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Brief paper

Dynamic output-feedback control of a chain of discrete-time integrators with arbitrary zeros and asymmetric input saturation ☆

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ABSTRACT

For a chain of discrete-time integrators with arbitrary zeros and asymmetric input saturation, this paper develops globally asymptotically stabilizing full-state-feedback and dynamic-output-feedback controllers. These controllers extend prior results on nested-saturation controllers for discrete-time chains of integrators, and they are applied to setpoint command following and rejection of constant disturbances.

1. Introduction

It is well known that an exponentially unstable system cannot be globally stabilized under bounded control. For such systems, the set of initial conditions that can be driven to zero is bounded. The situation is different, however, in the case of a system that is polynomially unstable. In the groundbreaking paper (Teel, 1992), a globally asymptotically stabilizing, full-state-feedback controller was constructed for a chain of integrators. Interestingly, the controller in Teel (1992) was not constructed by saturating a linear controller, but rather uses nested saturation functions. These functions act on the state of the system in a specific basis, successively reducing the magnitude of each state component so that the state reaches an invariant region in which the control input is not saturated and within which an asymptotically stabilizing linear controller drives the state to zero.

The nested-saturation controller of Teel (1992) has been extended and applied in various ways. For example, globally stabilizing controllers for linear systems subject to input saturation are constructed in Sussmann, Sontag, and Yang (1994). In Lauvdal, Murray, and Fossen (1997), a time-varying controller is developed for stabilizing a chain of integrators in the presence of magnitude and rate input saturation. Nested-saturation-based controllers are designed in Cao, and Lynch (2015), Castillo, Dzul, and Lozano (2004), Guerrero-Castellanos, Marchand, Hably, Lesecq, and Delamare (2011) and Patrikar, Makkapati, Pattanaik, Parwana, and Kothari (2019) for global stabilization of a quadrotor. Global stabilization of a chain of discrete-time integrators is considered in Amini, Ahi, and Haeri (2019), Marchand, Hably, and Chemori (2007), Yang, Sontag, and Sussmann (1997) and Zhou, and Yang (2018). All of these controllers employ full-state feedback.

For a discrete-time chain of integrators, the present paper extends the nested saturation controller in several ways. First, we consider the case where the input saturation is asymmetric. Asymmetric input saturation has been considered extensively, for example, Li, and Lin (2018, Chap. 9), Benzaouia, and Burgat (1988), Groff, da Silva, and Valmórbida (2019), Ma, Ge, Zheng, and Hu (2014), Mariano, Blanchini, Formentin, and Zaccarian (2020), Yuan, and Wu (2015) and Zheng, Jin, Zhu, and Sun (2017). This extension is advantageous in dealing with the asymmetry due to the control offset needed to follow a setpoint command and cancel a constant disturbance.

Next, using the realization given in Polderman (1989, Sec. 5), we transform the state-space basis representing the chain of integrators with arbitrary zeros so that the nested-saturation controller uses measurements of only the output of the last integrator of the chain. The resulting controller is thus a dynamic output-feedback controller rather than a static full-state-feedback controller. It turns out that the dynamic output-feedback controller is exactly proper with order $n-1$, where n is the number of integrators in the plant. An observer-based approach to output-feedback control of a chain of continuous-time integrators without zeros is presented in Mi, Yao, Deng, and Xie (2020).

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Next, we consider the case of setpoint command following and constant disturbance rejection. Because of the integral action of the plant, the controller is able to follow setpoint commands. However, rejection of a constant disturbance requires an appropriate offset of the control input, which, in the absence of saturation, is provided by a controller with integral action. In the presence of saturation, stabilization of a chain of integrators using a controller with integral action is an open problem; in the present paper, we assume that the disturbance is known. Cancellation of the disturbance thus requires a nonzero offset, which may not lie at the center of the input saturation function. Therefore, in order to take advantage of the full range of control inputs, the globally asymptotically stabilizing controller is extended to the case of asymmetric input saturation. To illustrate this point, consider the case of symmetric input saturation with bounds 1 and -1. If the disturbance is 0.9, then the constant control input that rejects the disturbance is $u = -0.9$, which implies that the effective range of control perturbations for stabilization is $[-0.1, 1.9]$. In the case of symmetric input saturation, the control perturbations would be confined to the symmetric range $[-0.1, 0.1]$. However, the globally asymptotically stabilizing controller in the present paper can utilize the full range of control perturbations $[-0.1, 1.9]$.

The contents of the paper are as follows. Section 2 presents the problem formulation for setpoint command following and constant disturbance rejection of a chain of discrete-time integrators with arbitrary zeros subject to asymmetric input saturation. Section 3 presents the integrator-state basis, while Section 4 presents the change of basis for full-state feedback. Section 5 develops a nested-saturation full-state-feedback controller. The full-state-feedback controller has the limitation of being dependent on measurements of all the states. To overcome this limitation, Section 7 presents the change of basis for output feedback, and Section 7 presents a dynamic-output-feedback controller that uses the measurements of only the output of the last integrator. Section 8 presents numerical examples using the dynamic-output-feedback controller. Finally, Section 9 presents conclusions and future research.

2. Problem formulation

Consider the discrete-time chain of integrators with arbitrary zeros and asymmetric input saturation given by

$$y_k = G(\mathbf{q})[\sigma_{u_{\min},u_{\max}}(u_k) + d], \tag{1}$$

where

$$G(\mathbf{q}) \triangleq \frac{a_{n-1}\mathbf{q}^{n-1} + \cdots + a_1\mathbf{q} + a_0}{(\mathbf{q}-1)^n}, \tag{2}$$

$y_k \in \mathbb{R}$ is the output, $u_k \in \mathbb{R}$ is the control, $a_0, \ldots, a_{n-1} \in \mathbb{R}$ are the numerator coefficients, $d \in \mathbb{R}$ is the constant disturbance, $\mathbf{q}$ is the forward-shift operator, $u_{\min} < 0 < u_{\max}$, and the asymmetric saturation function $\sigma_{\underline{u},\overline{u}} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\sigma_{\underline{u},\overline{u}}(p) \triangleq \begin{cases} \underline{u}, & p < \underline{u}, \\ p, & \underline{u} \le p \le \overline{u}, \\ \overline{u}, & p > \overline{u}, \end{cases} \tag{3}$$

where $\underline{u} < 0 < \overline{u}$. The controllable canonical realization of (2) yields the state-space representation of (1) given by

$$x_{k+1} = A_x x_k + B_x \sigma_{u_{\min},u_{\max}}(u_k) + B_x d, \tag{4}$$

$$y_k = C_x x_k, \tag{5}$$

where, for all $k \ge 0$, $x_k \in \mathbb{R}^n$ is the state at step k,

$$A_x \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\beta_{0,n} & -\beta_{1,n} & -\beta_{2,n} & \cdots & -\beta_{n-1,n} \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{6}$$

and, for each positive integer q and all $i = 0, 1, \ldots, q$,

$$\beta_{i,q} \triangleq (-1)^{q-i}\binom{q}{i}, \tag{7}$$

and

$$B_x \triangleq \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n, \tag{8}$$

$$C_x \triangleq \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix} \in \mathbb{R}^{1\times n}. \tag{9}$$

We assume that 1 is not a root of the numerator of (2), that is, $\sum_{i=0}^{n-1} a_i \neq 0$, which holds if and only if (A_x, B_x, C_x) is controllable and observable.

Let $r \in \mathbb{R}$ be a constant command, and, for all $k \ge 0$, define the error $e_k \in \mathbb{R}$ by

$$e_k \triangleq r - y_k. \tag{10}$$

The objective is to design a feedback controller such that, for all $x_0 \in \mathbb{R}^n$, $\lim_{k\to\infty} e_k = 0$, and, for all $k \ge 0$, $u_k \in [u_{\min}, u_{\max}]$. We assume that $-d \in (u_{\min}, u_{\max})$ so that there exists a constant control $u_* = -d$ that can reject d. Note that the control offset $u_k - u_* \in [u_{\min} + d, u_{\max} + d]$ can assume both positive and negative values.

3. Integrator-state basis

As an alternative realization of (2), cascading n integrators yields the dynamics and input matrices

$$A_\xi \triangleq \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{11}$$

$$B_\xi \triangleq \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n. \tag{12}$$

Define

$$\mathcal{K}_x \triangleq \begin{bmatrix} B_x & A_x B_x & \cdots & A_x^{n-1} B_x \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{13}$$

$$\mathcal{K}_\xi \triangleq \begin{bmatrix} B_\xi & A_\xi B_\xi & \cdots & A_\xi^{n-1} B_\xi \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{14}$$

$$S_{\xi x} \triangleq \mathcal{K}_\xi \mathcal{K}_x^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \beta_{0,1} & 1 & 0 & \cdots & 0 & 0 \\ \beta_{0,2} & \beta_{1,2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_{0,n-2} & \beta_{1,n-2} & \beta_{2,n-2} & \cdots & 1 & 0 \\ \beta_{0,n-1} & \beta_{1,n-1} & \beta_{2,n-1} & \cdots & \beta_{n-2,n-1} & 1 \end{bmatrix}, \tag{15}$$

which implies that

$$S_{\xi x}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \bar\beta_{0,1} & 1 & 0 & \cdots & 0 & 0 \\ \bar\beta_{0,2} & \bar\beta_{1,2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar\beta_{0,n-2} & \bar\beta_{1,n-2} & \bar\beta_{2,n-2} & \cdots & 1 & 0 \\ \bar\beta_{0,n-1} & \bar\beta_{1,n-1} & \bar\beta_{2,n-1} & \cdots & \bar\beta_{n-2,n-1} & 1 \end{bmatrix}, \tag{16}$$

where, for each nonnegative integer q and all $i = 0, 1, \ldots, q$,

$$\bar{\beta}_{i,q} \triangleq \binom{q}{i}. \tag{17}$$

Note that $\bar{\beta}_{0,q} = 1$, and, in particular, $\bar{\beta}_{0,0} = 1$. Since A_ξ is similar to A_x, Lemma 1 in Appendix A implies that (A_ξ, B_ξ, C_ξ) is a minimal realization of G, where

$$C_\xi \triangleq C_x S_{\xi x}^{-1} \in \mathbb{R}^{1\times n} = \begin{bmatrix}\mu_0 & \mu_1 & \cdots & \mu_{n-1}\end{bmatrix}, \tag{18}$$

and, for all $i = 0, 1, \ldots, n-1$,

$$\mu_i \triangleq \sum_{j=i}^{n-1} a_j \bar{\beta}_{i,j}. \tag{19}$$

Note that $\mu_0 = \sum_{i=0}^{n-1} a_i \neq 0$. We thus have the alternative state-space representation of (1) given by

$$\xi_{k+1} = A_\xi \xi_k + B_\xi \sigma_{u_{\min},u_{\max}}(u_k) + B_\xi d, \tag{20}$$

$$y_k = C_\xi \xi_k, \tag{21}$$

where, for all $k \geq 0$,

$$\xi_k \triangleq S_{\xi x} x_k \in \mathbb{R}^n \tag{22}$$

is the state at step k.

Note that, for all $i \in \{1, \ldots, n\}$, each component $\xi_{i,k}$ of ξ_k is the state of an integrator, and the output $y_k = \sum_{i=1}^n \mu_{i-1}\xi_{i,k}$ is a linear combination of all of the states of the integrators. In the special case where $\mu_1 = \cdots = \mu_{n-1} = 0$, the output $y_k = \mu_0 \xi_{1,k}$ depends on only the last integrator of the chain. In addition, note that (20) and (21) imply that

$$G(\mathbf{q}) = \frac{\mu_{n-1}(\mathbf{q}-1)^{n-1} + \cdots + \mu_1(\mathbf{q}-1) + \mu_0}{(\mathbf{q}-1)^n}. \tag{23}$$

For all $k \geq 0$, define the error state

$$\tilde{\xi}_k \triangleq \xi_k - \xi_* = S_{\xi x} x_k - \xi_* \in \mathbb{R}^n, \tag{24}$$

where $\xi_* \triangleq \begin{bmatrix} r/\mu_0 & 0 & 0 & \cdots & 0\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n$. Note that

$$e_k = r - y_k = r - C_\xi \xi_k = C_\xi(\xi_* - \xi_k) = -C_\xi \tilde{\xi}_k. \tag{25}$$

It thus follows from (20) that, for all $k \geq 0$,

$$\tilde{\xi}_{k+1} = A_\xi \tilde{\xi}_k + B_\xi \sigma_{u_{\min},u_{\max}}(u_k) + B_\xi d + (A_\xi - I_n)\xi_*, \tag{26}$$

which, since $(A_\xi - I_n)\xi_* = 0$, implies that $\tilde{\xi}_k$ satisfies

$$\tilde{\xi}_{k+1} = A_{\tilde{\xi}} \tilde{\xi}_k + B_{\tilde{\xi}} \sigma_{u_{\min},u_{\max}}(u_k) + B_{\tilde{\xi}} d, \tag{27}$$

$$e_k = C_{\tilde{\xi}} \tilde{\xi}_k, \tag{28}$$

where $A_{\tilde{\xi}} \triangleq A_\xi$, $B_{\tilde{\xi}} \triangleq B_\xi$, and $C_{\tilde{\xi}} \triangleq -C_\xi$. Since (A_ξ, B_ξ, C_ξ) is a minimal realization of G, it follows that $(A_{\tilde{\xi}}, B_{\tilde{\xi}}, C_{\tilde{\xi}})$ is a minimal realization of $-G$. Note that, if $\lim_{k\to\infty} \tilde{\xi}_k = 0$, then $\lim_{k\to\infty} e_k = 0$.

4. Computation of d

It follows from (1) and (2) that, for all $k \geq n$,

$$y_k = -\sum_{i=0}^{n-1} \beta_{i,n} y_{k-n+i} + \sum_{i=0}^{n-1} a_i[\sigma_{u_{\min},u_{\max}}(u_{k-n+i}) + d]. \tag{29}$$

Since $y_k = r - e_k$, (29) implies that

$$e_k = r(1 + \beta_{n-1,n} + \beta_{n-2,n} + \cdots + \beta_{1,n} + \beta_{0,n}) - \sum_{i=0}^{n-1} \beta_{i,n} e_{k-n+i} - \sum_{i=0}^{n-1} a_i[\sigma_{u_{\min},u_{\max}}(u_{k-n+i}) + d]. \tag{30}$$

Moreover, since $1 + \beta_{n-1,n} + \beta_{n-2,n} + \cdots + \beta_{1,n} + \beta_{0,n} = 0$, it follows from (30) that, for all $k \geq n$,

$$e_k = -\sum_{i=0}^{n-1} \beta_{i,n} e_{k-n+i} - \sum_{i=0}^{n-1} a_i[\sigma_{u_{\min},u_{\max}}(u_{k-n+i}) + d], \tag{31}$$

which implies that

$$d = \frac{e_k + \sum_{i=0}^{n-1}[\beta_{i,n} e_{k-n+i} + a_i \sigma_{u_{\min},u_{\max}}(u_{k-n+i})]}{-\sum_{i=0}^{n-1} a_i}. \tag{32}$$

Since d is constant, (32) needs to be computed only once. If, however, the disturbance is time-dependent, then (32) provides d_{k-1}, which is available at step k.

5. Change of basis for full-state feedback

For all $i \in \{1, \ldots, n\}$, let $\lambda_i \in (-1, 0)$, and define

$$A_z \triangleq \begin{bmatrix} 1 & -\lambda_2 & -\lambda_3 & \cdots & -\lambda_{n-1} & -\lambda_n \\ 0 & 1 & -\lambda_3 & \cdots & -\lambda_{n-1} & -\lambda_n \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\lambda_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{33}$$

$$B_z \triangleq \begin{bmatrix} 1 & 1 & \cdots & 1\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n. \tag{34}$$

Note that A_x, A_ξ, and A_z have a single Jordan block with the same eigenvalue, and thus they are similar. Define the controllability matrices

$$\mathcal{K}_z \triangleq \begin{bmatrix} B_z & A_z B_z & \cdots & A_z^{n-1} B_z \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{35}$$

$$\mathcal{K}_{\tilde{\xi}} \triangleq \mathcal{K}_\xi. \tag{36}$$

The PBH test implies that (A_z, B_z) is controllable. Furthermore, the basis change

$$A_z = S_{z\tilde{\xi}} A_{\tilde{\xi}} S_{z\tilde{\xi}}^{-1}, \tag{37}$$

$$B_z = S_{z\tilde{\xi}} B_{\tilde{\xi}}, \tag{38}$$

where

$$S_{z\tilde{\xi}} \triangleq \mathcal{K}_z \mathcal{K}_{\tilde{\xi}}^{-1}, \tag{39}$$

yields the transformed system

$$z_{k+1} = A_z z_k + B_z \sigma_{u_{\min},u_{\max}}(u_k) + B_z d, \tag{40}$$

$$e_k = C_z z_k, \tag{41}$$

where

$$z_k \triangleq S_{z\tilde{\xi}} \tilde{\xi}_k = S_{z\tilde{\xi}}(S_{\xi x} x_k - \xi_*) \in \mathbb{R}^n, \tag{42}$$

and

$$C_z \triangleq C_{\tilde{\xi}} S_{z\tilde{\xi}}^{-1}. \tag{43}$$

It thus follows from Lemma 1 in Appendix A that (A_z, B_z, C_z) is a minimal realization of $-G$. Moreover, note that, if $\lim_{k\to\infty} z_k = 0$, then $\lim_{k\to\infty} e_k = 0$.

6. Full-state-feedback controller

For all $i \in \{1, \ldots, n\}$, let $\underline{\varepsilon}_i < 0 < \bar{\varepsilon}_i$ satisfy the following conditions:

(C1) For all $i \in \{2, \ldots, n\}$,

$$\underline{\varepsilon}_i < \min\left\{-\sum_{j=1}^{i-1} \bar{\varepsilon}_j, \sum_{j=1}^{i-1} \underline{\varepsilon}_j\right\}, \tag{44}$$

$$\bar{\varepsilon}_i > \max\left\{-\sum_{j=1}^{i-1} \underline{\varepsilon}_j, \sum_{j=1}^{i-1} \bar{\varepsilon}_j\right\}. \tag{45}$$

(C2)

$$\sum_{i=1}^{n} \underline{\varepsilon}_i \geq u_{\min} + d, \tag{46}$$

$$\sum_{i=1}^{n} \overline{\varepsilon}_i \leq u_{\max} + d. \tag{47}$$

Furthermore, for all $k \geq 0$, define $\underline{\alpha}_{n,k} \triangleq \underline{\varepsilon}_n$ and $\overline{\alpha}_{n,k} \triangleq \overline{\varepsilon}_n$, and, for all $k \geq 0$ and $i \in \{n, n-1, \ldots, 2\}$, define

$$\underline{\alpha}_{i-1,k} \triangleq \underline{\varepsilon}_{i-1} + \begin{cases} \underline{\alpha}_{i,k} - \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}), & 0 > \lambda_i z_{i,k} \geq \underline{\varepsilon}_i, \\ 0, & \text{otherwise}, \end{cases} \tag{48}$$

$$\overline{\alpha}_{i-1,k} \triangleq \overline{\varepsilon}_{i-1} + \begin{cases} \overline{\alpha}_{i,k} - \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}), & 0 < \lambda_i z_{i,k} \leq \overline{\varepsilon}_i, \\ 0, & \text{otherwise}. \end{cases} \tag{49}$$

Finally, for all $k \geq n$, let the control u_k be given by the nested asymmetric-saturation controller

$$u_k = -d + \sum_{i=1}^{n} \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}). \tag{50}$$

Thus, the full-state-feedback controller is given by (32), (42), (48)–(50).

Remark. The case where $u_{\max} = -u_{\min}$, $a_{n-1} = \cdots = a_1 = 0$, $a_0 = 1$, and $d = r = 0$ is considered in Zhou and Yang (2018). Note that (32), (42), (48)–(50) define a full-state-feedback controller that depends on measurements of z_k. Using (24) and (42), this controller can be implemented using measurements of ξ_k. Furthermore, using (22), this controller can be implemented using measurements of x_k. Hence, (32), (42), (48)–(50) defines a full-state-feedback controller that depends on measurements of x_k.

The next result concerns control of the discrete-time chain of integrators (1) using the full-state-feedback controller (32), (42), (48)–(50). The proof is given in Appendix A.

Theorem 1. *For all $i \in \{1, \ldots, n\}$, let $\lambda_i \in (-1, 0)$ and let $\underline{\varepsilon}_i < 0 < \overline{\varepsilon}_i$ satisfy (C1) and (C2), and consider the discrete-time chain of integrators (1) with u_k given by the full-state-feedback controller (32), (42), (48)–(50). Then, the following statements hold:*

(i) *For all $k \geq 0$, $u_k \in [u_{\min}, u_{\max}]$.*
(ii) *The zero solution of the closed-loop system consisting of (4), (5), (32), (42), (48)–(50) is Lyapunov stable.*
(iii) *For all $i \in \{1, \ldots, n-1\}$, $\lim_{k\to\infty} \xi_{k,i} = 0$.*
(iv) *$\lim_{k\to\infty} y_k = \lim_{k\to\infty} \xi_{k,n}/\mu_0 = r$.*

7. Change of basis for output feedback

In this section, we construct an alternative realization of (2), each of whose states is a sum of present and past values of y_k and u_k. This realization is used in the next section to recast the full-state-feedback controller (42), (48)–(50) as an output-feedback controller.

The observable canonical form of (31) is given by

$$A_\eta \triangleq \begin{bmatrix} -\beta_{n-1,n} & 1 & 0 & \cdots & 0 \\ -\beta_{n-2,n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_{1,n} & 0 & 0 & 0 & 1 \\ -\beta_{0,n} & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$
$$= S_{\eta z} A_z S_{\eta z}^{-1}, \tag{51}$$

$$B_\eta \triangleq \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix}^{\mathrm{T}} = S_{\eta z} B_z \in \mathbb{R}^n, \tag{52}$$

$$C_\eta \triangleq \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}} = C_z S_{\eta z}^{-1} \in \mathbb{R}^{1\times n}, \tag{53}$$

where

$$S_{\eta z} \triangleq \mathcal{K}_\eta \mathcal{K}_z^{-1}, \tag{54}$$

$$\mathcal{K}_\eta \triangleq \begin{bmatrix} B_\eta & A_\eta B_\eta & \cdots & A_\eta^{n-1} B_\eta \end{bmatrix} \in \mathbb{R}^{n\times n}. \tag{55}$$

For all $k \geq n-1$, the state η_k is given by

$$\eta_k \triangleq \begin{bmatrix} \eta_{1,k} & \eta_{2,k} & \cdots & \eta_{n,k} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n, \tag{56}$$

where

$$\eta_{1,k} \triangleq -e_k, \tag{57}$$

and, for all $i \in \{2, \ldots, n\}$,

$$\eta_{i,k} \triangleq \sum_{j=1}^{n-i+1} \beta_{n-i-j+1,n} e_{k-j} + \sum_{j=1}^{n-i+1} a_{n-i-j+1}[\sigma_{u_{\min},u_{\max}}(u_{k-j}) + d]. \tag{58}$$

See Polderman (1989, Sec. 5) for more details. Note that the first component of η_k is $-e_k = y_k - r$, which depends on the output y_k of the last integrator. Furthermore, the remaining components of η_k depend on past values of u_k and e_k as well as d. Since d can be computed using (32), all of the components of η_k are known.

Thus, it follows from (31) that, for all $k \geq n-1$,

$$\eta_{k+1} = A_\eta \eta_k + B_\eta \sigma_{u_{\min},u_{\max}}(u_k) + B_\eta d, \tag{59}$$

$$e_k = C_\eta \eta_k. \tag{60}$$

Since A_η and A_z are similar, Lemma 1 in Appendix A implies that, for all $k \geq n$,

$$z_k = S_{z\eta} \eta_k, \tag{61}$$

where $S_{z\eta} \triangleq \mathcal{K}_z \mathcal{K}_\eta^{-1} = S_{\eta z}^{-1}$.

8. Dynamic output-feedback controller

This section uses the realization of the chain of integrators given by (1) to recast the full-state-feedback controller (32), (42), (48)–(50) as an output-feedback controller. Since each state of the realization (59) and (60) is a weighted sum of past and present measurements and control inputs, the resulting controller is a dynamic output-feedback compensator. In particular, the output-feedback controller is given by (32), (48)–(50), (61), where η_k is determined using (56)–(58). Note that, η_k depends on $u_{k-1}, \ldots, u_{k-n+1}$ as well as $e_k, \ldots, e_{k-n+1}$ and d, which themselves depends on $y_k, \ldots, y_{k-n+1}$. It thus follows that (32), (48)–(50), (61) is an exactly proper dynamic output-feedback controller of order $n-1$.

The next result concerns control of the discrete-time chain of integrators (1) using the output-feedback controller (32), (48)–(50), (61). The proof follows from Theorem 1.

Theorem 2. *For all $i \in \{1, \ldots, n\}$, let $\lambda_i \in (-1, 0)$, and let $\underline{\varepsilon}_i < 0 < \overline{\varepsilon}_i$ satisfy (C1) and (C2), and consider the discrete-time chain of integrators (1) with u_k given by the output-feedback controller*

$$u_k = -d + \sum_{i=1}^{n} \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}), \tag{62}$$

where $\underline{\alpha}_{n,k} \triangleq \underline{\varepsilon}_n$, $\overline{\alpha}_{n,k} \triangleq \overline{\varepsilon}_n$, *for all* $k \geq 0$ *and* $i \in \{n, n-1, \ldots, 2\}$,

$$\underline{\alpha}_{i-1,k} \triangleq \underline{\varepsilon}_{i-1} + \begin{cases} \underline{\alpha}_{i,k} - \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}), & 0 > \lambda_i z_{i,k} \geq \underline{\varepsilon}_i, \\ 0, & \text{otherwise}, \end{cases} \tag{63}$$

$$\overline{\alpha}_{i-1,k} \triangleq \overline{\varepsilon}_{i-1} + \begin{cases} \overline{\alpha}_{i,k} - \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}), & 0 < \lambda_i z_{i,k} \leq \overline{\varepsilon}_i, \\ 0, & \text{otherwise}, \end{cases} \tag{64}$$

and

$$z_k = \begin{bmatrix} z_{1,k} z_{2,k} & \cdots & z_{n,k} \end{bmatrix}^{\mathrm{T}} = S_{z\eta}\eta_k, \tag{65}$$

where, for all $k \geq n-1$, *the state* η_k *is given by*

$$\eta_k \triangleq \begin{bmatrix} \eta_{1,k} & \eta_{2,k} & \cdots & \eta_{n,k} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n, \tag{66}$$

where

$$\eta_{1,k} \triangleq -e_k, \tag{67}$$

and, for all $i \in \{2, \ldots, n\}$,

$$\eta_{i,k} \triangleq \sum_{j=1}^{n-i+1} \beta_{n-i-j+1,n} e_{k-j} + \sum_{j=1}^{n-i+1} a_{n-i-j+1}[\sigma_{u_{\min},u_{\max}}(u_{k-j}) + d]. \tag{68}$$

Then, the following statements hold:

(i) *For all* $k \geq 0$, $u_k \in [u_{\min}, u_{\max}]$.
(ii) *The zero solution of the closed-loop system consisting of* (59)–(68) *is Lyapunov stable.*
(iii) *For all* $i \in \{1, \ldots, n-1\}$, $\lim_{k\to\infty} \xi_{k,i} = 0$.
(iv) $\lim_{k\to\infty} y_k = \lim_{k\to\infty} \xi_{k,n}/\mu_0 = r$.

9. Numerical examples

Example 1. Consider the continuous-time chain of integrators

$$G_c(s) = \frac{1}{s^n}. \tag{69}$$

For all $t \geq 0$, let $u(t)$ and $y(t)$ denote the input and output of G_c, respectively. Let $T_s > 0$ be the sample time and, for all $k \geq 0$, define $y_k \triangleq y(kT_s)$. Moreover, for all $k \geq 1$ and $t \in [(k-1)T_s, kT_s)$, let $u(t) = u_k$. The discrete-time counterpart of G_c, which is the transfer function from u_k to y_k, is given by (2), where, for all $i \in \{0, 1, \ldots, n-1\}$,

$$a_i \triangleq \frac{T_s^n}{n!} \sum_{j=0}^{i} (-1)^j \binom{n+1}{j} (i+1-j)^n. \tag{70}$$

Note that $n!a_i/(T_s^n)$ are the coefficients of the Eulerian polynomial of order n (Weller, Moran, Ninness, & Pollington, 2001).

Let $n = 3$, $u_{\min} = -4$, $u_{\max} = 2$, and $T_s = 1$ s. For $t \in [0, 150]$ s, let $d = 1.5$, and, for $t \in (150, 300]$ s, let $d = -0.5$. The zeros of G are -0.268 and -3.732, and thus G is nonminimum phase. We use the output-feedback controller (32), (48)–(50), (61) with $u_0 = u_1 = u_2 = -1$, and

$$\lambda_1 = \lambda_2 = \lambda_3 = -0.1, \tag{71}$$

$$\underline{\varepsilon}_1 = -0.34, \quad \underline{\varepsilon}_2 = -0.35, \quad \underline{\varepsilon}_3 = -1.71, \tag{72}$$

$$\overline{\varepsilon}_1 = 0.34, \quad \overline{\varepsilon}_2 = 0.35, \quad \overline{\varepsilon}_3 = 0.71. \tag{73}$$

The command is $r = 500$, and we consider three initial conditions, namely, $x_0 = [10\ 20\ 40]^{\mathrm{T}}$, $x_0 = 0$, and $x_0 = -[10\ 20\ 40]^{\mathrm{T}}$. Fig. 1 shows y_k and u_k versus k. ◇

Example 2. We reconsider Example 1 but using the continuous-time nonminimum-phase triple integrator

$$G_c(s) = \frac{(s+1)(s-2)}{s^3}. \tag{74}$$

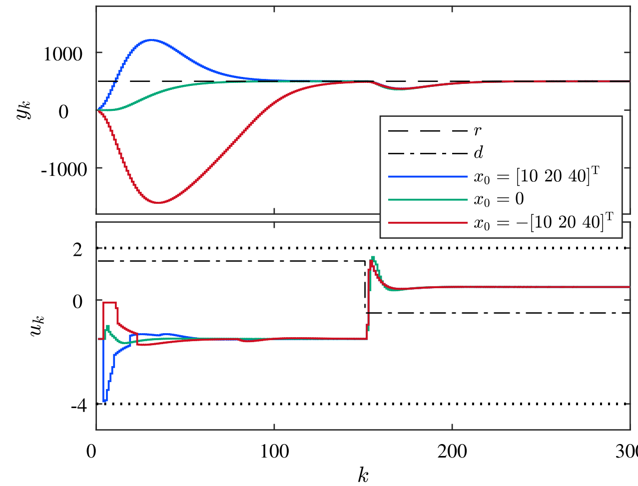

Fig. 1. Example 1. Command following and disturbance rejection for the sampled-data triple integrator with $G_c(s) = \frac{1}{s^3}$ using the output-feedback controller (32), (48)–(50), (61) for three initial conditions x_0. The dotted lines show the asymmetric saturation bounds $u_{\min}$ and $u_{\max}$. Note that, in all cases, u_k converges to $-d$.

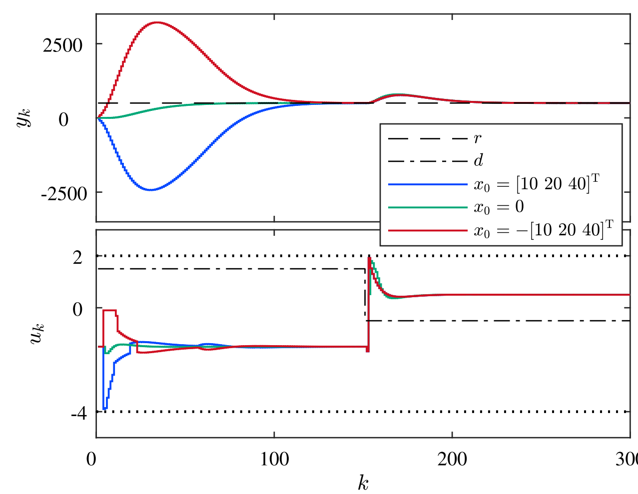

Fig. 2. Example 2. Command following and disturbance rejection for a sampled-data triple integrator with arbitrary zeros using the output-feedback controller (32), (48)–(50), (61) for three initial conditions x_0. The dotted lines show the asymmetric saturation bounds $u_{\min}$ and $u_{\max}$. Note that, in all cases, u_k converges to $-d$.

In this case, the discrete-time counterpart of G_c is given by (2), where

$$a_0 = 1/6, \quad a_1 = -20/6, \quad a_2 = 7/6. \tag{75}$$

Note that the zeros of G are 0.36 and 19.64, and thus G is nonminimum phase. Fig. 2 shows y_k and u_k versus k. ◇

10. Conclusions and future research

For a discrete-time chain of integrators with arbitrary zeros and asymmetric input saturation, this paper developed a globally asymptotically stabilizing output-feedback dynamic compensator for setpoint command following and rejection of constant disturbances. This controller was applied to two numerical examples, including sampled-data control of a chain of continuous-time integrators with zeros.

Various extensions of this work can be considered, such as the case where the command is not constant. For command following, the plant integrators provide asymptotic setpoint command following; the same principle applies to ramp commands assuming that the slope of the ramp does not violate the saturation level. The problem of unknown constant disturbance rejection is addressed by means of input estimation in Ding, and Zheng (2015). Since plant integrators do not provide disturbance rejection, it would be desirable to develop a nested-saturation globally stabilizing controller with integral action. The case where the zeros of the chain of integrators are uncertain may be amenable to robust and adaptive control techniques. Extensions to rate saturation and unknown delay are also of interest. Finally, it would be interesting to explore the use of the nonminimal state-space realization given in Taylor, Young, and Chotai (2013), which, like the minimal realization given in Polderman (1989), depends on only inputs and outputs but, unlike (Polderman, 1989), has the additional benefit that its states are independent of the plant parameters.

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Appendix A. Lemma 1

Lemma 1. *Let G be a SISO proper transfer function with nth-order minimal realization (A_1, B_1, C_1), let $A_2 \in \mathbb{R}^{n\times n}$, assume that A_1 and A_2 are similar, let $B_2 \in \mathbb{R}^n$, assume that (A_2, B_2) is controllable, and define S by*

$$S \triangleq \mathcal{K}_2\mathcal{K}_1^{-1}, \tag{76}$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are the controllability matrices of (A_1, B_1) and (A_2, B_2), respectively. Moreover, define $C_2 \triangleq C_1S^{-1}$. Then, (A_2, B_2, C_2) is a minimal realization of G, and

$$A_2 = SA_1S^{-1}, \quad B_2 = SB_1. \tag{77}$$

Proof. For all $i \in \{1, \ldots, n\}$, let e_i denote the ith column of I_n. Note that $B_1 = \mathcal{K}_1e_1$, which implies that $\mathcal{K}_1^{-1}B_1 = e_1$. Thus,

$$SB_1 = \mathcal{K}_2\mathcal{K}_1^{-1}B_1 = \mathcal{K}_2e_1 = B_2.$$

Next, note that, for all $i \in \{0, 1, \ldots, n-1\}$, $A_1^iB_1 = \mathcal{K}_1e_{i+1}$, and thus $\mathcal{K}_1^{-1}A_1^iB_1 = e_{i+1}$. Thus, for all $i \in \{0, 1, \ldots, n-1\}$,

$$SA_1^iB_1 = \mathcal{K}_2\mathcal{K}_1^{-1}A_1^iB_1 = \mathcal{K}_2e_{i+1} = A_2^iB_2, \tag{78}$$

which implies that, for all $i \in \{0, 1, \ldots, n-2\}$,

$$\begin{aligned} SA_1S^{-1}A_2^iB_2 &= SA_1A_1^iB_1 = SA_1^{i+1}B_1 \\ &= A_2^{i+1}B_2 = A_2A_2^iB_2. \end{aligned} \tag{79}$$

Now, let the characteristic polynomial χ of A_1 be written as $\chi(s) \triangleq s^n+a_{n-1}s^{n-1}+\cdots+a_0$, where, for all $i \in \{0, \ldots, n-1\}$, $a_i \in \mathbb{R}$. Since A_1 and A_2 are similar, it follows that $\chi(A_1) = \chi(A_2) = 0$, and thus

$$A_1^n = -\sum_{i=0}^{n-1} a_iA_1^i, \qquad A_2^n = -\sum_{i=0}^{n-1} a_iA_2^i.$$

Thus, using (78), it follows that

$$\begin{aligned} SA_1S^{-1}A_2^{n-1}B_2 &= SA_1A_1^{n-1}B_1 = SA_1^nB_1 \\ &= -S\sum_{i=0}^{n-1} a_iA_1^iB_1 = -\sum_{i=0}^{n-1} a_iA_2^iB_2 \\ &= A_2^nB_2 = A_2A_2^{n-1}B_2. \end{aligned} \tag{80}$$

Therefore, (79) and (80) imply that, for all $i \in \{0, 1, \ldots, n-1\}$,

$$SA_1S^{-1}A_2^iB_2 = A_2A_2^iB_2,$$

that is,

$$(SA_1S^{-1} - A_2)A_2^iB_2 = 0.$$

Therefore,

$$(SA_1S^{-1} - A_2)\mathcal{K}_2 = 0,$$

which, since $\mathcal{K}_2$ is nonsingular, implies that $SA_1S^{-1} = A_2$. Therefore, with C_2 defined by $C_2 = C_1S^{-1}$, it follows that (A_2, B_2, C_2) is a minimal realization of G. □

Appendix B. Proof of Theorem 1

Proof. To show (i), note that it follows from (48) and (49) that, for all $i \in \{n, n-1, \ldots, 2\}$ and $k \geq 0$,

$$\underline{\alpha}_{i-1,k} \geq \underline{\varepsilon}_{i-1} + \underline{\alpha}_{i,k} - \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_iz_{i,k}),$$
$$\overline{\alpha}_{i-1,k} \leq \overline{\varepsilon}_{i-1} + \overline{\alpha}_{i,k} - \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_iz_{i,k}),$$

which imply

$$\begin{aligned} \underline{\varepsilon}_{i-1} + \underline{\alpha}_{i,k} - \underline{\alpha}_{i-1,k} &\leq \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_iz_{i,k}) \\ &\leq \overline{\varepsilon}_{i-1} + \overline{\alpha}_{i,k} - \overline{\alpha}_{i-1,k}. \end{aligned} \tag{81}$$

Using (81), it follows from (50) that

$$\begin{aligned} u_k &= -d + \sigma_{\underline{\alpha}_{1,k},\overline{\alpha}_{1,k}}(\lambda_1z_{1,k}) + \sum_{i=2}^n \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_iz_{i,k}) \\ &\geq -d + \underline{\alpha}_{1,k} + \sum_{i=2}^n \underline{\varepsilon}_{i-1} + \underline{\alpha}_{i,k} - \underline{\alpha}_{i-1,k} \\ &\geq -d + \underline{\alpha}_{1,k} + \underline{\alpha}_{n,k} - \underline{\alpha}_{1,k} + \sum_{i=2}^n \underline{\varepsilon}_{i-1} \\ &= -d + \underline{\alpha}_{n,k} + \sum_{i=2}^n \underline{\varepsilon}_{i-1} \\ &= -d + \sum_{i=1}^n \underline{\varepsilon}_i \\ &\geq -d + \overline{d} + u_{\min} \\ &\geq u_{\min}, \end{aligned}$$

and

$$\begin{aligned} u_k &= -d + \sigma_{\underline{\alpha}_{1,k},\overline{\alpha}_{1,k}}(\lambda_1z_{1,k}) + \sum_{i=2}^n \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_iz_{i,k}) \\ &\leq -d + \overline{\alpha}_{1,k} + \sum_{i=2}^n \overline{\varepsilon}_{i-1} + \overline{\alpha}_{i,k} - \overline{\alpha}_{i-1,k} \\ &\leq -d + \overline{\alpha}_{1,k} + \overline{\alpha}_{n,k} - \overline{\alpha}_{1,k} + \sum_{i=2}^n \overline{\varepsilon}_{i-1} \\ &= -d + \overline{\alpha}_{n,k} + \sum_{i=2}^n \overline{\varepsilon}_{i-1} \end{aligned}$$

$$= -d + \sum_{i=1}^{n} \overline{\varepsilon}_i$$
$$\leq -d + \underline{d} + u_{\max}$$
$$\leq u_{\max},$$

which confirms (i).

To show (ii), note that using (50) and (50), it follows from (40) that

$$z_{n,k+1} = z_{n,k} + \sigma_{u_{\min},u_{\max}}(u_k) + d$$
$$= z_{n,k} + \sigma_{\underline{\alpha}_{n,k},\overline{\alpha}_{n,k}}(\lambda_n z_{n,k}) + \sum_{i=1}^{n-1} \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}). \quad (82)$$

Moreover, it follows from (81) that

$$\sum_{i=1}^{n-1} \underline{\varepsilon}_i \leq \sum_{i=1}^{n-1} \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}) \leq \sum_{i=1}^{n-1} \overline{\varepsilon}_i. \quad (83)$$

Furthermore, note that (44) and (45) imply that

$$\underline{\alpha}_{n,k} < \min\left\{-\sum_{i=1}^{n-1} \overline{\varepsilon}_i, \sum_{i=1}^{n-1} \underline{\varepsilon}_i\right\}, \quad (84)$$

$$\overline{\alpha}_{n,k} > \max\left\{-\sum_{i=1}^{n-1} \underline{\varepsilon}_i, \sum_{i=1}^{n-1} \overline{\varepsilon}_i\right\}. \quad (85)$$

Thus, using (83)–(85) and Lemma 2 in Appendix B, it follows from (82) that there exists $k_n \geq 0$ such that, for all $k \geq k_n$,

$$z_{n,k+1} = (1+\lambda_n) z_{n,k} + \sum_{i=1}^{n-1} \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}),$$

and

$$u_k = \lambda_n z_{n,k} + \sum_{i=1}^{n-1} \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}). \quad (86)$$

Next, substituting (86) into (40) yields that, for all $k \geq k_n$,

$$z_{k+1} = \begin{bmatrix} 1 & -\lambda_2 & -\lambda_3 & \cdots & -\lambda_{n-1} & 0 \\ 0 & 1 & -\lambda_3 & \cdots & -\lambda_{n-1} & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1+\lambda_n \end{bmatrix} z_k$$
$$+ B_z \sum_{i=1}^{n-1} \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}). \quad (87)$$

Note that (87) implies that, for all $k \geq k_n$,

$$z_{n-1,k+1} = z_{n-1,k} + \sum_{i=1}^{n-1} \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k})$$
$$= z_{n-1,k} + \sigma_{\underline{\alpha}_{n-1,k},\overline{\alpha}_{n-1,k}}(\lambda_{n-1} z_{n-1,k})$$
$$+ \sum_{i=1}^{n-2} \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}). \quad (88)$$

Moreover, it follows from (81) that

$$\sum_{i=1}^{n-2} \underline{\varepsilon}_i \leq \sum_{i=1}^{n-2} \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}) \leq \sum_{i=1}^{n-2} \overline{\varepsilon}_i. \quad (89)$$

Furthermore, note that (44) and (48) imply that

$$\underline{\alpha}_{n-1,k} \leq \underline{\varepsilon}_{n-1,k} < \min\left\{-\sum_{i=1}^{n-2} \overline{\varepsilon}_i, \sum_{i=1}^{n-2} \underline{\varepsilon}_i\right\}, \quad (90)$$

and (45) and (49) imply that

$$\overline{\alpha}_{n-1,k} \geq \overline{\varepsilon}_{n-1} > \max\left\{-\sum_{i=1}^{n-2} \underline{\varepsilon}_i, \sum_{i=1}^{n-2} \overline{\varepsilon}_i\right\}. \quad (91)$$

Thus, using (89)–(91) and Lemma 2 in Appendix C, it follows from (88) that there exists $k_{n-1} \geq k_n \geq 0$ such that, for all $k \geq k_{n-1}$,

$$z_{n-1,k+1} = (1+\lambda_{n-1}) z_{n-1,k} + \sum_{i=1}^{n-2} \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}),$$

and

$$u_k = \sum_{i=n-1}^{n} \lambda_i z_{i,k} + \sum_{i=1}^{n-2} \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}). \quad (92)$$

Substituting (92) into (40) implies that, for all $k > k_{n-1}$,

$$z_{k+1} = \begin{bmatrix} 1 & -\lambda_2 & \cdots & -\lambda_{n-2} & 0 & 0 \\ 0 & 1 & \cdots & -\lambda_{n-2} & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1+\lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_{n-1} & 1+\lambda_n \end{bmatrix} z_k$$
$$+ B_z \sum_{i=1}^{n-2} \sigma_{\underline{\alpha}_{i,k},\overline{\alpha}_{i,k}}(\lambda_i z_{i,k}). \quad (93)$$

Repeating the same process as in (88)–(93) implies that there exist $k_1 \geq k_2 \geq \cdots \geq k_{n-1} \geq k_n$ such that, for all $k \geq k_1$,

$$z_{k+1} = \begin{bmatrix} 1+\lambda_1 & 0 & \cdots & 0 & 0 \\ \lambda_1 & 1+\lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_1 & \lambda_2 & \cdots & 0 & 0 \\ \lambda_1 & \lambda_2 & \cdots & 1+\lambda_{n-1} & 0 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{n-1} & 1+\lambda_n \end{bmatrix} z_k. \quad (94)$$

Thus, since for all $i \in \{1, \ldots, n\}$, $\lambda_i \in (-1, 0)$, it follows that $z_k \equiv 0$ is an asymptotically stable equilibrium of (94), which confirms (ii). Finally, (42) and (ii) imply (iii) and (iv). □

Appendix C. Lemma used in proof of Theorem 1

Lemma 2. *Consider the scalar system*

$$s_{k+1} = s_k + u_k, \quad (95)$$

where, for all $k \geq 0$, $s_k \in \mathbb{R}$, $u_k \in \mathbb{R}$, and $s_0 \in \mathbb{R}$ is the initial condition. Let $\lambda \in (-1, 0)$, $u_{\min} < 0 < u_{\max}$, define $u_{\rm m} \triangleq \min\{u_{\max}, -u_{\min}\}$, and let $\underline{v}, \overline{v} \in \mathbb{R}$ satisfy

$$-u_{\rm m} < \underline{v} < 0 < \overline{v} < u_{\rm m}. \quad (96)$$

For all $k \geq 0$, let u_k be given by

$$u_k = \sigma_{u_{\min},u_{\max}}(\lambda s_k) + v_k, \quad (97)$$

where

$$\underline{v} < v_k < \overline{v}. \quad (98)$$

Then, the following statements hold:

(i) *For all $k \geq 0$, $u_{\min} + \underline{v} \leq u_k \leq u_{\max} + \overline{v}$.*
(ii) *There exists $k_0 \geq 0$ such that, for all $k \geq k_0$,*

$$\lambda s_k \in [u_{\min}, u_{\max}], \quad (99)$$

and thus

$$u_k = \lambda s_k + v_k. \quad (100)$$

Proof. Note that (i) follows from (97) and (98). To show (ii), define $V_k \triangleq s_k^2$. Evaluating $V_{k+1} - V_k$ along the trajectories of (95) and using (97) yields

$$\begin{aligned} V_{k+1} - V_k &= (s_k + u_k)^2 - s_k^2 = u_k(2s_k + u_k) \\ &= (\sigma_{u_{\min},u_{\max}}(\lambda s_k) + v_k)(2s_k + \sigma_{u_{\min},u_{\max}}(\lambda s_k) + v_k). \end{aligned} \tag{101}$$

First, consider the case where $k \in \{\kappa \geq 0 : \lambda s_\kappa < u_{\min}\}$, which implies that $\sigma_{u_{\min},u_{\max}}(\lambda s_k) = u_{\min}$. Thus, since $u_{\min} < \min\{-\overline{v}, \underline{v}\}$ and $\lambda \in (-1, 0)$, it follows from (98) and (101) that, for all $k \in \{\kappa \geq 0 : \lambda s_\kappa < u_{\min}\}$,

$$\begin{aligned} V_{k+1} - V_k &\leq (u_{\min} + \overline{v})(-2u_{\min} + u_{\min} + \underline{v}) \\ &= -(u_{\min} + \overline{v})(u_{\min} - \underline{v}). \end{aligned} \tag{102}$$

Next, consider the case where $k \in \{\kappa \geq 0 : \lambda s_\kappa > u_{\max}\}$, which implies that $\sigma_{u_{\min},u_{\max}}(\lambda s_k) = u_{\max}$. Thus, since $u_{\max} > \max\{-\underline{v}, \overline{v}\}$ and $\lambda \in (-1, 0)$, it follows from (98) and (101) that, for all $k \in \{\kappa \geq 0 \colon \lambda s_\kappa > u_{\max}\}$,

$$\begin{aligned} V_{k+1} - V_k &\leq (u_{\max} + \underline{v})(-2u_{\max} + u_{\max} + \overline{v}) \\ &= -(u_{\max} + \underline{v})(u_{\max} - \overline{v}). \end{aligned} \tag{103}$$

Therefore, (102) and (103) imply that, for all nonnegative $k \notin \{\kappa \geq 0 : u_{\min} \leq \lambda s_\kappa \leq u_{\max}\}$,

$$V_{k+1} - V_k < -c_0, \tag{104}$$

where $c_0 \triangleq \min\{(u_{\max} + \underline{v})(u_{\max} - \overline{v}), (u_{\min} + \overline{v})(u_{\min} - \underline{v})\}$. It follows from (96) that $c_0 > 0$.

Next, suppose for contradiction that $\{\kappa \geq 0 : u_{\min} \leq \lambda s_\kappa \leq u_{\max}\}$ is empty. It thus follows from (104) that, for all $k > 0$,

$$V_k - V_0 = \sum_{i=0}^{k-1} (V_{i+1} - V_i) < -kc_0,$$

which implies that

$$0 \leq \lim_{k\to\infty} V_k \leq V_0 - \lim_{k\to\infty} kc_0 = -\infty,$$

which is a contradiction. Therefore, $\{\kappa \geq 0 : u_{\min} \leq \lambda s_\kappa \leq u_{\max}\}$ is nonempty, and thus has a smallest nonnegative element k_0.

Next, since $\lambda s_{k_0} \in [u_{\min}, u_{\max}]$, it follows from (95) and (97) that

$$s_{k_0+1} = s_{k_0}(1 + \lambda) + v_k. \tag{105}$$

Using (96), $\lambda \in (-1, 0)$, and $s_{k_0} \in [u_{\max}/\lambda, u_{\min}/\lambda]$, it follows from (105) that

$$\begin{aligned} \frac{u_{\max}}{\lambda} &< \frac{u_{\max}}{\lambda} + u_{\max} + \underline{v} \leq s_{k_0+1} \\ &\leq \frac{u_{\min}}{\lambda} + u_{\min} + \overline{v} < \frac{u_{\min}}{\lambda}. \end{aligned}$$

Thus, $s_{k_0} \in [u_{\max}/\lambda, u_{\min}/\lambda]$ implies that $s_{k_0+1} \in [u_{\max}/\lambda, u_{\min}/\lambda]$. By induction, it follows that $[u_{\max}/\lambda, u_{\min}/\lambda]$ is an invariant set for (95), that is, for all $k \geq k_0$, $\lambda s_k \in [u_{\min}, u_{\max}]$. Therefore, (97) implies that, for all $k \geq k_0$, $u_k = \lambda s_k + v_k$, which confirms (ii). □

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