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What Is the Koopman Operator? A Simplified Treatment for Discrete-Time Systems

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***Abstract*— This paper provides an introduction to the discrete-time Koopman operator for nonexperts, including a treatment of the basic definitions and properties of the Koopman operator and a numerical method for approximating the Koopman spectrum. Furthermore, the paper stresses the role of compositional completeness for spaces of Koopman observables and gives conditions under which the L_p spaces are compositionally complete. Numerical examples are given to illustrate the basic concepts.**

I. Introduction

Stability analysis of nonlinear systems remains one of the most fundamental and important problems in systems theory. Lyapunov methods provide a powerful approach for analyzing the stability of nonlinear systems, and much of modern control theory is based on Lyapunov functions [1]. For stability analysis, as distinct from Lyapunov-based controller synthesis, converse Lyapunov theory guarantees the existence of a Lypunov function. However, construction of a Lyapunov function that guarantees asymptotic stability can be difficult in practice.

The Koopman operator provides an alternative approach to stability analysis of nonlinear systems. For discrete-time systems, the Koopman operator is linear in the space of observables, which can be viewed as a space of output mappings. The linearity is a simple consequence of the fact that the composition $g\circ f$ of the output map g and the vector field f is a linear function of the output map. The Koopman operator $\mathcal{K}(g) \stackrel{\triangle}{=} g\circ f$ is thus a linear operator on an infinite-dimensional space of observables.

The Koopman operator was introduced by B.O. Koopman [2] in 1931 and has developed in the intervening 8 decades (see, for example, the introductions of [3] or [4] for a historical discussion). Interest in the Koopman operator has increased dramatically in recent years, in particular for analyzing fluid motion [5], [6] as well as for estimation and control problems [7], [8], [9], [10]. The Koopman operator has also been applied in stochastic settings [11], [12], and to data driven applications [13], [14], [15]. The last of these is particularly important because it investigates the conditions under which finite dimensional matrix approximations of the Koopman operator converge to the infinite dimensional Koopman operator as the dimension of the projection goes to infinity.

The Koopman operator is useful for stability analysis. In particular, each eigenfunction of $\mathcal{K}$ provides a mode of the dynamics, and the corresponding eigenvalue determines whether that mode is stable or unstable. In effect, the Koopman operator provides a mode-by-mode analysis of system stability.

For practical purposes, it is necessary to compute eigenvalues and eigenvectors of an approximation $\mathcal{K}_p$ of the Koopman operator. This requires 1) a numerical technique for constructing the finite-dimensional approximation $\mathcal{K}_p$ of $\mathcal{K}$, and 2) analysis to show that the stability inferred from the eigenvalues of $\mathcal{K}_p$ is sufficient to guarantee stability of the underlying system. Results in this direction are given in [16].

The present paper has the following goals and contribution. First, we provide a succinct statement of the Koopman operator and its domain. To do this, we require the notion of a *compositionally complete* space of observables. To this end, we provide sufficient conditions for the space of observables to be compositionally complete, and as an application, we prove that if $\mathcal{D}\subseteq\mathbb{R}^n$, and f is a measurable, anti-Lipschitz, discrete-time mapping, then $\mathrm{L}_p(\mathcal{D},\mathbb{R})$ is compositionally complete for the Koopman operator of f. Next, we illustrate the Koopman operator spectrum and eigenfunctions by giving examples for several analytic cases. Although some of these cases have appeared in the literature before, there is no single source that contains all of them or stresses the space of observables and its compositional completeness. Next, we prove a result on Koopman eigenfunctions and stability. Finally, we introduce a simplified algorithm for approximating the Koopman spectrum and use this algorithm to numerically approximate the spectrum of one and two dimensional systems. Although this algorithm is related to the "Analytic EDMD" algorithm presented in [15, Section 7], it differs strongly from ordinary DMD and EDMD algorithms, since it requires no simulation data or ergodicity conditions.

The paper is organized as follows. Section II defines and reviews the basic properties of the Koopman operator. Section III proves a theorem on the compositional completeness of the L_p spaces. Section IV gives five analytic examples of the Koopman spectrum. Section V proves a result on Koopman eigenfunctions and stability. Section VI presents an Analytic-EDMD like method for numerically approximating the Koopman spectrum, and Section VII gives four numerical examples of the Koopman spectrum using this method.

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II. The Koopman Operator

Let $\mathcal{D} \subseteq \mathbb{R}^n$, let $f\colon \mathcal{D} \to \mathcal{D}$, and, for all $k \geq 0$, consider the discrete-time system

$$x(k+1) = f(x(k)), \quad x(0) = x_0, \tag{1}$$

where $x_0 \in \mathbb{R}^n$ is the initial condition. In addition, let $g\colon \mathcal{D} \to \mathbb{R}$ determine the output of (1) given by

$$y(k) = g(x(k)). \tag{2}$$

Note that, for all $k \geq 0$,

$$g(x(k)) = g(f^{(\circ k)}(x_0)) = (g \circ f^{(\circ k)})(x_0), \tag{3}$$

where $f^{(\circ 0)}$ is the identity map on $\mathcal{D}$, $f^{(\circ 1)} \stackrel{\triangle}{=} f$, $f^{(\circ 2)} \stackrel{\triangle}{=} f \circ f$, and $f^{(\circ k)}$ denotes the composition of f with itself k times.

Next, motivated by (3), let $\mathcal{G}$ be a set of real-valued functions on $\mathcal{D}$ which satisfy the following property:

$$\text{If } g \in \mathcal{G}, \text{ then } g \circ f \in \mathcal{G}. \tag{4}$$

If (4) holds, then $\mathcal{G}$ is *compositionally complete*. Note that, if $g \in \mathcal{G}$, then, for all $r \geq 0$, $g \circ f^{(\circ r)} \in \mathcal{G}$. If $\mathcal{G}$ is compositionally complete and $g \in \mathcal{G}$, then g is an *observable.*

Next, define the *Koopman operator* $\mathcal{K}\colon \mathcal{G} \to \mathcal{G}$ by

$$\mathcal{K}(g) \stackrel{\triangle}{=} g \circ f. \tag{5}$$

Note that, since $\mathcal{G}$ is compositionally complete, it follows that, if $g \in \mathcal{G}$, then $g \circ f \in \mathcal{G}$ and thus $\mathcal{K}(g) \in \mathcal{G}$.

We now assume that $\mathcal{G}$ is a compositionally complete vector space over $\mathbb{R}$. Then, for all $g_1, g_2 \in \mathcal{G}$ and all $a_1, a_2 \in \mathbb{R}$,

$$\begin{aligned} \mathcal{K}(a_1 g_1 + a_2 g_2) &= (a_1 g_1 + a_2 g_2) \circ f \\ &= a_1 g_1 \circ f + a_2 g_2 \circ f \\ &= a_1 \mathcal{K}(g_1) + a_2 \mathcal{K}(g_2). \end{aligned} \tag{6}$$

Therefore, $\mathcal{K}$ is a linear operator on $\mathcal{G}$.

In terms of powers of $\mathcal{K}$, note that, for all $g \in \mathcal{G}$, $\mathcal{K}^0(g) = g \circ f^{\circ 0} = g$, thus $\mathcal{K}^0$ is the identity map on $\mathcal{G}$. Furthermore, for all $g \in \mathcal{G}$, $\mathcal{K}^2(g) = \mathcal{K}(\mathcal{K}(g)) = \mathcal{K}(g \circ f) = (g \circ f) \circ f = g \circ (f \circ f) = g \circ f^{\circ 2}$, and thus, for all $k \geq 0$, $\mathcal{K}^k(g) = g \circ f^{\circ k}$. Hence, for all $k \geq 1$, the output (2) is given by

$$y(k) = \mathcal{K}^k(g)(x_0) = \mathcal{K}^{k-1}(g)(x_1). \tag{7}$$

The complexification of $\mathcal{G}$ is the vector space over $\mathbb{C}$ defined by $\mathcal{G}_{\mathbb{C}} \stackrel{\triangle}{=} \mathcal{G} + \jmath\mathcal{G}$. For all $g, h \in \mathcal{G}$, we define the *Koopman operator* $\mathcal{K}\colon \mathcal{G}_{\mathbb{C}} \to \mathcal{G}_{\mathbb{C}}$ by

$$\mathcal{K}(g + \jmath h) \stackrel{\triangle}{=} \mathcal{K}(g) + \jmath\mathcal{K}(h), \tag{8}$$

where, for convenience, $\mathcal{K}$ denotes the extension of $\mathcal{K}$ from $\mathcal{G}$ to $\mathcal{G}_{\mathbb{C}}$. Note that

$$\mathcal{K}(g + \jmath h) = g \circ f + \jmath h \circ f = (g + \jmath h) \circ f. \tag{9}$$

Since (6) holds for all $g_1, g_2 \in \mathcal{G}$ and all $a_1, a_2 \in \mathbb{C}$, it follows that $\mathcal{K}$ is a linear operator on $\mathcal{G}_{\mathbb{C}}$.

III. The Compositional Completeness of $\mathrm{L}_p(\mathcal{D}, \mathbb{R})$

In this section, we discuss properties of the inverse mapping f^{-1}. In order to simplify the discussion, we redefine the codomain of f to be $f(\mathcal{D}) \subseteq \mathcal{D}$. With this definition, $f\colon \mathcal{D} \to f(\mathcal{D})$ is surjective.

For all $p \in (1, \infty)$, let $\mathrm{L}_p(\mathcal{D}, \mathbb{R})$ denote the space of Lebesgue-measurable functions g such that $|g|^p$ is Lebesgue integrable on $\mathcal{D}$, and let $\mathrm{L}_\infty(\mathcal{D}, \mathbb{R})$ denote the space of Lebesgue-measurable functions g such that $|g|$ is essentially bounded on $\mathcal{D}$. We wish to view $\mathcal{K}$ as a function on $\mathrm{L}_p(\mathcal{D}, \mathbb{R})$. Unfortunately, $\mathcal{K}$ may fail to be well-defined in the sense that $g_1, g_2 \in \mathrm{L}_p(\mathcal{D}, \mathbb{R})$ may differ on only a set of measure zero but $\mathcal{K}(g_1)$ and $\mathcal{K}(g_2)$ may differ on a set of positive measure. This possibility is ruled out by the following definition.

Definition 3.1: f is *nonsingular* if, for all measurable $\mathcal{S} \subset \mathcal{D}$ with zero measure, $f^{-1}(\mathcal{S})$ has zero measure.

Theorem 3.2: $\mathcal{K}\colon \mathrm{L}_\infty(\mathcal{D}, \mathbb{R}) \to \mathrm{L}_\infty(\mathcal{D}, \mathbb{R})$ and is continuous if and only if f is nonsingular.

For $p \in [1, \infty)$, the analogous question for $\mathrm{L}_p(\mathcal{D}, \mathbb{R})$ requires a stronger condition than nonsingularity. Let μ denote the Lebesgue measure.

Definition 3.3: f is *strongly nonsingular* if there exists $b > 0$ such that, for all measurable $\mathcal{S} \subseteq \mathcal{D}$,

$$\mu(f^{-1}(\mathcal{S})) < b\mu(\mathcal{S}). \tag{10}$$

The following result follows from Theorem 2.1.1 in [17].

Theorem 3.4: The following statements are equivalent:

- $i)$ There exists $p \in [1, \infty)$ such that $\mathcal{K}\colon \mathrm{L}_p(\mathcal{D}, \mathbb{R}) \to \mathrm{L}_p(\mathcal{D}, \mathbb{R})$ and is continuous.
- $ii)$ For all $p \in [1, \infty)$, $\mathcal{K}\colon \mathrm{L}_p(\mathcal{D}, \mathbb{R}) \to \mathrm{L}_p(\mathcal{D}, \mathbb{R})$ and is continuous.
- $iii)$ f is strongly nonsingular.

If f is strongly nonsingular, then f is nonsingular. However, the converse is not true. For example, let $\mathcal{D} = \mathbb{R}$ and $f(x) = e^x$. Then f is nonsingular, but not strongly nonsingular, since there does not exist a $b > 0$ such that, for all $0 < \alpha < \beta < 1$, $\mu(f^{-1}(\alpha, \beta)) = \log\beta - \log\alpha \leq b(\beta - \alpha) = b\mu((\alpha, \beta))$.

Definition 3.5: Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and let $\phi\colon \mathcal{D} \to \mathcal{D}$. Then ϕ is *Lipschitz* with constant $L > 0$ if, for all $x, y \in \mathcal{D}$,

$$\|\phi(x) - \phi(y)\| \leq L\|x - y\|. \tag{11}$$

Furthermore, $\phi\colon \mathcal{D} \to \mathcal{D}$ is *anti-Lipschitz* with constant $L > 0$ if, for all $x, y \in \mathcal{D}$,

$$\|x - y\| \leq L\|\phi(x) - \phi(y)\|. \tag{12}$$

Theorem 3.6: Assume that f is anti-Lipschitz with constant L. Then f is injective and its inverse $h\colon f(\mathcal{D}) \to \mathcal{D}$ is Lipschitz with constant L. If, in addition, f is measurable, then f is strongly nonsingular.

Proof: The injectivity of f follows immediately from (12). Hence, f is invertible with inverse $h\colon f(\mathcal{D}) \to \mathcal{D}$. Next, let

$z, w \in f(\mathcal{D})$, and let $a, b \in \mathcal{D}$ satisfy $z = f(a)$ and $w = f(b)$. Since f is anti-Lipschitz with constant L, it follows that $\|h(z) - h(w)\| = \|a - b\| \le L\|f(a) - f(b)\| = L\|z - w\|$, and thus h is Lipschitz with constant L.

Next, assume that f is measurable, and let $\mathcal{S} \subseteq \mathcal{D}$. Equation (10) is immediate in the case where $\mu(\mathcal{S}) = \infty$. Hence, consider the case where $\mu(\mathcal{S}) < \infty$. For all $y \triangleq \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^{\mathrm{T}} \in \mathcal{D}$ and $r > 0$, define

$$Q(y, r) \triangleq [y_1, y_1 + r] \times \cdots \times [y_n, y_n + r],$$

and note that $\mu(Q(r, y)) = r^n$. Furthermore, define

$$\mathbb{Q} \triangleq \{Q(y, r) : y \in \mathcal{D} \text{ and } r > 0\}$$

and

$$\mathbb{Q}_{\mathcal{S}} \triangleq \left\{ \{Q_i\}_{i=1}^{\infty} : \text{for all } i \ge 1, Q_i \in \mathbb{Q},\ \mathcal{S} \subseteq \bigcup_{i=1}^{\infty} Q_i \right\}.$$

Note that

$$\mu(\mathcal{S}) = \inf \left\{ \sum_{i=1}^{\infty} \mu(Q_i) : \{Q_i\}_{i=1}^{\infty} \in \mathbb{Q}_{\mathcal{S}} \right\}. \tag{13}$$

Let $\varepsilon > 0$. From (13), it follows that, for all $b \in (0, \infty)$, there exists $\{\bar{Q}_i\}_{i=1}^{\infty} \in \mathbb{Q}_{\mathcal{S}}$ such that

$$\sum_{i=1}^{\infty} \mu(\bar{Q}_i) < \mu(\mathcal{S}) + \frac{\varepsilon}{b}. \tag{14}$$

Next, note that since f is injective on $\mathcal{D}$, it follows that, for all $y \in \mathcal{D}$,

$$f^{-1}(y) = \begin{cases} \{h(y)\}, & y \in f(\mathcal{D}), \\ \varnothing, & y \in \mathcal{D} - f(\mathcal{D}). \end{cases} \tag{15}$$

It now follows from (15) that, for all $i \ge 1$, $f^{-1}(\bar{Q}_i \cap \mathcal{D}) = f^{-1}(\bar{Q}_i \cap f(\mathcal{D}))$. Furthermore, since $\mathcal{S} \subseteq \mathcal{D}$ and $\mathcal{S} \subseteq \cup_{i=1}^{\infty} \bar{Q}_i$, it follows that $\mathcal{S} \subseteq (\cup_{i=1}^{\infty} \bar{Q}_i) \cap \mathcal{D} = \cup_{i=1}^{\infty} (\bar{Q}_i \cap \mathcal{D})$, and therefore

$$f^{-1}(\mathcal{S}) \subseteq f^{-1} \left(\bigcup_{i=1}^{\infty} (\bar{Q}_i \cap \mathcal{D}) \right) \subseteq \bigcup_{i=1}^{\infty} f^{-1}(\bar{Q}_i \cap f(\mathcal{D})). \tag{16}$$

Since $\mathcal{S}$ is measurable and f is measurable, it follows that $f^{-1}(\mathcal{S})$ is measurable. In addition, for all $i \ge 1$, since $\bar{Q}_i \cap \mathcal{D}$ is measurable, it follows that $f^{-1}(\bar{Q}_i \cap f(\mathcal{D}))$ is measurable. Thus, from (16),

$$\mu(f^{-1}(\mathcal{S})) \le \sum_{i=1}^{\infty} \mu(f^{-1}(\bar{Q}_i \cap f(\mathcal{D}))). \tag{17}$$

Next, let $\bar{r}_i$ be the side length associated with $\bar{Q}_i$ and let $\|\cdot\|_\infty$ be the infinity norm on $\mathbb{R}^n$. Since $h = f^{-1}$ is Lipschitz on $\mathcal{D}$, it follows that there exists a Lipschitz constant $L > 0$ and a positive number α such that, for all $x, y \in \bar{Q}_i \cap f(\mathcal{D})$,

$$\begin{aligned} \|h(x) - h(y)\| \le L\|x - y\| &\le L\alpha \|x - y\|_\infty \\ &\le L\alpha\sqrt{n}\bar{r}_i. \end{aligned} \tag{18}$$

Next, let $z = \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}^{\mathrm{T}} \in \bar{Q}_i \cap f(\mathcal{D})$ and let $\mathcal{B}_\infty$ denote the ball of the inifinity norm. It follows that

$$h(\bar{Q}_i \cap f(\mathcal{D})) \subseteq \mathcal{B}_\infty(z, L\alpha\sqrt{n}r_i), \tag{19}$$

hence

$$\mu(h(\bar{Q}_i \cap f(\mathcal{D}))) \le (2L\alpha\sqrt{n})^n \mu(\bar{Q}_i). \tag{20}$$

Finally, let $b \triangleq (2\alpha L\sqrt{n})^n$. From (14), (17), and (20), it follows that

$$\begin{aligned} \mu(f^{-1}(\mathcal{S})) &\le b \sum_{i=1}^{\infty} \mu(h(\bar{Q}_i \cap f(\mathcal{D}))) \\ &\le b \sum_{i=1}^{\infty} \mu(\bar{Q}_i) < b\mu(\mathcal{S}) + \varepsilon. \end{aligned} \tag{21}$$

Since ε is arbitrary, it follows from (21) that f is strongly nonsingular. □

IV. Illustrative Examples

This section provides some basic analytic examples which illustrate the Koopman spectrum in the discrete-time setting.

Example 1: Constant g. Let $\mathcal{D} \subseteq \mathbb{R}^n$, let $f\colon \mathcal{D} \to \mathbb{R}^n$, and let $\mathcal{G}$ be compositionally complete. If $g \in \mathcal{G}$ is a constant real-valued function on $\mathcal{D}$, then g is a Koopman eigenfunction with eigenvalue 1. ◇

Example 2: Linear f and g. Let $A \in \mathbb{R}^{n \times n}$ and, for all $x \in \mathbb{R}^n$, define $f(x) = Ax$. Furthermore, let $\mathcal{G} = \{g\colon \mathbb{R}^n \to \mathbb{R}\colon g(x) = cx, \text{ where } c \in \mathbb{R}^{1\times n}\}$. Now, let $c \in \mathbb{C}^{1 \times n}$ and, for all $x \in \mathbb{R}^n$, define $g(x) = cx$. Note that, for all $x \in \mathbb{R}^n$, $\mathcal{K}(g)(x) = (g \circ f)(x) = g(f(x)) = c(Ax) = (cA)(x)$. Since $cA \in \mathbb{R}^{1 \times n}$, it follows that $\mathcal{G}$ is compositionally complete.

Now, let $c \in \mathbb{C}^{1 \times n}$, define $g(x) = cx$, and let $\lambda \in \mathbb{C}$. Then, $cA = \lambda c$ if and only if $g \circ f = \lambda g$. Hence, g is a Koopman eigenfunction with eigenvalue λ if and only if c^{T} is an eigenvector of A^{T} with associated eigenvalue λ. ◇

Example 3: Linear f and polynomial g. Now, let $\mathcal{D} = \mathbb{R}^2$, let A be a 2×2 diagonalizable matrix with eigenvalues λ_1, λ_2, and let $f(x) = Ax$. Define $\mathcal{G}$ to be the space of real-valued polynomials in two variables. Let v_1 and v_2 be eigenvectors of A^{T} corresponding to λ_1 and λ_2, respectively, and, for all $i, j \ge 0$ and $x \in \mathbb{R}^2$, define the polynomial $g_{(i,j)}(x) = (v_1^{\mathrm{T}}x)^i (v_2^{\mathrm{T}}x)^j$. Then, for all $i, j \ge 0$ and $x \in \mathbb{R}^2$,

$$\begin{aligned} g_{(i,j)}(Ax) = (v_1^{\mathrm{T}}Ax)^i (v_2^{\mathrm{T}}Ax)^j &= ((A^{\mathrm{T}}v_1)^{\mathrm{T}}x)^i ((A^{\mathrm{T}}v_2)^{\mathrm{T}}x)^j \\ &= \lambda_1^i \lambda_2^j (v_1^{\mathrm{T}}x)^i (v_2^{\mathrm{T}}x)^j. \end{aligned}$$

Hence, $g_{(i,j)}$ is a Koopman eigenfunction with corresponding eigenvalue $\lambda_{(i,j)} = \lambda_1^i \lambda_2^j$. An analogous result holds for the case where A is a real $n \times n$ diagonalizable matrix. ◇

Example 4: Polynomial f with logarithmic g. Let $\mathcal{D} = (0, \infty)$, let p be a positive number, and, for all $x \in \mathcal{D}$, define $f\colon \mathcal{D} \to \mathcal{D}$ by $f(x) = x^p$. Let $\mathcal{G}$ be the vector space of continuous real-valued functions on $\mathcal{D}$. Next, let ℓ be a positive number, and, for all $x \in \mathcal{D}$, define $g_\ell(x) = (\log x)^\ell$. Since

$$(g_\ell \circ f)(x) = (\log x^p)^\ell = p^\ell (\log x)^\ell = p^\ell g_\ell(x), \tag{22}$$

it follows that g_ℓ is a Koopman eigenfunction with eigenvalue $\lambda_\ell = p^\ell$. ◇

Example 5: Affine f with exponential g. Let $\mathcal{G}$ be the L_2 space of real-valued functions on $\mathbb{R}^n$ with the inner product

$$\langle g_1, g_2 \rangle = \int_{\mathbb{R}^n} g_1(x) g_2(x) e^{-x^{\mathrm{T}} x}\, \mathrm{d}x. \tag{23}$$

Let $a \in \mathbb{R}^n$ and, for all $x \in \mathbb{R}^n$, define $f(x) = x + a$. For all $c \in \mathbb{R}^n$, define $g_c(x) = e^{c^{\mathrm{T}} x}$. Then,

$$\|g_c\| = \left(\int_{\mathbb{R}^n} e^{2c^{\mathrm{T}} x} e^{-x^{\mathrm{T}} x}\, \mathrm{d}x \right)^{1/2} = \sqrt{\pi^{n/2} e^{c^{\mathrm{T}} c}} \tag{24}$$

is finite, and thus $g_c \in \mathrm{L}_2(\mathbb{R}^n, \mathbb{R})$. Moreover, since

$$(g_c \circ f)(x) = e^{c^{\mathrm{T}}(x+a)} = e^{c^{\mathrm{T}} a} e^{c^{\mathrm{T}} x} = e^{c^{\mathrm{T}} a} g_c(x), \tag{25}$$

it follows that g_c is a Koopman eigenfunction with eigenvalue $\lambda_c = e^{c^{\mathrm{T}} a}$. This example shows that the set of Koopman eigenfunctions may be uncountable, whether or not the space of observables has a countable basis. ◇

V. Koopman Eigenfunctions and Stability

Theorem 5.1: Define

$$\mathcal{D}_0 \triangleq \{x \in \mathcal{D} \colon (f^{\circ k}(x))_{k=0}^{\infty} \text{ is bounded}\}. \tag{26}$$

Let λ be an eigenvalue of $\mathcal{K}$ with corresponding eigenfunction g, and assume that $|\lambda| > 1$ and g is bounded on every bounded subset of $\mathcal{D}$. Then, for all $x \in \mathcal{D}_0$, $g(x) = 0$.

Proof. Let $x \in \mathcal{D}_0$ and define $\mathcal{O}_x = \{f^{\circ 0}(x), f^{\circ 1}(x), \ldots\}$. Since $\mathcal{O}_x$ is bounded, it follows that g is bounded on $\mathcal{O}_x$. Let $\beta \triangleq \sup\{g(\xi) \colon \xi \in \mathcal{O}_x\}$. Since g is a Koopman eigenfunction, it follows that, for all $x \in \mathcal{D}$ and all $k \geq 0$,

$$|g(f^{\circ k}(x))| = |\lambda|^k |g(x)|,$$

and thus $|g(x)| \leq \beta / |\lambda|^k$. Since $|\lambda| > 1$, it follows that $g(x) = 0$. □

VI. Approximate Computation of the Spectrum

In this section, we give an algorithm for approximating the Koopman spectrum in the discrete time case. Unlike DMD or EDMD, no simulation data is required to compute the approximation.

Let $\mathcal{G}$ be a compositionally complete Hilbert space and let $(g_\ell)_{\ell=1}^{\infty}$ be an orthonormal basis of $\mathcal{G}$. Furthermore, for each positive integer p, define the finite-dimensional subspace $\mathcal{G}_p \triangleq \mathrm{span}\{g_1, \ldots, g_p\}$ of $\mathcal{G}$. Since $\mathcal{K}$ is linear, the projection of $\mathcal{K}$ onto $\mathcal{G}_p$ can be represented by the $p \times p$ matrix

$$\mathcal{K}_p \triangleq \begin{bmatrix} \langle \mathcal{K} g_1, g_1 \rangle & \langle \mathcal{K} g_2, g_1 \rangle & \ldots & \langle \mathcal{K} g_p, g_1 \rangle \\ \langle \mathcal{K} g_1, g_2 \rangle & \langle \mathcal{K} g_2, g_2 \rangle & \ldots & \langle \mathcal{K} g_p, g_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathcal{K} g_1, g_p \rangle & \langle \mathcal{K} g_2, g_p \rangle & \ldots & \langle \mathcal{K} g_p, g_p \rangle \end{bmatrix}, \tag{27}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{G}$. In the case where, $\mathcal{K}$ is compact, it follows that, in the operator topology, $\lim_{p\to\infty} \mathcal{K}_p = \mathcal{K}$, and, in fact, that the spectrum of $\mathcal{K}_p$ converges to the spectrum of $\mathcal{K}$ as $p \to \infty$ [18, Section 5.4]. We can thus approximate the eigenvalues and eigenvectors of $\mathcal{K}$ by computing the eigenvalues and eigenvectors of $\mathcal{K}_p$. In particular, the eigenfunctions of $\mathcal{K}$, which are projected onto $\mathcal{G}_p$, can be approximated by the eigenvectors of $\mathcal{K}_p$.

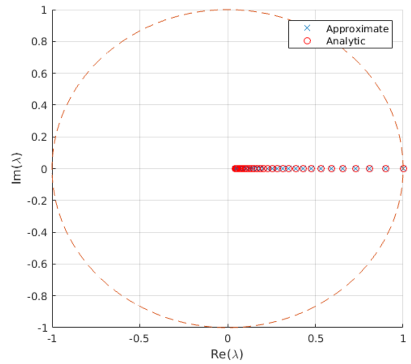

Fig. 1. Example 5: Comparison of the eigenvalues of $\mathcal{K}$ and $\mathcal{K}_p$ for $f(x) = 0.9x$, where x is scalar. The eigenvalues $\lambda_k = 0.9^k$ (red ∘) of $\mathcal{K}$ are plotted along with the eigenvalues (blue ×) for $k = 0$ to $k = 30$. The numerical agreement between the eigenvalues of $\mathcal{K}$ and $\mathcal{K}_p$ is within 10^{-6}, which suggests that the spectrum of $\mathcal{K}_p$ provides a good approximation of the spectrum of $\mathcal{K}$.

VII. Numerical Examples

Example 6. Scalar linear dynamics. Let $n = 1$, $a \in \mathbb{R}$, $f(x) = ax$, and $\mathcal{G} = \mathrm{L}_2([-1,1], \mathbb{R})$ with inner-product weighting $w(x) \triangleq \sqrt{1 - x^2}$. It follows from Example 3 that, for all $k \geq 0$, the Koopman eigenfunctions are $g_k(x) = x^k$ with eigenvalues $\lambda_k = a^k$. For $a = 0.9$, Figure 1 confirms the approximation method presented in Section VI. ◇

Example 7. Scalar cubic dynamics. Let $n = 1$ and let $f(x) = x - 0.1x^3$. Let $\mathcal{G}$ be as in Example 6. Figure 2 shows the spectrum of $\mathcal{K}_p$ obtained using 400 Chebyshev polynomials of the second kind. The middle plot shows the corresponding eigenfunctions. Let $\tilde{g}$ be an eigenfunction of $\mathcal{K}_p$ with corresponding eigenvalue $\tilde{\lambda}$, and define the error function by $\varepsilon(x) = (\tilde{g} \circ f)(x) - \tilde{\lambda} \tilde{g}(x)$. The lowest plot of Figure 2 shows this error function for each eigenvalue-eigenfunction pair over $\mathcal{D}$. Note that, for each eigenvector-eigenvalue pair, the largest error attained over $\mathcal{D}$ is on the order of 10^{-6}, which suggests good convergence for this number of basis functions. ◇

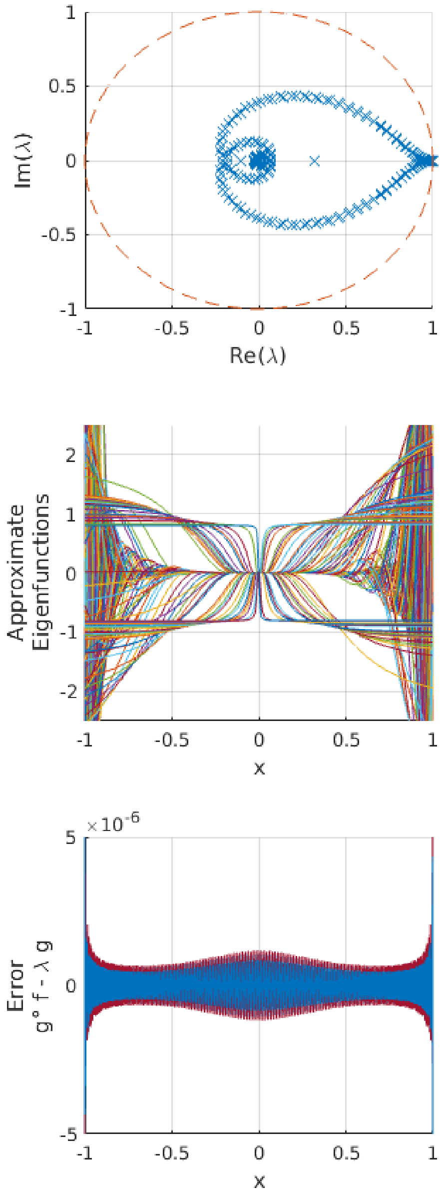

Fig. 2. Example 7: Approximate Koopman spectrum (topmost) and eigenfunctions (middle) using a basis of 400 Chebyshev polynomials of the second kind. The numerical error (lowest) is on the order of 10^{-6}, indicating good convergence for this number of basis functions.

Example 8. Two-dimensional linear dynamics. Let $n = 2$ and

$$f(x) = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.9 \end{bmatrix} x. \tag{28}$$

It follows from Example 3 that, for all $k \geq 0$ and $\ell \geq 0$, the Koopman eigenfunctions are $g_{(k,\ell)}(x_1, x_2) = x_1^k x_2^\ell$ with corresponding eigenvalues $\lambda_{(k,\ell)} = (1.1)^k (0.9)^\ell$. Figure 3 confirms the approximation method presented in Section VI.

If $\ell/k > (-\ln 0.9)/(\ln 1.1) = 1.1054$, then $\lambda_{(k,\ell)} > 1$. Hence, Theorem 5.1 implies $g_{(k,\ell)}(x_1, x_2) = 0$ on the set $\mathcal{D}_0$ defined by (26). For (28), $\mathcal{D}_0 = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 = 0\}$. It

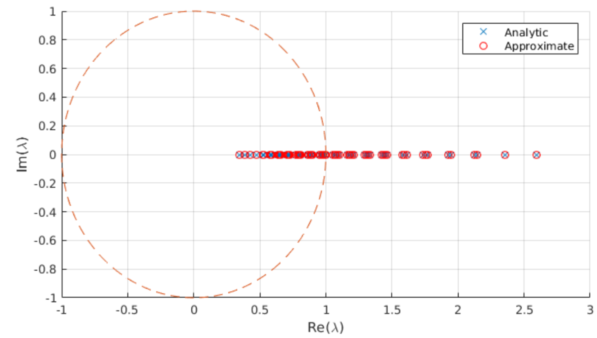

Fig. 3. Example 8: validation of $n = 2$ numerical approximation method for Equation (28). Analytic eigenvalues $\lambda_{(k,\ell)} = (0.9)^k (1.1)^\ell$ (red o's) are plotted along with numerically approximated eigenvalues (blue x's) for the index set $\{(k, \ell) \colon 0 \leq k \leq 10,\ 0 \leq \ell \leq 10,\ k + \ell \leq 10\}$. The numerical agreement between analytic and approximate eigenvalues is within 10^{-6}.

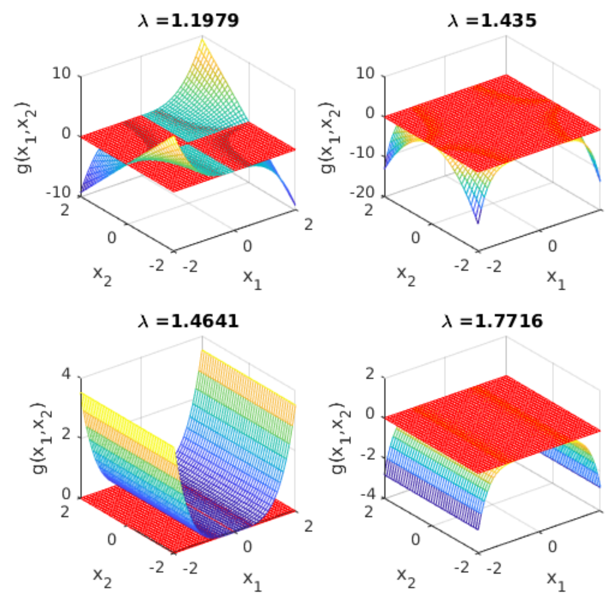

Fig. 4. Example 8: Eigenfunctions corresponding to unstable eigenvalues (gradient) along with the function $h(x_1, x_2) = 0$ (red). Each case shows that the corresponding eigenfunction is zero for $x_1 = 0$.

is easy to see the analytic solution satisfies this property. We may also use this fact to check the approximation. Figure 4 shows several eigenfunctions, each corresponding to an unstable eigenvalue, all of which are zero for $x_2 = 0$. ◇

Example 9. Lotka-Volterra dynamics. Let $n = 2$ and

$$f_1(x_1, x_2) \triangleq r x_1 - a x_1 x_2, \tag{29}$$

$$f_2(x_1, x_2) \triangleq -R x_2 + b x_1 x_2. \tag{30}$$

Equations (29) and (30) comprise the classical Lotka-Volterra predator-prey model, although the parameters we choose do not necessarily model a realistic ecosystem. Let $r = R = 0.9$ and $a = b = 10^{-3}$. Figure 5 shows the approximate Koopman spectrum obtained using a basis of 66 Chebyshev polynomials of the second kind. Approximate eigenfunctions corresponding to four eigenvalues are shown in Figure 6. The fact that all of these eigenvalues lie in the open unit disk

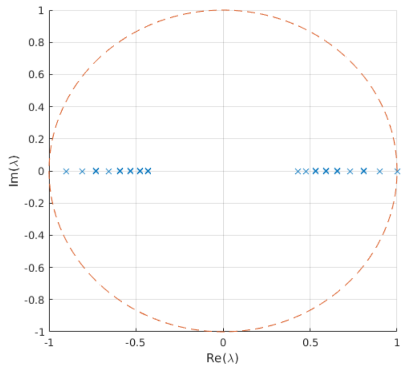

Fig. 5. Example 9: Approximate Koopman spectrum using a basis of 66 Chebyshev polynomials of the second kind in two variables corresponding to the index set $\{(k,\ell)\colon 0 \leq k \leq 10,\ 0 \leq \ell \leq 10,\ k+\ell \leq 10\}$.

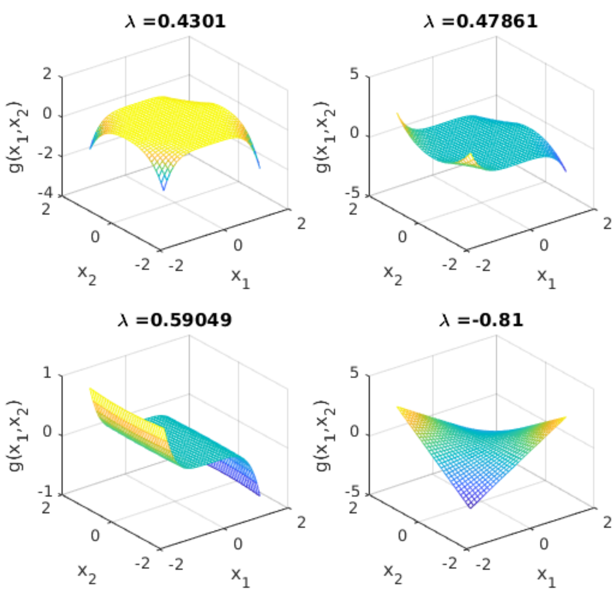

Fig. 6. Example 9: Approximate Koopman eigenfunctions corresponding to four eigenvalues.

suggests that this system is asymptotically stable. ◇

VIII. Conclusions and Future Research

The goal of this paper was to concisely present the essential properties of the Koopman operator in the discrete time setting, and introduce the novel concept of compositional completeness. We also provided a set of examples which illustrate the Koopmen spectrum for different mappings and spaces of observables, proved a new result on the stability of Koopman eigenfunctions, and provided a simplified method for approximating the Koopman spectrum. Finally, numerical examples of the approximation technique were given and shown to agree with analytical results for the cases in which analytical results were available.

This framework for the Koopman operator raises fundamental questions for future research. The highest priority is to determine the conditions under which other spaces of Koopman observables are compositionally complete. Another research goal is to provide error bounds on the accuracy of the approximate eigenvalues and eigenvectors. Beyond these questions, it is of special interest to extend the Koopman operator to systems with exogenous inputs, with potential application to feedback control.

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References

[1] H. K. Khalil, *Nonlinear Systems*. Pearson, 2013.

[2] B. O. Koopman, "Hamiltonian system and transformations in hilbert space," *Proc. N. A. S.*, vol. 17, pp. 315–318, 1931.

[3] R. M. Marko Budisić and I. Mezić, "Applied koopmanism," *Chaos*, vol. 22, no. 047510, 2012.

[4] H. Arbabi and I. Mezić, "Ergodic theory, dynamic mode decomposition, and computation of spectral properties of the koopman operator," *SIAM J. App. Dynamics*, vol. 16, no. 4, pp. 2096–2126, 2009.

[5] I. Mezić, "Analysis of fluid flows via spectral properties of the koopman operator," *Annual Review of Fluid Mechanics*, vol. 45, pp. 357–378, 2012.

[6] C. W. Rowley, I. Mezić, S. Bagheri, P. Schlatter, and D. S. Henningson, "Spectral analysis of nonlinear flows," *J. Fluid Mech.*, vol. 641, pp. 115–127, 2009.

[7] M. Korda and I. Mezić, "Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control," *Automatica*, 2018.

[8] I. Abraham, G. D. L. Torre, and T. D. Murphey, "Model-based control using koopman operators," *CoRR*, vol. abs/1709.01568, 2017.

[9] A. Surana, M. O. Williams, M. Morari, and A. Banaszuk, "Koopman operator framework for constrained state estimation," in *Proc. Conf. Dec. Contr.*, Dec 2017, pp. 94–101.

[10] M. Benosman, H. Mansour, and V. Huroyan, "Koopman-operator observer-based estimation of pedestrian crowd flows," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 14 028 – 14 033, 2017, 20th IFAC World Congress. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S2405896317332949

[11] N. Takeishi, Y. Kawahara, and T. Yairi, "Subspace dynamic mode decomposition for stochastic koopman analysis," *Phys. Rev. E*, vol. 96.3, pp. 033 310–1–11, 2017.

[12] I. M. Nelida Črnjarić Žic, Senka Maćešić, "Koopman operator spectrum for random dynamical systems," *arXiv:1711.03146*, 2017.

[13] M. Korda, M. Putinar, and I. Mezić, "Data-driven scpectral analysis of the koopman operator," *arXiv:1710.06532*, 2017.

[14] M. O. Williams, I. G. Kevrekidis, and y. . . v. . p. . .-. j. . J. Clarence W. Rowley, title = A data-driven approximation of the koopman operator: Extending dynamic mode decomposition.

[15] M. Korda and I. Mezić, "On the convergence of extended dynamic mode decomposition to the koopman operator," *J. Nonlinear Sci.*, vol. 28.2, pp. 689–710, 2018.

[16] A. Mauroy and I. Mezić, "Global stability analysis using the eigenfunctions of the koopman operator," *IEEE Trans. Autom. Contr.*, vol. 61, pp. 3356–3369, 2016.

[17] R. K. Singh, *Composition Operators on Function Spaces*. North-Holland, 1993.

[18] F. Chatelin, *Spectral Approximation of Linear Operators*. SIAM, 2011, reprint of the 1983 edition.