On the McMillan Degree of Full-Normal-Rank Transfer Functions

Khaled F. Aljanaideh [a], Ovidiu Furdui [b], and Dennis S. Bernstein[c]

[a]Aeronautical Engineering Department, Jordan University of Science and Technology, Irbid, Jordan, 22110; [b] Mathematics Department, Technical University of Cluj-Napoca, Cluj-Napoca, Romania [c]Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109

ARTICLE HISTORY
Compiled May 4, 2020

ABSTRACT

Computing the McMillan degree of a transfer function matrix G requires writing G in the Smith-McMillan form or computing all of the minors of G. This becomes more difficult and computationally expensive as the dimension of G increases. In this paper, we introduce a simple approach to compute the McMillan degree of G without the need to write G in the Smith-McMillan form and without the need to compute all of the minors of G. The presented approach requires less computations than the approaches in the literature and clarifies the relationship between the McMillan degree, the poles, and the transmission zeros of a MIMO system. We assume that the roots of the least common multiple of the denominators of the entries of G satisfy a multiplicity condition. This condition can be verified prior to the application of the algorithm. The proposed approach can be beneficial for systems with many inputs and many outputs.

KEYWORDS
McMillan degree and MIMO systems and minimal realizations and systems with many inputs and many outputs

1. Introduction

For a multi-input, multi-output (MIMO) transfer function G, the Smith-McMillan decomposition of G reveals the transmission zeros and poles through the numerator and denominator invariant polynomials of G, whose ratios appear on the diagonal of the Smith-McMillan form of G (Kailath, 1980; Vardulakis, 1991). Since the denominator invariant polynomials of G determine the McMillan degree of G, one method for determining the McMillan degree of G is to construct the Smith-McMillan form of G (Rosenbrock, 1967; Van Dooren et al., 1979).

An alternative method for determining the McMillan degree of G is to compute the least common multiple of the denominators of all of the minors of G (MacFarlane & Karcanias, 1976; Huang, 1974) and (Chen, 1999, p. 205). The required number of computations, however, increases rapidly with dimension.

Other approaches for computing the McMillan degree of G require constructing a controllable or observable realization of G (Roberts, 1969; Mayna, 1968; Munro &

CONTACT K. F. Aljanaideh Email: kfaljanaideh@just.edu.jo

McLeod, 1971). As shown in (Roberts, 1969; Mayna, 1968), a 5×5 transfer function matrix with transfer functions of order 6, can require constructing a state space realization of dimension 150. Moreover, in (Gupta & Fairman, 1974), to compute the McMillan degree of G, a block-Hankel matrix of size $lr \times mr$ is constructed from the Markov parameters of G, where m, l, and r denote the number of inputs, number of outputs, and the order of the least common multiple of the denominators of G, respectively. The McMillan degree of G is equal to the rank of the constructed block-Hankel matrix.

Other approaches to find the McMillan degree of G consider dyadic expansions of transfer-function matrices (Shaked & Dixon, 1977) and solving optimization problems of integer matrices (Karcanias et al., 2007).

The purpose of this paper is to present a technique for determining the McMillan degree of a transfer function G without knowledge of the Smith-McMillan form of G and without the need to compute all of the minors of G. The proposed method does not require constructing a state space realization of G, finding the Markov parameters or an expansion of G, or solving any optimization problems. The method presented in this paper is based on two assumptions, namely, 1) that G has full normal rank, and 2) that the roots of the least common multiple of the denominators of the entries of G satisfies a multiplicity condition. This condition can be verified prior to the application of the algorithm. The proposed approach can be beneficial for systems with many inputs and many outputs.

2. A Motivating Example

The following example shows that computing the McMillan degree of a MIMO transfer function using the approach in (Chen, 1999) can require a large number of computations, and thus a more efficient way might be preferable.

Consider the 10×10 transfer function matrix

$$G(s) = \frac{1}{q_1(s)} Q_0(s), \tag{1}$$

where

$$q_1(s) \stackrel{\triangle}{=} (s+1)(s+2)(s+3), \tag{2}$$

and

$$Q_0(s) = \begin{bmatrix} 1 & s-1 & s & -1 & 2 & 0 & 1 & s+2 & -1 & s \\ -1 & 1 & 0 & s & 1 & s & s+3 & s-2 & 1 & s-2 \\ 1 & 1 & 0 & -1 & s & s+2 & s-1 & 0 & -1 & s \\ s & 1 & 2 & 0 & -1 & s & 1 & 0 & s+1 & s-2 \\ 1 & s & 1 & s-1 & s & 2 & 1 & 0 & s-2 & s \\ s & s-2 & s+1 & s+3 & 1 & 2 & -1 & 0 & 1 & s-1 \\ s & 1 & s-1 & s-2 & s+1 & 1 & 3 & s-1 & 1 & s-1 \\ 1 & s-1 & s-2 & s+1 & 2 & -1 & s & 1 & s-2 & s \\ s & 1 & s+3 & s-2 & 1 & s & s & 2 & 0 & -1 \\ s+1 & s-2 & 1 & s & 1 & -1 & s+2 & 1 & s+1 & s \end{bmatrix}. \tag{3}$$

In order to determine the McMillan degree of G using the approach in (Chen, 1999, p. 205), all $184,755$ minors of G must be computed and their least common multiple must be determined. Alternatively, the McMillan degree n of G can be determined using the expression $n = mn_1 - k$ given by Theorem 4 in the present paper, where G is $m \times m$, n_1 is the degree of q_1, and k is the number of roots that are common to q_1 and $\det Q_0$. For this example, $m = 10$ and $n_1 = 3$. Furthermore, the degree of $\det Q_0$ is 10 and, since $\det Q_0(-1) \neq 0$, $\det Q_0(-2) \neq 0$, and $\det Q_0(-3) \neq 0$, it follows that $k = \operatorname{card}(\operatorname{roots} q_1 \cap \operatorname{roots} \det Q_0) = 0$. Hence, $n = mn_1 - k = 10(3) = 30$. The same value is obtained by using Matlab to compute a minimal realization of G.

Next, consider the 11×11 transfer function matrix

$$G(s) = \frac{1}{q_1(s)} Q(s), \tag{4}$$

where q_1 is given by (2), and

$$Q(s) = \begin{bmatrix} s+1 & 0_{1\times 10} \\ 0_{10\times 1} & Q_0(s) \end{bmatrix}. \tag{5}$$

The degree of $\det Q$ is 11 and, since $\det Q_0(-1) = 0$, $\det Q_0(-2) \neq 0$, and $\det Q_0(-3) \neq 0$, it follows that $k = \operatorname{card}(\operatorname{roots} q_1 \cap \operatorname{roots} \det Q) = 1$. Hence, $n = mn_1 - k = 11(3) - 1 = 32$.

3. The Smith-McMillan Form

The following theorem reviews the Smith-McMillan form of an $l \times m$ matrix $G \in \mathbb{R}(s)^{l\times m}$ whose entries are rational functions with coefficients in $\mathbb{R}$. Let $\operatorname{rank} G$ denote the normal rank of G. Note that G is not necessarily proper.

Theorem 1. *Let $G \in \mathbb{R}(s)^{l\times m}$ be nonzero, and define $r \triangleq \operatorname{rank} G$. Then, there exist unimodular matrices $S_1 \in \mathbb{R}[s]^{l\times l}$ and $S_2 \in \mathbb{R}[s]^{m\times m}$ and unique monic polynomials $p_1, \ldots, p_r, q_1, \ldots, q_r \in \mathbb{R}[s]$ such that p_i and q_i are coprime for all $i \in \{1, \ldots, r\}$, p_i divides p_{i+1} for all $i \in \{1 \ldots, r-1\}$, q_{i+1} divides q_i for all $i \in \{1, \ldots, r-1\}$, and*

$$G = S_1 \begin{bmatrix} p_1/q_1 & & & 0_{r\times(m-r)} \\ & \ddots & & \\ & & p_r/q_r & \\ 0_{(l-r)\times r} & & & 0_{(l-r)\times(m-r)} \end{bmatrix} S_2. \tag{6}$$

Proof: See (Kailath, 1980, pp. 443–445) and (Vardulakis, 1991, p. 99). □

Note that (6) can be rewritten as

$$G = \frac{1}{q_1} Q, \tag{7}$$

where

$$Q \triangleq S_1 \begin{bmatrix} p_1 & & & & 0_{r\times(m-r)} \\ & p_2 q_1/q_2 & & & \\ & & \ddots & & \\ & & & p_r q_1/q_r & \\ 0_{(l-r)\times r} & & & & 0_{(l-r)\times(m-r)} \end{bmatrix} S_2. \tag{8}$$

Since $q_2, \ldots, q_r$ divide q_1, it follows that Q is a polynomial matrix.

For convenience, we write

$$G = \begin{bmatrix} \frac{N_{1,1}}{D_{1,1}} & \cdots & \frac{N_{1,m}}{D_{1,m}} \\ \vdots & \cdots & \vdots \\ \frac{N_{l,1}}{D_{l,1}} & \cdots & \frac{N_{l,m}}{D_{l,m}} \end{bmatrix}, \tag{9}$$

where $D_{1,1}, \ldots, D_{l,m}$ are monic. The next theorem shows that, if every entry of G is written such that its numerator and denominator are coprime, then q_1 is the least common multiple of all of the denominators of the entries of G. Thus, q_1 can be determined without knowing (6).

The following result is proved in (Kailath, 1980, p. 444). We provide an alternative proof for completeness.

Theorem 2. *Assume that, for all $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, m\}$, $N_{i,j}$ and $D_{i,j}$ are coprime. Then, q_1 is the least common multiple of $\{D_{1,1}, \ldots, D_{l,m}\}$.*

Let q denote the least common multiple of $\{D_{1,1}, \ldots, D_{l,m}\}$. Equating (6) and (9) and multiplying by q_1 yields

$$S_1 \begin{bmatrix} p_1 & & & & 0_{r\times(m-r)} \\ & p_2\lambda_2 & & & \\ & & \ddots & & \\ & & & p_r\lambda_r & \\ 0_{(l-r)\times r} & & & & 0_{(l-r)\times(m-r)} \end{bmatrix} S_2 = \begin{bmatrix} \frac{N_{1,1}q_1}{D_{1,1}} & \cdots & \frac{N_{1,m}q_1}{D_{1,m}} \\ \vdots & \cdots & \vdots \\ \frac{N_{l,1}q_1}{D_{l,1}} & \cdots & \frac{N_{l,m}q_1}{D_{l,m}} \end{bmatrix}, \tag{10}$$

where, for all $i \in \{2, \ldots, r\}$, $\lambda_i \triangleq q_1/q_i \in \mathbb{R}[s]$. Since the left hand side of (10) is a polynomial matrix, it follows that the right hand side of (10) is also a polynomial matrix. Moreover, since, for all $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, m\}$, $N_{i,j}$ and $D_{i,j}$ are coprime, it follows that $D_{i,j}$ divides q_1. Hence, there exists $\mu \in \mathbb{R}[s]$ such that $q_1 = q\mu$, and thus μ divides q_1. Therefore, (10) implies

$$\begin{bmatrix} \frac{p_1}{\mu} & & & & 0_{r\times(m-r)} \\ & \frac{p_2\lambda_2}{\mu} & & & \\ & & \ddots & & \\ & & & \frac{p_r\lambda_r}{\mu} & \\ 0_{(l-r)\times r} & & & & 0_{(l-r)\times(m-r)} \end{bmatrix} = S_1^{-1} \begin{bmatrix} N_{1,1}\mu_{1,1} & \cdots & N_{1,m}\mu_{1,m} \\ \vdots & \cdots & \vdots \\ N_{l,1}\mu_{l,1} & \cdots & N_{l,m}\mu_{l,m} \end{bmatrix} S_2^{-1}, \tag{11}$$

where, for all $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, m\}$, $\mu_{i,j} \stackrel{\triangle}{=} q/D_{i,j} \in \mathbb{R}[s]$. Since the right hand side of (11) is a polynomial matrix, the left hand side of (11) is also a polynomial matrix, and thus μ divides p_1. Since p_1 and q_1 are coprime and since μ divides p_1 and q_1, it follows that $\mu = 1$. Hence, $q_1 = q$, and thus q_1 is the least common multiple of $\{D_{1,1}, \ldots, D_{l,m}\}$. □

4. Determining the McMillan Degree of *G*

Define

$$\delta \stackrel{\triangle}{=} \prod_{i=1}^{r} q_i. \tag{12}$$

Then, (6) can be written as

$$G = \frac{1}{\delta} N, \tag{13}$$

where N is the polynomial matrix

$$N \stackrel{\triangle}{=} S_1 \begin{bmatrix} p_1 \prod_{i=2}^{r} q_i & & & 0_{r\times(m-r)} \\ & \ddots & & \\ & & p_r \prod_{i=1}^{r-1} q_i & \\ 0_{(l-r)\times r} & & & 0_{(l-r)\times(m-r)} \end{bmatrix} S_2. \tag{14}$$

The McMillan degree n of G is defined by

$$n \stackrel{\triangle}{=} \deg \delta = \sum_{i=1}^{r} \deg q_i. \tag{15}$$

Let $\operatorname{lcm}(\mathcal{S})$ denote the least common multiple of the polynomials in the set $\mathcal{S}$.

Theorem 3. *(Chen, 1999, p. 205) Assume that each entry of G is written such that its numerator and denominator are coprime. Let $\mathcal{S}$ denote the set of denominators of all $i \times i$ minors of G for $i \in \{1, ..., \min\{l, m\}\}$. Then*

$$n = \deg \operatorname{lcm}(\mathcal{S}). \tag{16}$$

Theorem 3 requires computation of all $\sum_{i=1}^{\min\{l,m\}} \binom{l}{i}\binom{m}{i} = \binom{l+m}{l} - 1$ minors of G. This approach is applicable to transfer functions that may be either square or non-square and either full-normal-rank or rank-deficient. However, the number of required computations increases rapidly with $\min\{l, m\}$. In the next section we develop a more computationally efficient technique for computing the McMillan degree of square, full-normal-rank transfer functions.

5. Determining the McMillan Degree for Square, Full-Normal-Rank G

Define

$$\varepsilon \triangleq \prod_{i=1}^{r} p_i \in \mathbb{R}[s]. \tag{17}$$

Let mroots(p) denote the multiset of roots of $p \in \mathbb{R}[s]$ with multiplicity. Moreover, let $\mathbb{R}(s)_{\rm prop}^{l\times m}$ denote the set of $l \times m$ proper transfer functions with coefficients in $\mathbb{R}$.

For the rest of this section let $m \geq 2$, and let $G \in \mathbb{R}_{\rm prop}^{m\times m}(s)$. Consider the notation defined in Theorem 1 and (12)–(17). The following result follows immediately from Theorem 1.

Let $m \geq 2$ and $G \in \mathbb{R}(s)_{\rm prop}^{m\times m}$. Then, exactly one of the following statements is true:

i) $\det N = 0$.
ii) $\deg \det N \geq (m-1)n$.

In addition, *ii*) holds if and only if $\operatorname{rank} G = m$. In this case,

$$\det N = (\det S_1)(\det S_2)\delta^{m-1}\varepsilon. \tag{18}$$

Assume that $\operatorname{rank} G = m$, and define $\gamma \triangleq q_2 \cdots q_m \in \mathbb{R}[s]$. Then

$$\delta = q_1\gamma, \tag{19}$$

and thus (13) can be written as

$$G = \frac{1}{q_1\gamma}N. \tag{20}$$

Note from (7), (19), and (20) that

$$Q = \frac{1}{\gamma}N \in \mathbb{R}[s]^{m\times m}. \tag{21}$$

Since G is proper, it follows from (7) that each entry of Q is a polynomial of degree less than or equal to $n_1 \triangleq \deg q_1 \leq n$. Likewise, it follows from (13) that each entry of N has degree less than or equal to n. It follows from (7) and (13) that

$$\delta Q = q_1 N. \tag{22}$$

Thus, $Q = N$ if and only if $q_1 = \delta$, that is, $q_2 = \cdots = q_r = 1$.

Since, for all $i \in \{2, \ldots, m\}$, q_i divides q_1, it follows that γ divides q_1^{m-1}. Define the polynomial

$$\Gamma \triangleq \frac{q_1^{m-1}}{\gamma}. \tag{23}$$

The following lemma is used in the proof of Lemma 2. For convenience, we assume for the rest of the paper that $\det S_1 = \det S_2 = 1$.

Lemma 1.

$$\det Q = \Gamma\varepsilon. \tag{24}$$

Proof: Taking the determinant of both sides of (22) yields

$$\delta^m \det Q = q_1^m \det N. \tag{25}$$

Using (18), (25) implies

$$\delta \det Q = q_1^m \varepsilon. \tag{26}$$

Now (19) and (26) yield

$$\gamma \det Q = q_1^{m-1}\varepsilon. \tag{27}$$

Finally, (23) and (27) imply (24). □

For $p \in \mathbb{R}[s]$, let $\mathrm{mult}_p(z)$ denote the multiplicity of z as a root of p. Furthermore, let $\mathrm{roots}\,(p)$ denote the set of distinct roots of p.

The following assumption is needed for the main result below. This assumption concerns the multiplicity of the common roots between $\det(Q)$ and q_1. Since $\det(Q)$ and q_1 can be determined without knowledge of the McMillan form of G, it follows that Assumption A can be verified directly from the entries of G and thus prior to application of Theorem 4 below.

Assumption A: For all $z \in \mathrm{roots}(\det \mathrm{Q}) \cap \mathrm{roots}(q_1)$, $\mathrm{mult}_{\det Q}(z) \le \mathrm{mult}_{q_1}(z)$.

Assumption A states that if z is both a transmission zero and a pole of G, then the multiplicity of z as a transmission zero of G is less than or equal to the multiplicity of z as a pole of G. To illustrate a case where Assumption A does not hold, suppose that

$$G(s) = \frac{1}{(s+2)(s+3)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (s+1)(s+2) & 0 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}. \tag{28}$$

Then, $q_1(s) = (s+1)(s+2)$ and

$$Q(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (s+1)(s+2) & 0 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}. \tag{29}$$

Therefore, $\det Q(s) = (s+1)^2(s+2)^2$. Note that $-1 \in \mathrm{roots}(\det \mathrm{Q}) \cap \mathrm{roots}(q_1)$, and $\mathrm{mult}_{\det Q}(-1) > \mathrm{mult}_{q_1}(-1)$ and thus Assumption A does not hold. Note, however, that Assumption A holds for all cases where $\mathrm{roots}(\det \mathrm{Q})$ has no repeated elements.

The following lemma is used to prove Lemma 3.

Lemma 2. *If Assumption A holds, then* Γ *divides* q_1.

Proof: By Lemma 1, Γ divides $\det Q$. Moreover, (23) implies that $\mathrm{roots}(\Gamma) \subseteq \mathrm{roots}(q_1)$. Therefore, $\mathrm{roots}(\Gamma) \subseteq \mathrm{roots}(\det Q) \cap \mathrm{roots}(q_1)$. Next, Assumption A implies that, for all $z \in \text{roots}(\Gamma)$, $\text{mult}_{\det Q}(z) \leq \text{mult}_{q_1}(z)$. Since Γ divides $\det Q$, it follows that Γ divides q_1. $\square$

Using (19) and (23) we obtain

$$\Gamma\delta = \Gamma q_1 \gamma = q_1^m. \tag{30}$$

Lemma 2 implies that (30) can be written as

$$\delta = \frac{q_1^m}{\Gamma} = q_1^{m-1}\alpha, \tag{31}$$

where

$$\alpha \triangleq \frac{q_1}{\Gamma} \in \mathbb{R}[s]. \tag{32}$$

The following lemma is used in the proof of Theorem 4.

Lemma 3. *If Assumption A holds, then α and ε are coprime.*

Proof: Using (12) and (31),

$$q_2 \cdots q_m = q_1^{m-2}\alpha. \tag{33}$$

Therefore,

$$q_m = \alpha\beta, \tag{34}$$

where $\beta \in \mathbb{R}[s]$ is defined by

$$\beta \triangleq \begin{cases} 1, & m = 2, \\ \frac{q_1^{m-2}}{q_2 \cdots q_{m-1}}, & m \geq 3. \end{cases}$$

It follows from (34) that α divides q_m. Since q_m and p_m are coprime, it follows that α and p_m are coprime. Finally, since, for all $i \in \{1, \ldots, m-1\}$, p_i divides p_m, it follows that α and ε are coprime. $\square$

Let card denote the number of elements in a multiset including multiplicity. The following theorem determines the McMillan degree of G for square, full-normal-rank transfer functions.

Theorem 4. *Assume that Assumption A holds, and define* $k \triangleq \text{card}\,(\text{mroots}(q_1) \cap \text{mroots}(\det Q))$. *Then*

$$n = mn_1 - k. \tag{35}$$

Proof: Consider the Smith-McMillan form (6) of G, where $l = m = r$. It follows from (30) that

$$\deg \delta = m \deg q_1 - \deg \Gamma,$$

that is,

$$n = mn_1 - \deg \Gamma.$$

Note from (24) that $\mathrm{roots}(\Gamma) \subseteq \mathrm{roots}\,(\det Q) = \mathrm{roots}(\Gamma\varepsilon)$, and note from (32) that $\mathrm{roots}(\Gamma) \subseteq \mathrm{roots}(q_1) = \mathrm{roots}(\alpha\Gamma)$. Lemma 3 implies that α and ε are coprime. Thus, $\mathrm{roots}(\Gamma) = \mathrm{roots}(q_1) \cap \mathrm{roots}\,(\det Q)$. □

Theorem 4 provides an expression for the McMillan degree of a square, full-normal-rank transfer function G under Assumption A. This expression requires knowledge of only the least common multiple q_1 of the denominators of G and the matrix Q as given in (7), where each entry of G is expressed in coprime form. Theorem 4 thus makes it possible to determine the McMillan degree of G without knowledge of the decomposition (6).

Finally, the following result shows how to construct δ from knowledge of q_1, Q, and n.

Assume that Assumption A holds, and let $\mathrm{roots}(q_1) = \{s_1, \ldots, s_{m_1}\}$. For all $i \in \{1, \ldots, m_1\}$, define $\alpha_i \triangleq \mathrm{mult}_{q_1}(s_i)$ and $k_i \triangleq \mathrm{mult}_{\det Q}(s_i)$. Then,

$$\delta(s) = \prod_{i=1}^{m_1} (s - s_i)^{m\alpha_i - k_i}. \tag{36}$$

Substituting

$$q_1(s) = \prod_{i=1}^{m_1} (s - s_i)^{\alpha_i},$$
$$\Gamma(s) = \prod_{i=1}^{m_1} (s - s_i)^{k_i},$$

into (31) yields (36). □

6. Examples for Square *G*

Consider the transfer function

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)} \begin{bmatrix} 1 & s+3 & s+1 \\ s+2 & s+1 & 1 \\ s+5 & s+1 & s+3 \end{bmatrix}. \tag{37}$$

Then,

$$q_1(s) = (s+1)(s+2)(s+3), \tag{38}$$
$$Q(s) = \begin{bmatrix} 1 & s+3 & s+1 \\ s+2 & s+1 & 1 \\ s+5 & s+1 & s+3 \end{bmatrix}, \tag{39}$$

and $n_1 = 3$. Note that $\det Q(s) = (s+2)(s^2+7s+2)$, and thus $k = \text{card}\,(\{-1,-2,-3\} \cap \{-2, -\frac{7}{2} \pm \frac{\sqrt{41}}{2}\}) = 1$. Therefore, Theorem 4 implies $n = 3n_1 - k = 8$. In fact,

$$\delta(s) = (s+1)^3(s+2)^2(s+3)^3, \tag{40}$$

$$N(s) = (s+1)^2(s+2)(s+3)^2 Q(s). \tag{41}$$

By Theorem 3, the least common denominator of all 1×1, 2×2, and 3×3 minors of (37) is $(s+1)^3(s+2)^2(s+3)^3$, and thus the McMillan degree of (37) is $n = 8$, which confirms the result obtained using Theorem 4. ($\diamond$)

Consider the transfer function

$$G(s) = \tag{42}$$

$$\frac{1}{(s+1)^2(s+2)^2(s+3)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (s+1)(s+2) & 0 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}.$$

Then

$$q_1(s) = (s+1)^2(s+2)^2(s+3), \tag{43}$$

$$Q(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (s+1)(s+2) & 0 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}. \tag{44}$$

and $n_1 = 5$. Note that $\det Q(s) = (s+1)^2(s+2)^2$, and thus Assumption A is satisfied. In fact, $k = \text{card}\,(\{-1,-1,-2,-2,-3\} \cap \{-1,-1,-2,-2\}) = 4$, and thus $n = 3n_1 - k = 11$. In fact,

$$\delta(s) = (s+1)^4(s+2)^4(s+3)^3, \tag{45}$$

$$N(s) = (s+1)^2(s+2)^2(s+3) Q(s). \tag{46}$$

Using Theorem 3, the least common denominator of all 1×1, 2×2, and 3×3 minors of (42) is $(s+1)^4(s+2)^4(s+3)^3$, and thus the McMillan degree of (42) is $n = 11$, which is equivalent to the result obtained using Theorem 4. ($\diamond$)

Consider the transfer function

$$G(s) = \tag{47}$$

$$\frac{1}{(s+1)(s+2)(s+3)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (s+1)(s+2) & 0 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}.$$

Then q_1 is given by (38),

$$Q(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (s+1)(s+2) & 0 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}. \tag{48}$$

Note that $\det Q(s) = (s+1)^2(s+2)^2$, and thus Assumption A is not satisfied. In fact,

$k = \operatorname{card}(\{-1,-2,-3\} \cap \{-1,-1,-2,-2\}) = 2$, and thus $3n_1 - k = 7$. However,

$$\delta(s) = (s+1)(s+2)(s+3)^3, \tag{49}$$

and thus $n = 5$.

Using Theorem 3, the least common denominator of all 1×1, 2×2, and 3×3 minors of (47) is $(s+1)(s+2)(s+3)^3$, and thus the McMillan degree of (37) is $n = 5$, which is correct. $(\diamond)$

7. Determining the McMillan Degree for Rectangular, Full-Normal-Rank G

Let $G \in \mathbb{R}(s)^{l\times m}_{\rm prop}$, where l and m may be different, and assume G has full normal rank.

If $\operatorname{rank} G = l$, then

$$GG^{\rm T} = \begin{bmatrix} \frac{\bar N_{1,1}}{\bar D_{1,1}} & \cdots & \frac{\bar N_{1,l}}{\bar D_{1,l}} \\ \vdots & \ddots & \vdots \\ \frac{\bar N_{l,1}}{\bar D_{l,1}} & \cdots & \frac{\bar N_{l,l}}{\bar D_{l,l}} \end{bmatrix} \in \mathbb{R}[s]^{l\times l} \tag{50}$$

has full normal rank, whereas, if $\operatorname{rank} G = m$, then

$$G^{\rm T}G = \begin{bmatrix} \frac{\bar N_{1,1}}{\bar D_{1,1}} & \cdots & \frac{\bar N_{1,m}}{\bar D_{1,m}} \\ \vdots & \ddots & \vdots \\ \frac{\bar N_{m,1}}{\bar D_{m,1}} & \cdots & \frac{\bar N_{m,m}}{\bar D_{m,m}} \end{bmatrix} \in \mathbb{R}[s]^{m\times m} \tag{51}$$

has full normal rank.

Lemma 4. *The following statements hold:*

i) Suppose that $\operatorname{rank} G = l$, *and suppose that* q_1^2 *is the least common multiple of* $\{\bar D_{1,1}, \ldots, \bar D_{l,l}\}$. *Then, the McMillan degree of* $GG^{\rm T}$ *is* $2n$.

ii) Suppose that $\operatorname{rank} G = m$, *and suppose that* q_1^2 *is the least common multiple of* $\{\bar D_{1,1}, \ldots, \bar D_{m,m}\}$. *Then, the McMillan degree of* $G^{\rm T}G$ *is* $2n$.

i) The Smith-McMillan form of G is given by

$$G = S_1 \begin{bmatrix} p_1/q_1 & & & 0_{l\times(m-l)} \\ & \ddots & & \\ & & p_l/q_l & \end{bmatrix} S_2, \tag{52}$$

and the Smith-McMillan form of $GG^{\rm T}$ is given by

$$GG^{\rm T} = S_3 \begin{bmatrix} \bar p_1/\bar q_1 & & \\ & \ddots & \\ & & \bar p_l/\bar q_l \end{bmatrix} S_4, \tag{53}$$

where $S_3 \in \mathbb{R}[s]^{l\times l}$ and $S_4 \in \mathbb{R}[s]^{l\times l}$ are unimodular matrices and $\bar{p}_1,\ldots,\bar{p}_l$ and $\bar{q}_1,\ldots,\bar{q}_l$ are unique monic polynomials such that $\bar{p}_i$ and $\bar{q}_i$ are coprime and such that, for all $i \in \{1,\ldots,l\}$, $\bar{p}_i$ divides $\bar{p}_{i+1}$ and, for all $i \in \{1,\ldots,l-1\}$, $\bar{q}_{i+1}$ divides $\bar{q}_i$. Define $\bar{\delta} \triangleq \prod_{i=1}^{l} \bar{q}_i$. Then $\bar{n} \triangleq \deg \bar{\delta}$ is the McMillan degree of GG^{T}. Suppose that $\bar{\delta} \neq \delta^2$, then $\bar{\delta}$ divides δ^2, and thus

$$\delta^2 = \mu\bar{\delta}, \tag{54}$$

where $\mu \in \mathbb{R}[s]$. Moreover, suppose that μ divides q_1^2, then equating GG^{T} from (52) and (53) and multiplying by $\bar{q}_1$ yields

$$\begin{aligned}\frac{1}{\mu}S_1\begin{bmatrix} p_1\lambda_1 & & & 0_{l\times(m-l)} \\ & \ddots & & \\ & & p_l\lambda_l & \end{bmatrix} S_2S_2^{\mathrm{T}} \begin{bmatrix} p_1\lambda_1 & & \\ & \ddots & \\ & & p_l\lambda_l \\ 0_{l\times(m-l)} & & \end{bmatrix} S_1^{\mathrm{T}} \\ = S_3\begin{bmatrix} \bar{p}_1\bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{p}_l\bar{\lambda}_l \end{bmatrix} S_4,\end{aligned} \tag{55}$$

where, for all $i \in \{1,\ldots,l\}$, $\lambda_i \triangleq q_1/q_i \in \mathbb{R}[s]$ and $\bar{\lambda}_i \triangleq \bar{q}_1/\bar{q}_i \in \mathbb{R}[s]$. Moreover, since

$$GG^{\mathrm{T}} = \frac{1}{q_1^2}QQ^{\mathrm{T}}, \tag{56}$$

it follows that

$$QQ^{\mathrm{T}} = S_1\begin{bmatrix} p_1\lambda_1 & & & 0_{l\times(m-l)} \\ & \ddots & & \\ & & p_l\lambda_l & \end{bmatrix} S_2S_2^{\mathrm{T}} \begin{bmatrix} p_1\lambda_1 & & \\ & \ddots & \\ & & p_l\lambda_l \\ 0_{l\times(m-l)} & & \end{bmatrix} S_1. \tag{57}$$

Since the right hand side of (55) is a polynomial matrix, the left hand side is also a polynomial matrix. Thus, (55) and (57) imply that μ divides every entry of QQ^{T}. Since μ divides q_1^2 and every entry of QQ^{T}, it follows that q_1^2 is not the least common multiple of $\{\bar{D}_{1,1},\ldots,\bar{D}_{l,l}\}$. The proof of *ii)* is analogous. □

Assumption B: q_1^2 is the least common multiple of $\{\bar{D}_{1,1},\ldots,\bar{D}_{l,l}\}$.

Assumption C: If G has full row normal rank, then, for all $z \in \operatorname{roots}(\det QQ^{\mathrm{T}}) \cap \operatorname{roots}(q_1^2)$, $\operatorname{mult}_{\det QQ^{\mathrm{T}}}(z) \leq \operatorname{mult}_{q_1^2}(z)$. Moreover, if G has full column normal rank, then, for all $z \in \operatorname{roots}(\det \mathrm{Q}^{\mathrm{T}}\mathrm{Q}) \cap \operatorname{roots}(q_1^2)$, $\operatorname{mult}_{\det Q^{\mathrm{T}}Q}(z) \leq \operatorname{mult}_{q_1^2}(z)$.

Note that Assumptions B and C can be verified directly from the entries of GG^{T} or $G^{\mathrm{T}}G$ and thus prior to the application of Theorem 5 below.

The following theorem determines the McMillan degree of G for rectangular, full-normal-rank transfer functions.

Theorem 5. *If Assumptions B and C hold, then*

$$n = n_1\min\{m,l\} - \tfrac{1}{2}\bar{k}, \tag{58}$$

where

$$\bar{k} \triangleq \begin{cases} \operatorname{card}(\operatorname{roots}(q_1^2) \cap \operatorname{roots}(\det QQ^{\mathrm{T}})), & \operatorname{rank} G = l, \\ \operatorname{card}(\operatorname{roots}(q_1^2) \cap \operatorname{roots}(\det Q^{\mathrm{T}}Q)), & \operatorname{rank} G = m. \end{cases} \tag{59}$$

Suppose $\operatorname{rank} G = l$. Then,

$$GG^{\mathrm{T}} = \frac{1}{q_1^2} QQ^{\mathrm{T}} = \frac{1}{\delta^2} NN^{\mathrm{T}} \in \mathbb{R}(s)^{l\times l}_{\mathrm{prop}}.$$

Since q_1^2 is the least common multiple of $\bar{D}_{1,1}, \ldots, \bar{D}_{l,l}$, Lemma 4 implies that the McMillan degree of GG^{T} is $2n$. It follows from Theorem 4 that

$$2n = 2ln_1 - \operatorname{card}(\operatorname{roots}(q_1^2) \cap \operatorname{roots}(\det QQ^{\mathrm{T}})).$$

The proof for the case $\operatorname{rank} G = m$ is analogous. □

8. Examples for Nonsquare *G*

Consider the transfer function

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)} \begin{bmatrix} 1 & s+3 & s+1 \\ s+2 & s+1 & 1 \end{bmatrix}. \tag{60}$$

Then q_1 is given by (38),

$$Q(s) = \begin{bmatrix} 1 & s+3 & s+1 \\ s+2 & s+1 & 1 \end{bmatrix}, \tag{61}$$

$$Q(s)Q(s)^{\mathrm{T}} = \begin{bmatrix} 2s^2+8s+11 & s^2+6s+6 \\ s^2+6s+6 & 2s^2+6s+6 \end{bmatrix}, \tag{62}$$

and $n_1 = 3$. Note that q_1^2 is the least common multiple of the denominators of GG^{T} and thus Assumption B is satisfied. Moreover, note that $\det Q(s)Q(s)^{\mathrm{T}} = 3s^4+16s^3+34s^2+42s+30$, and thus Assumption C is satisfied. In fact, $\bar{k} = \operatorname{card}(\{-1,-2,-3\} \cap \{-2.144 \pm \jmath 0.3810, -0.5227 \pm 1.3549\}) = 0$. Therefore, using Theorem 5, $n = 2n_1 - \frac{1}{2}\bar{k} = 6$. In fact,

$$\delta(s) = (s+1)^2(s+2)^2(s+3)^2, \tag{63}$$

$$N(s) = (s+1)(s+2)(s+3)Q(s). \tag{64}$$

Using Theorem 3, the least common denominator of all 1×1 and 2×2 minors of (37) is $(s+1)^2(s+2)^2(s+3)^2$, and thus the McMillan degree of (60) is $n = 6$, which is equivalent to the result obtained using Theorem 5. (⋄)

Consider the transfer function (7), where

$$q_1(s) = (s+1)(s+2)(s+3)(s+4)^2(s+5)^2, \tag{65}$$

and

$$
Q(s) = \begin{bmatrix} 3(s+3)^2(s+5)^2 & 6(s+1)^2(s+3)(s+5) & (2s+7)(s+1)(s+2)(s+5) & (2s+5)(s+1)(s+4)(s+5) \\ 2(s+1)(s+2)(s+4) & (s+1)(s+2)(s+4)(s+5) & 2(s+4)(s+5)^2 & 8(s+2)^2(s+4) \\ 2(s^2+7s+18)(s+2)(s+4) & -2s(s+2)(s+4)(s+5) & (s+1)(s+2)(s+4)(s+5) & 2(5s+17)(s+2)^2(s+4) \end{bmatrix}, \tag{66}
$$

which is similar to an example considered in (Shaked & Dixon, 1977; Kalman, 1963).

Note that

$$
\begin{aligned}
\det(QQ^{\mathrm{T}}) = {} & (s+2)^2(s+3)^2(s+4)^4(s+5)^4(1001s^{12} + 29242s^{11} + 409370s^{10} \\
& + 3739262s^9 + 25007645s^8 + 125149220s^7 + 465212253s^6 + 1279084082s^5 \\
& + 2599011278s^4 + 3859646086s^3 + 4019012369s^2 + 2640388588s + 823122980).
\end{aligned} \tag{67}
$$

Using Theorem 5, we have $n_1 = 5$, $m = 3$, and $\bar{k} = 12$, and thus $n = 9$.

9. Comparison of Required Computation

To determine the McMillan degree of G, Theorem 4 requires computing one $m \times m$ minor, which is the determinant of the $m \times m$ matrix Q. On the other hand, Theorem 3 requires computing one $m \times m$ minor, which is the determinant of the transfer function G, as well as all other minors of G of size 1×1, 2×2, $\ldots$, $(m-1) \times (m-1)$.

For all $i \geq 2$, let N_i denote the number of addition and multiplication operations required to compute a minor of size $i \times i$. Then, a 1×1 minor requires $N_1 = 0$ addition and multiplication operations. For all $i \geq 2$, an $i \times i$ minor requires

$$
N_i = i(N_{i-1} + 1) + i - 1 = \sum_{k=0}^{i-1} \frac{i!}{k!} - 1 \tag{68}
$$

addition and multiplication operations. Then the number $\bar{C}_m$ of addition and multiplication operations required by Theorem 4 to compute the McMillan degree of an $m \times m$ transfer function is $\bar{C}_m = N_m$. Since

$$
\lim_{m \to \infty} \frac{N_m}{m!} = e, \tag{69}
$$

it follows that

$$
\bar{C}_m = O(m!). \tag{70}
$$

On the other hand, for all $i \in \{1, \ldots, m\}$, an $m \times m$ matrix has $\binom{m}{i}^2$ minors of dimension $i \times i$. The number C_m of addition and multiplication operations required by Theorem 3 to determine the McMillan degree of G is thus

$$C_m = \sum_{i=1}^{m} \binom{m}{i}^2 N_i. \tag{71}$$

For all $m \geq 2$, substituting (68) in (71) yields

$$\begin{aligned}\tilde{C}_m &\triangleq C_m - \bar{C}_m \\ &= \sum_{i=1}^{m-1} \binom{m}{i}^2 \left(\sum_{k=0}^{i-1} \frac{i!}{k!} - 1 \right) \\ &= \sum_{i=1}^{m-1} \binom{m}{i}^2 \sum_{k=0}^{i-1} \frac{i!}{k!} - \binom{2m}{m} + 2 \\ &= \sum_{i=1}^{m-1} \binom{m}{i}^2 \sum_{k=0}^{i} \frac{i!}{k!} - 2\binom{2m}{m} + 4. \end{aligned} \tag{72, 73}$$

Since

$$i! \sum_{k=0}^{i} \frac{1}{k!} = e \int_1^\infty x^i e^{-x} \mathrm{d}x, \tag{74}$$

it follows that

$$\tilde{C}_m = e \int_1^\infty \left(\sum_{i=1}^{m-1} \binom{m}{i}^2 x^i \right) e^{-x} \mathrm{d}x - 2\binom{2m}{m} + 4. \tag{75}$$

Furthermore, the arithmetic-mean–geometric-mean inequality implies that

$$\sum_{i=1}^{m-1} \binom{m}{i}^2 x^i \geq (m-1) \sqrt[m-1]{\prod_{j=1}^{m-1} \binom{m}{j}^2} \, x^{\frac{m}{2}}, \tag{76}$$

and thus

$$\tilde{C}_m \geq b_m, \tag{77}$$

where

$$b_m \triangleq e(m-1) \sqrt[m-1]{\prod_{j=1}^{m-1} \binom{m}{j}^2} \int_1^\infty x^{\frac{m}{2}} e^{-x} \mathrm{d}x - 2\binom{2m}{m} + 4. \tag{78}$$

Using (Furdui, 2013, Problem 1.5, p. 2), we have

$$\lim_{m\to\infty} \frac{\sqrt[m]{\prod_{j=1}^{m} \binom{m}{j}}}{e^{m/2} m^{-1/2}} = \frac{e}{\sqrt{2\pi}}, \tag{79}$$

and thus

$$(m-1)\sqrt[m-1]{\prod_{j=1}^{m-1}\binom{m}{j}^2} \sim e^m. \tag{80}$$

In addition,

$$\begin{aligned}
\int_1^\infty x^{\frac{m}{2}} e^{-x}\mathrm{d}x &= \int_0^\infty x^{\frac{m}{2}} e^{-x} - \int_0^1 x^{\frac{m}{2}} e^{-x}\mathrm{d}x \\
&= \Gamma\left(\frac{m+2}{2}\right) - O\left(\frac{1}{m}\right) \\
&\sim \left(\frac{m}{2}\right)! \\
&\sim \sqrt{m}\left(\frac{m}{2e}\right)^{\frac{m}{2}}.
\end{aligned} \tag{81}$$

Using Stirling's formula $\binom{2m}{m} \sim \frac{4^m}{\sqrt{m}}$ and

$$\lim_{m\to\infty} \frac{4^m}{m\left(\frac{m}{2}\right)^{\frac{m}{2}} e^{\frac{m}{2}}} = 0, \tag{82}$$

it follows that

$$\begin{aligned}
b_m &\sim e^m \left(\frac{m}{2}\right)! - 2\binom{2m}{m} \\
&= \sqrt{m}\left(\frac{m}{2}\right)^{\frac{m}{2}} e^{\frac{m}{2}} - 2\frac{4^m}{\sqrt{m}} \\
&\sim \sqrt{m}\left(\frac{m}{2}\right)^{\frac{m}{2}} e^{\frac{m}{2}}.
\end{aligned} \tag{83}$$

It thus follows from (77) and (83) that $C_m - \bar{C}_m$ asymptotically grows more quickly than $e^{m/2}$. Figure 1 shows exact values of C_m and $\bar{C}_m$ for $m = 2, \ldots, 10$ plotted logarithmically.

Remark: Note that computing the McMillan degree for nonsquare transfer functions is performed by first multiplying the nonsquare transfer function by its transpose to obtain a square transfer function. Therefore, the computational cost for computing the McMillan degree of nonsquare transfer functions has the same order as the computational cost of computing the McMillan degree of square transfer functions.

10. Conclusions

This paper introduced a method to determine the McMillan degree of a full-normal-rank transfer function matrix G without knowledge of the invariant polynomials of G and without the need to compute all of the minors of G. We assumed that the roots of the least common multiple of the denominators of the entries of G satisfy a multiplicity condition. This condition can be verified prior to the application of the algorithm. We showed that the proposed method outperformes the standard choice

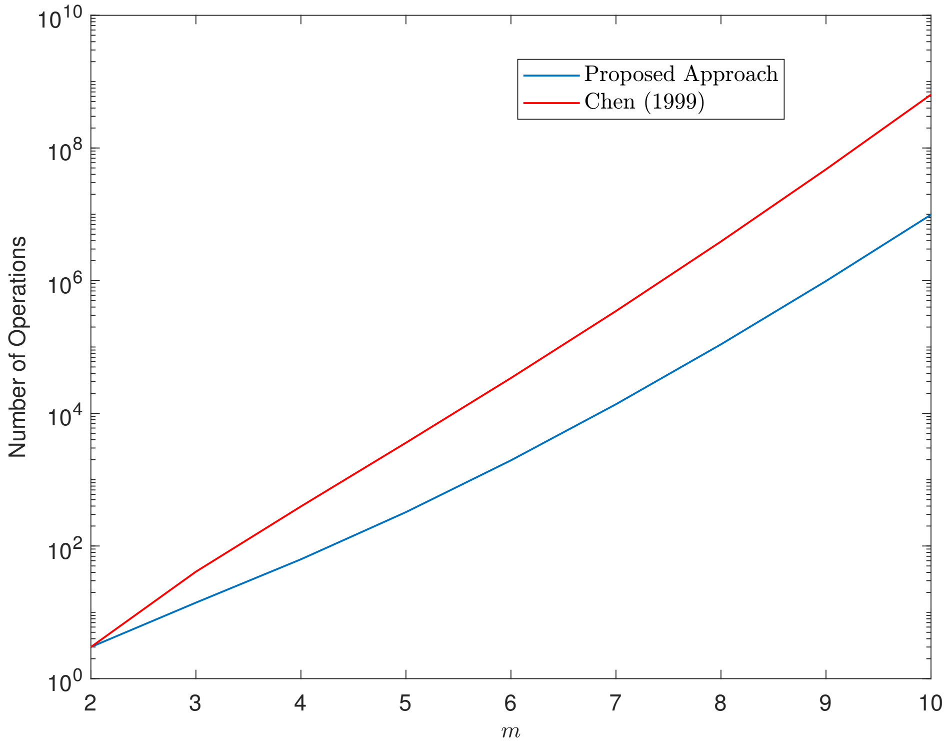

Figure 1. Number of addition and multiplication operations C_m required by Theorem 3 (Chen, 1999) and $\bar{C}_m$ required by Theorem 4 (the proposed approach) to compute the McMillan degree of a transfer function of size $m \times m$.

used in the literature to compute the McMillan degree in terms of the number of required computations.

Acknowledgments

This research was supported in part by NSF grant CMMI 1536834.

References

Chen, C.-T. (1999). *Linear System Theory and Design*. Oxford University Press.

Furdui, O. (2013). *Limits, Series, and Fractional Part Integrals. Problems in Mathematical Analysis*. New York: Springer.

Gupta, R., & Fairman, F. (1974). Derivation of the ho algorithm. *IEEE Transactions on Automatic Control*, *19*, 270–271.

Huang, H.-L. (1974). A generalized-jordan-form-approach one-step irreducible realization of matrices. *IEEE Transactions on Automatic Control*, *19*, 271–272.

Kailath, T. (1980). *Linear Systems*. NJ: Prentice-Hall.

Kalman, R. E. (1963). Mathematical description of linear dynamical systems. *Journal* *of the Society for Industrial and Applied Mathematics, Series A: Control*, *1*, 152–192.

Karcanias, N., Sagianos, E., & Milonidis, E. (2007). Structured transfer function matrices and integer matrices: the computation of the generic mcmillan degree and infinite zero structure. *International Journal of Control*, *80*, 1404–1420.

MacFarlane, A., & Karcanias, N. (1976). Poles and zeros of linear multivariable systems: a survey of the algebraic, geometric and complex-variable theory. *International Journal of Control*, *24*, 33–74.

Mayna, D. (1968). Computational procedure for the minimal realisation of transfer-function matrices. In *Proceedings of the Institution of Electrical Engineers* (pp. 1363–1368). IET volume 115.

Munro, N., & McLeod, R. (1971). Minimal realisation of transfer-function matrices using the system matrix. In *Proceedings of the Institution of Electrical Engineers* (pp. 1298–1301). IET volume 118.

Roberts, D. (1969). A computer program for the minimal realization of transfer function matrices. *IEEE Transactions on Automatic Control*, *14*, 412–413.

Rosenbrock, H. (1967). On linear system theory. In *Proceedings of the Institution of Electrical Engineers* (pp. 1353–1359). IET volume 114.

Shaked, U., & Dixon, M. (1977). Generalized minimal realization of transfer-function matrices. *International Journal of Control*, *25*, 785–803.

Van Dooren, P., Dewilde, P., & Vandewalle, J. (1979). On the determination of the smith-macmillan form of a rational matrix from its laurent expansion. *IEEE Transactions on Circuits and systems*, *26*, 180–189.

Vardulakis, A. I. G. (1991). *Linear Multivariable Control: Algebraic Analysis and Synthesis Method*. Chichester: Wiley.