+ Z(\mathrm{j}\omega))^{-1} \operatorname{Inc}[S_{ww}(\omega)](L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}$$

$$+ (L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1} \operatorname{Inc}[S_{ww}(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}L^*(\mathrm{j}\omega)]]. \tag{A1}$$

For convenience, define the matrix $T(\omega)$ by

$$T(\omega) \triangleq (L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}(L(\mathrm{j}\omega) - Z^*(\mathrm{j}\omega)). \tag{A2}$$

The matrix $T(\omega)$ satisfies

$$(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1} = \tfrac{1}{2}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}[L(\mathrm{j}\omega) + Z(\mathrm{j}\omega) - (L(\mathrm{j}\omega) - Z^*(\mathrm{j}\omega))]C_d^{-1}(\omega)$$

$$= \tfrac{1}{2}[I - (L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}(L(\mathrm{j}\omega) - Z^*(\mathrm{j}\omega))]C_d^{-1}(\omega)$$

$$= \tfrac{1}{2}(I - T(\omega))C_d^{-1}(\omega), \tag{A3}$$

and

$$L(\mathrm{j}\omega) = 2C_d(\omega)(I - T(\omega))^{-1} - Z(\mathrm{j}\omega)$$

$$= 2C_d(\omega)(I - T(\omega))^{-1} - Z(\mathrm{j}\omega)(I - T(\omega))(I - T(\omega))^{-1}$$

$$= (2C_d(\omega) - Z(\mathrm{j}\omega) + Z(\mathrm{j}\omega)T(\omega))(I - T(\omega))^{-1}$$

$$= (Z(\mathrm{j}\omega) + Z^*(\mathrm{j}\omega) - Z(\mathrm{j}\omega) + Z(\mathrm{j}\omega)T(\omega))(I - T(\omega))^{-1}$$

$$= (Z^*(\mathrm{j}\omega) + Z(\mathrm{j}\omega)T(\omega))(I - T(\omega))^{-1}. \tag{A4}$$

By substituting equations (43), (A3) and (A4) into equation (A1), and using the diagonality of $Z(j\omega)$, $C_d(j\omega)$ and Inc $[S_{ww}(\omega)]$, the left side of equation (A1) can be rewritten as

$$\sum_{i=1}^{r}\frac{E^c_{Inc,i}(\omega)}{E^{th}_i(\omega)}=\frac{1}{4\pi}\operatorname{tr}[\mathrm{Inc}\,[S_{ww}(\omega)]^{-1}U(\omega)\,\mathrm{Inc}\,[S_{ww}(\omega)]U^*(\omega)-I]$$

$$=\frac{1}{4\pi}\operatorname{tr}[(\mathrm{Inc}\,[S_{ww}(\omega)]^{-1/2}U(\omega))\,\mathrm{Inc}\,[S_{ww}(\omega)][\mathrm{Inc}\,[S_{ww}(\omega)]^{-1/2}U(\omega)]^*-I), \tag{A5}$$

where $U(\omega)\triangleq C_d(\omega)T(\omega)C_d^{-1}(\omega)$. It follows from equation (A5) that

$$\sum_{i=1}^{r}\frac{E^c_{Inc,i}(\omega)}{E^{th}_i(\omega)}=\frac{r}{4\pi}\left[\left(\frac{1}{r}\sum_{i=1}^{r}\lambda_i\right)-1\right], \tag{A6}$$

where λ_i is an eigenvalue of $[\mathrm{Inc}\,[S_{ww}(\omega)]^{-1/2}U(\omega)]\,\mathrm{Inc}\,[S_{ww}(\omega)][\mathrm{Inc}\,[S_{ww}(\omega)]^{-1/2}U(\omega)]^*$. Furthermore, since

$$\begin{aligned}T(\omega)&=(L(j\omega)+Z(j\omega))^{-1}(L(j\omega)-Z^*(j\omega))\\&=(L(j\omega)+Z(j\omega))^{-1}(-L^*(j\omega)-Z^*(j\omega))\\&=-(L(j\omega)+Z(j\omega))^{-1}(L(j\omega)+Z(j\omega))^*,\end{aligned}$$

it follows that $T(\omega)$ is non-singular and thus $U(\omega)$ is non-singular. Therefore $[\mathrm{Inc}\,[S_{ww}(\omega)]^{-1/2}U(\omega)]\,\mathrm{Inc}\,[S_{ww}(\omega)][\mathrm{Inc}\,[S_{ww}(\omega)]^{-1/2}U(\omega)]^*$ is positive definite, and therefore all λ_i's are real and positive. The standard inequality between the arithmetic and geometric means, given by

$$\frac{1}{r}\left[\sum_{i=1}^{r}\lambda_i\right]\geqslant\left[\prod_{i=1}^{r}\lambda_i\right]^{1/r},$$

leads to

$$\sum_{i=1}^{r}\frac{E^c_{Inc,i}(\omega)}{E^{th}_i(\omega)}\geqslant\frac{r}{4\pi}[(\det[\mathrm{Inc}\,[S_{ww}(\omega)]^{-1}U(\omega)\,\mathrm{Inc}\,[S_{ww}(\omega)]U^*(\omega)])^{1/r}-1]$$

$$=\frac{r}{4\pi}([\det(U(\omega)U^*(\omega))]^{1/r}-1)=\frac{r}{4\pi}([\det(T(\omega)T^*(\omega))]^{1/r}-1)$$

$$=\frac{r}{4\pi}([\det((L(j\omega)+Z(j\omega))^{-1}(L^*(j\omega)+Z^*(j\omega))(L(j\omega)+Z(j\omega))$$

$$\times(L^*(j\omega)+Z^*(j\omega))^{-1})]^{1/r}-1)=0 \qquad \square$$

APPENDIX B: PROOF OF THEOREM 3.2

First we consider $E^c_{Inc,i}(\omega)$ and $E^d_{Inc,i}(\omega)$. Define

$$R(\omega)\triangleq(L(j\omega)+Z(j\omega))^{-1}, \tag{B1}$$

which satisfies the identities

$$L(j\omega)R(\omega)=I-Z(j\omega)R(\omega),\qquad R(\omega)=R^*(\omega)(L(j\omega)+Z(j\omega))^*R(\omega).$$

By using these identities, $E^c_{Inc}(\omega)$ given by equation (44) can be rewritten as

$$\begin{aligned}E^c_{Inc}(\omega)&=-(1/\pi)\,\mathrm{Re}\,[L(j\omega)R(\omega)C_d(\omega)E^{th}(\omega)R^*(\omega)]\\&=-(1/2\pi)[L(j\omega)R(\omega)C_d(\omega)E^{th}(\omega)R^*(\omega)+R(\omega)E^{th}(\omega)C_d(\omega)R^*(\omega)L^*(j\omega)]\end{aligned}$$

$$= -(1/2\pi)[(I - Z(\mathrm{j}\omega)R(\omega))C_d(\omega)E^{th}(\omega)R^*(\omega) + R(\omega)E^{th}(\omega)C_d(\omega)$$
$$\times (I - R^*(\omega)Z^*(\mathrm{j}\omega))]$$
$$= (1/2\pi)[Z(\mathrm{j}\omega)R(\omega)C_d(\omega)E^{th}(\omega)R^*(\omega) + R(\omega)E^{th}(\omega)C_d(\omega)R^*(\omega)Z^*(\mathrm{j}\omega)$$
$$- C_d(\omega)E^{th}(\omega)R^*(\omega) - R(\omega)E^{th}(\omega)C_d(\omega)]$$
$$= (1/2\pi)[Z(\mathrm{j}\omega)R(\omega)C_d(\omega)E^{th}(\omega)R^*(\omega) + R(\omega)E^{th}(\omega)C_d(\omega)R^*(\omega)Z^*(\mathrm{j}\omega)$$
$$- C_d(\omega)E^{th}(\omega)R^*(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))R(\omega) - R^*(\omega)(L(\mathrm{j}\omega)$$
$$+ Z(\mathrm{j}\omega))^*R(\omega)E^{th}(\omega)C_d(\omega)].$$

Thus, $E^c_{Inc,i}(\omega)$ satisfies

$$E^c_{Inc,i}(\omega) = (1/2\pi)e_i^{\mathrm{T}}[Z(\mathrm{j}\omega)R(\omega)C_d(\omega)E^{th}(\omega)R^*(\omega) + R(\omega)E^{th}(\omega)C_d(\omega)R^*(\omega)Z^*(\mathrm{j}\omega)$$
$$-C_d(\omega)E^{th}(\omega)R^*(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))R(\omega) - R^*(\omega)(L(\mathrm{j}\omega)$$
$$+Z(\mathrm{j}\omega))^*R(\omega)E^{th}(\omega)C_d(\omega)]e_i$$
$$= (1/2\pi)\ \mathrm{tr}\ [e_i^{\mathrm{T}}[Z(\mathrm{j}\omega)R(\omega)C_d(\omega)E^{th}(\omega)R^*(\omega) + R(\omega)E^{th}(\omega)C_d(\omega)R^*(\omega)Z^*(\mathrm{j}\omega)$$
$$- C_d(\omega)E^{th}(\omega)R^*(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))R(\omega) - R^*(\omega)(L(\mathrm{j}\omega)$$
$$+ Z(\mathrm{j}\omega))^*R(\omega)E^{th}(\omega)C_d(\omega)]e_i$$
$$= (1/2\pi)\ \mathrm{tr}\ [R^*(\omega)e_ie_i^{\mathrm{T}}Z(\mathrm{j}\omega)R(\omega)C_d(\omega)E^{th}(\omega)$$
$$+ R^*(\omega)Z^*(\mathrm{j}\omega)e_ie_i^{\mathrm{T}}R(\omega)C_d(\omega)E^{th}(\omega) - R^*(\omega)Z(\mathrm{j}\omega)R(\omega)e_ie_i^{\mathrm{T}}C_d(\omega)E^{th}(\omega)$$
$$-R^*(\omega)Z^*(\mathrm{j}\omega)R(\omega)E^{th}(\omega)C_d(\omega)e_ie_i^{\mathrm{T}} - R^*(\omega)(L(\mathrm{j}\omega)+L^*(\mathrm{j}\omega))R(\omega)e_ie_i^{\mathrm{T}}$$
$$\times E^{th}(\omega)C_d(\omega)]$$
$$= (1/2\pi)\mathrm{tr}[R^*(\omega)(e_ie_i^{\mathrm{T}}Z(\mathrm{j}\omega) + Z^*(\mathrm{j}\omega)e_ie_i^{\mathrm{T}})R(\omega)C_d(\omega)E^{th}(\omega)$$
$$- R^*(\omega)(Z(\mathrm{j}\omega) + Z^*(\mathrm{j}\omega))R(\omega)e_ie_i^{\mathrm{T}}C_d(\omega)E^{th}(\omega)].$$

The last equation holds because $L(\mathrm{j}\omega) + L^*(\mathrm{j}\omega) = 0$ and $e_ie_i^{\mathrm{T}}$, $C_d(\omega)$ and $E^{th}(\omega)$ are diagonal. Finally, we obtain

$$E^c_{Inc,i}(\omega) = \frac{1}{\pi}\ \mathrm{tr}\ [R^*(\omega)e_ie_i^{\mathrm{T}}C_d(\omega)R(\omega)C_d(\omega)E^{th}(\omega) - R^*(\omega)C_d(\omega)R(\omega)e_ie_i^{\mathrm{T}}C_d(\omega)E^{th}(\omega)]$$

$$= \frac{1}{\pi}\ c_i(\omega)\sum_{j=1}^{r} c_j(\omega)R^*_{(j,i)}(\omega)R_{(i,j)}(\omega)E^{th}_j(\omega) - \frac{1}{\pi}\ c_i(\omega)\sum_{j=1}^{r} c_j(\omega)R^*_{(i,j)}(\omega)R_{(j,i)}(\omega)E^{th}_i(\omega)$$

$$= \sum_{j=1}^{r} \delta_{ij}(\omega)E^{th}_j(\omega) - \sum_{j=1}^{r} \delta_{ji}(\omega)E^{th}_i(\omega) = \sum_{\substack{j=1\\ j\neq i}}^{r} [\delta_{ij}(\omega)E^{th}_j(\omega) - \delta_{ji}(\omega)E^{th}_i(\omega)],$$

which proves equation (49).

Next we consider equation (51). Since $Z(\mathrm{j}\omega)$ is diagonal, it follows from equation (45) that

$$E^d_{Inc,i}(\omega) = -\frac{1}{\pi}\ \mathrm{Re}\ [e_i^{\mathrm{T}}Z(\mathrm{j}\omega)R(\omega)C_d(\omega)E^{th}(\omega)R^*(\omega)e_i]$$
$$= -\frac{1}{\pi}\ \mathrm{tr}\ [\mathrm{Re}\ [e_i^{\mathrm{T}}Z(\mathrm{j}\omega)R(\omega)C_d(\omega)E^{th}(\omega)R^*(\omega)e_i]]$$

$$= -\frac{1}{\pi}\,\mathrm{tr}\,[\mathrm{Re}\,[R^*(\omega)e_ie_i^{\mathrm{T}}Z(\mathrm{j}\omega)R(\omega)C_d(\omega)E^{th}(\omega)]]$$

$$= -\frac{1}{\pi}\sum_{j=1}^{r} c_i(\omega)c_j(\omega)R^*_{(j,i)}(\omega)R_{(i,j)}(\omega)E_j^{th}(\omega)$$

$$= -\delta_{ii}(\omega)E_i^{th}(\omega) - \sum_{\substack{j=1\\ j\neq i}}^{r}\delta_{ij}(\omega)E_j^{th}(\omega),$$

which proves equation (51).

Now we consider $E^c_{Coh,i}(\omega)$ and $E^d_{Coh,i}(\omega)$. In the similar manner to E^c_{Inc}, we obtain from equation (38),

$$E^c_{Coh}(\omega) = \frac{1}{4\pi}[Z(\mathrm{j}\omega)R(\omega)\,\mathrm{Coh}\,[S_{ww}]R^*(\omega) + R(\omega)\,\mathrm{Coh}\,[S_{ww}]R^*(\omega)Z^*(\omega) - \mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega) - R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]].$$

Thus $E^c_{Coh,i}(\omega)$ satisfies

$$\begin{aligned}
E^c_{Coh,i}(\omega) &= \frac{1}{4\pi}\,\mathrm{tr}\,[e_i^{\mathrm{T}}(Z(\mathrm{j}\omega)R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega)\\
&\quad + R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega)Z^*(\mathrm{j}\omega))e_i\\
&\quad - e_i^{\mathrm{T}}(\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega) + R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)])e_i]\\
&= \frac{1}{4\pi}\,\mathrm{tr}\,[R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega)e_ie_i^{\mathrm{T}}(Z(\mathrm{j}\omega) + Z^*(\mathrm{j}\omega))\\
&\quad - (e_ie_i^{\mathrm{T}}\,\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega) + R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]e_ie_i^{\mathrm{T}})]\\
&= \frac{1}{4\pi}\,\mathrm{tr}\,[2R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega)e_ie_i^{\mathrm{T}}C_d(\omega)\\
&\quad - (e_ie_i^{\mathrm{T}}\,\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))R(\omega)\\
&\quad + R^*(\omega)(L(\omega) + Z(\omega))^*R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]e_ie_i^{\mathrm{T}})]\\
&= \frac{1}{4\pi}\Bigg[2\sum_{p=1}^{r}\sum_{\substack{q=1\\ q\neq p}}^{r} R(\omega)_{(i,p)}R^*(\omega)_{(q,i)}\,\mathrm{Coh}\,[S_{ww}(\omega)]_{(p,q)}c_i(\omega)\\
&\quad - \sum_{\substack{q=1\\ q\neq i}}^{r} R(\omega)_{(p,i)}R^*_{(q,p)}(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]_{(i,q)}c_p(\omega))\\
&\quad - \sum_{s=1}^{r}[R^*(\omega)(L(\mathrm{j}\omega) + L^*(\mathrm{j}\omega))R(\omega)]_{(i,s)}\,\mathrm{Coh}\,[S_{ww}(\omega)]_{(s,i)}].
\end{aligned}$$

By using the above expression and $L(\mathrm{j}\omega) + L^*(\mathrm{j}\omega) = 0$ with the definition of the cross-energy in equations (40), equation (50) can be obtained. In a similar manner,

$$\begin{aligned}
E^d_{Coh,i}(\omega) &= \frac{-1}{2\pi}\,\mathrm{Re}\,[Z(\mathrm{j}\omega)R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega)]_{(i,i)}\\
&= \frac{-1}{4\pi}[Z(\mathrm{j}\omega)R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega) + R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega)Z^*(\mathrm{j}\omega)]_{(i,i)}
\end{aligned}$$

$$= \frac{-1}{4\pi}\,\mathrm{tr}\,[e_i^{\mathrm{T}} Z(\mathrm{j}\omega)R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega)e_i$$
$$+ e_i^{\mathrm{T}} R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega)Z^*(\mathrm{j}\omega)e_i]$$
$$= \frac{-1}{4\pi}\,\mathrm{tr}\,[(e_i e_i^{\mathrm{T}} Z(\mathrm{j}\omega) + Z^*(\mathrm{j}\omega)e_i e_i^{\mathrm{T}})R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega)]$$
$$= \frac{-1}{2\pi} c_i(\omega)[R(\omega)\,\mathrm{Coh}\,[S_{ww}(\omega)]R^*(\omega)]_{(i,i)} = -c_i(\omega)\sum_{p=1}^{r}\sum_{\substack{q=1\\ q\neq p}}^{r}\delta_{ipq}(\omega)E_{pq}^{th}(\omega). \qquad \square$$

APPENDIX C: PROOF OF COROLLARY 3.4

Since $L(\mathrm{j}\omega) + L^*(\mathrm{j}\omega) = 0$ and $Z(\mathrm{j}\omega) + Z^*(\mathrm{j}\omega) = 2C_d(\omega)$, it follows that

$$(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))C_d^{-1}(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^* = (L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^* C_d^{-1}(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega)). \tag{C1}$$

Then, from equations (54) and (C1), and the diagonality of $C_d(\omega)$, we obtain

$$\sum_{\substack{j=1\\ j\neq i}}^{r}[\delta_{ij}(\omega) - \delta_{ji}(\omega)] = \frac{1}{\pi}c_i(\omega)\sum_{\substack{j=1\\ j\neq i}}^{r}[c_j(\omega)]|[(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}]_{(i,j)}|^2$$
$$- c_j(\omega)|[(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}]_{(j,i)}|^2]$$
$$= \frac{1}{\pi}c_i(\omega)([(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}C_d(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}]_{(i,i)}$$
$$- [(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega)^{-*}C_d(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega)^{-1}]_{(i,i)})$$
$$= \frac{1}{\pi}c_i(\omega)[(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}C_d(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}$$
$$- (L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}C_d(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}]_{(i,i)}$$
$$= \frac{1}{\pi}c_i(\omega)[(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}C_d(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}$$
$$\times ((L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))C_d^{-1}(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^*$$
$$- (L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^* C_d^{-1}(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega)))$$
$$\times (L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}C_d(\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}]_{(i,i)}$$
$$= 0. \qquad \square$$

APPENDIX D: PROOF OF THEOREM 4.1

As shown in equation (12), P^c does not depend on $\mathrm{Im}\,[S_{vy}(\omega)]$. By using Parseval's theorem we have

$$P^c = \int_{-\infty}^{\infty} E^c(\omega)\,\mathrm{d}\omega$$
$$= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\,[L(\mathrm{j}\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}S_{ww}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}]\,\mathrm{d}\omega$$
$$= \frac{-1}{2\pi}\int_{-\infty}^{\infty} L(\mathrm{j}\omega)(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}DD^{\mathrm{T}}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}\,\mathrm{d}\omega$$
$$= \frac{-1}{2\pi}\int_{-\infty}^{\infty}[C_2(\mathrm{j}\omega I - \tilde{A})^{-1}\tilde{B}][C_1(\mathrm{j}\omega I - \tilde{A})^{-1}\tilde{B}]^*\,\mathrm{d}\omega = -C_2\tilde{Q}C_1^{\mathrm{T}},$$

which proves equation (74). Since $w_i(t)$ is white noise, we have

$$\begin{aligned}\mathscr{E}[w(t)x^{\mathrm{T}}(t)] &= \mathscr{E}\left[w(t)\left\{\mathrm{e}^{\tilde{A}t}x_0 + \int_0^t \mathrm{e}^{\tilde{A}(t-s)}Bw(s)\,\mathrm{d}s\right\}^{\mathrm{T}}\right] \\ &= \mathscr{E}\left[\int_0^t w(t)w^{\mathrm{T}}(s)B^{\mathrm{T}}\,e^{\tilde{A}^{\mathrm{T}}(t-s)}\,\mathrm{d}s\right] = \int_0^t \delta(t-s)S_{ww}B^{\mathrm{T}}\,\mathrm{e}^{\tilde{A}^{\mathrm{T}}(t-s)}\,\mathrm{d}s \\ &= \tfrac{1}{2}\,DD^{\mathrm{T}}B^{\mathrm{T}} = \tfrac{1}{2}\,D\tilde{B}^{\mathrm{T}},\end{aligned} \tag{D1}$$

where $\delta(t)$ is the symmetric delta function and

$$B \triangleq \begin{bmatrix} B_Z \\ 0 \end{bmatrix}.$$

By multiplying equation (D1) on the right side by C_1^T and using equation (73), we obtain

$$\tfrac{1}{2}\,D\tilde{B}^{\mathrm{T}}C_1^{\mathrm{T}} = \mathscr{E}[w(t)x^{\mathrm{T}}(t)C_1^{\mathrm{T}}] = \mathscr{E}[w(t)y^{\mathrm{T}}(t)] = P^e,$$

which proves equation (75). Equation (76) follows immediately from equation (31) in Corollary 3.2. □

APPENDIX E: PROOF OF COROLLARY 4.1

In a similar manner to P^c, equation (78) can be obtained as

$$\begin{aligned}P_i^d &= \int_{-\infty}^{\infty} E_i^d(\omega)\,\mathrm{d}\omega \\ &= \frac{-1}{2\pi}\left[\int_{-\infty}^{\infty} \mathrm{Re}\,[Z(\mathrm{j}\omega)(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-1}S_{ww}(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-*}]\,\mathrm{d}\omega\right]_{(i,i)} \\ &= \frac{-1}{2\pi}\left[\int_{-\infty}^{\infty} \mathrm{C_d}(\mathrm{L}(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-1}S_{ww}(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-*}\,\mathrm{d}\omega\right]_{(i,i)} \\ &= -\left[C_d\frac{1}{2\pi}\int_{-\infty}^{\infty}(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-1}DD^{\mathrm{T}}(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-*}\,\mathrm{d}\omega\right]_{(i,i)} \\ &= -\left(C_d\frac{1}{2\pi}\int_{-\infty}^{\infty}[C_1(\mathrm{j}\omega I-\tilde{A})^{-1}\tilde{B}][C_1(\mathrm{j}\omega I-\tilde{A})^{-1}\tilde{B}]^*\,\mathrm{d}\omega\right)_{(i,i)} \\ &= -(C_dC_1\tilde{Q}C_1^{\mathrm{T}})_{(i,i)}.\end{aligned}$$

□

APPENDIX F: NOTATION

$\mathscr{R}$	real numbers	A^{T}, A^*	transpose, complex conjugate transpose of A
$\mathscr{E}$	expectation		
R_{xy}	cross-correlation function matrix of x and y	$A > (\geqslant) 0$	positive (non-negative) definite matrix
S_{xx}	power spectral density matrix of x	e_i	ith column of I
		tr $[A]$	trace of A
S_{xy}	cross-spectral density matrix of x and y	$\{A\}, \langle A\rangle$	digonal, off-diagonal portion of A
I	identity matrix	Inc $[S]$	incoherent (diagonal) portion of spectral density matrix S
j	$=\sqrt{-1}$		
$A_{(k,l)}$	(k, l)-element of A	Coh [S]	coherent (off-diagonal) portion of spectral density matrix S
Re $[A]$, Im $[A]$	real, imaginary part of A		
diag $(a_1, \ldots, a_r)$	diagonal matrix ith diagonal element of which is a_i	e	$[1\ \ 1\ \ \cdots\ \ 1]^{\mathrm{T}}$

$G(s) \sim \left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	$= C(sI - A)^{-1}B + D$, state space realization of the transfer function $G(s)$
A	coefficient matrix of compartmental model
C_d	resistance or damping matrix
C_L	stiffness part of coupling matrix $L(s)$
c_i	resistance or damping of ith subsystem
D	disturbance matrix
G_L	gyroscopic part of coupling matrix $L(s)$
E	column vector with components E_i^{th}
E^{th}	thermodynamic energy matrix
E_i^{th}	steady state thermodynamic energy
E_{ij}^{th}	steady state thermodynamic cross energy
E_i^{bl}	steady state blocked energy
E_i^{mec}	steady state coupled mechanical energy
E_i^u	steady state uncoupled mechanical energy
$E^c(\omega)$	average coupling energy flow matrix per unit bandwidth
$E^d(\omega)$	average energy dissipation rate matrix per unit bandwidth
$E^e(\omega)$	average external power matrix per unit bandwidth
$E_i^c(\omega)$	average coupling energy flow of ith subsystem per unit bandwidth
$E_i^d(\omega)$	average energy dissipation rate of ith subsystem per unit bandwidth
E_i^e	average external power of ith subsystem per unit bandwidth
$E_{Coh(Inc),i}^c(\omega)$	average (in)coherent coupling energy flow of ith subsystem per unit bandwidth
$E_{Coh(Inc),i}^d(\omega)$	average (in)coherent energy dissipation rate of ith subsystem per unit bandwidth
$E_{Coh(Inc),i}^e(\omega)$	average (in)coherent external power of ith subsystem per unit bandwidth
$\mathscr{S}$	matrix involving σ_{ij}
$\mathscr{H}(\omega)$	matrix involving $\delta_{ij}(\omega)$
k_i	stiffness of ith subsystem
$L(s)$	linear time-invariant coupling matrix
m_i	mass of ith subsystem
P^c	steady state average coupling energy flow matrix
P^d	steady state average energy dissipation rate matrix
P^e	steady state average external power matrix
P_e	column vector comprised of $P_{Inc,i}^e$
P_i^c	steady state average coupling energy flow of ith subsystem
P_i^d	steady state average energy dissipation rate of ith subsystem
P_i^e	steady state average external power of ith subsystem
$P_{Coh(Inc),i}^c$	steady state (in)coherent average coupling energy flow of ith subsystem
$P_{Coh(Inc),i}^d$	steady state (in)coherent average energy dissipation rate of ith subsystem
$P_{Coh(Inc),i}^e$	steady state (in)coherent average external power of ith subsystem
$\tilde{Q}$	steady state covariance for feedback representation of interconnected system
$\tilde{Q}_j$	steady state covariance for coupling coefficient
$\tilde{w}_i(t)$	normalized white noise disturbance with unit intensity
$Z(s)$	subsystem impedance matrix
$z_i(s)$	subsystem (impedance transfer function)
$\delta_i(\omega)$	dissipative coefficient of ith subsystem
$\delta_{ij}(\omega)$	coupling coefficient between ith and jth subsystems
$\delta_{ipq}(\omega)$	coupling coefficient between ith, pth and qth subsystems
σ_i	dissipative coefficient of ith subsystem (time domain)
σ_{ij}	coupling coefficient between ith and jth subsystems (time domain)