Revisiting Minimal Realizations

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Revisiting Minimal Realizations

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Let G be a $p \times m$ transfer function with McMillan degree n, and let (A_1, B_1, C_1, D) be a minimal realization of G. Consider a matrix A_2 that is similar to A_1 and an *arbitrary* $n \times m$ matrix B_2 such that (A_2, B_2) is controllable. Since B_1 and B_2 are unrelated, it would seem unlikely that a $p \times n$ matrix C_2 could be found such that (A_2, B_2, C_2, D) is also a minimal realization of G. This article shows that this is always possible in the single-input, multiple-output (SIMO) case, that is, when $m = 1$ (see "Summary"). This result was implicitly used in the groundbreaking article [1] to facilitate the construction of a globally asymptotically stabilizing controller for a chain of integrators under bounded control inputs. A dual result holds in the multiple-input, single-output (MISO) case, where $p = 1$ and C_1 is replaced by an arbitrary matrix C_2 such that (A_2, C_2) is observable. The purpose of this "Lecture Note" is to prove these results. For the multiple-input, multiple-output (MIMO) case, necessary and sufficient conditions are given under which an analogous result holds.

The results in this article are formulated in terms of transfer functions depending on the variable s. However, since the time-domain response is not considered, all the results in this article can be interpreted in either the Laplace domain for continuous-time systems or the z-transform domain for discrete-time systems, where it is customary to write z in place of s.

Summary

Although realizations of transfer functions are a standard topic in textbooks about linear systems theory, this article focuses on several points that are not widely treated. First, it is shown that, for a given dynamics matrix A, the smallest number of sensors and actuators that can be used to form a minimal realization is determined by properties of the Jordan form of A. This leads to the notions of minimally sensed and minimally actuated systems. Next, the rank of a transfer function is related to properties of the Jordan form of A and the rank of the Rosenbrock system matrix.

The rest of the article is devoted to the following question. Given a minimal realization (A_1, B_1, C_1, D) of G and a controllable pair (A_2, B_2), such that A_1 and A_2 are similar and B_1 and B_2 have the same number of columns, under what conditions does there exist a matrix C_2 that has the same number of rows as C_1 such that (A_2, B_2, C_2, D) is also a minimal realization of G? Perhaps surprisingly, it is shown that this is always possible in the case where G has a single input, and a dual result holds in the case where G has a single output. Finally, necessary and sufficient conditions are given in the case where G has multiple inputs and multiple outputs. This "Lecture Note" provides a convenient resource for graduate students who are learning the elements of linear systems theory and instructors who may wish to demonstrate interesting features of minimal realizations.

PROPERTIES OF MINIMAL REALIZATIONS

Let $\mathbb{R}(s)^{p\times m}$ denote the set of $p \times m$ matrices (each of whose entries is a rational function with real coefficients), and let $\mathbb{R}(s)^{p\times m}_{\text{prop}}$ denote the proper transfer functions in $\mathbb{R}(s)^{p\times m}$. Let $G \in \mathbb{R}(s)^{p\times m}_{\text{prop}}$, and let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$. Then, (A, B, C, D) is a *realization* of G if $G(s) = C(sI - A)^{-1}B + D$. Note that $D = G(\infty)$. Now, assume that (A, B, C, D) is a realization of G. Then, (A, B, C, D) is a *minimal realization* of G, and n is the *McMillan degree* of G if there do not exist $\hat{A} \in \mathbb{R}^{(n-1)\times(n-1)}$, $\hat{B} \in \mathbb{R}^{(n-1)\times m}$, and $\hat{C} \in \mathbb{R}^{p\times(n-1)}$ such that $(\hat{A}, \hat{B}, \hat{C}, D)$ is a realization of G.

For $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, define the controllability matrix

$$\mathcal{C}(A, B) \triangleq [B \; AB \cdots A^{n-1}B] \in \mathbb{R}^{n\times nm}.$$

The pair (A, B) is *controllable* if rank $\mathcal{C}(A, B) = n$. The following result is the *Popov–Belevitch–Hautus (PBH) test for controllability* [2, Th. 16.6.19].

Theorem 1

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Then, (A, B) is controllable if and only if, for all $\lambda \in \mathbb{C}$,

$$\text{rank}[\lambda I_n - A \;\; B] = n. \tag{1}$$

For $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$, define the observability matrix

$$\mathcal{O}(A,C) \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{np\times n}.$$

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The pair (A, C) is *observable* if $\operatorname{rank} \mathcal{O}(A,C) = n$. The following result is the *PBH test for observability* [2, Th. 16.3.19].

Theorem 2

Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$. Then, (A, C) is observable if and only if, for all $\lambda \in \mathbb{C}$,

$$\operatorname{rank}\begin{bmatrix} \lambda I_n - A \\ C \end{bmatrix} = n. \tag{2}$$

The triple (A, B, C) is *controllable and observable* if (A, B) is controllable and (A, C) is observable.

MINIMALLY SENSED AND MINIMALLY ACTUATED TRANSFER FUNCTIONS

If $\operatorname{rank} C < p$, then at least one of the sensors that defines C is *redundant*; that is, the output of C from that sensor can be reconstructed by forming a linear combination of the remaining sensor outputs. Likewise, if $\operatorname{rank} B < m$, then at least one of the actuators that defines B is *redundant*; that is, the input to B from that actuator can be reproduced by forming a linear combination of the remaining actuator inputs.

For $A \in \mathbb{R}^{n\times n}$, let $\mu(A)$ denote the maximum over all eigenvalues λ of A of the geometric multiplicity (that is, the number of associated Jordan blocks) of λ. For the $p \times m$ proper transfer function G with minimal realization (A, B, C, D), define $\mu(G) \triangleq \mu(A)$.

The following result, which follows from Theorem 1 and Theorem 2, shows that, for a given matrix A, the smallest values of p and m such that (A, B, C) is controllable and observable are $p = m = \mu(A)$.

Proposition 1

Let $A \in \mathbb{R}^{n\times n}$. Then, the following statements hold:

1) Let $B \in \mathbb{R}^{n\times m}$, and assume that (A, B) is controllable. Then, $\mu(A) \le \operatorname{rank} B$.
2) Let $B \in \mathbb{R}^{n\times \mu(A)}$, and assume that (A, B) is controllable. Then, $\mu(A) = \operatorname{rank} B$.
3) There exists $B \in \mathbb{R}^{n\times \mu(A)}$ such that (A, B) is controllable.
4) Let $C \in \mathbb{R}^{p\times n}$, and assume that (A, C) is observable. Then, $\mu(A) \le \operatorname{rank} C$.
5) Let $C \in \mathbb{R}^{\mu(A)\times n}$, and assume that (A, C) is observable. Then, $\mu(A) = \operatorname{rank} C$.
6) There exists $C \in \mathbb{R}^{\mu(A)\times n}$ such that (A, C) is observable.
7) There exist $B \in \mathbb{R}^{n\times \mu(A)}$ and $C \in \mathbb{R}^{\mu(A)\times n}$ such that (A, B, C) is controllable and observable.

Statement 2 of Proposition 1 implies that if $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n}$, and (A, B) is controllable, then the geometric multiplicity of every eigenvalue of A is one, and thus the Jordan form of A has, at most, one block corresponding to each distinct eigenvalue of A. Matrices of this type are called *cyclic*. If $A \in \mathbb{R}^{n\times n}$ is cyclic, then there exists $B \in \mathbb{R}^{n}$ such that (A, B) is controllable. For example,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{3}$$

is controllable. Note that $\mu(A) = 1$, and thus A is cyclic. However, A has a 2×2 Jordan block and thus is not diagonalizable.

Statement 7 of Proposition 1 shows that, for a given dynamics matrix A, a minimal realization (A, B, C, D) can be constructed from $\mu(A)$ inputs and $\mu(A)$ outputs but not fewer. A $p \times m$ proper transfer function G is *minimally sensed* if $p = \mu(G)$; in this case, up to $p - \mu(A)$ sensors may be redundant. Analogously, G is *minimally actuated* if $m = \mu(G)$; in this case, up to $m - \mu(A)$ actuators may be redundant. For example,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{4}$$

is controllable and minimally actuated, and thus it follows from statement 2 of Proposition 1 that (A, B) has no redundant actuators. On the other hand,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

is controllable and has no redundant actuators, but it is not minimally actuated.

CHARACTERIZING THE RANK OF *G*

Let G be a $p \times m$ proper transfer function. Then, the *rank* of G is the maximum value of $\operatorname{rank} G(s)$ taken over the set of complex numbers s such that for all $i = 1,\dots,p$ and $j = 1,\dots,m$, s is not a pole of the (i, j) entry of G. Note that $\operatorname{rank} G \le \min\{p, m\}$. Furthermore, G has a *full row rank* if $\operatorname{rank} G = p$, and G has a *full column rank* if $\operatorname{rank} G = m$. Let $\operatorname{rank}_{\mathbb{R}(s)} G$ denote the number of linearly independent columns of G over the field $\mathbb{R}(s)$ of rational functions with real coefficients. Finally, let $\operatorname{colrank}_{\mathbb{R}} G$ denote the number of linearly independent columns of G over the field $\mathbb{R}$, and let $\operatorname{rowrank}_{\mathbb{R}} G$ denote the number of linearly independent rows of G over the field $\mathbb{R}$.

The polynomial matrix $P \in \mathbb{R}[s]^{n\times n}$ is *unimodular* if $\det P$ is a nonzero real number. The following result given by [3, p. 443] and [2, Th. 6.7.5, p. 514] presents the *Smith–McMillan form* S of G.

Theorem 3

Let $G \in \mathbb{R}(s)^{p\times m}_{\text{prop}}$, and let $\rho \triangleq \operatorname{rank} G$. Then, there exist unimodular matrices $S_1 \in \mathbb{R}[s]^{p\times p}$ and $S_2 \in \mathbb{R}[s]^{m\times m}$ and unique monic polynomials $p_1,\dots,p_\rho, q_1,\dots,q_\rho \in \mathbb{R}[s]$ such that for all $i \in \{1,\dots,\rho\}$, p_i and q_i are coprime for all $i \in \{1,\dots,\rho - 1\}$, p_i divides p_{i+1} and q_{i+1} divides q_i, and $G = S_1 S S_2$, where

$$S \triangleq \begin{bmatrix} p_1/q_1 & & & 0_{\rho\times(m-\rho)} \\ & \ddots & & \\ & & p_\rho/q_\rho & \\ 0_{(p-\rho)\times\rho} & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix}. \tag{6}$$

Note that the roots of the polynomial $q_1 q_2 \cdots q_\rho$ are the poles of G, and the roots of the polynomial $p_1 p_2 \cdots p_\rho$ are the transmission zeros of G.

Theorem 4

Let $G \in \mathbb{R}(s)_{\text{prop}}^{p\times m}$ with McMillan degree n and minimal realization (A, B, C, D), and let $\rho \triangleq \operatorname{rank} G$. Let $\operatorname{spec}(A) = \{\lambda_1, \ldots, \lambda_r\}$, where $\lambda_1, \ldots, \lambda_r$ are distinct, and let $q_1, \ldots, q_\rho$ be as defined in (6). Then, for all $k = 1, \ldots, \rho$, $q_k = (s - \lambda_1)^{b_{1k}}(s - \lambda_2)^{b_{2k}} \cdots (s - \lambda_r)^{b_{rk}}$, where for all $i = 1, \ldots, r$, $b_{i1} \geq b_{i2} \geq \cdots \geq b_{in_i} > 0 = b_{i,n_i+1} = \cdots = b_{i\rho}$, n_i is the geometric multiplicity of λ_i, and, for all $j = 1, \ldots, n_i$, the jth Jordan block associated with λ_i is $b_{ij} \times b_{ij}$.

Proof

It follows from Theorem 3 that, for all $k = 1, \ldots, \rho - 1$, q_{k+1} divides q_k, and the roots of the polynomial $q_1 q_2 \ldots q_\rho$ are the poles of G (and thus the eigenvalues of A). Hence, for all $i = 1, \ldots, r$, there exist $\hat{n}_i \leq \rho$ and $c_{i1} \geq c_{i2} \geq \cdots \geq c_{i\hat{n}_i} > 0 = c_{i,\hat{n}_i+1} = \cdots = c_{i\rho}$ such that, for all $k = 1, \ldots, \rho$, $q_k = (s - \lambda_1)^{c_{1k}}(s - \lambda_2)^{c_{2k}} \cdots (s - \lambda_r)^{c_{rk}}$. Next, it follows from the construction in sections 7 and 8 in [4] that there exists a minimal realization $(\bar{A}, \bar{B}, \bar{C}, D)$ of G such that $\bar{A}$ is in Jordan form, where, for all $i = 1, \ldots, r$, $\bar{A}$ has $\hat{n}_i$ Jordan blocks associated with λ_i, and, for all $j = 1, \ldots, \hat{n}_i$, the jth Jordan block associated with λ_i is $c_{ij} \times c_{ij}$. Since $\bar{A}$ is the Jordan form of A, it follows that, for all $i = 1, \ldots, r$, $\hat{n}_i = n_i$ and, for all $j = 1, \ldots, \rho$, $c_{ij} = b_{ij}$. ☐

The following result provides lower and upper bounds for $\operatorname{rank} G$. This result uses the *Rosenbrock system matrix*

$$\mathcal{Z}(s) \triangleq \begin{bmatrix} sI_n - A & B \\ C & -D \end{bmatrix} \in \mathbb{R}[s]^{(n+p)\times(n+m)}, \tag{7}$$

which can be used to characterize the invariant zeros of a realization (A, B, C, D) of a transfer function G [2, Sec. 16.10].

Proposition 2

Let $G \in \mathbb{R}(s)_{\text{prop}}^{p\times m}$ with McMillan degree n and minimal realization (A, B, C, D). Then,

$$\mu(G) \leq \operatorname{rank} G \tag{8}$$
$$= \operatorname{rank}_{\mathbb{R}(s)} G \tag{9}$$
$$= \operatorname{rank} \mathcal{Z} - n \tag{10}$$
$$\leq \begin{cases} \operatorname{rank} B \leq \operatorname{colrank}_{\mathbb{R}} G \leq m \\ \operatorname{rank} C \leq \operatorname{rowrank}_{\mathbb{R}} G \leq p \end{cases} \tag{11}$$

and

$$\operatorname{rank} D \leq \min\{\operatorname{colrank}_{\mathbb{R}} G, \operatorname{rowrank}_{\mathbb{R}} G\}. \tag{12}$$

Proof

Let $\operatorname{spec}(A) = \{\lambda_1, \ldots, \lambda_r\}$, where $\lambda_1, \ldots, \lambda_r$ are distinct, and, for all $i = 1, \ldots, r$, let n_i be the geometric multiplicity of λ_i. Theorem 4 implies that, for all $i = 1, \ldots, r$, $n_i \leq \operatorname{rank} G$. Since $\mu(G) = \max_i n_i$, it follows that $\mu(G) \leq \operatorname{rank} G$. Next, (9) follows from [2, Prop. 6.7.9, p. 515], and (10) follows from [2, Prop. 16.10.3, p. 1280]. Let $G = [G_1 \ \cdots \ G_m]$, $B = [B_1 \ \cdots \ B_m]$, and $D = [D_1 \ \cdots \ D_m]$, where $G_1, \ldots, G_m, D_1, \ldots, D_m$ are $p \times 1$ and $B_1, \ldots, B_m$ are $n \times 1$. Suppose that $G_1 = \alpha_2 G_2 + \cdots + \alpha_m G_m$, where $\alpha_2, \ldots, \alpha_m$ are real numbers. Then,

$$C(sI - A)^{-1}\bar{B} + \bar{D} = 0, \tag{13}$$

where $\bar{B} = B_1 - (\alpha_2 B_2 + \cdots + \alpha_m B_m)$ and $\bar{D} = D_1 - (\alpha_2 D_2 + \cdots + \alpha_m D_m)$. Taking the inverse Laplace transform of (13) implies that, for all $t \geq 0$, $Ce^{At}\bar{B} + \delta(t)\bar{D} = 0$. Hence, for all $t > 0$,

$$Ce^{At}\bar{B} = 0. \tag{14}$$

Taking the limit of (14) as $t \to 0$ yields $C\bar{B} = 0$. Furthermore, differentiating (14) and taking the limit as $t \to 0$ yields $CA\bar{B} = 0$. Proceeding similarly, it follows that $\mathcal{O}_n \bar{B} = 0$, where $\mathcal{O}_n$ is the observability matrix obtained from (A, C), and thus $\bar{B} = 0$. Hence, $B_1 = \alpha_2 B_2 + \cdots + \alpha_m B_m$, and thus $\operatorname{rank} B \leq \operatorname{colrank}_{\mathbb{R}} G$. Next, taking the limit of (13) as $s \to \infty$ yields $\bar{D} = 0$. Therefore, $D_1 = \alpha_2 D_2 + \cdots + \alpha_m D_m$, and thus $\operatorname{rank} D \leq \operatorname{colrank}_{\mathbb{R}} G$. The proofs of $\operatorname{rank} C \leq \operatorname{rowrank}_{\mathbb{R}} G$ and $\operatorname{rank} D \leq \operatorname{rowrank}_{\mathbb{R}} G$ follow in a similar way. ☐

The following three examples demonstrate Proposition 2.

Example 1

Let

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ \frac{1}{s-1} & \frac{s+1}{s(s-1)} \end{bmatrix}. \tag{15}$$

A minimal realization of G is

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = 0. \tag{16}$$

Note that $\mu(G) = \operatorname{rank} G = \operatorname{rank}_{\mathbb{R}(s)} G = \operatorname{rank} \mathcal{Z} - 3 = 1 \leq \operatorname{rank} B = \operatorname{colrank}_{\mathbb{R}} G = \operatorname{rank} C = \operatorname{rowrank}_{\mathbb{R}} G = 2$. ◊

Example 2

Let

$$G(s) = \begin{bmatrix} \frac{5s}{s^3 - 2s} & \frac{10s^2}{s^3 - 2s} \\ \frac{6}{s^3 - 2s} & \frac{12s}{s^3 - 2s} \end{bmatrix}. \tag{17}$$

A minimal realization of G is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad D = 0. \tag{18}$$

The eigenvalues of A are 0 and $\pm\sqrt{2}$. Note that $\mu(G) = \operatorname{rank} G = \operatorname{rank}_{\mathbb{R}(s)} G = \operatorname{rank} \mathcal{Z} - 3 = 1 \le \operatorname{rank} B = \operatorname{colrank}_{\mathbb{R}} G = \operatorname{rank} C = \operatorname{rowrank}_{\mathbb{R}} G = 2.$ ◊

Example 3

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and note that the geometric multiplicity of the eigenvalue zero of A is 2. Next, choose

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = 0.$$

Then, (A, B, C, D) is a minimal realization of

$$G(s) = \begin{bmatrix} \frac{1}{s^2} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}.$$

Note that $\mu(G) = \operatorname{rank} G = \operatorname{rank} G_{\mathbb{R}(s)} = \operatorname{rank} \mathcal{Z} - 3 = \operatorname{rank} B = \operatorname{colrank}_{\mathbb{R}} G = \operatorname{rank} C = \operatorname{rowrank}_{\mathbb{R}} G = 2.$ ◊

CHANGING THE BASIS OF A REALIZATION

Theorem 5

Let $G \in \mathbb{R}(s)^{p\times m}_{\text{prop}}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$, and assume that (A, B, C, D) is a realization of G. Then, the following statements hold:

1) If $S \in \mathbb{R}^{n\times n}$ is nonsingular, then $(SAS^{-1}, SB, CS^{-1}, D)$ is a realization of G.
2) Assume that (A, B, C, D) is a minimal realization of G, and let $\hat{A} \in \mathbb{R}^{n\times n}$, $\hat{B} \in \mathbb{R}^{n\times m}$, and $\hat{C} \in \mathbb{R}^{p\times n}$. Then, $(\hat{A}, \hat{B}, \hat{C}, D)$ is a minimal realization of G if and only if there exists a nonsingular matrix $S \in \mathbb{R}^{n\times n}$ such that $\hat{A} = SAS^{-1}$, $\hat{B} = SB$, and $\hat{C} = CS^{-1}$.
3) (A, B, C, D) is a minimal realization of G if and only if (A, B, C) is controllable and observable.
4) Assume that (A, B, C, D) and $(\hat{A}, \hat{B}, \hat{C}, D)$ are minimal realizations of G. Then, $\mathcal{C}(A, B)\mathcal{C}(\hat{A}, \hat{B})^T$ and $\mathcal{O}(\hat{A}, \hat{C})^T \mathcal{O}(\hat{A}, \hat{C})$ are nonsingular, and $\hat{A} = SAS^{-1}$, $\hat{B} = SB$, and $\hat{C} = CS^{-1}$, where

$$S \triangleq \mathcal{C}(\hat{A}, \hat{B})\, \mathcal{C}(\hat{A}, \hat{B})^{\mathrm{T}} [\mathcal{C}(A, B)\, \mathcal{C}(\hat{A}, \hat{B})^{\mathrm{T}}]^{-1} \tag{19}$$

$$= [\mathcal{O}(\hat{A}, \hat{C})^{\mathrm{T}}\, \mathcal{O}(\hat{A}, \hat{C})]^{-1}\, \mathcal{O}(\hat{A}, \hat{C})^{\mathrm{T}}\, \mathcal{O}(A, C). \tag{20}$$

The following result shows that two realizations of a transfer function have the same Markov parameters and vice versa.

Theorem 6

Let $G \in \mathbb{R}(s)^{p\times m}_{\text{prop}}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$, and assume that (A, B, C, D) is a realization of G. Furthermore, let $\hat{A} \in \mathbb{R}^{\hat{n}\times \hat{n}}$, $\hat{B} \in \mathbb{R}^{\hat{n}\times m}$, and $\hat{C} \in \mathbb{R}^{p\times \hat{n}}$. Then, $(\hat{A}, \hat{B}, \hat{C}, D)$ is a realization of G if and only if, for all $i = 0, 1, \ldots, n-1$, $\hat{C}\hat{A}^i\hat{B} = CA^iB$.

Proof

Necessity is given by [2, Lemma 16.9.7]. Sufficiency follows from the Ho–Kalman realization [2, Prop. 16.9.15]. □

MINIMAL REALIZATIONS IN SINGLE-INPUT, MULTIPLE-OUTPUT AND MULTIPLE-INPUT, SINGLE-OUTPUT CASES

Proposition 3

Let A_1 and A_2 be real $n \times n$ matrices that are similar, let B_1 and B_2 be $n \times 1$ matrices, assume that (A_1, B_1) and (A_2, B_2) are controllable, and define

$$S \triangleq \mathcal{C}(A_2, B_2)\, \mathcal{C}(A_1, B_1)^{-1}. \tag{21}$$

Then,

$$A_2 = SA_1S^{-1}, \tag{22}$$

$$B_2 = SB_1. \tag{23}$$

Proof

For all $i \in \{1, \ldots, n\}$, let e_i denote the ith column of I_n. Note that $B_1 = \mathcal{C}(A_1, B_1)e_1$, which implies that $\mathcal{C}(A_1, B_1)^{-1}B_1 = e_1$. Thus,

$$\begin{aligned} SB_1 &= \mathcal{C}(A_2, B_2)\, \mathcal{C}(A_1, B_1)^{-1} B_1 \\ &= \mathcal{C}(A_2, B_2) e_1 \\ &= B_2. \end{aligned} \tag{24}$$

Next note that, for all $i \in \{0, 1, \ldots, n-1\}$, $A_1^i B_1 = \mathcal{C}(A_1, B_1)\, e_{i+1}$, and thus $\mathcal{C}(A_1, B_1)^{-1} A_1^i B_1 = e_{i+1}$. Therefore, for all $i \in \{0, 1, \ldots, n-1\}$,

$$\begin{aligned} SA_1^i B_1 &= \mathcal{C}(A_2, B_2)\, \mathcal{C}(A_1, B_1)^{-1} A_1^i B_1 \\ &= \mathcal{C}(A_2, B_2) e_{i+1} \\ &= A_2^i B_2, \end{aligned} \tag{25}$$

which implies that, for all $i \in \{0, 1, \ldots, n-2\}$,

$$\begin{aligned} SA_1 S^{-1} A_2^i B_2 &= SA_1 A_1^i B_1 \\ &= SA_1^{i+1} B_1 \\ &= A_2^{i+1} B_2 \\ &= A_2 A_2^i B_2. \end{aligned} \tag{26}$$

Now, let the characteristic polynomial χ of A_1 be written as $\chi(s) \triangleq s^n + a_{n-1}s^{n-1} + \cdots + a_0$, where, for all $i \in \{0, \ldots, n-1\}$, $a_i \in \mathbb{R}$. Since A_1 and A_2 are similar, it follows that $\chi(A_1) = \chi(A_2) = 0$, and thus

$$A_1^n = -\sum_{i=0}^{n-1} a_i A_1^i, \qquad A_2^n = -\sum_{i=0}^{n-1} a_i A_2^i. \tag{27}$$

Using (25), it follows that

$$
\begin{aligned}
SA_1S^{-1}A_2^{n-1}B_2 &= SA_1A_1^{n-1}B_1\\
&= SA_1^nB_1\\
&= -S\sum_{i=0}^{n-1} a_iA_1^iB_1\\
&= -\sum_{i=0}^{n-1} a_iA_2^iB_2\\
&= A_2^nB_2\\
&= A_2A_2^{n-1}B_2. \qquad (28)
\end{aligned}
$$

Therefore, (26) and (28) imply that, for all $i \in \{0, 1, \ldots, n-1\}$,

$$SA_1S^{-1}A_2^iB_2 = A_2A_2^iB_2;$$

that is,

$$(SA_1S^{-1} - A_2)A_2^iB_2 = 0.$$

Hence,

$$(SA_1S^{-1} - A_2)\,\mathcal{C}(A_2, B_2) = 0,$$

which, since $\mathcal{C}(A_2, B_2)$ is nonsingular, implies that $SA_1S^{-1} = A_2$. □

The following result is an immediate consequence of Proposition 3.

Theorem 7

Let $G \in \mathbb{R}(s)_{\text{prop}}^{p\times 1}$ with nth-order minimal realization (A_1, B_1, C_1, D), let $A_2 \in \mathbb{R}^{n\times n}$, assume that A_1 and A_2 are similar, let $B_2 \in \mathbb{R}^{n\times 1}$, assume that (A_2, B_2) is controllable, and define S by (21) and $C_2 \triangleq C_1S^{-1}$. Then, (A_2, B_2, C_2, D) is a minimal realization of G, and

$$A_2 = SA_1S^{-1}, \qquad (29)$$

$$B_2 = SB_1. \qquad (30)$$

Furthermore, $\mathcal{O}(A_2, C_2)^{\mathrm{T}}\mathcal{O}(A_2, C_2)$ is nonsingular, and

$$S = [\mathcal{O}(A_2, C_2)^{\mathrm{T}}\mathcal{O}(A_2, C_2)]^{-1}\mathcal{O}(A_2, C_2)^{\mathrm{T}}\mathcal{O}(A_1, C_1). \qquad (31)$$

Proof

With S defined by (21), Proposition 1 implies that $A_2 = SA_1S^{-1}$ and that $B_2 = SB_1$. Therefore, with C_2 defined by $C_2 \triangleq C_1S^{-1}$, it follows that (A_2, B_2, C_2, D) is a minimal realization of G. Finally, since (A_1, B_1, C_1, D) and (A_2, B_2, C_2, D) are both minimal realizations of G, (31) follows from [2, Prop. 16.9.8]. □

Replacing G by G^{T} yields the following dual version of Theorem 7.

Theorem 8

Let $G \in \mathbb{R}(s)_{\text{prop}}^{1\times m}$ with nth-order minimal realization (A_1, B_1, C_1, D), let $A_2 \in \mathbb{R}^{n\times n}$, assume that A_1 and A_2 are similar, let $C_2 \in \mathbb{R}^{1\times n}$, and assume that (A_2, C_2) is observable. Define S by

$$S \triangleq \mathcal{O}(A_2, C_2)^{-1}\mathcal{O}(A_1, C_1), \qquad (32)$$

and $B_2 \triangleq SB_1$. Then, (A_2, B_2, C_2, D) is a minimal realization of G, and

$$A_2 = SA_1S^{-1}, \qquad (33)$$

$$C_2 = C_1S^{-1}. \qquad (34)$$

Furthermore, $\mathcal{C}(A_1, B_1)\,\mathcal{C}(A_2, B_2)^{\mathrm{T}}$ is nonsingular, and

$$S = \mathcal{C}(A_2, B_2)\,\mathcal{C}(A_2, B_2)^{\mathrm{T}}[\mathcal{C}(A_1, B_1)\,\mathcal{C}(A_2, B_2)^{\mathrm{T}}]^{-1}. \qquad (35)$$

Proof

The result follows from Theorem 7 by noting that $(A_1^{\mathrm{T}}, C_1^{\mathrm{T}}, B_1^{\mathrm{T}}, D^{\mathrm{T}})$ is a minimal realization of G^{T}, $\mathcal{C}(A_1^{\mathrm{T}}, C_1^{\mathrm{T}}) = \mathcal{O}(A_1, C_1)^{\mathrm{T}}$, $\mathcal{C}(A_2^{\mathrm{T}}, C_2^{\mathrm{T}}) = \mathcal{O}(A_2, C_2)^{\mathrm{T}}$, $\mathcal{O}(A_1^{\mathrm{T}}, B_1^{\mathrm{T}}) = \mathcal{C}(A_1, B_1)^{\mathrm{T}}$, and $\mathcal{O}(A_2^{\mathrm{T}}, B_2^{\mathrm{T}}) = \mathcal{C}(A_2, B_2)^{\mathrm{T}}$. □

The following example uses Theorem 7 to construct an alternative realization of a SIMO transfer function.

Example 4

Consider the 2×1 SIMO transfer function

$$G(s) = \begin{bmatrix} \dfrac{s^2 + 17s + 34}{s^3 + 4s^2 - 4s - 16} \\[2mm] \dfrac{2s^2 + 19s + 26}{s^3 + 4s^2 - 4s - 16} \end{bmatrix}, \qquad (36)$$

which has the minimal realization (A_1, B_1, C_1, D), where

$$A_1 = \begin{bmatrix} -6 & 5 & 3 \\ -4 & 3 & -3 \\ 0 & 3 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D = 0. \qquad (37)$$

Let A_2 and B_2 be given by

$$A_2 = \begin{bmatrix} -2 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 2 & -4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \qquad (38)$$

Since A_2 and A_1 have the same eigenvalues (namely, -2, 2, and -4), it follows that they are similar. Furthermore, note that

$$\mathcal{C}(A_1, B_1) = \begin{bmatrix} 1 & 13 & -14 \\ 2 & 11 & -10 \\ 3 & 3 & 30 \end{bmatrix}, \quad \mathcal{C}(A_2, B_2) = \begin{bmatrix} 1 & -2 & 4 \\ 1 & 6 & 4 \\ 1 & 3 & -10 \end{bmatrix} \qquad (39)$$

are nonsingular, and define

$$S \triangleq \mathcal{C}(A_2, B_2)\,\mathcal{C}(A_1, B_1)^{-1} = \frac{1}{18}\begin{bmatrix} -18 & 18 & 0 \\ 12 & -6 & 6 \\ -15 & 24 & -5 \end{bmatrix}, \qquad (40)$$

$$C_2 \triangleq C_1S^{-1} = \frac{1}{14}\begin{bmatrix} -19 & 15 & 18 \\ -5 & 15 & 18 \end{bmatrix}. \qquad (41)$$

Theorem 7 implies that $A_2 = SA_1S^{-1}$, $B_2 = SB_1$, and (A_2, B_2, C_2, D) is a minimal realization of G. It can then be shown numerically that (31) holds. ◊

The following example uses Theorem 8 to construct an alternative realization of a MISO transfer function.

Example 5

Consider the 1×2 MISO transfer function

$$G(s) = \begin{bmatrix} \frac{s^2+20s+27}{s^3+3s^2-13s-15} & \frac{5s^2+8s-5}{s^3+3s^2-13s-15} \end{bmatrix}, \tag{42}$$

which has the minimal realization (A_1, B_1, C_1, D) where

$$A_1 = \begin{bmatrix} -7 & 6 & 4 \\ -6 & 5 & 4 \\ 2 & 2 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 5 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}, \quad C_1 = [1 \ \ 0 \ \ 0], \quad D = 0. \tag{43}$$

Let A_2 and C_2 be given by

$$A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 4 & 3 & 0 \\ 5 & 2 & -5 \end{bmatrix}, \quad C_2 = [0 \ \ 0 \ \ 1]. \tag{44}$$

Since A_2 and A_1 have the same eigenvalues (namely, -1, 3, and -5), it follows that they are similar. Furthermore, note that

$$\mathcal{O}(A_1, C_1) = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 6 & 4 \\ 21 & -4 & -80 \end{bmatrix}, \quad \mathcal{O}(A_2, C_2) = \begin{bmatrix} 0 & 0 & 1 \\ 5 & 2 & -5 \\ 22 & -4 & 25 \end{bmatrix} \tag{45}$$

are nonsingular, and define

$$S \triangleq \mathcal{O}(A_2, C_2)^{-1}\mathcal{O}(A_1, C_1) = \frac{1}{3}\begin{bmatrix} 2 & -2 & 0 \\ -8 & 14 & 6 \\ 3 & 0 & 0 \end{bmatrix}, \tag{46}$$

$$B_2 \triangleq SB_1 = \frac{1}{3}\begin{bmatrix} -2 & 2 \\ 38 & 22 \\ 3 & 15 \end{bmatrix}. \tag{47}$$

Theorem 8 implies that $A_2 = SA_1S^{-1}$, $C_2 = C_1S^{-1}$, and (A_2, B_2, C_2, D) is a minimal realization of G. It can then be shown numerically that (35) holds. ◊

MULTIPLE-INPUT, MULTIPLE-OUTPUT EXAMPLES

This section considers the case where a minimal realization (A_1, B_1, C_1, D) of a MIMO transfer function G is given and an alternative minimal realization (A_2, B_2, C_2, D) of G is of interest. The following example is analogous to Example 4. However, the SIMO transfer function is replaced by a MIMO transfer function. The effectiveness of (19) is investigated.

Example 6

Consider the 2×2 MIMO transfer function

$$G(s) = \begin{bmatrix} \frac{s-1}{(s-2)^2} & \frac{s-1}{(s-2)^2} \\ \frac{2}{s-2} & \frac{1}{s-2} \end{bmatrix}, \tag{48}$$

which has the minimal realization (A_1, B_1, C_1, D), where

$$A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = 0. \tag{49}$$

Let

$$A_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 4 & 4 \\ 8 & 0 \\ 1 & 5 \end{bmatrix}, \tag{50}$$

and note that A_2 is similar to A_1. Using (19), define

$$S \triangleq \mathcal{C}(A_2, B_2)\,\mathcal{C}(A_2, B_2)^{\mathrm{T}}[\mathcal{C}(A_1, B_1)\,\mathcal{C}(A_2, B_2)^{\mathrm{T}}]^{-1} \tag{51}$$

$$= \begin{bmatrix} 0 & 0 & 4 \\ 8 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix}, \tag{52}$$

$$C_2 \triangleq C_1S^{-1} = \frac{1}{16}\begin{bmatrix} 1 & 2 & 4 \\ 4 & 2 & 0 \end{bmatrix}. \tag{53}$$

It can be shown that $C_2(sI_3 - A_2)^{-1}B_2 = G(s)$, which implies that (A_2, B_2, C_2, D) is a minimal realization of $G(s)$. In this case, $\mathcal{C}(A_1, B_1)\,\mathcal{C}(A_2, B_2)^{\mathrm{T}}$ is nonsingular, and the matrix S given by (19) satisfies $A_2 = SA_1S^{-1}$, $B_2 = SB_1$. Now, redefine B_2 as

$$B_2 = \begin{bmatrix} 0 & 4 \\ 8 & 0 \\ 1 & 5 \end{bmatrix}, \tag{54}$$

and define

$$\begin{aligned} S &\triangleq \mathcal{C}(A_2, B_2)\,\mathcal{C}(A_2, B_2)^{\mathrm{T}}[\mathcal{C}(A_1, B_1)\,\mathcal{C}(A_2, B_2)^{\mathrm{T}}]^{-1} \\ &= \begin{bmatrix} -4 & 0 & 4 \\ 8 & 0 & 0 \\ -3.43 & 4 & 1 \end{bmatrix}, \end{aligned} \tag{55}$$

$$C_2 \triangleq C_1S^{-1} = \frac{1}{16}\begin{bmatrix} -1 & 3.21 & 4 \\ 4 & 4 & 0 \end{bmatrix}. \tag{56}$$

Note that

$$C_2(sI_3 - A_2)^{-1}B_2 + D = \begin{bmatrix} \frac{1.86s - 3.71}{(s-2)^2} & \frac{s-1}{(s-2)^2} \\ \frac{2}{s-2} & \frac{1}{s-2} \end{bmatrix}, \tag{57}$$

which is not equal to G. Therefore, although $\mathcal{C}(A_1, B_1)$ $\mathcal{C}(A_2, B_2)^{\mathrm{T}}$ is nonsingular, the matrix S given by (19) does not satisfy $A_2 = SA_1S^{-1}$, $B_2 = SB_1$. ◊

Example 6 shows that the matrix S given by (19) may not be useful for constructing new realizations. In addition, (19) may not be useful in the case where $\mathcal{C}(A, B_1)$ $\mathcal{C}(A, B_2)^{\mathrm{T}}$ is singular, which occurs in the following example.

Example 7

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $\mu(A)=2$, the transfer function of the smallest size whose minimal realization includes A is of size 2×2. Letting

$$B_1=\begin{bmatrix}0&0\\1&1\\0&1\end{bmatrix},\quad C_1=\begin{bmatrix}0&0&1\\1&-1&-1\end{bmatrix},\quad D=0,$$

it follows that (A, B_1, C_1, D) is a minimal realization of

$$G(s)=\begin{bmatrix}0 & \frac{1}{s}\\ \frac{1-s}{s^2} & \frac{1-2s}{s^2}\end{bmatrix}. \tag{58}$$

Next, let

$$B_2=\begin{bmatrix}0&0\\1&-1\\0&-1\end{bmatrix},$$

and note that

$$\mathcal{C}(A,B_1)\,\mathcal{C}(A,B_2)^{\mathrm{T}}=\begin{bmatrix}0&0&0\\0&0&-1\\0&-1&-1\end{bmatrix}.$$

Since $\mathcal{C}(A,B_1)\,\mathcal{C}(A,B_2)^{\mathrm{T}}$ is singular, statement 4 of Theorem 5 implies that (19) cannot be used to construct a nonsingular matrix $S\in\mathbb{R}^{3\times 3}$ such that (A, B_2, C_2, D) is a minimal realization of G, where $B_2=SB_1$ and $C_2=C_1S^{-1}$. ◊

CONSTRUCTING MINIMAL REALIZATIONS IN THE MULTIPLE-INPUT, MULTIPLE-OUTPUT CASE

Example 6 shows that in the case where $\mathcal{C}(A_1,B_1)\,\mathcal{C}(A_2,B_2)^{\mathrm{T}}$ is nonsingular, (19) does not necessarily yield a matrix S such that $A_2=SA_1S^{-1}$, $B_2=SB_1$, and $C_2=C_1S^{-1}$. Furthermore, Example 7 shows that $\mathcal{C}(A_1,B_1)\,\mathcal{C}(A_2,B_2)^{\mathrm{T}}$ may be singular, and thus S given by (19) may not exist. However, the possibility remains that a state transformation S of another form may yet exist.

Let $\oplus$ and $\otimes$ denote the Kronecker sum and product, respectively. For all $A\in\mathbb{R}^{n\times m}$, let $\operatorname{vec} A\in\mathbb{R}^{nm}$ denote the *vectorization* of A, that is, the vector formed by stacking columns of A. We recover A by writing $A=\operatorname{vec}^{-1}(\operatorname{vec} A)$. Moreover, the *range* of $A\in\mathbb{R}^{n\times m}$ is defined by $\mathcal{R}(A)=\{Ax\colon x\in\mathbb{R}^m\}$.

Proposition 4

Let $A_1, A_2\in\mathbb{R}^{n\times n}$, assume that A_1 and A_2 are similar, let $B_1, B_2\in\mathbb{R}^{n\times m}$, assume that (A_1, B_1) and (A_2, B_2) are controllable, and define

$$X\triangleq\begin{bmatrix}(-A_1^{\mathrm{T}})\oplus A_2\\ B_1^{\mathrm{T}}\otimes I_n\end{bmatrix}\in\mathbb{R}^{(n^2+nm)\times n^2}. \tag{59}$$

Then, the following statements hold:

1) rank $X=n^2$.
2) The following conditions are equivalent:
   a) There exists a matrix $S\in\mathbb{R}^{n\times n}$ such that

$$A_2S=SA_1,\quad B_2=SB_1. \tag{60}$$

   b) There exists a unique matrix $S\in\mathbb{R}^{n\times n}$ satisfying (60).
   c) There exists a nonsingular matrix $S\in\mathbb{R}^{n\times n}$ such that

$$A_2=SA_1S^{-1},\quad B_2=SB_1. \tag{61}$$

   d) There exists a unique nonsingular matrix $S\in\mathbb{R}^{n\times n}$ satisfying (61).
   e)

$$\begin{bmatrix}0\\ \operatorname{vec} B_2\end{bmatrix}\in\mathcal{R}(X). \tag{62}$$

   f) If these conditions hold, then

$$S\triangleq\operatorname{vec}^{-1}\left(X^{+}\begin{bmatrix}0\\ \operatorname{vec} B_2\end{bmatrix}\right), \tag{63}$$

$$=\mathcal{C}(A_2,B_2)\,\mathcal{C}(A_2,B_2)^{\mathrm{T}}[\mathcal{C}(A_1,B_1)\,\mathcal{C}(A_2,B_2)^{T}]^{-1}, \tag{64}$$

   where

$$X^{+}\triangleq(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}. \tag{65}$$

3) If $m=1$, then a)–e) hold.

Proof

To prove statement 1, suppose that rank $X<n^2$, which implies that there exists a nonzero vector $v\in\mathbb{R}^{n^2}$ such that

$$[(-A_1^{T})\oplus A_2]v=0,$$
$$(B_1^{T}\otimes I_n)v=0.$$

Using the identity $\operatorname{vec} YZW=(W^{T}\otimes Y)\operatorname{vec} Z$ [2, Prop. 9.1.1], it follows that

$$A_2V-VA_1=0, \tag{66}$$
$$VB_1=0, \tag{67}$$

where $V\triangleq\operatorname{vec}^{-1}v\in\mathbb{R}^{n\times n}$. Right-multiplying (66) by $A_1^iB_1$ implies that, for all $i\geq 0$,

$$A_2VA_1^iB_1-VA_1^{i+1}B_1=0. \tag{68}$$

Setting $i=0$ in (68) and using (67) yields

$$VA_1B_1=0. \tag{69}$$

Setting $i=1$ in (68) and using (69) yields

$$VA_1^2B_1=0.$$

Similarly, it follows that, for all $i\geq 0$,

$$VA_1^iB_1=0,$$

which implies that $V\mathcal{C}(A_1,B_1)=0$. Since (A_1,B_1) is controllable, it follows that $V=0$, which is a contradiction. Thus, rank $X=n^2$.

To prove statement 2, note that d) implies b), d) implies c), and b) implies a). To show that a) and e) are equivalent,

The first question concerns the minimal number of sensors and actuators required to construct a minimal realization for a given dynamics matrix.

note that, using $\operatorname{vec} YZW = (W^T \otimes Y) \operatorname{vec} Z$, (60) is equivalent to

$$X \operatorname{vec} S = \begin{bmatrix} 0 \\ \operatorname{vec} B_2 \end{bmatrix}.$$

To show that c) implies e), let $S \in \mathbb{R}^{n\times n}$ satisfy (61). It thus follows that $A_2 S = SA_1$ and $B_2 = SB_1$, which, using $\operatorname{vec} YZW = (W^{\mathrm{T}} \otimes Y) \operatorname{vec} Z$, implies that

$$X \operatorname{vec} S = \begin{bmatrix} 0 \\ \operatorname{vec} B_2 \end{bmatrix},$$

which confirms (62). To show that e) implies d), note that it follows from (62), statement 1, and [2, Prop. 8.1.9] that there exists a unique $s \in \mathbb{R}^{n^2}$ satisfying

$$Xs = \begin{bmatrix} 0 \\ \operatorname{vec} B_2 \end{bmatrix},$$

and which is given by

$$s = X^{+} \begin{bmatrix} 0 \\ \operatorname{vec} B_2 \end{bmatrix}.$$

Using $\operatorname{vec} YZW = (W^{\mathrm{T}} \otimes Y) \operatorname{vec} Z$, it follows that $S \triangleq \operatorname{vec}^{-1} s \in \mathbb{R}^{n\times n}$ is the unique matrix satisfying

$$A_2 S = SA_1, \quad B_2 = SB_1,$$

which implies that, for all $i \geq 1$,

$$A_2^i S = SA_1^i,$$

and thus, for all $i \geq 1$,

$$A_2^i B_2 = A_2^i SB_1 = SA_1^i B_1.$$

Hence,

$$\mathcal{C}(A_2, B_2) = S\mathcal{C}(A_1, B_1).$$

Since (A_1, B_1) and (A_2, B_2) are controllable, it follows that $S \in \mathbb{R}^{n\times n}$ is nonsingular. Moreover, letting $C_1 \in \mathbb{R}^{p\times n}$, it follows that (A_1, B_1, C_1) and $(A_2, B_2, C_1 S^{-1}) = (SA_1 S^{-1}, SB_1, C_1 S^{-1})$ are realizations of the same transfer function. Therefore, statement 4 of Theorem 5 yields (64).

To prove statement 3, defining $S \triangleq \mathcal{C}(A_2, B_2)\mathcal{C}(A_1, B_1)^{-1}$, it follows from Proposition 3 that $A_2 = SA_1 S^{-1}$ and $B_2 = SB_1$. It thus follows that a) holds. Thus, statement 2 implies that a)–e) hold. □

The equality between (63) and (64) yields the following result in the case where $m = 1$.

Corollary 1

Let $A_1, A_2 \in \mathbb{R}^{n\times n}$, assume that A_1 and A_2 are similar, let $B_1, B_2 \in R^n$, and assume that (A_1, B_1) and (A_2, B_2) are controllable. Then,

$$\begin{bmatrix} (-A_1^T) \oplus A_2 \\ B_1^{\mathrm{T}} \otimes I_n \end{bmatrix} \operatorname{vec}[\mathcal{C}(A_2, B_2)\mathcal{C}(A_1, B_1)^{-1}] = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}. \tag{70}$$

Proof

Using statements 2 and 3 in Proposition 4, it follows that $S \triangleq \mathcal{C}(A_2, B_2)\mathcal{C}(A_2, B_2)^{\mathrm{T}}[\mathcal{C}(A_1, B_1)\mathcal{C}(A_2, B_2)^{\mathrm{T}}]^{-1} = \mathcal{C}(A_2, B_2)\mathcal{C}(A_1, B_1)^{-1}$ satisfies (60), which, using $\operatorname{vec} YZW = (W^{\mathrm{T}} \otimes Y) \operatorname{vec} Z$, is equivalent to (70). Alternatively, using Proposition 4, it follows from (63)–(65) that

$$X\operatorname{vec}(\mathcal{C}(A_2, B_2)\mathcal{C}(A_1, B_1)^{-1}) = X(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}} \begin{bmatrix} 0 \\ B_2 \end{bmatrix}. \tag{71}$$

Since $\left[\begin{smallmatrix} 0 \\ B_2 \end{smallmatrix}\right] \in \mathcal{R}(X)$ and $X(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}$ is the projector onto $\mathcal{R}(X)$ (see [2, Prop. 8.1.7]), it follows that $X(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}\left[\begin{smallmatrix} 0 \\ B_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ B_2 \end{smallmatrix}\right]$. Hence, (71) implies (70). □

Corollary 1 also follows from Proposition 3. Specifically, defining $S \triangleq \mathcal{C}(A_2, B_2)\mathcal{C}(A_1, B_1)^{-1}$ and using $\operatorname{vec} YZW = (W^{\mathrm{T}} \otimes Y) \operatorname{vec} Z$, it follows from (22) and (23) that

$$\begin{bmatrix} (-A_1^{\mathrm{T}}) \oplus A_2 \\ B_1^{\mathrm{T}} \otimes I_n \end{bmatrix} \operatorname{vec} S = \begin{bmatrix} \operatorname{vec}(A_2 S - SA_1) \\ SB_1 \end{bmatrix} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \tag{72}$$

which confirms (70).

Proposition 4 provides a necessary and sufficient condition for the existence of $S \in \mathbb{R}^{n\times n}$ such that $A_2 = SA_1 S^{-1}$ and $B_2 = SB_1$. If (A_1, B_1, C_1, D) is a minimal realization of a MIMO transfer function G, then $(A_2, B_2, C_1 S^{-1}, D)$ is also a minimal realization of G, provided that a)–e) hold. Moreover, Proposition 4 implies that, if $\left[\begin{smallmatrix} 0 \\ \operatorname{vec} B_2 \end{smallmatrix}\right] \notin \mathcal{R}(X)$, then there does not exist a nonsingular $S \in \mathbb{R}^{n\times n}$ such that $A_2 = SA_1 S^{-1}$ and $B_2 = SB_1$. Consequently, if $\left[\begin{smallmatrix} 0 \\ \operatorname{vec} B_2 \end{smallmatrix}\right] \notin \mathcal{R}(X)$, then there do not exist $C_1, C_2 \in \mathbb{R}^{p\times n}$ such that (A_1, B_1, C_1, D) and (A_2, B_2, C_2, D) are realizations of the same transfer function, which is demonstrated in the following example.

Example 8

Let

$$A_1 = A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}. \tag{73}$$

The denominator polynomials of the Smith–McMillan form are directly related to the Jordan form of the dynamics matrix of a minimal realization.

It thus follows that

$$X=\begin{bmatrix}0&0&-1&0\\1&0&0&-1\\0&0&0&0\\0&0&1&0\\1&0&1&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad \begin{bmatrix}0\\ \mathrm{vec}\,B_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\2\\0\\1\\1\end{bmatrix}. \tag{74}$$

Note that both (A_1, B_1) and (A_2, B_2) are controllable. Moreover, note that the first row of X is minus the seventh row of X, while the first element of $\left[\begin{smallmatrix}0\\ \mathrm{vec}\,B_2\end{smallmatrix}\right]$ is not minus the seventh element of $\left[\begin{smallmatrix}0\\ \mathrm{vec}\,B_2\end{smallmatrix}\right]$. Therefore, $\left[\begin{smallmatrix}0\\ \mathrm{vec}\,B_2\end{smallmatrix}\right]\in\mathcal{R}(X)$ is not satisfied, and thus Proposition 4 implies that there do not exist $C_1, C_2\in\mathbb{R}^{p\times n}$ such that (A_1, B_1, C_1, D) and (A_2, B_2, C_2, D) are the realizations of the same transfer function. ◊

The following result is a dual version of Proposition 4.

Proposition 5

Let $A_1, A_2\in\mathbb{R}^{n\times n}$, assume that A_1 and A_2 are similar, let $C_1, C_2\in\mathbb{R}^{p\times n}$, assume that (A_1, C_1) and (A_2, C_2) are observable, and define

$$X\triangleq\begin{bmatrix}(-A_1^{\mathrm T})\oplus A_2\\ I_n\otimes C_2\end{bmatrix}\in\mathbb{R}^{(n^2+np)\times n^2}. \tag{75}$$

Then, the following statements hold:

1) rank $X=n^2$.
2) The following conditions are equivalent:
   a) There exists a matrix $S\in\mathbb{R}^{n\times n}$ such that
   $$A_2S=SA_1,\quad C_2S=C_1. \tag{76}$$
   b) There exists a unique matrix $S\in\mathbb{R}^{n\times n}$ satisfying (76).
   c) There exists a nonsingular matrix $S\in\mathbb{R}^{n\times n}$ such that
   $$A_2=SA_1S^{-1},\quad C_2=C_1S^{-1}. \tag{77}$$
   d) There exists a unique nonsingular matrix $S\in\mathbb{R}^{n\times n}$ satisfying (77).
   e) $$\begin{bmatrix}0\\ \mathrm{vec}\,C_1\end{bmatrix}\in\mathcal{R}(X). \tag{78}$$

   If these conditions hold, then

$$S\triangleq\mathrm{vec}^{-1}\left(X^+\begin{bmatrix}0\\ \mathrm{vec}\,C_1\end{bmatrix}\right) \tag{79}$$

$$=[\mathcal{O}(A_2,C_2)^{\mathrm T}\mathcal{O}(A_2,C_2)]^{-1}\mathcal{O}(A_2,C_2)^{\mathrm T}\mathcal{O}(A_1,C_1), \tag{80}$$

where

$$X^+\triangleq(X^{\mathrm T}X)^{-1}X^{\mathrm T}. \tag{81}$$

3) If $p=1$, then a)–e) hold.

The equality between (79) and (80) yields the following result in the case where $p=1$.

Corollary 2

Let $A_1, A_2\in\mathbb{R}^{n\times n}$, assume that A_1 and A_2 are similar, let $C_1, C_2\in\mathbb{R}^{1\times n}$, and assume that (A_1, C_1) and (A_2, C_2) are observable. Then,

$$\begin{bmatrix}(-A_1^{\mathrm T})\oplus A_2\\ I_n\otimes C_2\end{bmatrix}\mathrm{vec}[\mathcal{O}(A_2,C_2)^{-1}\mathcal{O}(A_1,C_1)]=\begin{bmatrix}0\\ \mathrm{vec}\,C_1\end{bmatrix}. \tag{82}$$

The following example considers a MIMO system, for which conditions a)–e) of Proposition 4 hold.

Example 9

Let (A,B_1,C) be a minimal realization of the $\mu(A)\times\mu(A)$ transfer function G with McMillan degree $\mu(A)$, which implies that A, B_1, C are $\mu(A)\times\mu(A)$ matrices. Hence, $A=\lambda I_{\mu(A)}$, where the algebraic and geometric multiplicities of the eigenvalue $\lambda\in\mathbb{R}$ are both $\mu(A)$. Note that since $A=\lambda I_{\mu(A)}$ and rank $\mathcal{C}(A,B_1)=\mu(A)$, it follows that B_1 is nonsingular. To obtain another minimal realization of G, let $B_2\in\mathbb{R}^{\mu(A)\times\mu(A)}$ be nonsingular. Thus, $S\triangleq B_2B_1^{-1}$ is the unique matrix $S\in\mathbb{R}^{\mu(A)\times\mu(A)}$ that satisfies $B_2=SB_1$. Moreover, since, for all nonsingular $S_0\in\mathbb{R}^{\mu(A)\times\mu(A)}$, $S_0AS_0^{-1}=A$, it follows that the only minimal realization of G with input matrix B_2 is $(A,B_2,CB_1B_2^{-1})$. Finally, note that $S=B_2B_1^{-1}$ can also be obtained from (63) or (64). Specifically, note that using $\mathrm{vec}\,YZW=(W^T\otimes Y)\mathrm{vec}\,Z$, it follows from (63) that

$$\begin{aligned}\mathrm{vec}\,S&=(X^{\mathrm T}X)^{-1}X^{\mathrm T}\begin{bmatrix}0\\ \mathrm{vec}\,B_2\end{bmatrix}\\&=\left([0\ \ B_1\otimes I_n]\begin{bmatrix}0\\ B_1^{\mathrm T}\otimes I_n\end{bmatrix}\right)^{-1}[0\ \ B_1\otimes I_n]\begin{bmatrix}0\\ \mathrm{vec}\,B_2\end{bmatrix}\\&=[(B_1B_1^{\mathrm T})\otimes I_n]^{-1}(B_1\otimes I_n)\,\mathrm{vec}\,B_2\\&=[(B_1^{-\mathrm T}B_1^{-1})\otimes I_n]\,\mathrm{vec}\,B_2B_1^{\mathrm T}\\&=\mathrm{vec}\,B_2B_1^{\mathrm T}B_1^{-\mathrm T}B_1^{-1}\\&=\mathrm{vec}\,B_2B_1^{-1}.\end{aligned}$$

The second question relates to the existence of realizations with alternative input and output matrices.

The following example considers a MIMO system, where conditions a)–e) of Proposition 4 do not hold.

Example 10

Let (A_1, B_1, C) be a minimal realization of the 2×2 transfer function G with $\mu(G) = 2$ and McMillan degree 5, where

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{83}$$

Let

$$A_2 = \frac{1}{294}\begin{bmatrix} 476 & 182 & 56 & 224 & -336 \\ 932 & 974 & -4 & -268 & -480 \\ 970 & 844 & 334 & 34 & -786 \\ 508 & 382 & 124 & 664 & -702 \\ 500 & 626 & -40 & -34 & -96 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \tag{84}$$

and note that (A_2, B_2) is controllable and that $A_2 = S_2 A_1 S_2^{-1}$, where

$$S_2 \triangleq \begin{bmatrix} 2 & 1 & 3 & 1 & 2 \\ 2 & 5 & 1 & 4 & 2 \\ 5 & 1 & 1 & 5 & 4 \\ 3 & 5 & 4 & 2 & 4 \\ 5 & 5 & 5 & 4 & 2 \end{bmatrix}. \tag{85}$$

Define

$$\begin{aligned} S &\triangleq \mathcal{C}(A, B_2)\mathcal{C}(A, B_2)^{\mathrm{T}}[\mathcal{C}(A, B_1)\mathcal{C}(A, B_2)^{\mathrm{T}}]^{-1} \\ &= \begin{bmatrix} -4.49 & -7.30 & 5.20 & 3.95 & 0.02 \\ 3.14 & -6.73 & -1.65 & 4.08 & 0.35 \\ -3.14 & -15.45 & 4.86 & 8.40 & 0.36 \\ -7.18 & -13.20 & 7.08 & 7.55 & 0.07 \\ -2.24 & -8.95 & 3.29 & 5.16 & 0.30 \end{bmatrix}. \end{aligned} \tag{86}$$

In this case, $B_2 \neq SB_1$ and $A_2 \neq SA_1S^{-1}$. Therefore, (A^2, B^2, CS^{-1}) is not a minimal realization of G. ◊

CONCLUDING REMARKS AND OPEN QUESTIONS

This article focused on two questions relating to minimal realizations of MIMO transfer functions. The first question concerns the minimal number of sensors and actuators required to construct a minimal realization for a given dynamics matrix, while the second question relates to the existence of realizations with alternative input and output matrices.

To support the development in this article, the Smith–McMillan form was used to characterize the rank of a transfer function. Using a construction from [4], it was shown that the denominator polynomials of the Smith–McMillan form are directly related to the Jordan form of the dynamics matrix of a minimal realization. No characterization was given, however, concerning the numerator polynomials of the Smith–McMillan form. In the case where G is square and has a minimal realization (A,B,C,D) with nonsingular D, it can be seen that the numerator polynomials of the Smith–McMillan form are directly related to the Jordan form of $A - BD^{-1}C$. A characterization of the numerator polynomials in the case where G is arbitrary remains an open problem, however. This fundamental problem warrants future investigation.

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