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Finite-Delay Input Reconstruction for Left-Invertible Discrete-Time Systems with Zero Nonzero Zeros

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***Abstract*— This paper considers the problem of finite-delay input reconstruction for discrete-time systems whose transfer functions have full column normal rank in the case where the initial condition is unknown. The main result states that, if the transfer function has a finite-impulse-response (FIR) delayed left inverse, then input reconstruction is possible. It is also shown that an FIR delayed left inverse with minimum possible delay exists if and only if the system has zero nonzero zeros. These results provide conditions under which finite-delay input reconstruction with unknown initial conditions is possible for discrete-time linear time-invariant systems.**

I. INTRODUCTION

In many applications, it is useful to be able to determine the input to a system based on knowledge of its output. An estimate of the input can be used, for example, to assess the health of actuators, determine the features of disturbances, or for tracking and navigation [1]–[3]. For a static map, determining the input based on knowledge of the output is equivalent to inverting the map. If the map is one-to-one, then the input is unique; otherwise, there may be two or more possible inputs.

For dynamical systems, there has been longstanding interest in the existence of inverses. This problem has been studied within the context of linear systems [4]–[7] as well as nonlinear systems [8]–[10]. The problem of determining the input to a dynamical system is often considered in conjunction with state estimation. Consequently, the problem of determining the input is known as either input estimation or input reconstruction depending on whether the setting is stochastic or deterministic, respectively. The literature on input estimation and input reconstruction has attracted increasing attention over the last few years [11]–[14].

A fundamental problem in input estimation is the effect of the initial condition. In the simpler case, the initial condition is assumed to be known [5], [6]. However, in practice, it is more realistic to assume that the initial condition is unknown, in which case the response of the system depends on both the input and initial condition, making it more difficult to estimate the input [11], [14].

Another difficulty in input reconstruction is the presence of zeros in the transfer function from the input to the response. Any such zero opens up the possibility that, for a suitable initial condition, there exists a nonzero input such that the response is identically zero. When this case does not occur, the system is called *input observable* [15, p. 162]. In the absence of input observability, it is impossible to unambiguously determine the input, although input reconstruction may be possible asymptotically. Note that input reconstruction may be possible in the presence of zeros in the case where the initial condition is known.

The goal of the present paper is to determine the class of inverses that achieve input reconstruction in the case where the initial condition is unknown. The main result, which is given by Theorem 3.1, states that input reconstruction is possible in the case where the transfer function has a finite-impulse-response delayed left inverse. Proposition 4.2 shows that an FIR delayed left inverse with minimum possible delay exists if and only if the system has zero nonzero zeros. These results thus provide definitive conditions under which input reconstruction is possible for discrete-time linear time-invariant systems. In the case where the system has zero nonzero zeros, a technique for constructing an FIR delayed left inverse is given in Section VI based on the Smith-McMillan form. However, this FIR delayed left inverse does not necessarily provide the minimal possible delay.

II. INPUT RECONSTRUCTION

Definition 2.1: Let $G \in \mathbb{R}(\mathbf{z})_{\rm prop}^{m\times p}$, and let d be a nonnegative integer. Then, G is *delayed left invertible with delay* d if there exists $H \in \mathbb{R}(\mathbf{z})_{\rm prop}^{m\times p}$ such that $H(\mathbf{z})G(\mathbf{z}) = \mathbf{z}^{-d}I_m$. H is a *delayed left inverse of* G *with delay* d. Furthermore, G is *delayed left invertible* if there exists $d \geq 0$ such that G is delayed left invertible with delay d, and H is a *delayed left inverse of* G if there exists $d \geq 0$ such that H is a delayed left inverse of G with delay d.

If H is a delayed left inverse of G with delay d, then the output of HG is equal to the d-step-delayed input of HG. However, HG does not account for the free response of the state space model formed by cascading state space models of G and H. The missing free response can be accounted for by specifying initial conditions of the realizations of G and H. Let

$$G \overset{\min}{\sim} \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right], \quad H \overset{\min}{\sim} \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right],$$

and, for all $k \geq 0$, consider the state space equations

$$x_G(k+1) = A_G x_G(k) + B_G u(k), \quad (1)$$
$$y(k) = C_G x_G(k) + D_G u(k), \quad (2)$$

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and

$$x_H(k+1) = A_H x_H(k) + B_H y(k), \tag{3}$$
$$z(k) = C_H x_H(k) + D_H y(k). \tag{4}$$

Then, the state space realization of the cascade HG is given by

$$x(k+1) = Ax(k) + Bu(k), \tag{5}$$
$$z(k) = Cx(k) + Du(k), \tag{6}$$

where

$$x \triangleq \begin{bmatrix} x_G \\ x_H \end{bmatrix}, \; A \triangleq \begin{bmatrix} A_G & 0 \\ B_H C_G & A_H \end{bmatrix}, \quad B \triangleq \begin{bmatrix} B_G \\ B_H D_G \end{bmatrix},$$
$$C \triangleq \begin{bmatrix} D_H C_G & C_H \end{bmatrix}, \quad D \triangleq D_H D_G.$$

Note that the realization (5), (6) of HG is not necessarily minimal. Alternatively, replacing $\mathbf{z}$ by the forward shift operator $\mathbf{q}$ yields the time-domain transfer function representation [16]

$$z(k) = H(\mathbf{q})G(\mathbf{q})u(k),$$

which also accounts for nonzero initial conditions.

Example 2.1: Let

$$G(\mathbf{z}) = \begin{bmatrix} \dfrac{1}{\mathbf{z}^2} \\ \dfrac{1}{\mathbf{z}+1} \end{bmatrix}, \quad H(\mathbf{z}) = \begin{bmatrix} \dfrac{\mathbf{z}}{\mathbf{z}+1} & \dfrac{1}{\mathbf{z}^2} \end{bmatrix},$$

so that $H(\mathbf{z})G(\mathbf{z}) = \mathbf{z}^{-2}$ and thus H is a delayed left inverse of G with delay 2. Figure 1 shows the input and output of a state space realization of HG with zero initial conditions and with nonzero initial conditions. Note that H fails to reconstruct the input in the case where the initial conditions are nonzero.

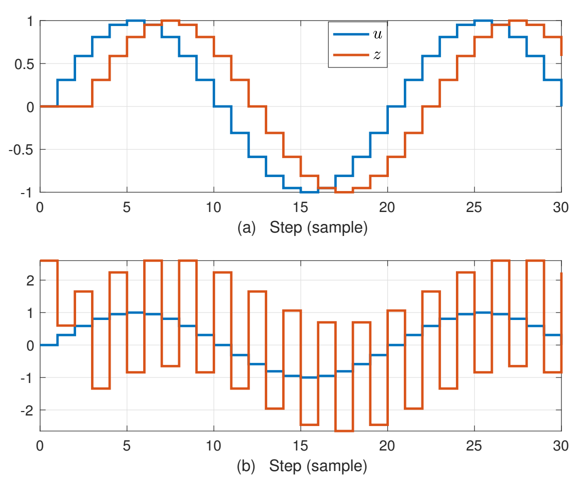

Fig. 1. (a) shows the input and output of a state space realization of HG with zero initial conditions. (b) shows the input and output of a state space realization of HG with nonzero initial conditions.

Example 2.2: Let G be as in Example 2.1, and let $H(\mathbf{z}) = \mathbf{z}^{-1}(G(\mathbf{z})^{\mathrm{T}}G(\mathbf{z}))^{-1}G(\mathbf{z})^{\mathrm{T}}$, that is,

$$H(\mathbf{z}) = \begin{bmatrix} \dfrac{\mathbf{z}^3 + 2\mathbf{z}^2 + \mathbf{z}}{\mathbf{z}^4 + \mathbf{z}^2 + 2\mathbf{z} + 1} & \dfrac{\mathbf{z}^4 + \mathbf{z}^3}{\mathbf{z}^4 + \mathbf{z}^2 + 2\mathbf{z} + 1} \end{bmatrix}.$$

Hence, H is a delayed left inverse of G with delay 1. Figure 2 shows the input and output of a state space realization of HG with zero and nonzero initial conditions. Note that H fails to reconstruct the input in the case where the initial conditions are nonzero.

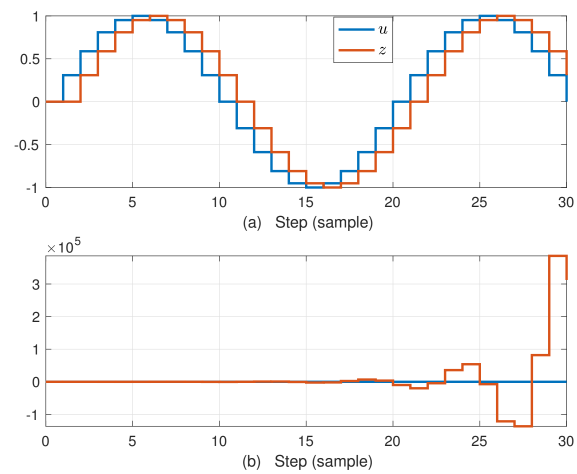

Fig. 2. (a) shows the input and output of a state space realization of HG with zero initial conditions. (b) shows the input and output of a state space realization of HG with nonzero initial conditions. The reconstructed input is unbounded since H is unstable.

Example 2.3: Let G be as in Example 2.1, and let

$$H(\mathbf{z}) = \begin{bmatrix} 0 & \dfrac{\mathbf{z}+1}{\mathbf{z}} \end{bmatrix},$$

so that $H(\mathbf{z})G(\mathbf{z}) = \mathbf{z}^{-1}$, and thus H is a delayed left inverse of G with delay 1. Figure 3 shows the input and output of a state space realization of HG with zero and nonzero initial conditions. Note that H correctly reconstructs the input in the case where the initial conditions are nonzero.

Note that the left inverses considered in Example 2.1 and Example 2.2 are infinite impulse response (IIR) transfer functions, whereas the left inverse in Example 2.3 is an FIR transfer function. This distinction suggests that FIR left inverses provide correct input reconstruction in the presence of nonzero initial conditions. Theorem 3.1 in the next section shows that this is indeed the case.

III. FIR LEFT INVERSE WITH DELAY d

Definition 3.1 ([17]): Let $A \in \mathbb{R}^{n \times n}$. Then, the *index* of A, denoted by $\operatorname{ind} A$, is the smallest nonnegative integer ν such that rank A^ν = rank $A^{\nu+1}$.

Note that, if A is nilpotent, then $\operatorname{ind} A$ is the smallest positive integer ν such that $A^\nu = 0$.

Definition 3.2: Let $G \in \mathbb{R}(\mathbf{z})^{p \times m}_{\text{prop}}$, where $G \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $A \in \mathbb{R}^{n \times n}$. Then, the *index* of G, denoted by $\operatorname{ind} G$, is $\operatorname{ind} A$.

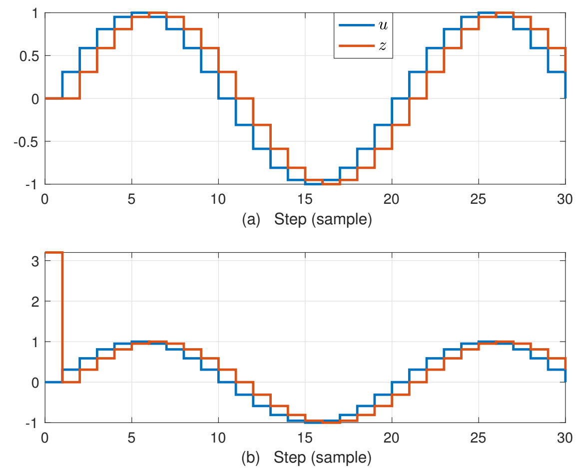

Fig. 3. (a) shows the input and output of a state space realization of HG with zero initial conditions. (b) shows the input and output of a state space realization of HG with nonzero initial conditions.

The following result shows that an FIR delayed left inverse of G provides input reconstruction after a finite number of steps despite an arbitrary unknown initial condition in a state space realization of G.

Theorem 3.1: Let $G \in \mathbb{R}(\mathbf{z})_{\rm prop}^{p\times m}$ and $H \in \mathbb{R}(\mathbf{z})_{\rm prop}^{m\times p}$, with minimal state space realizations given by (1)-(4). Assume that H is FIR and that H is a delayed left inverse of G with delay d. Define $K(\mathbf{z}) \triangleq \big(C_H(\mathbf{z}I - A_H)^{-1}B_H + D_H\big)C_G(\mathbf{z}I - A_G)^{-1}$. Then, for all $k \geq \max\{\operatorname{ind} H, \operatorname{ind} K, d\}$, $z(k) = u(k-d)$. If, in addition, $x_H(0) = 0$, then, for all $k \geq \max\{\operatorname{ind} K, d\}$, $z(k) = u(k-d)$.

Remark 3.1: Theorem 3.1 shows that, for all $k \geq \max\{\operatorname{ind} H, \operatorname{ind} K, d\}$, the output z is equal to the input u delayed by d steps. Note, however, that if $\max\{\operatorname{ind} H, \operatorname{ind} K\} > d$, then, for all $k = 0, \ldots, \max\{\operatorname{ind} H, \operatorname{ind} K\} - d - 1$, the input $u(k)$ is not reconstructed.

IV. Zero Nonzero Zeros

Note that Theorem 3.1 does not impose any constraints on the transmission zeros of G. However, if G has a transmission zero, then it follows from [18, p. 398] that there exist an initial condition and input, not both zero, such that the response of a state space realization of G is identically zero. The following proposition provides a statement of this result.

Proposition 4.1: Let $G \in \mathbb{R}(\mathbf{z})_{\rm prop}^{p\times m}$, where $G \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, and, for all $k \geq 0$, consider

$$x(k+1) = Ax(k) + Bu(k), \tag{7}$$
$$y(k) = Cx(k) + Du(k). \tag{8}$$

Assume that (A, B, C, D) has an invariant zero $\mathbf{z}_0 \in \mathbb{C}$, and let $\begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathbf{z}_0))$ have nonzero real part, where

$$\mathcal{Z}(\mathbf{z}) \triangleq \begin{bmatrix} \mathbf{z}I - A & -B \\ C & D \end{bmatrix}.$$

Define the initial state

$$x(0) \triangleq \operatorname{Re}(\bar{x}), \tag{9}$$

and, for all $k \geq 0$, define the input sequence

$$u(k) \triangleq \operatorname{Re}(\mathbf{z}_0^k \bar{u}). \tag{10}$$

Then, for all $k \geq 0$, $y(k) = 0$.

In the case where $\mathbf{z}_0 = 0$ and $k = 0$, it follows that $\mathbf{z}_0^k = 1$, which follows from $\lim_{x\to 0, x\in\mathbb{C}\backslash\{0\}} x^0 = 1$.

The following result shows that the assumptions of Theorem 3.1 imply that G has zero nonzero zeros and hence input reconstruction is possible for systems with this property.

Proposition 4.2: Let $G \in \mathbb{R}(\mathbf{z})_{\rm prop}^{p\times m}$ and assume that G has full column normal rank. Let η be the smallest nonnegative integer d for which there exists a delayed left inverse of G with delay d. Then, for all $d \geq \eta$, there exists an FIR $H \in \mathbb{R}(\mathbf{z})_{\rm prop}^{m\times p}$ such that H is a delayed left inverse of G with delay d if and only if G has zero nonzero transmission zeros.

Consider the case where G has at least one zero at zero and zero nonzero zeros. With $\mathbf{z}_0 = 0$, it follows from Proposition 4.1 that, if $y \equiv 0$, then either u is an impulse or $u \equiv 0$. Hence, the initial input $u(0)$ cannot be reconstructed. However, the inability to reconstruct the initial input cannot be inferred from Theorem 3.1. Nevertheless, the following conjecture, if true, can be used to strengthen Theorem 3.1 and would imply that $u(0)$ cannot be reconstructed.

Conjecture 4.1: Let G, H, and K be as defined in Theorem 3.1. Assume that $d = 0$ and that G has at least one zero zero. Then $K \neq 0$.

It can be noted from Theorem 3.1 that, if $d \geq 1$, then it is not possible to reconstruct $u(0)$. In the case where $d = 0$ and G has at least one zero zero, Conjecture 4.1 (if true) implies that $\operatorname{ind} K \geq 1$ and thus it follows from Theroem 3.1 that $u(0)$ cannot be reconstructed.

V. Examples

Example 5.1: Let

$$G(\mathbf{z}) = \begin{bmatrix} \dfrac{\mathbf{z}}{\mathbf{z}^2+1} \\ \dfrac{\mathbf{z}}{\mathbf{z}^2+3} \end{bmatrix}, \quad H(\mathbf{z}) = \begin{bmatrix} \dfrac{\mathbf{z}^2+1}{\mathbf{z}^2} & 0 \end{bmatrix},$$

so that $H(\mathbf{z})G(\mathbf{z}) = \mathbf{z}^{-1}$ and thus H is an FIR delayed left inverse of G with delay 1. Furthermore,

$$A_G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.7321 \\ 0 & 0 & -1.7321 & 0 \end{bmatrix}, \quad B_G = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix},$$

$$C_G = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad D_G = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

is a minimal realization of G, and

$$A_H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_H = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$
$$C_H = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_H = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

is a minimal realization of H. Note that $d = 1$ and $\operatorname{ind} H = 2$. It can be shown that $\operatorname{ind} K = 2$, where K is defined in Theorem 3.1. Therefore, Theorem 3.1 implies that, for all $k \geq 2$, $z(k) = u(k-1)$. Figure 4 shows the input and output of (5), (6) with $x_H(0) \neq 0$ and with $x_H(0) = 0$. Note that G has one zero at zero. Consequently, the initial input $u(0)$ cannot be reconstructed.

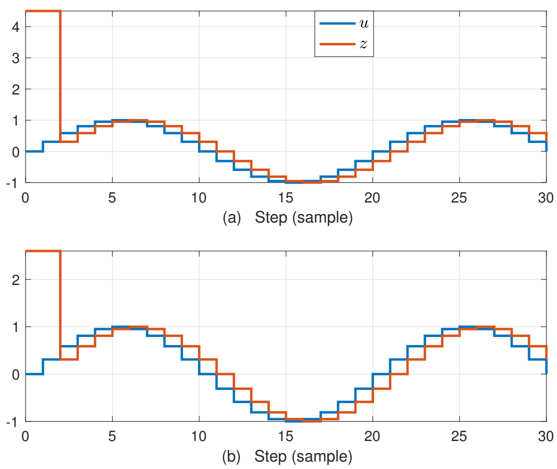

Fig. 4. (a) shows the input and output of (5), (6) with $x_H(0) \neq 0$. (b) shows the input and output of (5), (6) with $x_H(0) = 0$.

Example 5.2: Let

$$G(\mathbf{z}) = \begin{bmatrix} \dfrac{\mathbf{z}+2}{\mathbf{z}(\mathbf{z}+1)} & \dfrac{1}{\mathbf{z}} \\ 1 & 1 \end{bmatrix}, \quad H(\mathbf{z}) = \begin{bmatrix} \dfrac{\mathbf{z}+1}{\mathbf{z}} & -\dfrac{\mathbf{z}+1}{\mathbf{z}^2} \\ -\dfrac{\mathbf{z}+1}{\mathbf{z}} & \dfrac{\mathbf{z}+2}{\mathbf{z}^2} \end{bmatrix},$$

so that $H(\mathbf{z})G(\mathbf{z}) = \mathbf{z}^{-2}I_2$ and thus H is an FIR delayed left inverse of G with delay 2. Note that since G is square, H is the unique delayed left inverse of G with delay 2. Furthermore,

$$A_G = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_G = \begin{bmatrix} 2.2361 & 0 \\ 0.8944 & 0.4472 \end{bmatrix},$$
$$C_G = \begin{bmatrix} -0.4472 & 2.2361 \\ 0 & 0 \end{bmatrix}, \quad D_G = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

is a minimal realization of G, and

$$A_H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_H = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix},$$
$$C_H = \begin{bmatrix} 0.5 & -0.5 & -0.5 \\ -0.5 & 1 & 0.5 \end{bmatrix}, \quad D_H = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

is a minimal realization of H. Note that $d = 2$ and $\operatorname{ind} H = 2$. It can be shown that $\operatorname{ind} K = 2$, where K is defined in Theorem 3.1. Therefore, it follows from Theorem 3.1 that, for all $k \geq 2$, $z(k) = u(k-2)$. Figure 5 shows the input and output of (5),(6) with $x_H(0) \neq 0$ and with $x_H(0) = 0$.

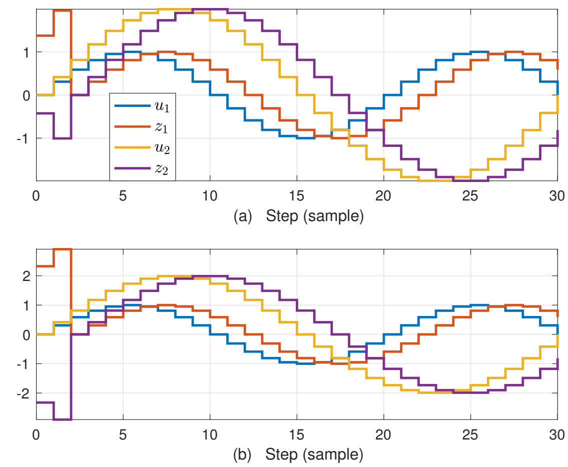

Fig. 5. (a) shows the input and output of (5), (6) with $x_H(0) \neq 0$. (b) shows the input and output of (5), (6) with $x_H(0) = 0$. Note that $u = [u_1\ u_2]^{\mathrm{T}}$ and $z = [z_1\ z_2]^{\mathrm{T}}$.

Example 5.3: Let

$$G(\mathbf{z}) = \begin{bmatrix} \dfrac{\mathbf{z}}{\mathbf{z}+1} \\ 0 \\ \dfrac{\mathbf{z}}{\mathbf{z}+2} \end{bmatrix}, \quad H(\mathbf{z}) = \begin{bmatrix} \dfrac{\mathbf{z}+1}{\mathbf{z}} & \dfrac{1}{\mathbf{z}^3} & 0 \end{bmatrix},$$

so that $H(\mathbf{z})G(\mathbf{z}) = 1$ and thus H is an FIR delayed left inverse of G with delay 0. Furthermore,

$$A_G = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B_G = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},$$
$$C_G = \begin{bmatrix} -2 & 0 \\ 0 & 0 \\ 0 & -2 \end{bmatrix}, \quad D_G = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

is a minimal realization of G, and

$$A_H = \begin{bmatrix} 0 & 0.7071 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_H = \begin{bmatrix} 0.7071 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix},$$
$$C_H = \begin{bmatrix} 1.4142 & 0 & 0 \end{bmatrix}, \quad D_H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

is a minimal realization of H. Note that $d = 0$ and $\operatorname{ind} H = 3$. It can be shown that $\operatorname{ind} K = 1$, where K is defined in Theorem 3.1. Therefore, Theorem 3.1 implies that, for all $k \geq 3$, $z(k) = u(k)$. In addition, if $x_H(0) = 0$, then for all $k \geq 1$, $z(k) = u(k)$. Figure 6 shows the input and output of (5), (6) with $x_H(0) \neq 0$ and with $x_H(0) = 0$. Note that G has one zero at zero. Consequently, the initial input $u(0)$ cannot be reconstructed.

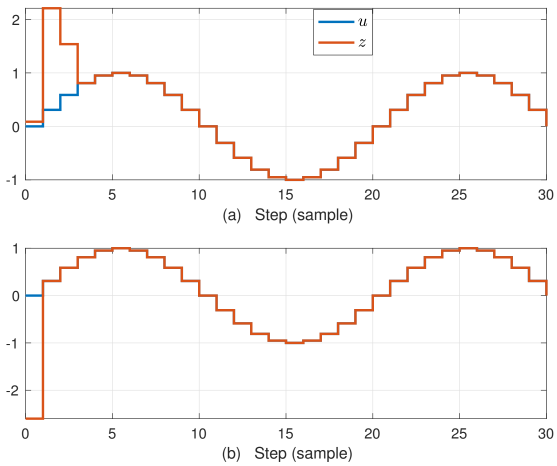

Fig. 6. (a) shows the input and output of (5), (6) with $x_H(0) \neq 0$. (b) shows the input and output of (5), (6) with $x_H(0) = 0$.

VI. SMITH-MCMILLAN CONSTRUCTION OF A DELAYED LEFT INVERSE

In this section, we use the *Smith-Mcmillan form* [17] to construct an FIR delayed left inverse for systems with zero nonzero zeros.

Theorem 6.1: Let $G \in \mathbb{R}(\mathbf{z})^{p\times m}_{\rm prop}$, and let ρ be the normal rank of G. Then there exist unimodular matrices $S_1 \in \mathbb{R}(\mathbf{z})^{p\times p}$ and $S_2 \in \mathbb{R}(\mathbf{z})^{m\times m}$ and unique monic polynomials $p_1,\ldots,p_\rho,q_1,\ldots,q_\rho \in \mathbb{R}(\mathbf{z})$ such that p_i and q_i are coprime for all $i\in\{1,\ldots,\rho\}$, p_i divides p_{i+1} for all $i\in\{1,\ldots,\rho-1\}$, q_{i+1} divides q_i for all $i\in\{1,\ldots,\rho-1\}$, and $G = S_1 S S_2$, where

$$S = \begin{bmatrix} p_1/q_1 & & & 0_{\rho\times(m-\rho)} \\ & \ddots & & \\ & & p_\rho/q_\rho & \\ 0_{(p-\rho)\times\rho} & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix}. \quad (11)$$

S is the *Smith-McMillan form* of G.

Proposition 6.1: Let $G \in \mathbb{R}(\mathbf{z})^{p\times m}_{\rm prop}$, assume that G has full column normal rank, and assume that G has zero nonzero zeros. Then, the left inverse $H_s \overset{\triangle}{=} S_2^{-1}S^{+}S_1^{-1}$ has zero nonzero poles, where S, S_1, and S_2 are defined in Theorem 6.1 and $S^{+} \overset{\triangle}{=} (S^{\rm T}S)^{-1}S^{\rm T}$.

Corollary 6.1: Let G and H_s be as defined in Proposition 6.1, and let d_0 be the smallest nonnegative integer such that $H(\mathbf{z}) = \mathbf{z}^{-d_0}H_s(\mathbf{z})$ is a proper transfer function. Then, H is an FIR delayed left inverse of G.

Example 6.1: Let $G(\mathbf{z}) = \begin{bmatrix} \frac{1}{\mathbf{z}} \\ \frac{1}{\mathbf{z}^2} \end{bmatrix}$. Then

$$S(\mathbf{z}) = \begin{bmatrix} \frac{1}{\mathbf{z}^2} \\ 0 \end{bmatrix}, \quad S_1(\mathbf{z}) = \begin{bmatrix} \mathbf{z} & 1 \\ 1 & 0 \end{bmatrix}, \quad S_2(\mathbf{z}) = 1,$$

such that S is the Smith-Mcmillan form of G and $G = S_1 S S_2$. Evaluating the expression for H given in Corollary 6.1 yields $H(\mathbf{z}) = \begin{bmatrix} 0 & 1 \end{bmatrix}$ such that $H(\mathbf{z})G(\mathbf{z}) = \mathbf{z}^{-2}$. Hence, H is an FIR delayed left inverse of G with delay 2. Furthermore,

$$A_G = \begin{bmatrix} 0 & 0.7071 \\ 0 & 0 \end{bmatrix}, \quad B_G = \begin{bmatrix} 0 \\ 1.4142 \end{bmatrix},$$
$$C_G = \begin{bmatrix} 0 & 0.7071 \\ 1 & 0 \end{bmatrix}, \quad D_G = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is a minimal realization of G. Note that $d = 2$ and, since H is a static gain matrix, we set ${\rm ind}\, H = 0$. It can be shown that ${\rm ind}\, K = 2$, where K is defined in Theorem 3.1. Therefore, Theorem 3.1 implies that, for all $k \geq 2$, $z(k) = u(k-2)$. Figure 7 shows the input and output of (5), (6) with $x_H(0) \neq 0$ and with $x_H(0) = 0$.

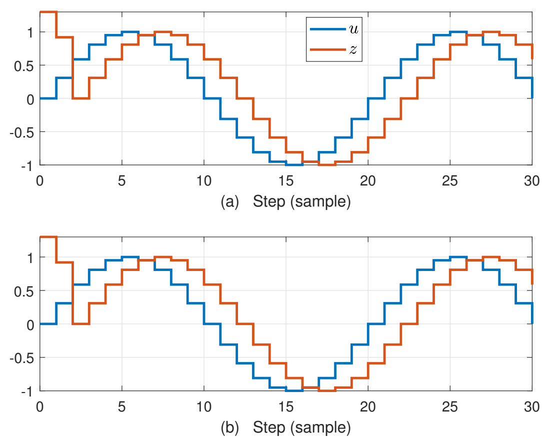

Fig. 7. (a) shows the input and output of (5), (6) with $x_H(0) \neq 0$. (b) shows the input and output of (5), (6) with $x_H(0) = 0$.

Example 6.2: Let $G(\mathbf{z}) = \begin{bmatrix} \frac{1}{\mathbf{z}} & 0 \\ \frac{3}{\mathbf{z}} & \frac{1}{\mathbf{z}+2} \\ \frac{1}{\mathbf{z}} & 0 \end{bmatrix}$. Then

$$S(\mathbf{z}) = \begin{bmatrix} \frac{1}{\mathbf{z}(\mathbf{z}+2)} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, S_1(\mathbf{z}) = \begin{bmatrix} \mathbf{z}+2 & -1 & 0 \\ 6+\frac{5\mathbf{z}}{2} & -\frac{5}{2} & 0 \\ \mathbf{z}+2 & -1 & 1 \end{bmatrix},$$

$$S_2(\mathbf{z}) = \begin{bmatrix} \frac{\mathbf{z}+2}{2} & \mathbf{z} \\ \frac{1}{2} & 1 \end{bmatrix},$$

such that S is the Smith-Mcmillan form of G and $G = S_1 S S_2$. Evaluating the expression for H given in Corollary 6.1 yields

$$H(\mathbf{z}) = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3(\mathbf{z}+2)}{\mathbf{z}} & \frac{\mathbf{z}+2}{\mathbf{z}} & 0 \end{bmatrix}$$

such that $H(\mathbf{z})G(\mathbf{z}) = \mathbf{z}^{-1}I_2$. Hence, H is an FIR delayed left inverse of G with delay 1. Furthermore,

$$A_G = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \quad B_G = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C_G = \begin{bmatrix} 0.5 & 0 \\ 1.5 & 1 \\ 0.5 & 0 \end{bmatrix}, \quad D_G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is a minimal realization of G, and

$$A_H = 0, \quad B_H = \begin{bmatrix} -1.5 & 0.5 & 0 \end{bmatrix},$$
$$C_H = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \quad D_H = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \end{bmatrix},$$

is a minimal realization of H. Note that $d = 1$ and $\operatorname{ind} H = 1$. It can be shown that $\operatorname{ind} K = 1$, where K is defined in Theorem 3.1. Therefore, Theorem 3.1 implies that, for all $k \geq 1$, $z(k) = u(k-1)$. Figure 8 shows the input and output of (5), (6) with $x_H(0) \neq 0$ and with $x_H(0) = 0$.

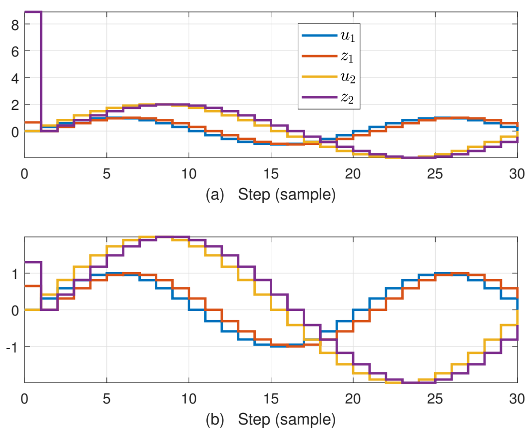

Fig. 8. (a) shows the input and output of (5), (6) with $x_H(0) \neq 0$. (b) shows the input and output of (5), (6) with $x_H(0) = 0$. Note that $u = [u_1\ u_2]^{\mathrm{T}}$ and $z = [z_1\ z_2]^{\mathrm{T}}$.

VII. CONCLUSIONS

It was shown that input reconstruction can be achieved in the case where the system has an FIR delayed left inverse. It was also shown that an FIR delayed left inverse with minimum possible delay exists for systems with full column normal rank if and only if the system has zero nonzero zeros. A procedure for constructing an FIR delayed left inverse using the Smith-McMillan form was presented. Examples were provided for illustration.

This development leads to several open problems. Example 2.2 as well as additional examples not shown suggest that the delayed left inverse based on the generalized inverse expression possesses the minimal delay. Proof of this conjecture is open. Conditions under which this delayed left inverse is FIR are also of interest. Further, proof of Conjecture 4.1 is also open. Finally, a related problem is to determine conditions under which the Smith-McMillan construction of an FIR delayed left inverse possesses the minimal delay.

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